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A NOTE ON THE CONVERGENCE OF A CRANK-NICOLSON SCHEME FOR THE KDV EQUATION

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Abstract. The aim of this paper is to establish the convergence of a fully discrete Crank-Nicolson type Galerkin scheme for the Cauchy problem associated to the KdV equation. The convergence is achieved for initial data in L^2, and we show that the scheme converges strongly in $L^2(0,T;L^2_{\text{loc}}(\mathbb{R}))$ to a weak solution for some $T>0$. Finally, the convergence is illustrated by a numerical example.

Key words. Crank-Nicolson scheme, KdV equation.

1. Introduction

In this paper we analyze a fully discrete Crank-Nicolson second order accurate scheme for the initial value problem associated to the KdV equation

$$
\begin{cases} u_t + (\frac{u^2}{2})_x + u_{xxx} = 0, & x \in \mathbb{R}\times(0,T) \\ u(x,0) = u_0(x), & x\in\mathbb{R}, \end{cases} \tag{1}
$$

where $T>0$ is fixed, $u:\mathbb{R}\times[0,T)\to\mathbb{R}$ is the unknown, and u_0 is the initial data.

This equation originally arose as a model for shallow water waves, but it has later been used for models of varying phenomena, such as magneto-acoustic waves in plasmas, lattice waves etc. It has also been widely studied from the purely mathematical side, the delicate balance between nonlinear convection and dispersion allows for a rich family of explicit solutions called solitons. Solitons were originally discovered by Zabusky and Kruskal using numerical methods [17]. To obtain explicit, but complicated, formulas for solitons, one can use the Bäcklund transform. Solitons are localized, meaning that they tend rapidly to a constant for large $|x|$, and they interact in a particle like manner.

Despite the fact that solitons were discovered using numerical methods, it is quite difficult to approximate solutions to the KdV equations numerically. A numerical method must take into account both the nonlinear convection coming from the term uu_x and the (hard to compute) dispersive waves originating from u_{xxx}. When approximating smooth solutions, to the best our knowledge, spectral methods [9, 13, 11] or discontinuous Galerkin methods [16, 15, 3] most efficiently produce accurate approximations. These methods are essentially semi-discrete, where the time variable is kept as a continuous variable, and their fully discrete counterparts are hard to analyze, see however [9] in which a very efficient fully discrete version is presented.

Regarding fully discrete methods, a simple first order method (which is a discretization of the semi-discrete method used by Sjöberg to first give an existence proof for the Cauchy problem for the KdV equation [12]) is analyzed and shown to converge to a solution [7]. However in practice this method requires a very fine grid, and correspondingly large computational effort, to produce acceptable

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solutions. By using a higher order approximation in space and fully implicit time stepping [5], the efficiency improves slightly, while the resulting scheme is shown to converge for initial data in L^2. The purpose of this note is to analyze a second-order-in-time version of the scheme presented in [5], and to show that one still has convergence for general L^2 initial data, while in practice the scheme is second order accurate, and comparable with the second order discontinuous Galerkin scheme of [6].

We shall now briefly and informally explain our strategy. Define, *for the moment*, a weak solution to the KdV equation to be a function $u(t,x)$ such that $u \in C^1([0,\infty); H^2(\mathbb{R}))$ and that for all $v \in H^2(\mathbb{R})$,

$$(u_t, v) + (uu_x, v) + (u_x, v_{xx}) = 0, \tag{2}$$

where $(\cdot,\cdot)$ denotes the usual L^2 inner product. We propose a Crank-Nicolson discretization of this equation. Let Δt be some small positive number, and set $u^n \approx u(n\Delta t, \cdot)$, $u^{n+\frac{1}{2}} = (u^{n+1} + u^n)/2$. Given u^0, we define u^n to be the solution of

$$\left(u^{n+1}, v\right) + \Delta t \left(u^{n+\frac{1}{2}} u^{n+\frac{1}{2}}{}_x, v\right) + \Delta t \left(u^{n+\frac{1}{2}}{}_x, v_{xx}\right) = \left(u^n, v\right), \tag{3}$$

for all $v \in H^2(\mathbb{R})$ and $n \geq 0$. Assuming that this equation has a unique solution u^{n+1}, we can choose $v = u^{n+1} + u^n$ to get

$$\left\|u^{n+1}\right\|_{L^2(\mathbb{R})} = \left\|u^n\right\|_{L^2(\mathbb{R})} = \left\|u^0\right\|_{L^2(\mathbb{R})}. \tag{4}$$

Furthermore, by using a clever trick taken from Kato [10], we can get an á priori H^1 bound on u^n. Let R denote a positive constant, and introduce a smooth function φ satisfying;

- **a** $1 \leq \varphi(x) \leq 2R + 2$,
- **b** $\varphi'(x) = 1$ for $|x| < R$,
- **c** $\varphi'(x) = 0$ for $|x| \geq R + 1$
- **d** $0 \leq \varphi'(x) \leq 1$ for all x, and
- **e** $\left|\varphi^{(k)}(x)\right| \leq C\varphi(x)$ for all x and $k = 1, 2, 3$, and some constant C independent of R.

Assuming that u^n and u^{n+1} are in $H^2(\mathbb{R})$, $u^{n+\frac{1}{2}}\varphi$ is an admissible test function in (3), testing with this function yields

$$\begin{aligned}\frac{1}{2}\left\|u^{n+1}\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})} &+ \Delta t \left(u^{n+\frac{1}{2}} u^{n+\frac{1}{2}}{}_x, u^{n+\frac{1}{2}}\varphi\right) \\ &+ \Delta t \left(u^{n+\frac{1}{2}}{}_x, \left(u^{n+\frac{1}{2}}\varphi\right)_{xx}\right) = \frac{1}{2}\left\|u^n\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{5}$$

To save space, we write $w = u^{n+\frac{1}{2}}$, then

$$\begin{aligned}\left(u^{n+\frac{1}{2}} u^{n+\frac{1}{2}}{}_x, u^{n+\frac{1}{2}}\varphi\right) &= -\frac{1}{2}\int_{\mathbb{R}} w^2 \left(w\varphi\right)_x \, dx \\ &= -\frac{1}{3}\int_{\mathbb{R}} w^3 \varphi_x \, dx.\end{aligned}$$

Also, by using the Poincaré's and Cauchy-Schwartz inequalities,

$$\begin{aligned}\sup|w\sqrt{\varphi_x}| &\le \sqrt{2}\Big(\int_{\mathbb{R}}(w\sqrt{\varphi_x})\,(w\sqrt{\varphi_x})_x\,dx\Big)^{1/2}\\ &\le \sqrt{2}\Big(\int_{\mathbb{R}}|ww_x\varphi_x|\,dx\Big)^{1/2}+\Big(\int_{\mathbb{R}}w^2\,|\varphi_{xx}|\,dx\Big)^{1/2}\\ &\le \sqrt{2}\Big(\int_{\mathbb{R}}w_x^2\varphi_x\,dx\Big)^{1/4}\Big(\int_{\mathbb{R}}w^2\varphi_x\,dx\Big)^{1/4}+\Big(\int_{\mathbb{R}}w^2\,|\varphi_{xx}|\,dx\Big)^{1/2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}}w^3\varphi_x\,dx &\le \sup|w\sqrt{\varphi_x}|\int_{\mathbb{R}}w^2\sqrt{\varphi_x}\,dx\\ &\le \sqrt{2}\Big(\int_{\mathbb{R}}w_x^2\varphi_x\,dx\Big)^{1/4}\Big(\int_{\mathbb{R}}w^2\varphi_x\,dx\Big)^{1/4}\Big(\int_{\mathbb{R}}w^2\sqrt{\varphi_x}\,dx\Big)\\ &\quad+\Big(\int_{\mathbb{R}}w^2\sqrt{\varphi_x}\,dx\Big)\Big(\int_{\mathbb{R}}w^2\,|\varphi_{xx}|\,dx\Big)^{1/2}.\end{aligned}$$

Next, we use Young's inequality $ab\le \frac14 a^4+\frac34 b^{\frac43}$, to get

$$\begin{aligned}\frac13\Big|\int_{\mathbb{R}}w^3\varphi_x\,dx\Big| &\le \frac{\sqrt2}{12}\int_{\mathbb{R}}w_x^2\varphi_x\,dx+\frac{\sqrt2}{4}\Big(\int_{\mathbb{R}}w^2\varphi_x\,dx\Big)^{1/3}\Big(\int_{\mathbb{R}}w^2\sqrt{\varphi_x}\,dx\Big)^{4/3}\\ &\quad+\frac13\Big(\int_{\mathbb{R}}w^2\sqrt{\varphi_x}\,dx\Big)\Big(\int_{\mathbb{R}}w^2\,|\varphi_{xx}|\,dx\Big)^{1/2}\\ &\le \frac{\sqrt2}{12}\int_{\mathbb{R}}w_x^2\varphi_x\,dx+C_{u^0},\end{aligned}\tag{6}$$

for some constant C_{u^0} depending only on $\left\|u^0\right\|_{L^2(\mathbb{R})}$. For the third term on the right hand side of (5) we use the equality

$$\left(u^{n+\frac12}{}_x,\left(u^{n+\frac12}\varphi\right)_{xx}\right)=\frac32\int_{\mathbb{R}}w_x^2\varphi_x\,dx-\frac12\int_{\mathbb{R}}w^2\varphi_{xxx}\,dx.$$

Using this and (6), the identity (5) gives

$$\begin{aligned}\frac12\left\|u^{n+1}\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})}+\Delta t\frac{36-2\sqrt2}{24}\int_{-R}^{R}\left(u^{n+\frac12}{}_x\right)^2\,dx&\\ \le \frac12\left\|u^n\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})}+C_{u^0}\Delta t.&\end{aligned}\tag{7}$$

This yields the estimates

$$\left\|u^m\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})}\le\left\|u^0\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})}+C_{u^0}m\Delta t$$

and

$$\Delta t\sum_{n=0}^{m}\int_{-R}^{R}\left(u^{n+\frac12}{}_x\right)^2\,dx\le\frac{24}{33}\left(2\left\|u^0\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})}+C_{u^0}m\Delta t\right).$$

This means that if the initial data u^0 are in L^2, then $u^{n+\frac12}$ is is in $\ell^2([0,m\Delta t];H^1(\mathbb{R}))$. This is enough, see Section 3, to prove the compactness of the sequence $\left\{u^{n+\frac12}\right\}$ via the Simon-Aubin compactness lemma. Also it turns out that any limit (as $\Delta t\to 0$) of this sequence solves the KdV equation with initial data u^0.

Furthermore, the local truncation error of the Crank-Nicolson method is of third order, so that for sufficiently regular solutions the global error is of order Δt^2.

In order to define a fully discrete method, we propose a finite element discretization of (2) using test functions of the form φv, where v is in some finite element space. This has the advantage that (5) will hold automatically, and thus hopefully lead to a H^1 type bound on the finite element approximation. The downside of this is that we no longer have an á priori L^2 estimate, and consequentially must assume a relation between the space discretization and Δt in order to bound the approximations in L^2.

Standard Galerkin type approximations, using smooth splines on a uniform mesh, to periodic solutions of KdV equation are analyzed in [1, 2, 14] . All these works aimed at deriving optimal rate of convergence estimate for Galerkin approximations. The discontinuous Galerkin method has been used to approximate the solution of (1) and rate of convergence analysis has been presented for both periodic and full line case in [16, 8]. Also a comprehensive analysis of discontinuous Galerkin methods for generalized KdV equations is given in [3].

All the above mentioned references use the well posedness theory for the KdV equation to prove convergence, and convergence rates. Therefore, by themselves, they do not yield the existence of a solution by furnishing constructive existence proofs.

There are however a few results regarding proof of convergence of numerical methods for the KdV equation, which also give a direct and constructive existence theorem. Indeed, the first proof of existence and uniqueness of solutions to the KdV equation for initial data in $H^3(\mathbb{R}/\mathbb{Z})$ is based on a semi-discrete difference approximation [12]. The corresponding fully discrete scheme, which incidentally coincides with a fully discrete splitting scheme, was analyzed in [7], and it was shown that the scheme converges to the classical solution if the initial data is in $H^3(\mathbb{R})$, and to the weak solution if the initial data lies in $L^2(\mathbb{R})$.

Regarding the constructive existence proof for the KdV equation, in [5], we have established the convergence of a higher order finite element Galerkin type scheme for (1). The corresponding approximation is generated by an implicit Euler discretization of a Galerkin scheme. In this paper, we also consider a similar Galerkin type discretization, but for the time variable we use Crank-Nicolson type discretization instead of implicit Euler discretization used in [5]. The advantage of using Crank-Nicolson type discretization is that it provides an approximate solution which is second order in time. But, using implicit Euler for time discretization yields an approximate solution which is first order in time. Thus, the results of this paper can be seen as a generalization of the work [5] in the context of higher order approximation methods in time.

The rest of the paper is organized as follows: In Section 2, we present the necessary notation and define the fully-discrete finite element Galerkin type numerical scheme. Since the fully-discrete scheme is implicit in nature, the solvability of the scheme cannot be taken for granted and this is addressed in Section 2.3. In Section 3, we show the convergence to a weak solution if the initial data is in $L^2(\mathbb{R})$ and finally in Section 4, we exhibit a numerical experiment illustrating the convergence.

2. Numerical scheme

In this section, we define the finite element Crank-Nicolson-Galerkin type numerical scheme for the KdV equation. We start by introducing the necessary notations.

2.1. Notation. Let Δt and Δx denote the approximation parameters corresponding to time and space discretization respectively. For $j \in \mathbb{Z}$, we set $x_j = j\Delta x$, and for $n = 0, 1, \cdots, N$, where $(N + \frac{1}{2})\Delta t = T$ for some fixed time horizon $T > 0$, we set $t_n = n\Delta t$ and $t_{n+\frac{1}{2}} = (t_n + t_{n+1})/2$. Furthermore, we introduce the spatial grid cells $I_j = [x_{j-1}, x_j]$.

Moreover given $R > 0$, we define the cut off function φ as $\varphi(x) = \overline{\varphi} * w(x)$ where $\overline{\varphi}(x) = \max\{1, \min\{1 + x + R, 1 + 2R\}\}$ and w is a symmetric positive function with integral one and support in $[-1, 1]$. Let C_R be defined as

$$C_R = \max\left\{\|\varphi\|_{L^\infty(\mathbb{R})}, \|\varphi_x\|_{L^\infty(\mathbb{R})}, \|\varphi_{xx}\|_{L^\infty(\mathbb{R})}, \|\varphi_{xxx}\|_{L^\infty(\mathbb{R})}\right\}. \tag{8}$$

We define the weighted L^2 inner product as

$$\langle u, v\rangle_\varphi = (u, v\varphi)$$

where $(\cdot, \cdot)$ denotes the usual L^2 inner product, and the associated weighted norm by $\|u\|_{2,\varphi}^2 = \langle u, u\rangle_\varphi$.

2.2. Finite element scheme. Our proposed scheme is a finite element approximation to (2). We start with the description of finite dimensional space of test functions. Assume that r is a fixed integer ≥ 2 and let $\mathbb{P}_r(I)$ denote the space of polynomials on the interval I of degree $\leq r$. We seek an approximation $u^{\Delta x}$ to the solution of (1) such that for each $t \in [0, T]$, $u\Delta x$ belongs to the finite dimensional space

$$S_{\Delta x} = \left\{v \in H^2(\mathbb{R}) \mid v \in \mathbb{P}_r(I_j) \text{ for all } j\right\}.$$

Define P to be the L^2 orthogonal projection onto $S_{\Delta x}$. Let the sequence $\{u^n_{\Delta x}\}_{n\in\mathbb{N}}$ be define by the following: Given $u^0_{\Delta x} = Pu_0$, find $u^{n+1}_{\Delta x} \in S_{\Delta x}$ such that

$$\left(u^{n+1}_{\Delta x}, \varphi v\right) - \Delta t\left(\frac{(u^{n+\frac{1}{2}}_{\Delta x})^2}{2}, (\varphi v)_x\right) + \Delta t\left(\left(u^{n+\frac{1}{2}}_{\Delta x}\right)_x, (\varphi v)_{xx}\right) = \left(u^n_{\Delta x}, \varphi v\right), \tag{9}$$

for all $v \in S_{\Delta x}$ and for $n = 0, 1, \ldots, N$. Recall that

$$u^{n+\frac{1}{2}}_{\Delta x} = \frac{u^n_{\Delta x} + u^{n+1}_{\Delta x}}{2}.$$

Observe that, (9) is an implicit scheme, and in order to calculate $u^{n+1}_{\Delta x}$ given $u^n_{\Delta x}$ one must solve a non-linear equation.

2.3. Solvability for one time step. In order to show the existence of $u^n_{\Delta x}$ we define the following iteration scheme:

$$\begin{cases} \left(w^{\ell+1}, \varphi v\right) - \frac{1}{2}\Delta t\left(\left(\frac{u^n_{\Delta x} + w^\ell}{2}\right)^2,\ (\varphi v)_x\right) + \Delta t\left(\frac{(u^n_{\Delta x})_x + w^{\ell+1}_x}{2}, (\varphi v)_{xx}\right) = (u^n_{\Delta x}, \varphi v), \\ w^0 = u^n_{\Delta x}, \end{cases} \tag{10}$$

this is to hold for all test functions $v \in S_{\Delta x}$. The following lemma guarantee the solvability of the implicit scheme (9).

Lemma 2.1. *Choose a constant L such that $0 < L < 1$ and set*

$$K = \frac{7 - L}{1 - L} > 7.$$

We consider the iteration (10) *with* $w^0 = u^n_{\Delta x}$*, and assume that the following CFL condition holds*

$$\lambda \leq \frac{L}{\sqrt{C_R} 2\sqrt{2} K \left\| u^n_{\Delta x} \right\|_{2,\varphi}}, \tag{11}$$

where C_R *is defined by* (8) *and* λ *is given by*

$$\lambda^2 = \frac{\Delta t^2}{\Delta x^3}. \tag{12}$$

Then there exists a function $u^{n+1}_{\Delta x}$ *which solves* (9)*, and* $\lim_{\ell \to \infty} w^\ell = u^{n+1}_{\Delta x}$*. Furthermore*

$$\left\| u^{n+1}_{\Delta x} \right\|_{2,\varphi} \leq K \left\| u^n_{\Delta x} \right\|_{2,\varphi}. \tag{13}$$

Proof. First note that the scheme (10) can be written as

$$\begin{aligned} \left(w^{\ell+1}, \varphi v\right) + \frac{\Delta t}{4} \left((u^n_{\Delta x} w^\ell)_x, \varphi v\right) + \frac{\Delta t}{4} \left(w^\ell w^\ell_x, \varphi v\right) \\ + \frac{\Delta t}{2} \left(w^{\ell+1}_x, (\varphi v)_{xx}\right) = \mathcal{F}(u^n_{\Delta x}, \varphi v) \end{aligned} \tag{14}$$

for all $v \in S_{\Delta x}$, with

$$\mathcal{F}(u^n_{\Delta x}, \varphi v) = (u^n_{\Delta x}, \varphi v) + \frac{\Delta t}{8} \left((u^n_{\Delta x})^2, (\varphi v)_x\right) - \frac{\Delta t}{2} \left((u^n_{\Delta x})_x, (\varphi v)_{xx}\right).$$

From (14) we have

$$\begin{aligned} \left(w^{\ell+1} - w^\ell, \varphi v\right) + \frac{\Delta t}{4} \left((u^n_{\Delta x}(w^\ell - w^{\ell-1}))_x, \varphi v\right) + \frac{\Delta t}{4} \left(w^\ell w^\ell_x - w^{\ell-1} w^{\ell-1}_x, \varphi v\right) \\ + \frac{\Delta t}{2} \left((w^{\ell+1} - w^\ell)_x, (\varphi v)_{xx}\right) = 0. \end{aligned} \tag{15}$$

We choose $v = w^{\ell+1} - w^\ell =: w$ in (15)

$$\begin{aligned} (w, \varphi w) + \frac{\Delta t}{2} \left(w_x, (\varphi w)_{xx}\right) \\ = \underbrace{-\frac{\Delta t}{4} \left(\left(u^n_{\Delta x} \left(w^\ell - w^{\ell-1}\right)\right)_x, \varphi w\right)}_{\mathcal{A}_1} - \underbrace{\frac{\Delta t}{4} \left(w^\ell w^\ell_x - w^{\ell-1} w^{\ell-1}_x, \varphi w\right)}_{\mathcal{A}_2} \end{aligned} \tag{16}$$

To estimate the terms in the above expression we need the following identity

$$\int_{\mathbb{R}} w_x \left(\varphi w\right)_{xx} \, dx = \frac{3}{2} \int_{\mathbb{R}} w_x^2 \varphi_x \, dx - \frac{1}{2} \int_{\mathbb{R}} w^2 \varphi_{xxx} \, dx. \tag{17}$$

We also use the following inverse inequality

$$\left\| z_x \right\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\Delta x^{1/2}} \left\| z_x \right\|_{L^2(\mathbb{R})} \leq \frac{C}{\Delta x^{3/2}} \left\| z \right\|_{L^2(\mathbb{R})}, \tag{18}$$

where the constant C is independent of z and Δx. Now, using the above identities, we turn to estimate the terms $\mathcal{A}_1$, $\mathcal{A}_2$ of (16). We proceed as follows: Applying

Cauchy-Schwartz inequality repeatedly, we get

$$
\begin{aligned}
\mathcal{A}_2 =& \frac{\Delta t}{4} \int_{\mathbb{R}} (w^\ell w_x^\ell - w^{\ell-1} w_x^{\ell-1}) \varphi w \, dx \\
\leq& \frac{\Delta t^2}{8} \int_{\mathbb{R}} (w^\ell w_x^\ell - w^{\ell-1} w_x^{\ell-1})^2 \varphi \, dx + \frac{1}{8} \int_{\mathbb{R}} \varphi w^2 \, dx \\
\leq& \frac{\Delta t^2}{8} \int_{\mathbb{R}} ((w^\ell - w^{\ell-1}) w_x^\ell + w^{\ell-1} (w^\ell - w^{\ell-1})_x)^2 \varphi \, dx + \frac{1}{8} \int_{\mathbb{R}} \varphi w^2 \, dx \\
\leq& \frac{\Delta t^2}{4} \int_{\mathbb{R}} (w^\ell - w^{\ell-1})^2 (w_x^\ell)^2 \varphi \, dx \\
&+ \frac{\Delta t^2}{4} \int_{\mathbb{R}} (w^{\ell-1})^2 (w_x^\ell - w_x^{\ell-1})^2 \varphi \, dx + \frac{1}{8} \int_{\mathbb{R}} \varphi w^2 \, dx \\
\leq& \frac{\Delta t^2}{4} \left\| w_x^\ell \right\|_{L^\infty(\mathbb{R})}^2 \int_{\mathbb{R}} \left(w^\ell - w^{\ell-1} \right)^2 \varphi \, dx \\
&+ \frac{\Delta t^2}{4} \left\| w_x^\ell - w_x^{\ell-1} \right\|_{L^\infty(\mathbb{R})}^2 \int_{\mathbb{R}} \left(w^{\ell-1} \right)^2 \varphi \, dx + \frac{1}{8} \int_{\mathbb{R}} \varphi w^2 \, dx \\
\leq& \frac{C \Delta t^2}{\Delta x^3} \left\| w^\ell \right\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} \left(w^\ell - w^{\ell-1} \right)^2 \varphi \, dx \\
&+ \frac{C \Delta t^2}{\Delta x^3} \left\| w^\ell - w^{\ell-1} \right\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} \left(w^{\ell-1} \right)^2 \varphi \, dx + \frac{1}{8} \int_{\mathbb{R}} \varphi w^2 \, dx.
\end{aligned}
$$

In the last step we have used the inverse inequality (18). Now, using the definition of λ, we conclude

$$
\mathcal{A}_2 \leq \frac{1}{8} \int_{\mathbb{R}} \varphi w^2 \, dx + C_R \lambda^2 \max \left\{ \left\| w^\ell \right\|_{2,\varphi}^2, \left\| w^{\ell-1} \right\|_{2,\varphi}^2 \right\} \int_{\mathbb{R}} \left(w^\ell - w^{\ell-1} \right)^2 \varphi \, dx,
$$

Next we estimate $\mathcal{A}_1$ as follows

$$
\begin{aligned}
\mathcal{A}_1 &= \frac{\Delta t}{4} \int_{\mathbb{R}} \left(u_{\Delta x}^n (w^{\ell-1} - w^\ell) \right)_x \varphi w \, dx \\
&\leq \frac{\Delta t^2}{8} \int_{\mathbb{R}} \left(\left(u_{\Delta x}^n (w^{\ell-1} - w^\ell) \right)_x \right)^2 \varphi \, dx + \frac{1}{8} \int_{\mathbb{R}} w^2 \varphi dx.
\end{aligned}
$$

Thus, using the inverse inequality (18), we obtain

$$
\mathcal{A}_1 \leq \frac{1}{8} \int_{\mathbb{R}} \varphi w^2 \, dx + C_R \lambda^2 \left\| u_{\Delta x}^n \right\|_{2,\varphi}^2 \int_{\mathbb{R}} \left(w^\ell - w^{\ell-1} \right)^2 \varphi \, dx.
$$

Combining above two estimates

$$
\begin{aligned}
\mathcal{A}_1 + \mathcal{A}_2 \leq& \frac{1}{4} \int_{\mathbb{R}} w^2 \varphi \, dx \\
&+ C_R \lambda^2 \max \left\{ \left\| w^\ell \right\|_{2,\varphi}^2, \left\| w^{\ell-1} \right\|_{2,\varphi}^2, \left\| u_{\Delta x}^n \right\|_{2,\varphi}^2 \right\} \int_{\mathbb{R}} \left(w^\ell - w^{\ell-1} \right) \varphi \, dx,
\end{aligned}
$$

For the second term before the equality sign in (16), we use the identity (17). Since $\varphi_x \geq 0$, from (17) we obtain

$$
\Delta t \left(w_x, (w\varphi)_{xx} \right) \geq -C_R \Delta t \left\| w \right\|_{L^2(\mathbb{R})}^2 \geq -C_R \Delta t \left\| w \right\|_{2,\varphi}^2,
$$

where the constant C_R depends on the φ_{xxx}. Collecting these bounds, and assuming Δt small enough that we have for $\ell \geq 1$

(19)
$$\begin{aligned}&\left\|(w^{\ell+1}-w^{\ell})\right\|_{2,\varphi}^2\\&\qquad\leq 2C_R\lambda^2 \max\left\{\|u_{\Delta x}^n\|_{2,\varphi}^2, \left\|w^{\ell}\right\|_{2,\varphi}^2, \left\|w^{\ell-1}\right\|_{2,\varphi}^2\right\} \left\|(w^{\ell}-w^{\ell-1})\right\|_{2,\varphi}^2.\end{aligned}$$

For w^1, setting $\ell = 0$ in (10), we obtain

$$\begin{aligned}&\left(w^1 - u_{\Delta x}^n, \varphi v\right) + \Delta t\left(\left(\frac{u_{\Delta x}^n + w^1}{2}\right)_x, (\varphi v)_{xx}\right)\\&=\frac{\Delta t}{2}\left((u_{\Delta x}^n)^2, (\varphi v)_x\right) = -\Delta t\left(u_{\Delta x}^n (u_{\Delta x}^n)_x, \varphi v\right).\end{aligned}$$

Setting $v = \frac{u_{\Delta x}^n + w^1}{2}$ yields,

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}}\left((w^1)^2 - (u_{\Delta x}^n)^2\right)\,dx + \Delta t\int_{\mathbb{R}}\left(\frac{u_{\Delta x}^n + w^1}{2}\right)_x\left(\varphi\frac{u_{\Delta x}^n + w^1}{2}\right)_{xx} dx\\ = -\Delta t\int_{\mathbb{R}} u_{\Delta x}^n (u_{\Delta x}^n)_x \frac{u_{\Delta x}^n + w^1}{2}\varphi\,dx.\end{aligned}$$

For the term in right hand side we use Cauchy-Schwartz inequality and the inverse inequality (18) and, for the term before the equality sign we use the identity (17) and estimate as before to conclude

$$\left(\frac{1}{4} - \frac{C_R\Delta t}{2}\right)\int_{\mathbb{R}}(w^1)^2\varphi\,dx \leq \left(\frac{1}{2} + \frac{1}{4} + \frac{C_R\Delta t}{2}\right)\|u_{\Delta x}^n\|_{2,\varphi}^2 + \frac{C_R\lambda^2}{2}\|u_{\Delta x}^n\|_{2,\varphi}^4.$$

We can always assume that $\frac{1}{4} - \frac{C_R\Delta t}{2} > \frac{1}{8}$. Thus, finally, we derive

$$\int_{\mathbb{R}}(w^1)^2\varphi\,dx \leq 8\left(1 + C_R\lambda^2\|u_{\Delta x}^n\|_{2,\varphi}^2\right)\|u_{\Delta x}^n\|_{2,\varphi}^2 \tag{20}$$

Then we claim that the following holds for $\ell \geq 1$

$$\left\|w^{\ell+1} - w^{\ell}\right\|_{2,\varphi} \leq L\left\|w^{\ell} - w^{\ell-1}\right\|_{2,\varphi}, \tag{21a}$$

$$\left\|w^{\ell}\right\|_{2,\varphi} \leq K\|u_{\Delta x}^n\|_{2,\varphi}, \tag{21b}$$

$$\left\|w^1\right\|_{2,\varphi} \leq 5\|u_{\Delta x}^n\|_{2,\varphi}, \tag{21c}$$

for $\ell = 1, 2, 3, \ldots$. To prove these claims, we argue by induction. From (20) and (11), we obtain

$$\begin{aligned}\left\|w^1\right\|_{2,\varphi} &\leq \left(2\sqrt{2} + 2\sqrt{2}\sqrt{C_R}\lambda\|u_{\Delta x}^n\|_{2,\varphi}\right)\|u_{\Delta x}^n\|_{2,\varphi}\\&\leq\left(2\sqrt{2} + \frac{L}{K}\right)\|u_{\Delta x}^n\|_{2,\varphi} \leq 5\|u_{\Delta x}^n\|_{2,\varphi} \leq K\|u_{\Delta x}^n\|_{2,\varphi}.\end{aligned}$$

This proves the claim (21c) and (21b) for $l = 1$. Next, Setting $\ell = 1$ in (19) and using (11) gives

$$\begin{aligned}\left\|w^2 - w^1\right\|_{2,\varphi} &\leq \sqrt{2C_R}\,\lambda \max\left\{\left\|w^1\right\|_{2,\varphi}, \|u^n_{\Delta x}\|_{2,\varphi}\right\}\left\|w^1 - u^n_{\Delta x}\right\|_{2,\varphi} \\ &\leq \left(5\sqrt{2C_R}\,\lambda \|u^n_{\Delta x}\|_{2,\varphi}\right)\left\|w^1 - u^n_{\Delta x}\right\|_{2,\varphi} \\ &\leq \frac{5L}{2K}\left\|w^1 - u^n_{\Delta x}\right\|_{2,\varphi} \leq L\left\|w^1 - u^n_{\Delta x}\right\|_{2,\varphi}.\end{aligned}$$

This shows that (21a) holds for $\ell = 1$. Next assume that (21a) and (21b) hold for $\ell = 1, \dots, m$, then

$$\begin{aligned}\left\|w^{m+1}\right\|_{2,\varphi} &\leq \sum_{\ell=0}^{m}\left\|w^{\ell+1} - w^\ell\right\|_{2,\varphi} + \left\|w^0\right\|_{2,\varphi} \\ &\leq \left\|w^1 - w^0\right\|_{2,\varphi}\sum_{\ell=0}^{m} L^\ell + \left\|w^0\right\|_{2,\varphi} \\ &\leq 6\|u^n_{\Delta x}\|_{2,\varphi}\frac{1}{1-L} + \|u^n_{\Delta x}\|_{2,\varphi} \\ &= \frac{7-L}{1-L}\|u^n_{\Delta x}\|_{2,\varphi} = K\|u^n_{\Delta x}\|_{2,\varphi}.\end{aligned}$$

Hence, (21b) holds for all ℓ. Using (19), this implies that (21a) holds as well. Using (19), one can show that $\{w^\ell\}$ is Cauchy, hence $\{w^\ell\}$ converges. This completes the proof of Lemma 2.1. □

3. Convergence

As we mentioned earlier, the convergence analysis exploits the fact that the solution operator of the KdV equation possesses an inherent smoothing effect. In particular, we shall use the $H^1_{\mathrm{loc}}(\mathbb{R})$ estimate of the approximate solution generated by the scheme (9). We proceed with the following Lemma.

Lemma 3.1. *Let K and L be as in Lemma 2.1 and let $u^n_{\Delta x}$ be the solution of the scheme* (9). *Assume Δt is such that*

$$\lambda \leq \frac{L}{\sqrt{C_R}2\sqrt{2}K\sqrt{y_T}} \tag{22}$$

for some y_T which depends only on $\|u_0\|_{L^2(\mathbb{R})}$. [1]

Then there exists a positive time T and a constant C, both depending only on $\|u_0\|_{L^2(\mathbb{R})}$, such that for all n satisfying $n\Delta t \leq T$, the following estimate holds

$$\|u^n_{\Delta x}\|_{L^2(\mathbb{R})} \leq C\left(\|u_0\|_{L^2(\mathbb{R})}\right). \tag{23}$$

Furthermore, the approximation $u^n_{\Delta x}$ satisfies the H^1 estimate

(24)

$$\Delta t \sum_{(n+\frac{1}{2})\Delta t \leq T}\left\|\left(u^{n+\frac{1}{2}}_{\Delta x}\right)_x\right\|^2_{L^2([-R,R])} \leq C\left(\left\|u^0\right\|_{L^2(\mathbb{R})}, R\right), \quad \text{for } (n+\tfrac{1}{2})\Delta t < T,$$

where the constant C depends only on R and $\|u_0\|_{L^2(\mathbb{R})}$.

---

[1] The number y_T, is given by $y(T)$, where y is defined in (28).

Proof. Starting with (9), it follows that (5) holds, and consequently

$$
\begin{aligned}
&\int_{\mathbb{R}} \left(u^{n+1}_{\Delta x}\right)^2 \varphi\, dx + \Big(\frac{72-2\sqrt{2}}{24}\Big)\Delta t \int_{\mathbb{R}} \left(u^{n+\frac12}_{\Delta x}\right)^2_x \varphi_x\, dx \\
&\qquad \le \int_{\mathbb{R}} \left(u^{n}_{\Delta x}\right)^2 \varphi\, dx + \frac{\Delta t\sqrt{2}}{4}\Big(\int_{\mathbb{R}} \left(u^{n+\frac12}_{\Delta x}\right)^2 \varphi_x\, dx\Big)^{\frac13}\Big(\int_{\mathbb{R}} \left(u^{n+\frac12}_{\Delta x}\right)^2 \sqrt{\phi_x}\, dx\Big)^{\frac43} \\
&\qquad\quad + \frac{\Delta t}{3}\Big(\int_{\mathbb{R}} \left(u^{n+\frac12}_{\Delta x}\right)^2 \sqrt{\phi_x}\, dx\Big)\Big(\int_{\mathbb{R}} \left(u^{n+\frac12}_{\Delta x}\right)^2 |\varphi_{xx}|\, dx\Big)^{\frac12} \\
&\qquad\quad + \Delta t \int_{\mathbb{R}} \left(u^{n+\frac12}_{\Delta x}\right)^2 |\varphi_{xxx}|\, dx.
\end{aligned}
\tag{25}
$$

When we considered the fully-discrete scheme (3) we had an á priori L^2 bound on $u^n_{\Delta x}$ to proceed. Now we shall instead use this estimate to obtain such a bound.

As $\phi_x \ge 0$ we ignore the second term in (25). The derivatives $\varphi^{(k)}(x) \le \varphi(x)$ for $j = 1, 2, 3\ldots$. Thus, applying Young's inequality yields

$$
\begin{aligned}
\int_{\mathbb{R}} \left(u^{n+1}_{\Delta x}\right)^2 \varphi\, dx \le \int_{\mathbb{R}} \left(u^{n}_{\Delta x}\right)^2 \varphi\, dx + \Delta t\, C_R\Big[\Big(\int_{\mathbb{R}} \left(u^{n+\frac12}_{\Delta x}\right)^2 \varphi\, dx\Big)^{\frac53} \\
+ \Big(\int_{\mathbb{R}} \left(u^{n+\frac12}_{\Delta x}\right)^2 \varphi\, dx\Big)^{\frac32} + \Big(\int_{\mathbb{R}} \left(u^{n+\frac12}_{\Delta x}\right)^2 \varphi\, dx\Big)\Big].
\end{aligned}
\tag{26}
$$

Setting $a_n = \int_{\mathbb{R}} (u^n_{\Delta x})^2 \varphi\, dx$ and $a_{n+\frac12} = \int_{\mathbb{R}} (u^{n+\frac12}_{\Delta x})^2 \varphi\, dx$ gives

$$
a_{n+1} \le a_n + \Delta t\, f(a_{n+\frac12}) \tag{27}
$$

where the function f is given by

$$
f(a) = C_R\left[a^{\frac53} + a^{\frac32} + a\right].
$$

It is easy to see that $a_{n+\frac12} \le (a_n + a_{n+1})/2$. Therefore, $\{a_n\}$ solves the Crank-Nicolson method for the following differential inequality

$$
\frac{da}{dt} \le f(a).
$$

Thus we consider the following ordinary differential equation

$$
\begin{cases}
\frac{dy}{dt} = f\left(\frac{K^2+1}{2} y\right), & t > 0, \\
y(0) = a_0.
\end{cases}
\tag{28}
$$

Since the function f is locally Lipschitz continuous for positive arguments, this differential equation has a unique solution which blows up at some finite time, say at $t = T^\infty$. We choose $T = T^\infty/2$. Also, note that the solution $y(t)$ of the above differential equation is *strictly-increasing* and *convex*. Next we compare the solution of this ODE with (27) under the assumption that (22) holds.

Next we claim that $a_n \le y(t_n)$ for all $n \ge 0$. We argue by induction. Since $y(0) = a_0$, the claim follows for $n = 0$. We assume that the claim holds for $n = 0, 1, 2, \ldots, m$. As $0 < a_m \le y(T)$, (22) implies that λ satisfies the CFL condition (11). So, from Lemma 2.1, we have $a_{m+1} \le K^2 a_m$. Thus

$$
a_{m+\frac12} \le (a_m + a_{m+1})/2 \le \left(\frac{K^2+1}{2}\right) a_m.
$$

Then, using the convexity of f we have

$$\begin{aligned} a_{m+1} &\leq a_m + \Delta t\, f\left(\frac{K^2+1}{2} a_m\right) \\ &\leq y(t_m) + \Delta t f\left(\frac{K^2+1}{2} y(t_m)\right) \\ &\leq y(t_m) + \Delta t \frac{dy}{dt}\Big|_{t=t_m} \leq y(t_{m+1}). \end{aligned}$$

This proves the claim. Therefore, as $\varphi \geq 1$, we have the required L^2-stability estimate

$$\|u^n_{\Delta x}\|_{L^2(\mathbb{R})} \leq \sqrt{y(T)} \leq C\left(\left\|u^0_{\Delta x}\right\|_{L^2(\mathbb{R})}, R\right).$$

Therefore, summing (25) over n, we obtain

$$\Delta t \sum_{n\Delta t \leq T} \int_{-R}^{R} \left|\left(u^{n+\frac{1}{2}}_{\Delta x}\right)_x\right|^2 dx \leq C(R, \|u_0\|_{L^2(\mathbb{R})}).$$

This proves (24) and completes the proof of Lemma 3.1. $\square$

3.1. Bounds on temporal derivative. In this section, we obtain the bounds on time derivative of the approximate solution given by the scheme (9). Following [5], we obtain the estimate on the temporal derivative stated as follows:

Lemma 3.2. *Let $\{u^n_{\Delta x}\}$ be the solution of the scheme* (9)*. We also assume that the hypothesis of Lemma 3.1 hold. Then the following estimate holds*

$$\left\|D^+_t(u^n_{\Delta x}\varphi)\right\|_{H^{-2}([-R,R])} \leq C(\|u_0\|_{L^2(\mathbb{R})}, R)\left(\left\|\left(u^{n+\frac{1}{2}}_{\Delta x}\right)_x\right\|_{L^2([-R,R])} + 1\right), \tag{29}$$

where $D^+_t u^n_{\Delta x}$ is the forward time difference given by

$$D^+_t u^n_{\Delta x} = \frac{u^{n+1}_{\Delta x} - u^n_{\Delta x}}{\Delta t}.$$

Proof. See the proof of Lemma 3.3 in [5]. $\square$

Before stating the theorem of convergence, we define the weak solution of the Cauchy problem for (1) as follows.

Definition 3.1. *Let Q be a given positive number. Then $u \in L^2(0,T;H^1(-Q,Q))$ is said to be a weak solution of* (1) *in the region $[0,T)\times(-Q,Q)$ if*

$$\int_0^T \int_{-\infty}^{\infty} \left(\phi_t u + \phi_x \frac{u^2}{2} - \phi_{xx} u_x\right) dxdt + \int_{-\infty}^{\infty} \phi(x,0) u_0(x)\, dx = 0. \tag{30}$$

for all $\phi \in C^\infty_c((-Q,Q)\times[0,T))$.

Next we define the approximate solution $u^{\Delta x}$ by the following interpolation formula

(31)

$$u^{\Delta x}(t,x) = \begin{cases} u^{n-\frac{1}{2}}_{\Delta x}(x) + (t - t_{n-\frac{1}{2}}) D^t_+ u^{n-\frac{1}{2}}_{\Delta x}(x), & \text{for } t \in [t_{n-\frac{1}{2}}, t_{n+\frac{1}{2}}) \text{ with } n \geq 1, \\ u^0_{\Delta x}(x) + 2t \frac{u^{\frac{1}{2}}_{\Delta x}(x) - u^0_{\Delta x}(x)}{\Delta t} & \text{for } t \in [t_0, t_{\frac{1}{2}}). \end{cases}$$

Then we have the following theorem of convergence.

Theorem 3.1. *Let $\{u^n_{\Delta x}\}_{n\in\mathbb{N}}$ be a sequence of functions defined by the scheme (9), and assume that $\|u_0\|_{L^2(\mathbb{R})}$ is finite. Assume furthermore that $\Delta t = \mathcal{O}(\Delta x^2)$, then there exists a a positive time T and a constant C, depending only on T, R and $\|u_0\|_{L^2(\mathbb{R})}$ such that*

$$\left\|u^{\Delta x}\right\|_{L^\infty(0,T;L^2([-R,R]))} \le C(R, \|u_0\|_{L^2(\mathbb{R})}), \tag{32}$$

$$\left\|u^{\Delta x}\right\|_{L^2(0,T;H^1([-R,R]))} \le C(R, \|u_0\|_{L^2(\mathbb{R})}), \tag{33}$$

$$\left\|\partial_t(u^{\Delta x}\varphi)\right\|_{L^2(0,T;H^{-2}([-R,R]))} \le C(R, \|u_0\|_{L^2(\mathbb{R})}) \tag{34}$$

where $u^{\Delta x}$ is given by (31). Moreover, there exists a sequence $\{\Delta x_j\}_{j=1}^\infty$ and a function $u \in L^2(0,T;L^2([-R,R]))$ such that

$$u^{\Delta x_j} \to u \text{ strongly in } L^2(0,T;L^2([-R,R])), \tag{35}$$

as $j \to \infty$. The function u is a weak solution of the Cauchy problem for (1), that is, it satisfies (30) with $Q = R - 1$.

Proof. For the simplicity we assume that $T = (N + \frac{1}{2})\Delta t$ for some natural number N. We write the approximation $u^{\Delta x}$ as, for $t_{n-\frac{1}{2}} \le t < t_{n+\frac{1}{2}}$,

$$u^{\Delta x}(x,t) = (1-\alpha_n(t))u^{n-\frac{1}{2}}_{\Delta x}(x) + \alpha_n(t)u^{n+\frac{1}{2}}_{\Delta x}(x),$$

where $\alpha_n(t) = (t - t_n)/\Delta t \in [0,1]$. Therefore, using (23), for $n = 1, 2, \dots, N$ we have

$$\begin{aligned}
\left\|u^{\Delta x}\right\|_{L^2(\mathbb{R})} &\le \|1-\alpha_n(t)\|_\infty \left\|u^{n-\frac{1}{2}}_{\Delta x}\right\|_{L^2(\mathbb{R})} + \|\alpha_n(t)\|_\infty \left\|u^{n+\frac{1}{2}}_{\Delta x}\right\|_{L^2(\mathbb{R})} \\
&\le \frac{1}{2}\left(\|u^n_{\Delta x}\|_{L^2(\mathbb{R})} + \left\|u^{n-1}_{\Delta x}\right\|_{L^2(\mathbb{R})} + \|u^n_{\Delta x}\|_{L^2(\mathbb{R})} + \left\|u^{n+1}_{\Delta x}\right\|_{L^2(\mathbb{R})}\right) \\
&\le C(\|u_0\|_{L^2(\mathbb{R})}, R),
\end{aligned}$$

and for $t \in [0, t_{1/2})$,

$$\begin{aligned}
\left\|u^{\Delta x}\right\|_{L^2(\mathbb{R})} &\le \|1-(2t/\Delta t)\|_\infty \left\|u^0_{\Delta x}\right\|_{L^2(\mathbb{R})} + \|2t/\Delta t\|_\infty \left\|u^{\frac{1}{2}}_{\Delta x}\right\|_{L^2(\mathbb{R})} \\
&\le C(\|u_0\|_{L^2(\mathbb{R})}, R).
\end{aligned}$$

Thus, collecting the above bounds we conclude that (32) holds.

Next we derive the estimate on spatial derivative

$$\begin{aligned}
&\int_0^T \left\|u^{\Delta x}_x\right\|^2_{L^2([-R,R])} dt \\
&\le 2\left\|(u^0_{\Delta x})_x\right\|^2_{L^2([-R,R])} \int_0^{t_{\frac{1}{2}}} \left(1 - \frac{2t}{\Delta t}\right)^2 dt + 2\left\|(u^{\frac{1}{2}}_{\Delta x})_x\right\|^2_{L^2([-R,R])} \int_0^{t_{\frac{1}{2}}} \left(\frac{2t}{\Delta t}\right)^2 dt \\
&\quad + 2\sum_{n=1}^N \left\|\left(u^{n-\frac{1}{2}}_{\Delta x}\right)_x\right\|^2_{L^2([-R,R])} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} (1-\alpha_n(t))^2\, dt \\
&\qquad + 2\sum_{n=1}^N \left\|\left(u^{n+\frac{1}{2}}_{\Delta x}\right)_x\right\|^2_{L^2([-R,R])} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} (\alpha_n(t))^2\, dt \\
&\le \Delta t\left\|\left(u^0_{\Delta x}\right)_x\right\|^2_{L^2([-R,R])} + 2\Delta t\sum_{n=0}^N \left\|\left(u^{n+\frac{1}{2}}_{\Delta x}\right)_x\right\|^2_{L^2([-R,R])}.
\end{aligned}$$

Now applying inverse inequality (18) and using (24), we conclude the proof of (33).

Note that

$$u_t^{\Delta x} = \begin{cases} D_+^t u_{\Delta x}^{n-\frac{1}{2}} & \text{for } (t,x) \in \mathbb{R} \times [t_{n-\frac{1}{2}}, t_{n+\frac{1}{2}}), \\ \frac{u_{\Delta x}^{\frac{1}{2}} - u_{\Delta x}^0}{\Delta t/2} & \text{for } (t,x) \in \mathbb{R} \times [0, t_{1/2}), \end{cases}$$
$$= \begin{cases} \frac{D_+^t u_{\Delta x}^n + D_+^t u_{\Delta x}^{n-1}}{2} & \text{for } (t,x) \in \mathbb{R} \times [t_{n-\frac{1}{2}}, t_{n+\frac{1}{2}}), \\ D_+^t u_{\Delta x}^0 & \text{for } (t,x) \in \mathbb{R} \times [0, t_{1/2}). \end{cases}$$

Thus, using (29), it is easy to see that (34) holds.

Since φ is a positive and bounded smooth function, using (32), (33) we have

$$\left\|\varphi u^{\Delta x}\right\|_{L^\infty(0,T;L^2([-R,R]))} \leq C(\|u_0\|_{L^2(\mathbb{R})}, R), \tag{36a}$$

$$\left\|\varphi u^{\Delta x}\right\|_{L^2(0,T;H^1([-R,R]))} \leq C(\|u_0\|_{L^2(\mathbb{R})}, R). \tag{36b}$$

Based on the bounds (36) and (34), applying Aubin-Simon compactness lemma (see [7]) to the set $\left\{\varphi u^{\Delta x}\right\}_{\Delta x>0}$, we conclude that there exists a sequence $\{\Delta x_j\}_{j\in\mathbb{N}}$ such that $\Delta x_j \to 0$, and a function $\tilde{u}$ such that

$$u^{\Delta x_j}\varphi \to \tilde{u} \qquad \text{strongly in } L^2(0,T;L^2([-R,R])), \tag{37}$$

as j goes to infinity. As $\varphi \geq 1$, (37) implies that there exists a u such that (35) holds.

This strong convergence allows passage to the limit in nonlinearity. However, it remains to prove that u is a weak solution of (1). In what follows, we will consider the standard L^2-projection of a function ψ with $k+1$ continuous derivatives into space $S_{\Delta x}$, denoted by $\mathcal{P}$, i.e.,

$$\int_{\mathbb{R}} (\mathcal{P}\psi(x) - \psi(x))\, v(x) = 0, \quad \forall v \in S_{\Delta x}.$$

For the projection mentioned above we have that (for a proof, see the monograph of Ciarlet [4])

$$\|\psi(x) - \mathcal{P}\psi(x)\|_{H^k(\mathbb{R})} \leq C\Delta x \,\|\psi\|_{H^{k+1}(\mathbb{R})},$$

where C is a constant independent of Δx.

We also need the following inequality:

$$\left\|u_{\Delta x}^{n+\frac{1}{2}}\right\|_{L^\infty[-R+1,R-1]} \leq C(R)\left\|u_{\Delta x}^{n+\frac{1}{2}}\right\|_{H^1(-R,R)} \tag{38}$$

where C_R is some positive constant depends only on R.

From (9), we have, for $n \geq 1$

$$\left(D_+^t u_{\Delta x}^n, \varphi v\right) - \left(\frac{(u_{\Delta x}^{n+\frac{1}{2}})^2}{2}, (\varphi v)_x\right) + \left(\left(u_{\Delta x}^{n+\frac{1}{2}}\right)_x, (\varphi v)_{xx}\right) = 0,$$
$$\left(D_+^t u_{\Delta x}^{n-1}, \varphi v\right) - \left(\frac{(u_{\Delta x}^{n-\frac{1}{2}})^2}{2}, (\varphi v)_x\right) + \left(\left(u_{\Delta x}^{n-\frac{1}{2}}\right)_x, (\varphi v)_{xx}\right) = 0.$$

Taking the average of the above two relations gives, for $n \geq 1$

$$\mathcal{F}_n(\varphi v) = 0, \tag{39}$$

where

$$
\begin{aligned}
\mathcal{F}_n(\varphi v) = \left(D_+^t u_{\Delta x}^{n-\frac{1}{2}}, \varphi v\right) - \frac{1}{2}\left(\frac{(u_{\Delta x}^{n+\frac{1}{2}})^2 + (u_{\Delta x}^{n-\frac{1}{2}})^2}{2}, (\varphi v)_x\right) \\
+ \quad \left(\left(\frac{u_{\Delta x}^{n+\frac{1}{2}} + u_{\Delta x}^{n-\frac{1}{2}}}{2}\right)_x, (\varphi v)_{xx}\right)
\end{aligned} \tag{40}
$$

We first show that

$$
\int_0^T \int_{\mathbb{R}} u_t^{\Delta x}\varphi v - \frac{(u^{\Delta x})^2}{2}(\varphi v)_x + (u^{\Delta x})_x(\varphi v)_{xx}\, dx\, dt = \mathcal{O}(\Delta x), \tag{41}
$$

for any test function $v \in C_c^\infty\left((-R+1, R-1)\times[0,T)\right)$, where φ is specified in the beginning of Section 2.

We proceed as follows

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} u_t^{\Delta x}\varphi v - \frac{(u^{\Delta x})^2}{2}(\varphi v)_x + (u^{\Delta x})_x(\varphi v)_{xx}\, dx\, dt \\
&= \int_0^{t_{1/2}} \int_{\mathbb{R}} u_t^{\Delta x}\varphi v - \frac{(u^{\Delta x})^2}{2}(\varphi v)_x + (u^{\Delta x})_x(\varphi v)_{xx}\, dx\, dt \\
&\quad + \sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{\mathbb{R}} u_t^{\Delta x}\varphi v - \frac{(u^{\Delta x})^2}{2}(\varphi v)_x + (u^{\Delta x})_x(\varphi v)_{xx}\, dx\, dt \\
&=: I + II.
\end{aligned}
$$

Let $v^{\Delta x} = \mathcal{P}v$, then from the definition of $u_{\Delta x}$ (c.f. (31)), we write II as follows.

$$
\begin{aligned}
II = &\underbrace{\sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \mathcal{F}_n(\varphi v^{\Delta x}) dt}_{=0\ by\ (39)} + \underbrace{\sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{\mathbb{R}} \left(D_+^t u_{\Delta x}^{n-\frac{1}{2}}\right)\left(\varphi(v - v^{\Delta x})\right)\, dt\, dx}_{\mathcal{E}_1^{\Delta x}} \\
&+ \underbrace{\sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{\mathbb{R}} \frac{1}{2}\frac{(u_{\Delta x}^{n+\frac{1}{2}})^2 + (u_{\Delta x}^{n-\frac{1}{2}})^2}{2}\left(\varphi(v - v^{\Delta x})\right)_x\, dt\, dx}_{\mathcal{E}_2^{\Delta x}} \\
&+ \underbrace{\sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{\mathbb{R}} \frac{\left(u_{\Delta x}^{n+\frac{1}{2}}\right)_x + \left(u_{\Delta x}^{n-\frac{1}{2}}\right)_x}{2}\left(\varphi(v - v^{\Delta x})\right)_{xx}\, dt\, dx}_{\mathcal{E}_3^{\Delta x}} \\
&- \underbrace{\sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{\mathbb{R}} \frac{1}{2}\left[(u^{\Delta x})^2 - \frac{(u_{\Delta x}^{n+\frac{1}{2}})^2 + (u_{\Delta x}^{n-\frac{1}{2}})^2}{2}\right](\varphi v)_x\, dt\, dx}_{\mathcal{E}_4^{\Delta x}} \\
&+ \underbrace{\sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{\mathbb{R}} \left[u_x^{\Delta x} - \frac{(u_{\Delta x}^{n+\frac{1}{2}})_x + (u_{\Delta x}^{n-\frac{1}{2}})_x}{2}\right](\varphi v)_{xx}\, dt\, dx}_{\mathcal{E}_5^{\Delta x}}.
\end{aligned}
$$

Similarly I can be written as

$$I = \underbrace{\int_0^{t_{1/2}} \int_{\mathbb{R}} \left[(D_+^t u_{\Delta x}^0)(\varphi v^{\Delta x}) - \frac{1}{2}(u_{\Delta x}^{\frac{1}{2}})^2(\varphi v^{\Delta x})_x + (u_{\Delta x}^{\frac{1}{2}})_x(\varphi v^{\Delta x})_{xx} \right] dt\, dx}_{=0 \, by\, (9)}$$

$$+ \underbrace{\int_0^{t_{1/2}} \int_{\mathbb{R}} (D_+^t u_{\Delta x}^0)\varphi(v - v^{\Delta x})\, dt\, dx}_{\mathcal{C}_1^{\Delta x}} + \underbrace{\int_0^{t_{1/2}} \int_{\mathbb{R}} \frac{1}{2}(u_{\Delta x}^{\frac{1}{2}})^2 \left(\varphi(v - v^{\Delta x})\right)_x \, dt\, dx}_{\mathcal{C}_2^{\Delta x}}$$

$$+ \underbrace{\int_0^{t_{1/2}} \int_{\mathbb{R}} (u_{\Delta x}^{\frac{1}{2}})_x \left(\varphi(v - v^{\Delta x})\right)_{xx} \, dt\, dx}_{\mathcal{C}_3^{\Delta x}}$$

$$- \underbrace{\int_0^{t_{1/2}} \int_{\mathbb{R}} \frac{1}{2} \left[(u^{\Delta x})^2 - (u_{\Delta x}^{\frac{1}{2}})^2 \right] (\varphi v)_x \, dt\, dx}_{\mathcal{C}_4^{\Delta x}}$$

$$+ \underbrace{\int_0^{t_{1/2}} \int_{\mathbb{R}} \left[u_x^{\Delta x} - (u_{\Delta x}^{\frac{1}{2}})_x \right] (\varphi v)_{xx} \, dt\, dx}_{\mathcal{C}_5^{\Delta x}}.$$

Using the similar argument as in the proof of (Theorem 3.1, [5]), one can show that, for $j = 1, \dots, 5$

$$\left|\mathcal{E}_j^{\Delta x}\right|, \ \left|\mathcal{C}_j^{\Delta x}\right| \to 0, \text{ as } \Delta x \downarrow 0.$$

Thus, we conclude that (41) holds. Furthermore, passing limit as $\Delta x \to 0$, we conclude that

$$\int_0^T \int_{\mathbb{R}} u_t \varphi v - \frac{u^2}{2}(\varphi v)_x + u_x(\varphi v)_{xx} \, dx\, dt = 0, \tag{42}$$

for any test function $v \in C_c^\infty([-R+1, R-1] \times [0, T))$. Finally, we choose $v = \phi/\varphi$ in (42) with $\phi \in C_c^\infty([-R+1, R-1] \times [0, T))$ and integrate-by-parts to conclude that (30) holds, i.e. that

$$\int_0^T \int_{-\infty}^{\infty} \Big(\phi_t u + \phi_x \frac{u^2}{2} - \phi_{xx} u_x \Big)\, dx dt + \int_{-\infty}^{\infty} \phi(x, 0) u_0(x)\, dx = 0.$$

This finishes the proof of the Theorem 3.1 □

4. Numerical experiments

The fully-discrete scheme given by (9) has been tested on two numerical experiments in order to investigate how well this method works in practice.

We let $S_{\Delta x}$ consist of piecewise cubic splines defined as follows: Let f and g be the functions

$$f(y) = 1 + y^2\,(2\,|y| - 3)\,,$$

$$g(y) = \begin{cases} y(y+1)^2 & y \le 0, \\ -y(y-1)^2 & y > 0, \end{cases}$$

and we define $f(y) = g(y) = 0$ for $|y| > 1$. For $j \in \mathbb{Z}$ we define

$$v_{2j}(x) = f\left(\frac{x - x_j}{\Delta x}\right), \qquad v_{2j+1}(x) = g\left(\frac{x - x_j}{\Delta x}\right),$$

where $x_j = j\Delta x$. The space spanned by $\{v_j\}_{j=-M}^{M}$ is a $4M + 2$ dimensional subspace of $H^2(\mathbb{R})$. In our numerical examples, we used periodic boundary conditions. In the two-soliton example, the exact solution, as well as the numerical approximations, are all very close to zero at the boundary. Regarding the weight function, we chose this to be $\varphi(x) = 50 + x$ in the intervals under consideration in both our examples. In the Newton iteration to obtain u^{n+1}, (10), we terminated the iteration if $\left\|w^{\ell+1} - w^{\ell}\right\|_{L^2} \le 0.002\Delta x \left\|u^n\right\|_{L^2}$, something which typically required 3-5 iterations. Furthermore, we observed that setting $\Delta t = \mathcal{O}(\Delta x)$, as opposed to $\Delta t = \mathcal{O}\left(\Delta x^2\right)$ as warranted by the theory, did not effect the quality of the approximate solutions. In our computations, we therefore use $\Delta t = 0.5\Delta x$.

For $t = n\Delta t$, we set $u_{\Delta x}(x, t) = u^n(x, t) = \sum_{j=-M}^{M} u_j^n v_j(x)$. We measured the percentage L^2 error, defined as

$$E = 100 \, \frac{\left\|u - u_{\Delta x}\right\|_{L^2}}{\left\|u\right\|_{L^2}},$$

where the norms were computed using the trapezoid rule on the points x_j. The map taking initial data to the solution of the KdV equation at a time $t > 0$ is known to preserve an infinite number of integrals, the first two of which are $\int_{\mathbb{R}} u \, dx$ and $\int_{\mathbb{R}} u^2 \, dx$. It has been observed that numerical methods which preserve discrete variants of some of these integrals, generate more accurate approximations than methods which preserve fewer. We therefore computed how well the two first integral were preserved, measured by

$$I_1 = \frac{\int u_{\Delta x} \, dx}{\int u_0 \, dx} \quad \text{and} \quad I_2 = \frac{\left\|u_{\Delta x}\right\|_{L^2}}{\left\|u_0\right\|_{L^2}}.$$

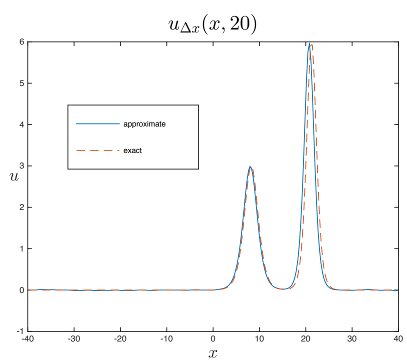

FIGURE 1. The exact and numerical solutions at $t = 20$ with initial data $w_2(x, -10)$ with $M = 128$.

TABLE 1. Relative percentage L^2 errors, and I_1 and I_2 for the two-soliton solution measured at $t = 20$.

M	E	rate	I_1	I_2
64	83.62		0.99	1.09
128	27.70	1.6	1.00	1.02
256	7.91	1.8	1.00	1.02
512	1.91	2.0	1.00	1.00
1024	0.47	2.0	1.00	1.00
2048	0.12	2.0	1.00	1.00
4096	0.03	2.0	1.00	1.00

A two-soliton solution. We tested the method on the so-called two-soliton solution. This is a family of exact solutions given by the formula

(43)
$$w_2(x,t) = 6(b-a)\frac{b\,\mathrm{csch}^2\left(\sqrt{b/2}(x-2bt)\right) + a\,\mathrm{sech}^2\left(\sqrt{a/2}(x-2at)\right)}{\left(\sqrt{a}\tanh\left(\sqrt{a/2}(x-2at)\right) - \sqrt{b}\coth\left(\sqrt{b/2}(x-2bt)\right)\right)^2},$$

for any real numbers a and b. We have used $a = 0.5$ and $b = 1$, and set $u_0(x) = w_2(x, -10)$. This solution represents two waves that "collide" at $t = 10$ and separate for $t > 10$. Computationally, this is a challenging problem. We computed the approximate solution at $t = 20$, and the exact solution in this case is $w_2(x, 10)$. In Figure 1 we show the exact and numerical solutions at $t = 20$.

The computed solution in Figure 1 looks "right", in the sense that the two bumps in the solution have separated well and passed through each other. Nevertheless, the error is about 30%. This is due to an error in the position of the larger bump, which again is due to a much smaller error in the height of the bump. This error causes the speed of the wave to be slightly smaller than the speed of the corresponding wave in the exact solution. Since the wave is quite narrow, this causes the L^2 error to be large. In Table 1 we show the percentage errors for the two-soliton simulation.

What is notable here is that once we are in the asymptotic regime, the errors here are *much* smaller than the errors found using a fully implicit method, reported in [5], and that the rate seems to converge to 2.

TABLE 2. Relative percentage L^2 errors at $t = 0.1$, with initial data given by (44).

$\log_2(M)$	E	rate
11	48	
12	48	0.0
13	41	0.2
14	41	0.0
15	37	0.2
16	33	0.2

Irregular initial data. Since the theory states that the method converges for initial data in L^2, we tested the scheme for an example with initial data in L^2 but

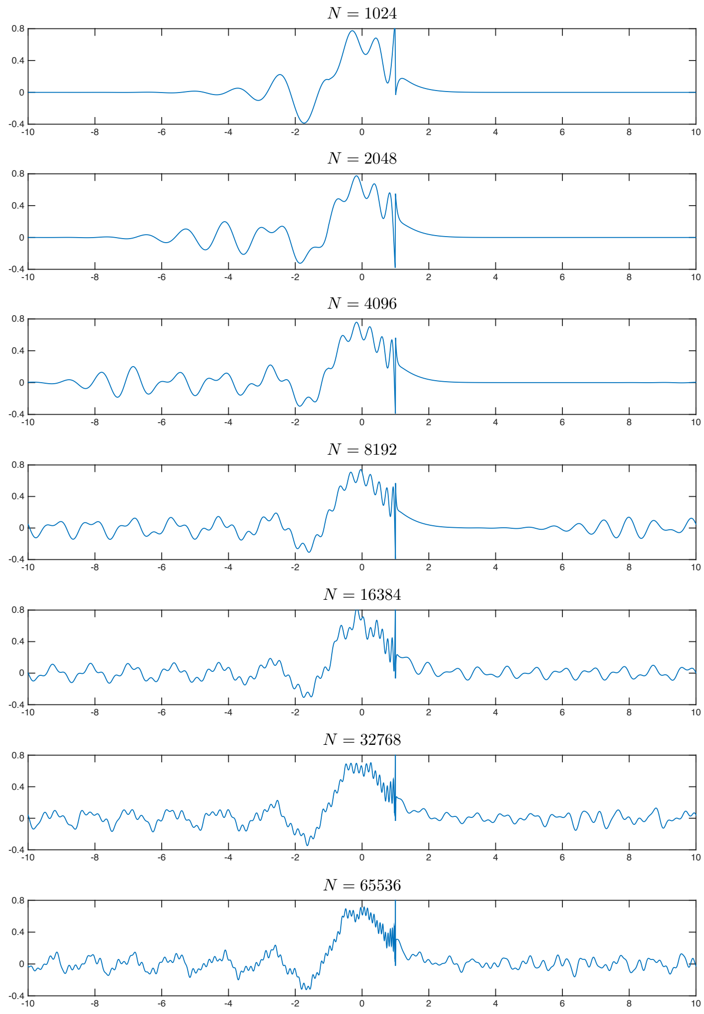

FIGURE 2. Approximate solutions at $t = 0.1$ using initial data given by (44) with $N = 1024,\ldots,65536$ grid points in the interval $[-10, 10]$.

not in H^1. For this test case we used

$$u(x,0) = \begin{cases} \frac{1}{2}(x+1) & |x| \le 1, \\ 0 & \text{otherwise}, \end{cases} \tag{44}$$

for $x \in [-10, 10]$ and periodically extended outside this interval. In this case we do not have a reliable reference solution, so we considered the self-convergence of the scheme. In Figure 2 we show the computed solution using $2^{10}, 2^{11}, \ldots, 2^{16}$ grid points in the interval $[-10, 10]$.

This solution looks much more complicated than the two-soliton solution, and it is perhaps not apparent from the figure whether the approximations converge.

In Table 2 we show the relative L^2 errors using the approximate solution with 2^{16} grid points as a reference solution.

This indicates that the convergence rate, if indeed there is such a rate, is small. This is similar to numerical results [7] where the numerical convergence rate for a different example with L^2 initial data was found to be small. It is not surprising that less regular initial data gives a lower convergence rate.

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