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A NEW PARALLEL FINITE ELEMENT ALGORITHM BASED ON TWO-GRID DISCRETIZATION FOR THE GENERALIZED STOKES PROBLEM

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Abstract. Based on two-grid discretization, a new parallel finite element algorithm for the generalized Stokes problem is proposed and analyzed. Motivated by the observation that for a solution to the generalized Stokes problem, low frequency components can be approximated well by a relatively coarse grid and high frequency components can be computed on a fine grid, this algorithm first solves the generalized Stokes problem on a coarse grid, and then corrects the resulted residual by standard additive Schwarz method on a fine grid. Under some regular assumptions, error estimates of the approximate solutions are provided. Numerical results are also given to illustrate the effectiveness of the algorithm.

Key words. Generalized Stokes problem, finite element, parallel algorithm, Schwarz method, two-grid method.

1. Introduction

The generalized Stokes problem arises naturally in the time discretization of non-stationary Navier-Stokes equations which mathematically model the flow motion of an incompressible Newtonian viscous fluid. It consists of the key and most time-consuming part of the solving process of time-dependent Navier-Stokes equations at each nonlinear iteration. Therefore, the development of efficient algorithms for the generalized Stokes problem is very important.

Recently, based on local finite element discretizations, an approach to local and parallel finite element computations is proposed for a class of linear and nonlinear elliptic boundary value problems in [28–30]. Based on this approach, some new local and parallel algorithms have been proposed and analyzed for the steady Stokes equations [11, 20], the stationary Navier-Stokes equations [9, 10, 14, 21], the stream function form of Navier-Stokes equations [15], and the transient Stokes equations [22]. These algorithms have low communication complexity. They only require existing sequential solver as subproblems solver and hence allow existing sequential PDE codes to run in a parallel environment with a little investment in recoding.

However, based on our analysis and numerical tests, there is still room to improve some of the above mentioned algorithms for some incompressible flow problems. First and foremost, although the coarse grid size is suitably chosen, the finite element approximations obtained from the algorithms are much less precise for some problems compared with the global standard Galerkin finite element solution, especially, when the overlapping size of subdomains is small. Secondly, the accuracy

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of the finite element approximations not only depends on the overlapping size of subdomains, but also depends on the shapes of subdomains and hence depends on the way to decomposing the solution domain into subdomains. Finally, the approximate solutions are piecewise defined and hence are generally discontinuous, making the algorithms not applicable for these problems that continuity of the solutions is required.

The motivation of this work is to overcome the above mentioned weakness of the local and parallel algorithms and present a new improved algorithm for the generalized Stokes problem. Based on our understanding of the local and global properties of a finite element solution to the generalized Stokes problem, i.e., the global behavior of a solution to the generalized Stokes problem is mostly governed by low frequency components while local behavior is mostly governed by high frequency components, we first approximate the low frequency components of the solution on a coarse grid, then use a standard additive Schwarz method on a fine grid to correct the resulted residual (which contains mostly high frequencies). This new algorithm is an improvement of the parallel algorithm proposed in [11] for the steady Stokes problem in the sense that continuous and more precise solutions can be obtained; see Section 3.

It is noted that unlike the standard multigrid and domain decomposition methods where the two-grid method is used to devise iterative methods for solving a given discretization scheme (see, e.g., Bank [4], Hackbusch [8], Simth, Bjørstad and Gropp [23] , Quarteroni and Valli [18], Toselli and Widlund [25]), our algorithm is to design a discretization scheme. Moreover, in our algorithm, the global coarse grid problem needs to be solved only once and it does not have to be coupled with the subsequent parallel solvers.

The rest of this paper is organized as follows. In the next section, the generalized Stokes problem and its mixed finite element approximations are provided. In Section 3, a parallel algorithm based on local finite element computations proposed in [11] is reviewed. Analysis of improvement for this parallel algorithm is performed and a new improved algorithm is devised and analyzed. In Section 4, two numerical tests are carried out to illustrate the effectiveness of the new algorithm. Finally, conclusions are drawn in Section 5.

2. The generalized Stokes problem and its mixed finite element approximations

Let Ω be a bounded domain with Lipschitz-continuous boundary $\partial\Omega$ in R^d ($d = 2, 3$). We shall use the standard notations for Sobolev spaces $W^{s,p}(\Omega)$, $W^{s,p}(\Omega)^d$ and their associated norms and seminorms; see, e.g., [1, 5]. For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega), H^s(\Omega)^d = W^{s,2}(\Omega)^d$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega} = 0$ is in the sense of trace, $\|\cdot\|_{s,\Omega} = \|\cdot\|_{s,2,\Omega}$. For a subdomain $\Omega_0 \subset \Omega$, we view $H_0^1(\Omega_0)$ as a subspace of $H_0^1(\Omega)$ by extending the functions in $H_0^1(\Omega_0)$ to be functions in $H_0^1(\Omega)$ with zero outside of Ω_0.

We consider the following generalized Stokes problem

$$\alpha u - \nu\Delta u + \nabla p = f \quad \text{in } \Omega, \tag{1}$$

$$\text{div } u = 0 \quad \text{in } \Omega, \tag{2}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where $u = (u_1, \cdots, u_d)$ is the velocity, p the pressure, $f = (f_1, \cdots, f_d)$ the prescribed body force, ν the kinematic viscosity and α a positive parameter proportional to the inverse of time-step size.

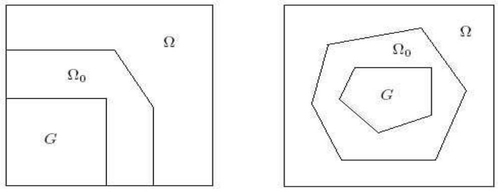


FIGURE 1. Subdomains.

The weak form of (1)-(3) reads: find a pair $(u,p)\in H_0^1(\Omega)^d\times L_0^2(\Omega)$ such that

$$a(u,v)-b(v,p)=(f,v),\quad \forall v\in H_0^1(\Omega)^d, \tag{4}$$

$$b(u,q)=0,\quad \forall q\in L_0^2(\Omega), \tag{5}$$

where

$$L_0^2(\Omega)=\{q\in L^2(\Omega):\int_\Omega q dx=0\},$$

$(\cdot,\cdot)$ is the standard inner-product of $L^2(\Omega)^l$ $(l=1,2,3)$ and $a(\cdot,\cdot),\ b(\cdot,\cdot)$ are defined as

$$a(u,v)=(\alpha u,v)+\nu(\nabla u,\nabla v),\ b(v,q)=(\mathrm{div} v,q),\quad \forall u,\ v\in H_0^1(\Omega)^d, q\in L_0^2(\Omega).$$

As for the existence, uniqueness, and regularity of a global strong solution to the generalized Stokes problem, we have the following well-known result [17,24].

Lemma 1. *Let Ω be a C^{t+1}-smooth bounded domain in R^d for $t\ge 1$ or a bounded convex polygonal or polyhedral domain in R^d for $t=1$. Then for any given function $f\in H^{t-1}(\Omega)^d$, there exists a uniquely determined functions pair*

$$(u,p)\in(H^{t+1}(\Omega)\cap H_0^1(\Omega))^d\times(H^t(\Omega)\cap L_0^2(\Omega))$$

satisfying (4)-(5) *and*

$$\|u\|_{t+1,\Omega}+\|p\|_{t,\Omega}\le c\|f\|_{t-1,\Omega}. \tag{6}$$

Here and hereafter, c is a generic positive constant which is independent of mesh parameter and may stand for different values at its different occurrences.

To describe the mixed finite element approximations of (4)-(5), let us assume $T^h(\Omega)=\{K\}$ be a regular triangulation (see, e.g., [5,7]) of Ω into triangles or quadrilaterals (when $d=2$), or tetrahedrons or hexahedrons (when $d=3$) with mesh size $h(x)$ whose value is the diameter of the element K containing x, $X_h(\Omega)\subset H^1(\Omega)^d, M_h(\Omega)\subset L^2(\Omega)$ be two finite element subspaces associated with the mesh $T^h(\Omega)$ and

$$X_h^0(\Omega)=X_h(\Omega)\cap H_0^1(\Omega)^d,\ M_h^0(\Omega)=M_h(\Omega)\cap L_0^2(\Omega).$$

Given $G\subset\subset\Omega_0\subset\Omega$ (here $G\subset\subset\Omega_0$ means that $\mathrm{dist}(\partial G\backslash\partial\Omega,\partial\Omega_0\backslash\partial\Omega)>0$; see Figure 1), we define $X_h(G),\ M_h(G)$, and $T^h(G)$ to be the restriction of $X_h(\Omega),\ M_h(\Omega)$ and $T^h(\Omega)$ to G, respectively, and

$$X_0^h(G)=\{v\in X_h(\Omega):\mathrm{supp}\ v\subset\subset G\},\ M_0^h(G)=\{q\in M_h(\Omega):\mathrm{supp}\ q\subset\subset G\}.$$

Some basic assumptions on the mesh and mixed finite element spaces are needed, namely (cf. [3,11,16,19,26,28–30])

*A*0. **Triangulation.** There exists $\theta \geq 1$ such that

$$h_{\Omega}^{\theta} \leq ch(x), \ \forall x \in \Omega, \tag{7}$$

where $h_{\Omega} = \max_{x \in \Omega} h(x)$ is the largest mesh size of $T^h(\Omega)$. Sometimes, we shall drop the subscript in h_{Ω} and use h for the mesh size on a domain that is clear from the context.

*A*1. **Approximation.** For each $(u, p) \in H^{t+1}(G)^d \times H^t(G)(t \geq 1)$, there exists an approximation $(\pi_h u, \rho_h p) \in X_h(G) \times M_h(G)$ such that

$$\|h^{-1}(u - \pi_h u)\|_{0,G} + \|u - \pi_h u\|_{1,G} \leq ch_G^s \|u\|_{1+s,G}, \ 0 \leq s \leq t, \tag{8}$$

$$\|h^{-1}(p - \rho_h p)\|_{-1,G} + \|p - \rho_h p\|_{0,G} \leq ch_G^s \|p\|_{s,G}, \ 0 \leq s \leq t. \tag{9}$$

*A*2. **Inverse estimate.** For any $(v, q) \in X_h(G) \times M_h(G)$, there hold

$$\|v\|_{1,G} \leq c\|h^{-1}v\|_{0,G}, \ \|q\|_{0,G} \leq c\|h^{-1}q\|_{-1,G}. \tag{10}$$

*A*3. **Superapproximation.** For $G \subset \Omega$, let $\omega \in C_0^{\infty}(\Omega)$ with supp $\omega \subset\subset G$. Then for any $(u, p) \in X_h(G) \times M_h(G)$, there is $(v, q) \in X_0^h(G) \times M_0^h(G)$ such that

$$\|h^{-1}(\omega u - v)\|_{1,G} \leq c\|u\|_{1,G}, \ \|h^{-1}(\omega p - q)\|_{0,G} \leq c\|p\|_{0,G}. \tag{11}$$

*A*4. **Stability.** There exists a constant $\beta > 0$ such that

$$\beta\|q\|_{0,G} \leq \sup_{v \in X_h^0(G), v \neq 0} \frac{(\text{div } v, q)}{\|v\|_{1,G}}, \quad \forall q \in M_h^0(G). \tag{12}$$

With the above notations, the mixed finite element approximation of problem (4)-(5) reads: find a solution pair $(u_h, p_h) \in X_h^0(\Omega) \times M_h^0(\Omega)$ such that

$$a(u_h, v) - b(v, p_h) = (f, v), \quad \forall v \in X_h^0(\Omega), \tag{13}$$

$$b(u_h, q) = 0, \quad \forall q \in M_h^0(\Omega). \tag{14}$$

It is well known that under the conditions of Lemma 1, if $f \in H^{t-1}(\Omega)^d$, the finite element solution pair (u_h, p_h) of problem (13)-(14) have the following error estimate (cf. [2, 7])

$$\|u - u_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq ch^s(\|u\|_{s+1,\Omega} + \|p\|_{s,\Omega}), \quad 1 \leq s \leq t. \tag{15}$$

3. Parallel finite element algorithms

In this section, we shall firstly recall a parallel algorithm based on local finite element computations proposed in [11] for the steady Stokes problem and extend it to the generalized Stokes problem, then give analysis of improvement and introduce our new parallel algorithm.

Let us firstly divide Ω into a number of disjoint subdomains $D_1, \cdots, D_m$, then enlarge each D_j to obtain Ω_j such that $D_j \subset\subset \Omega_j \subset \Omega$; see Figure 2. These Ω_j $(j = 1, 2, \cdots, m)$ compose an overlapping decomposition of Ω. Assume $T^H(\Omega)$ to be a shape-regular coarse grid of size $H \gg h$, $T^h(\Omega_j)$ be a local shape-regular fine grid on subdomain Ω_j and $T^h(\Omega)$ be a globally fine grid which coincides with the local fine grid in subdomain Ω_j; see Figure 3. We set $\Gamma_j = \partial\Omega_j \backslash \partial\Omega$ and $M_h^{\Gamma_j}(\Omega_j) = \{q \in M_h(\Omega_j) : q|_{\Gamma_j} = 0\}$. Assume that $X_H^0(\Omega), M_H^0(\Omega)$ are two finite element subspaces associate with the coarse mesh $T^H(\Omega)$, and $X_h^0(\Omega_j), M_h^0(\Omega_j)$ are finite element subspaces associate with the meshes $T^h(\Omega_j)$ $(j = 1, 2, \cdots, m)$.

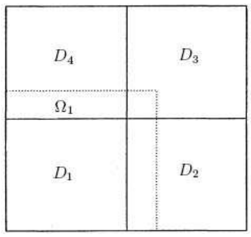


FIGURE 2. A decomposition of Ω into subdomains.

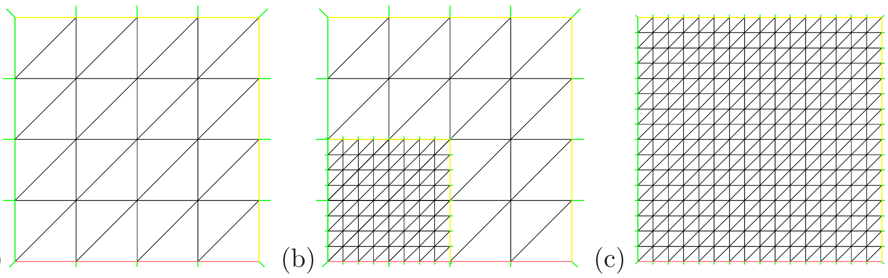


FIGURE 3. Meshes: (a) a global coarse mesh; (b) a locally refined mesh; and (c) a global fine mesh.

3.1. A parallel finite element algorithm. The parallel algorithm proposed in [11] applied to the generalized Stokes problem reads:

Algorithm 1. Parallel finite element algorithm.

1. Find a global coarse grid solution pair $(u_H, p_H) \in X_H^0(\Omega) \times M_H^0(\Omega)$ such that

$$a(u_H, v) - b(v, p_H) = (f, v), \quad \forall v \in X_H^0(\Omega),$$
$$b(u_H, q) = 0, \quad \forall q \in M_H^0(\Omega).$$

2. Find local fine grid corrections $(\gamma_{h,j}, \eta_{h,j}) \in X_h^0(\Omega_j) \times M_h^0(\Omega_j)$ $(j = 1, 2, \cdots, m)$ in parallel:

$$a(\gamma_{h,j}, v) - b(v, \eta_{h,j}) = (f, v) - a(u_H, v) + b(v, p_H), \quad \forall v \in X_h^0(\Omega_j),$$
$$b(\gamma_{h,j}, q) = -b(u_H, q), \quad \forall q \in M_h^0(\Omega_j).$$

3. Set $(u^h, p^h) = (u_H, p_H) + (\gamma_{h,j}, \eta_{h,j})$ in D_j $(j = 1, 2, \cdots, m)$.

Define piecewise norms

$$|||u - u^h|||_{1,\Omega} = \left(\sum_{j=1}^{m} \|u - u^h\|_{1,D_j}^2 \right)^{1/2},$$
$$|||p - p^h|||_{0,\Omega} = \left(\sum_{j=1}^{m} \|p - p^h\|_{0,D_j}^2 \right)^{1/2}.$$

A simple modification to the arguments for Theorem 4.3 in [11] leads to the following error estimate.

Theorem 1. *Assume that $D_j \subset\subset \Omega_j \subset \Omega$ $(j = 1, 2, \cdots, m)$, Assumptions A0-A4 and Lemma 1 hold. Then the approximate solution pair (u^h, p^h) obtained from Algorithm 1 satisfies*

(16)
$$|||u - u^h|||_{1,\Omega} + |||p - p^h|||_{0,\Omega} \le c(h^s + H^{s+1})(\|u\|_{s+1,\Omega} + \|p\|_{s,\Omega}), \quad 1 \le s \le t.$$

Theorem 1 shows that if the proportion of coarse mesh size H to fine mesh size h is suitably chosen, Algorithm 1 can yield a convergence rate of the same order as the usual global standard Galerkin finite element method and may obtain asymptotically optimal errors. For example, if $(u, p) \in (H^3(\Omega) \cap H_0^1(\Omega))^d \times (H^2(\Omega) \cap L_0^2(\Omega))$ for $t = 2$ and a second order finite element discretization is employed for the velocity both on coarse and fine grids, this estimate means that an asymptotically optimal error can be obtained by taking $H = O(h^{2/3})$.

However, detailed analysis and numerical tests show that there is still room to improve the above algorithm. To begin with, let us consider Step 2 of Algorithm 1. It is the mixed finite element approximation of the following local generalized Stokes problem defined in $\Omega_j (j = 1, 2, \cdots, m)$:

$$\alpha\gamma_j - \nu\Delta\gamma_j + \nabla\eta_j = f - \alpha u_H + \nu\Delta u_H - \nabla p_H \quad \text{in } \Omega_j, \tag{17}$$
$$\text{div } \gamma_j = -\text{div } u_H \quad \text{in } \Omega_j, \tag{18}$$
$$\gamma_j = 0 \quad \text{on } \partial\Omega_j. \tag{19}$$

Due to the coarse grid approximation u_H generally does not satisfy $\int_{\Omega_j} \text{div} u_H dx = 0$ $(j = 1, 2, \cdots, m)$, we can seen from (17)-(19) that simply setting $\gamma_j = 0$ on the boundary $\partial\Omega_j$ leads to the invalidity of an important compatibility condition for the local generalized Stoke problem, i.e.,

$$-\int_{\Omega_j} \text{div } u_H dx = \int_{\partial\Omega_j} \gamma_j \cdot n_j ds,$$

where n_j denotes the unit outward normal to $\partial\Omega_j$.

In fact, if we were able to substitute the exact solution (u, p) of (1)-(3) into Algorithm 1, we would find that the local generalized Stokes problem in $\Omega_j (j = 1, 2, \cdots, m)$ should be given by

$$\alpha\gamma_j - \nu\Delta\gamma_j + \nabla\eta_j = f - \alpha u_H + \nu\Delta u_H - \nabla p_H \quad \text{in } \Omega_j, \tag{20}$$
$$\text{div } \gamma_j = -\text{div } u_H \quad \text{in } \Omega_j, \tag{21}$$
$$\gamma_j = u - u_H \quad \text{on } \partial\Omega_j \backslash \partial\Omega, \tag{22}$$
$$\gamma_j = 0 \quad \text{on } \partial\Omega \cap \partial\Omega_j. \tag{23}$$

Therefore, the accuracy of the approximate solution (u^h, p^h) obtained from Algorithm 1 largely depends on the precision of u_H on the artificial boundary $\partial\Omega_j \backslash \partial\Omega$, or equivalently, the values of $\int_{\Omega_j} \text{div } u_H dx$ $(j = 1, 2, \cdots, m)$; a good approximation u_H of u on $\partial\Omega_j \backslash \partial\Omega$ or small enough $\int_{\Omega_j} \text{div } u_H dx$ may yield a satisfactory approximate solution, while a poor approximation u_H on $\partial\Omega_j \backslash \partial\Omega$ will result in an unacceptable solution. This not only depends on the coarse grid size H, but also depends on the shapes and sizes of subdomains $\Omega_j (j = 1, 2, \cdots, m)$ and hence depends on the decomposition of Ω into subdomains Ω_j $(j = 1, 2, \cdots, m)$ (both the way to decomposition and the overlapping size of subdomains). Our numerical tests justified this observation. Moreover, from Algorithm 1 we can see that the approximate solution is piecewise defined and hence is generally discontinuous leading to that Algorithm 1 is unapplicable for these problems that a continuous solution is required.

3.2. New parallel finite element algorithm. Our new algorithm are motivated by the above observation. We just modify the boundary conditions on the artificial boundary $\Gamma_j = \partial\Omega_j \backslash \partial\Omega$ $(j = 1, 2, \cdots, m)$ of the local generalized Stokes problems at Step 2 of Algorithm 1, so as to better approximate the local fine grid corrections $(\gamma_{h,j}, \eta_{h,j})$ $(j = 1, 2, \cdots, m)$. However, as we do not know the exact solution u, we cannot obtain the boundary condition (22); we instead employ a standard additive Schwarz method to approximate the fine grid corrections $(\gamma_{h,j}, \eta_{h,j})$ $(j = 1, 2, \cdots, m)$. This leads to our new algorithm for the generalized Stokes problem, i.e., we first approximate the low frequency components of the finite element solution to the generalized Stokes equations on a coarse grid , then use a standard additive Schwarz method on a fine grid to correct the resulted residual. After the Schwarz sequence converges, we use these converged corrections $(\gamma_{h,j}, \eta_{h,j})$ $(j = 1, 2, \cdots, m)$ to update the solution in disjoint subdomains $D_1, \cdots, D_m$. We note that if the Schwarz sequence converges, both the continuity and higher accuracy of the finite element approximation (u^h, p^h) are achieved. The convergence of the overlapping Schwarz methods for the Stokes problem (see, e.g., [12, 13, 27]) guarantees the effectiveness of our algorithm. The new algorithm is given as follows.

Algorithm 2. Parallel Schwarz correction algorithm.

1. Find a global coarse grid solution $(u_H, p_H) \in X_H^0(\Omega) \times M_H^0(\Omega)$ such that

$$a(u_H, v) - b(v, p_H) = (f, v), \quad \forall v \in X_H^0(\Omega),$$
$$b(u_H, q) = 0, \quad \forall q \in M_H^0(\Omega).$$

2. Find fine grid correction $(\gamma_h, \eta_h) \in X_h^0(\Omega) \times M_h^0(\Omega)$ by the following iterative procedure:

1). Find $(\gamma_{h,j}^0, \eta_{h,j}^0) \in X_h^0(\Omega_j) \times M_h^0(\Omega_j)$ $(j = 1, 2, \cdots, m)$ in parallel:

$$a(\gamma_{h,j}^0, v) - b(v, \eta_{h,j}^0) = (f, v) - a(u_H, v) + b(v, p_H), \quad \forall v \in X_h^0(\Omega_j),$$
$$b(\gamma_{h,j}^0, q) = -b(u_H, q), \quad \forall q \in M_h^0(\Omega_j),$$

then set $(\gamma_h^0, \eta_h^0) = \sum_{j=1}^{m} \omega_j (\widetilde{\gamma_{h,j}^0}, \widetilde{\eta_{h,j}^0})$.

2). For $n = 1, 2, \cdots$, until convergence:

i). Find $(d_{h,j}^n, r_{h,j}^n) \in X_h^0(\Omega_j) \times M_h^{\Gamma_j}(\Omega_j)$ $(j = 1, 2, \cdots, m)$ in parallel such that

$$a(d_{h,j}^n, v) - b(v, r_{h,j}^n) = (f, v) - a(u_H + \gamma_h^{n-1}, v) + b(v, p_H + \eta_h^{n-1}), \quad \forall v \in X_h^0(\Omega_j),$$
$$b(d_{h,j}^n, q) = -b(u_H + \gamma_h^{n-1}, q), \quad \forall q \in M_h^{\Gamma_j}(\Omega_j).$$

ii). $(\gamma_h^n, \eta_h^n) = (\gamma_h^{n-1}, \eta_h^{n-1}) + \sum_{j=1}^{m} \omega_j (\widetilde{d_{h,j}^n}, \widetilde{r_{h,j}^n})$.

3. Set $(u^h, p^h) = (u_H, p_H) + (\gamma_h, \eta_h)$.

In the above algorithm, $\omega_j = \omega_j(x)$ $(j = 1, 2, \cdots, m)$ are relaxation parameters satisfying $0 \le \omega_j \le 1$ and $\sum_{j=1}^{m} \omega_j \equiv 1$. $(\widetilde{\gamma_{h,j}^0}, \widetilde{\eta_{h,j}^0})$ and $(\widetilde{d_{h,j}^n}, \widetilde{r_{h,j}^n})$ denote the extensions of $(\gamma_{h,j}^0, \eta_{h,j}^0)$ and $(d_{h,j}^n, r_{h,j}^n)$ by zero in $\Omega \backslash \Omega_j$, respectively, and $M_h^{\Gamma_j}(\Omega_j) = \{q \in M_h(\Omega_j) : q|_{\Gamma_j} = 0\}$ for $j = 1, 2, \cdots, m$.

Theorem 2. *Under the conditions of Theorem 1, the approximate solution pair* (u^h, p^h) *obtained from Algorithm 2 satisfies the following error estimate:*

$$\|u - u^h\|_{1,\Omega} + \|p - p^h\|_{0,\Omega} \le c(h^s + H^{s+1})(\|u\|_{s+1,\Omega} + \|p\|_{s,\Omega}), \quad 1 \le s \le t. \tag{24}$$

Proof. At Step 2 of Algorithm 2, if we discard Sub-step 2) and set $\omega_j \equiv 1$ in D_j and $\omega_j \equiv 0$ in $\Omega\backslash D_j$, Algorithm 2 is exactly Algorithm 1. Noting that Sub-step 2) of Algorithm 2 is precisely the additive Schwarz sequence applied to the generalized Stokes problem with the residual of (u_H, p_H) as its right-hand side and taking (γ_h^0, η_h^0) as the initial guess for the Schwarz iteration process, the convergence of the overlapping Schwarz methods for the generalized Stokes equations (see, e.g., [12, 13] where the convergence of the overlapping additive and multiplicative Schwarz methods was proved in divergence-free subspaces for two subdomains, and [27] where the convergence proof was performed in general mixed spaces for multi-subdomains) and (16) immediately yield the error estimate (24). □

Comparing Algorithm 2 with Algorithm 1, we can see that the difference between the two algorithms lies in Step 2. Algorithm 2 uses a standard overlapping additive Schwarz method to correct the residual of (u_H, p_H) computed at Step 1. Numerical tests show that several Schwarz iterations are enough to achieve an ideal continuous solution with small overlapping size of subdomains. Although the formulae of error estimates are formally the same between the approximate solutions from Algorithm 1 and Algorithm 2 (see (16) and (24), respectively), the Schwarz iteration at Step 2 of Algorithm 2 yields a much smaller constant c in (24) than that in (16), and hence leads to a better solution than Algorithm 1 (from Algorithms 1 and 2 we can see that the correction yielded at Step 2 of Algorithm 1 is just the initial guess of the Schwarz iteration process at Step 2 of Algorithm 2. Therefore, by a Schwarz iteration process, Algorithm 2 can yield a higher accurate fine grid correction than Algorithm 1).

4. Numerical results

In this section, we shall report some numerical results to illustrate the effectiveness of our new Algorithm 2 and compare it with Algorithm 1. The routine UMFPACK [6] is used to solve the linear systems arising from the coarse grid problem and local fine grid problems. In our numerical experiments, Ω is the unit square $[0, 1] \times [0, 1]$ in R^2. The mesh consists of triangular elements which are obtained by dividing Ω (or $\Omega_j,\ j = 1, 2, \cdots, m$) into sub-squares of equal size and then drawing the diagonal in each sub-square; see Figure 3.

4.1. Analytic solution.

For this test case we set f and the boundary conditions such that the exact solution of the generalized Stokes problem is given by

$$\begin{aligned}
&u = (u_1, u_2),\ u_1 = 256x^2(x-1)^2 y(y-1)(2y-1),\\
&u_2 = -256y^2(y-1)^2 x(x-1)(2x-1),\\
&p = x^2 + y^2 - 2/3.
\end{aligned}$$

The parameters in the generalized Stokes problem are set as $\alpha = 100,\ \nu = 0.1$. We divide $\Omega = [0, 1] \times [0, 1]$ into four disjoint subdomains

$$\begin{aligned}
&D_1 = (0, 1/2) \times (0, 1/2), \qquad D_2 = (1/2, 1) \times (0, 1/2),\\
&D_3 = (1/2, 1) \times (1/2, 1), \qquad D_4 = (0, 1/2) \times (1/2, 1),
\end{aligned}$$

then extend each D_j $(j = 1, 2, 3, 4)$ outside the current local mesh with one layer of fine grid elements to get Ω_j $(j = 1, 2, 3, 4)$. These Ω_j $(j = 1, 2, 3, 4)$ compose an overlapping decomposition of $\Omega : \overline{\Omega} = \bigcup_{j=1}^{4} \overline{\Omega}_j$; see Figure 2.

The stable Taylor-Hood mixed finite element spaces

$$X_H^0(\Omega) = \{v \in H_0^1(\Omega)^2 : v|_K \in \big(P_2\big)^2, \quad \forall K \in T^H(\Omega)\},$$
$$M_H^0(\Omega) = \{q \in L_0^2(\Omega) \cap C^0(\Omega) : q|_K \in P_1, \quad \forall K \in T^H(\Omega)\},$$
$$X_h^0(\Omega_j) = \{v \in H_0^1(\Omega_j)^2 : v|_K \in \big(P_2\big)^2, \quad \forall K \in T^h(\Omega_j)\} \quad j = 1, 2, 3, 4,$$
$$M_h^0(\Omega_j) = \{q \in L_0^2(\Omega_j) \cap C^0(\Omega_j) : q|_K \in P_1, \quad \forall K \in T^h(\Omega_j)\} \quad j = 1, 2, 3, 4,$$
$$M_h^{\Gamma_j}(\Omega_j) = \{q \in C^0(\Omega_j) : q|_{\Gamma_j} = 0,\ q|_K \in P_1, \quad \forall K \in T^h(\Omega_j)\} \quad j = 1, 2, 3, 4$$

are employed, where P_1 or P_2 is the space of complete linear or quadratic polynomials. The stopping criterion for the Schwarz iteration in Algorithm 2 is

$$\max_{j=1,2,3,4} \left\{ \frac{\|\nabla(\gamma_h^{n+1} - \gamma_h^n)\|_{0,\Omega_j}}{\|\nabla \gamma_h^{n+1}\|_{0,\Omega_j}} \right\} < TOL. \tag{25}$$

In general, due to that the Schwarz iteration is applied to the residual equations of (u_H, p_H) in Algorithm 2, the value of TOL in (25) does not need to be too small. To check this, we performed a series of experiments to compute the approximate solution by Algorithm 2 with $TOL = 10^{-k}$ $(k = 1, 2, 3, 4, 5, 6)$. It is found that when TOL decreases from 10^{-2} to 10^{-6}, no obvious improvement has been observed in accuracy of the solutions. Consequently, in our subsequent experiments, the parameter TOL is set as 10^{-2}.

Secondly, to test the asymptotical errors of the solutions provided by the algorithms, we compute the finite element solutions by Algorithms 1-2 with fine meshes of sizes $h = n^{-3}$ $(n = 2, 3, 4, 5)$ and corresponding coarse meshes of size H satisfying $H^3 = h^2$. In addition to the errors, we compute the convergence rates by the formula $\frac{log(E_i/E_{i+1})}{log(h_i/h_{i+1})}$, where E_i and E_{i+1} are the relative errors $\frac{\||\nabla(u-u^h)\||_{0,\Omega}+\||p-p^h\||_{0,\Omega}}{\|\nabla u\|_{0,\Omega}+\|p\|_{0,\Omega}}$ corresponding to the fine meshes of sizes h_i and h_{i+1}, respectively. The numerical results are listed in Tables 1-2, in which it denotes the Schwarz iterations count satisfying the stopping criterion, and

$$K_{div} = \max_{K \in T^h(\Omega)} |\int_K \text{div } u^h dx|$$

which measures the approximation of the incompressibility of the fluid.

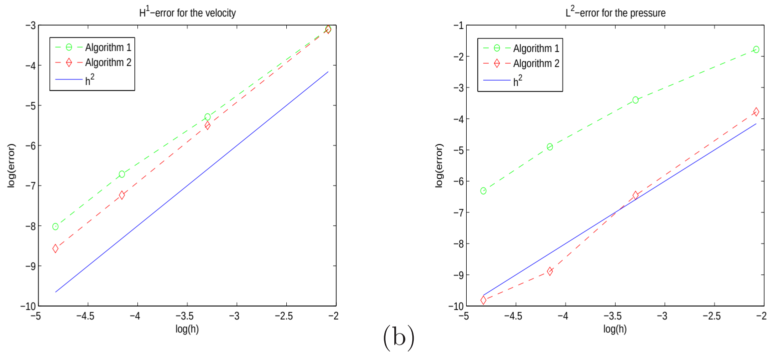
TABLE 1. Errors of the solutions by Algorithm 1.

h	H	$\frac{\||\nabla(u-u^h)\||_{0,\Omega}}{\|\nabla u\|_{0,\Omega}}$	$\frac{\||p-p^h\||_{0,\Omega}}{\|p\|_{0,\Omega}}$	K_{div}	rate
1/8	1/4	0.0451015	0.168485	0.000525346	
1/27	1/9	0.00505172	0.0334438	3.87397e-005	1.69431
1/64	1/16	0.00121176	0.00745499	9.80859e-007	1.67706
1/125	1/25	0.000328927	0.00181646	1.45937e-007	1.98850

It can be seen from Tables 1-2 that Algorithm 2 has an obviously better performance than Algorithm 1 either in terms of accuracy of the approximate solution, the convergence order or the approximation of the incompressibility of the fluid. Compared with Algorithm 1, several Schwarz iterations of our new Algorithm 2 largely improved the accuracy of the approximate solutions.

TABLE 2. Errors of the solutions by Algorithm 2.

h	H	it	$\frac{\lVert\lvert\nabla(u-u^h)\rvert\rVert_{0,\Omega}}{\lVert\nabla u\rVert_{0,\Omega}}$	$\frac{\lVert\lvert p-p^h\rvert\rVert_{0,\Omega}}{\lVert p\rVert_{0,\Omega}}$	K_{div}	rate
1/8	1/4	3	0.0447399	0.0228861	0.000320316	
1/27	1/9	4	0.00409212	0.00157115	2.64174e-005	1.97220
1/64	1/16	6	0.000720161	0.000137735	4.77246e-008	2.02570
1/125	1/25	7	0.000190035	5.46893e-005	6.27145e-008	1.98195



(a) (b)

FIGURE 4. Errors of the solutions with $\alpha = 100, \nu = 0.1$: (a) H^1-error for the velocity; and (b) L^2-error for the pressure.

According to the mixed finite element spaces we chosen and the relationship between the mesh sizes H and h, i.e., $H = O(h^{2/3})$, by Theorems 1-2, we obtain

$$\lVert\lvert u - u^h\rvert\rVert_{1,\Omega} + \lVert\lvert p - p^h\rvert\rVert_{0,\Omega} \approx ch^2.$$

The results shown in Tables 2 support the above estimate for our new parallel algorithm. However, a big difference has been observed between the numerical results and the theoretical analysis for Algorithm 1; see Figure 4.

Table 3 reports the maximal "error constants" of the two algorithms from which we can see that better "error constants" are obtained by our new algorithm. This explains why higher accurate solutions can be obtained by our new algorithm.

TABLE 3. Maximum of the "error constants".

Method	Algorithm 1	Algorithm 2
$\max\left\{\frac{\lVert\lvert\nabla(u-u^h)\rvert\rVert_{0,\Omega}}{h^2}\right\}$	37.5916	21.8197
$\max\left\{\frac{\lVert\lvert p-p^h\rvert\rVert_{0,\Omega}}{h^2}\right\}$	12.8749	0.617577

Finally, to investigate the effect of the zero-order term coefficient α in the generalized Stokes equations on the methods, we set $\nu = 0.1,\ h = 1/125,\ H = 1/25$ and then compute the finite element solution by Algorithms 1-2 with different value of α. Table 4 lists the numerical results which show that as α increases, the accuracy of the pressure both from Algorithms 1 and 2 drops. However, for all values of α being tested, our new algorithm with several Schwarz iterations for the correction problem yielded a much higher accurate solution than the original algorithm.

4.2. Lid-driven cavity flow problem. For this test, we consider the incompressible lid-driven cavity flow problem defined on the unit square. In this problem, the

TABLE 4. Errors of the solutions: $\nu = 0.1,\ h = 1/125,\ H = 1/25.$

α	Algorithm 1			Algorithm 2			
	$\frac{\lVert\lvert\nabla(u-u^h)\rvert\rVert_{0,\Omega}}{\lVert\nabla u\rVert_{0,\Omega}}$	$\frac{\lVert\lvert p-p^h\rvert\rVert_{0,\Omega}}{\lVert p\rVert_{0,\Omega}}$	K_{div}	it	$\frac{\lVert\lvert\nabla(u-u^h)\rvert\rVert_{0,\Omega}}{\lVert\nabla u\rVert_{0,\Omega}}$	$\frac{\lVert\lvert p-p^h\rvert\rVert_{0,\Omega}}{\lVert p\rVert_{0,\Omega}}$	K_{div}
200	0.000329079	0.00323433	1.4525e-007	6	0.00018994	5.56798e-005	6.27952e-008
500	0.000328336	0.00739821	1.43207e-007	5	0.000189957	0.000161907	6.32209e-008
1000	0.000326492	0.014212	1.39913e-007	6	0.000190303	0.000958141	6.43804e-008
2000	0.000323832	0.0276608	1.33712e-007	8	0.000190841	0.00304023	6.53567e-008

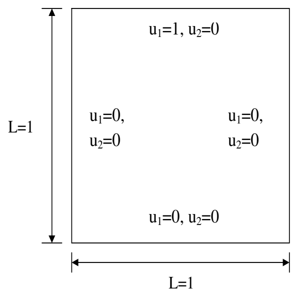
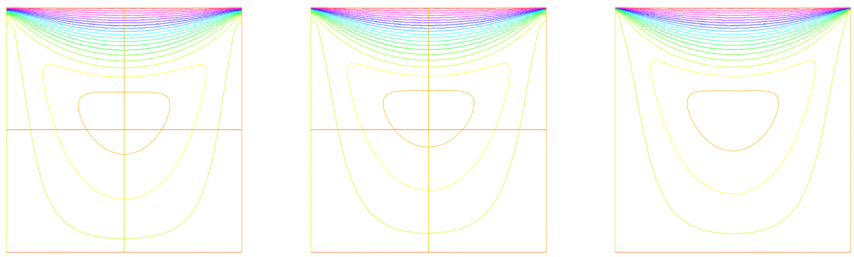


FIGURE 5. Lid-driven cavity flow.

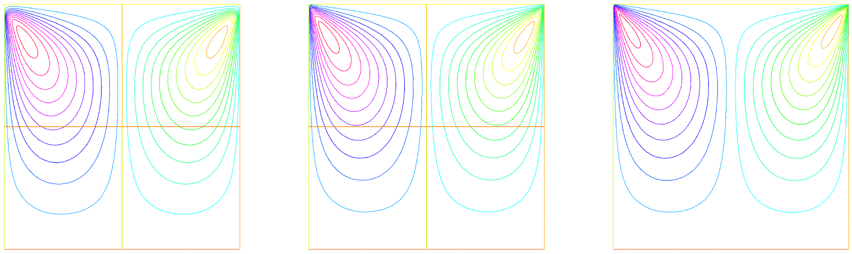


(a) Algorithm 1 (b) Algorithm 2 (c) Global standard FE

FIGURE 6. Contours of u_1^h-velocity for the driven cavity flow: (a) Algorithm 1; (b) Algorithm 2 (it=20); and (c) global standard finite element method.

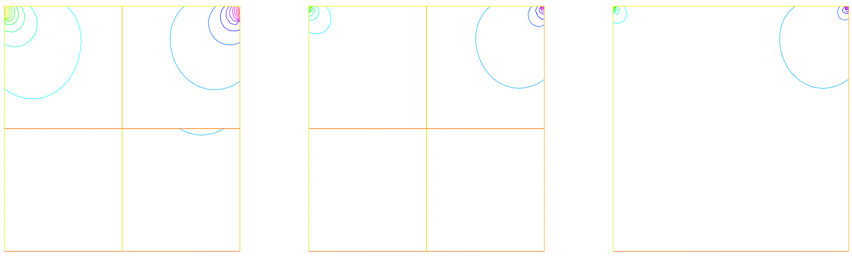
body force $f = 0$ and the boundary conditions are shown in Figure 5. The parameters are set as $\alpha = 10,\ \nu = 1$. The subdomains D_j $(j = 1, 2, 3, 4)$ are the same as the first test case and the overlapped subdomains Ω_j $(j = 1, 2, 3, 4)$ are obtained by extending each D_j $(j = 1, 2, 3, 4)$ outside the current local mesh with one layer of coarse grid elements. The mesh consists of triangular elements and the mesh sizes $H = 1/16, h = 1/64$. The stable Taylor-Hood mixed finite elements are also employed.

Figures 6-8 depict the contours of the approximate velocities and pressures computed by Algorithms 1-2 and the standard finite element method, respectively. Note that the horizontal and vertical lines at the middle of the left two sub-figures in Figures 6-8 is the artificial boundaries of D_j $(j = 1, 2, 3, 4)$. Figures 6-8 show that the approximate pressure obtained from Algorithm 1 is poor, while 20 Schwarz iterations of our new Algorithm 2 obtained a solution with an accuracy comparable to the standard finite element solution. This test further illustrated the effectiveness of our new algorithm.



(a) Algorithm 1 (b) Algorithm 2 (c) Global standard FE

FIGURE 7. Contours of u_2^h-velocity for the driven cavity flow: (a) Algorithm 1; (b) Algorithm 2 (it=20); and (c) global standard finite element method.



(a) Algorithm 1 (b) Algorithm 2 (c) Global standard FE

FIGURE 8. Contours of pressure p^h for the driven cavity flow: (a) Algorithm 1; (b) Algorithm 2 (it=20); and (c) global standard finite element method.

5. Conclusions

In this work we extended the parallel finite element algorithm proposed in [11] for the standard steady Stokes problem to the generalized Stokes problem. Analysis of improvement on the algorithm was carried out and a new improved algorithm was proposed, analyzed and compared with the original algorithm. By additive overlapping Schwarz iterations applied to solving the correction problem on fine grid, the new algorithm can yield a better solution than the original algorithm. Numerical results denominated the effectiveness of the new algorithm.

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