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A PRIORI ERROR ESTIMATES OF A SIGNORINI CONTACT PROBLEM FOR ELECTRO-ELASTIC MATERIALS

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Abstract. We consider a mathematical model for a static process of frictionless unilateral contact between a piezoelectric body and a conductive foundation. A variational formulation of the model, in the form of a coupled system for the displacements and the electric potential, is derived. The existence of a unique weak solution for the problem is established. We use the penalty method applied to the frictionless unilateral contact model to replace the Signorini contact condition, we show the existence of a unique solution, and derive error estimates. Moreover, under appropriate regularity assumptions of the solution, we have the convergence of the continuous penalty solution as the penalty parameter ϵ vanishes. Then, the numerical approximation of a penalty problem by using the finite element method is introduced. The error estimates are derived and convergence of the scheme is deduced under suitable regularity conditions.

Key words. Piezoelectric, variational inequality, Signorini condition, penalty method, fixed point process, finite element approximation, error estimates.

1. Introduction

In recent years, piezoelectric materials have triggered intensive studies to fulfill their potential applications in a variety of fields due to include the coupling between the mechanical and electrical material properties. Indeed, there is a considerable interest in frictional or frictionless contact problems involving piezoelectric materials, see, e.g., [1, 2, 4, 7, 8, 9, 11] and the references therein. Here, we consider a mathematical model which describes the frictionless contact between an piezoelectric body and a foundation, within the framework of small deformations theory. The material's behavior is modeled with a linear electroelastic constitutive law, the process is static and the foundation is assumed to be electrically conductive. Contact is described with the Signorini contact conditions and a regularized electrical conductivity condition. The numerical approximation of a static unilateral contact problems with or without friction for piezoelectric materials can be found in [1, 2, 5, 7].

In the present work, the numerical approximations were based on variational inequalities modeling unilateral contact in piezoelectricity. Here, a penalty method is employed to replace the Signorini contact condition. This approach was used previously by F. Chouly and P. Hild [3] to numerically approximate the solution of contact problems in linear elasticity. The novelty of the paper is in dealing with a model which couples the piezoelectric properties of the material with the electrical conductivity conditions on the contact surface. Consideration of the electrical contact condition leads to nonstandard boundary conditions on the contact surface and supplementary nonlinearities in the problem. Because of the latter and piezoelectric effect, the mathematical problem is formulated as a coupled system of the variational inequality for the displacement field and non-linear variational equation for the electric potential. In this paper, We analyze both the continuous and discrete (using continuous conforming piecewise linear finite element methods)

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problems. We show that the theoretical convergence of the penalty method gives the best results when $\epsilon = h$, where ϵ is the penalty parameter, and h is the mesh size. We note that the convergence is limited by the same terms involved when considering the direct approximation of the variational inequality without penalty.

The paper is organized as follows. In Section 2 we present the models of electroelastic frictionless unilateral contact with the electrical contact condition. list the assumptions on the data, derive the variational formulation of each model, and state the existence and uniqueness result. In Section 3 we introduce the penalty problem and show that it has a unique solution. In Section 4, we describe the finite element approximation of the penalty problem and we present the results of some error estimates for the numerical approximation. finally, The proof of the main result is provided in Section 5.

2. Setting of the problem and variational formulation

2.1. The contact problem. In this section we describe the problem of unilateral frictionless contact between a piezoelectric body and a conductive foundation.
The physical setting is the following : we consider an elasto-piezoelectric body which initially occupies an open bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with a sufficiently smooth boundary $\partial\Omega = \Gamma$. The body is acted upon by a volume forces of density f_0 and has volume electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe these constraints we decompose Γ into three mutually disjoint open parts Γ_D, Γ_N and Γ_C, on the one hand, and a partition of $\Gamma_D \cup \Gamma_N$ into two open parts Γ_a and Γ_b, on the other hand, such that $meas(\Gamma_D) > 0$ and $meas(\Gamma_a) > 0$. The body is clamped on Γ_D and a surface tractions of density f_2 act on Γ_N. Moreover, the electric potential vanishes on Γ_a and the surface electric charge of density q_2 is prescribed on Γ_b. On Γ_C the body may come into contact with a conductive obstacle, the so called foundation. We assume that the foundation is electrically conductive and its potential is maintained at φ_F. The contact is frictionless unilateral and there may be electrical charges on the contact surface. The indices i, j, k, l run between 1 and d. The summation convention over repeated indices is adopted and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable, e.g., $u_{i,j} = \partial u_i/\partial x_j$. Everywhere below we use $\mathbb{S}^d$ to denote the space of second order symmetric tensors on $\mathbb{R}^d$ while " $\cdot$ " and $\|\cdot\|$ will represent the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, that is $\forall u,\, v \in \mathbb{R}^d$, $\forall \sigma,\, \tau \in \mathbb{S}^d$,

$$u \cdot v = u_i \cdot v_i, \quad \|v\| = (v \cdot v)^{\frac{1}{2}}, \quad \text{and} \quad \sigma \cdot \tau = \sigma_{ij} \cdot \tau_{ij}, \quad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}}.$$

We denote by $u : \Omega \to \mathbb{R}^d$ the displacement field, by $\sigma : \Omega \to \mathbb{S}^d$, $\sigma = (\sigma_{ij})$ the stress tensor and by $D : \Omega \to \mathbb{R}^d$, $D = (D_i)$ the electric displacement field. We also denote $E(\varphi) = (E_i(\varphi))$ the electric vector field, where $\varphi : \Omega \to \mathbb{R}$ is an electric potential such that $E(\varphi) = -\nabla\varphi$. We shall adopt the usual notations for normal and tangential components of displacement vector and stress : $v_n = v \cdot n$, $\quad v_\tau = v - v_n n$, $\quad \sigma_n = (\sigma n) \cdot n$, $\quad \sigma_\tau = \sigma n - \sigma_n n$, where n denote the outward normal vector on Γ. Moreover, let $\varepsilon(u) = (\varepsilon_{ij}(u))$ denote the linearized strain tensor given by $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, and " Div ", " div " denote respectively the divergence operators for tensor and vector valued functions, i.e. $\operatorname{Div}\sigma = (\sigma_{ij,j})$, $\quad \operatorname{div} D = (D_{j,j})$.

Under the previous assumption, the classical model for this process is the following.
Problem P. Find a displacement field $u : \Omega \to \mathbb{R}^d$, a stress field $\sigma : \Omega \to \mathbb{S}^d$, an

electric potentiel $\varphi : \Omega \to \mathbb{R}$ and an electric displacement field $D : \Omega \to \mathbb{R}^d$ such that

(1) $$\sigma = \mathfrak{F}\varepsilon(u) - \mathcal{E}^* E(\varphi) \quad \text{in } \Omega,$$

(2) $$D = \mathcal{E}\varepsilon(u) + \beta E(\varphi) \quad \text{in } \Omega,$$

(3) $$Div\sigma(u, \varphi) + f_0 = 0 \quad \text{in } \Omega,$$

(4) $$\operatorname{div} D = q_0 \quad \text{in } \Omega,$$

(5) $$u = 0 \quad \text{on } \Gamma_D,$$

(6) $$\sigma n = f_2 \quad \text{on } \Gamma_N,$$

(7) $$\sigma_n \leq 0,\ u_n \leq 0,\ \sigma_n u_n = 0,\ \sigma_\tau = 0 \quad \text{on } \Gamma_C,$$

(8) $$\varphi = 0 \quad \text{on } \Gamma_a,$$

(9) $$D \cdot n = q_2 \quad \text{on } \Gamma_b,$$

(10) $$D \cdot n = \psi(u_n)\phi_L(\varphi - \varphi_F) \quad \text{on } \Gamma_C,$$

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \overline{\Omega}$. Equations (1) and (2) represent the electro-elastic constitutive law of the material in which $\mathfrak{F}$ denotes the elasticity operator, $\mathcal{E}$ represents the third order piezoelectric tensor, $\mathcal{E}^*$ is its transpose and β denotes the electric permittivity tensor. Equations (3) and (4) represent the equilibrium equations for the stress and electric displacement fields, respectively. Relations (5) and (6) are the displacement and traction boundary conditions, respectively, and (8), (9) represent the electric boundary conditions. The frictionless unilateral boundary conditions (7) represent the Signorini law. Finally, (10) represent the regularization electrical contact condition on Γ_C, which was considered in [6], where ψ and ϕ are a regularization function and the truncation function, respectively, such that

$$\phi_L(s) = \begin{cases} -L & \text{if } s < -L, \\ s & \text{if } -L \leq s \leq L, \\ L & \text{if } s > L, \end{cases} \qquad \psi(r) = \begin{cases} 0 & \text{if } r < 0, \\ k\delta r & \text{if } 0 \leq r \leq \frac{1}{\delta}, \\ k & \text{if } r > \frac{1}{\delta}, \end{cases}$$

in which L is a large positive constant, $\delta > 0$ denotes a small parameter and $k \geq 0$ is the electrical conductivity coefficient. Note also that when $\psi \equiv 0$, then (10) leads to

(11) $$D \cdot n = 0 \quad \text{on } \Gamma_C.$$

The condition (11) models the case when the obstacle is a perfect insulator.

2.2. Variational formulation. To present the variational formulation of Problem P we need some additional notation and preliminaries. We start by introducing the spaces

$$H = L^2(\Omega)^d, \quad H_1 = H^1(\Omega)^d,$$
$$\mathcal{H} = \{\tau = (\tau_{ij}) \,|\, \tau_{ij} = \tau_{ji} \in L^2(\Omega)\}, \quad \mathcal{H}_1 = \{\sigma \in \mathcal{H} \,|\, \operatorname{Div}\sigma \in H\}.$$

These are real Hilbert spaces endowed with the inner products

$$(u, v)_H = \int_\Omega u_i v_i \, dx, \quad (u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}},$$

$$(\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \, dx, \quad (\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (\operatorname{Div}\sigma, \operatorname{Div}\tau)_{\mathcal{H}},$$

and the associated norms $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively.

Let $H_\Gamma = H^{1/2}(\Gamma)^d$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $v \in H_1$, we also use the notation v to denote the trace γv of v on Γ.

Let $H_\Gamma^{'}$ be the dual of H_Γ and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $H_\Gamma^{'}$ and H_Γ. For every $\sigma \in \mathcal{H}_1$, σn can be defined as the element in $H_\Gamma^{'}$ which satisfies

$$\langle \sigma n, \gamma v \rangle = (\sigma, \varepsilon(v))_{\mathcal{H}} + (\operatorname{Div} \sigma, v)_H, \quad \forall v \in H_1. \tag{12}$$

Moreover, If σ is continuously differentiable on $\overline{\Omega}$, then

$$\langle \sigma n, \gamma v \rangle = \int_\Gamma \sigma n \cdot v \; da. \tag{13}$$

for all $v \in H_1$, where da is the surface measure element. Keeping in mind the boundary condition (5), we introduce the closed subspace of H_1 defined by

$$V = \{v \in H_1 \,|\, v = 0 \text{ on } \Gamma_D\},$$

and K be the set of admissible displacements

$$K = \{v \in V \,|\, v_n \leq 0 \text{ on } \Gamma_C\}.$$

Since $meas(\Gamma_D) > 0$ and Korn's inequality (see, e.g., [10]) holds,

$$\|\varepsilon(v)\|_{\mathcal{H}} \geq c_k \|v\|_{H_1}, \quad \forall v \in V, \tag{14}$$

where $c_k > 0$ is a constant which depends only on Ω and Γ_D. Over the space V we consider the inner product given by

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \quad \|u\|_V = (u, u)_V^{\frac{1}{2}}, \tag{15}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (14) that the norms $\|\cdot\|_{H_1}$and $\|\cdot\|_V$ are equivalent on V. Therefore $(V, \|\cdot\|_V)$ is a Hilbert space. Moreover, by the Sobolev trace theorem, (14) and (15) there exists a constant $c_0 > 0$ which only depends on the domain Ω, Γ_C and Γ_D such that

$$\|v\|_{L^2(\Gamma)^d} \leq c_0 \|v\|_V, \quad \forall v \in V. \tag{16}$$

We also introduce the spaces

$$\begin{aligned} W &= \{\psi \in H^1(\Omega) \,|\, \psi = 0 \text{ on } \Gamma_a\}, \\ \mathcal{W} &= \{D = (D_i) \in H^1(\Omega) \,|\, (D_i) \in L^2(\Omega), \ \operatorname{div} D \in L^2(\Omega)\}. \end{aligned}$$

The spaces W and $\mathcal{W}$ are real Hilbert spaces with the inner products

$$(\varphi, \psi)_W = (\varphi, \psi)_{H^1(\Omega)}, \quad (D, E)_{\mathcal{W}} = (D, E)_{L^2(\Omega)^d} + (\operatorname{div} D, \operatorname{div} E)_{L^2(\Omega)}.$$

The associated norms will be denoted by $\|\cdot\|_W$ and $\|\cdot\|_{\mathcal{W}}$, respectively. Notice also that, since $meas(\Gamma_a) > 0$, the following Friedrichs-Poincar inequality holds:

$$\|\nabla \psi\|_{\mathcal{W}} \geq c_F \|\psi\|_W, \quad \forall \psi \in W, \tag{17}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. Moreover, by the Sobolev trace theorem, there exists a constant c_1, depending only on Ω, Γ_a and Γ_C, such that

$$\|\xi\|_{L^2(\Gamma_C)} \leq c_1 \|\xi\|_W, \quad \forall \xi \in W. \tag{18}$$

When $D \in \mathcal{W}$ is a sufficiently regular function, the following Green's type formula holds,

$$(D, \nabla \xi)_{L^2(\Omega)^d} + (\operatorname{div} D, \xi)_{L^2(\Omega)} = \int_\Gamma D \cdot \nu \xi \; da, \quad \forall \xi \in H^1(\Omega). \tag{19}$$

As usual, we denote by $(H^s(\Omega))^d, s \in \mathbb{R}, d = 1, 2, 3$, the Sobolev spaces in one, two or three space dimensions. The Sobolev norm of $(H^s(\Omega))^d$ (dual norm if $s < 0$) is denoted by $\| \cdot \|_{s,\Omega}$ and we keep the same notation when $d = 1, 2$ or 3.

Recall also that the transposite $\mathcal{E}^*$ is given by

$$\mathcal{E}^* = (e^*_{ijk}), \quad \text{where} \;\; e^*_{ijk} = e_{kij},$$

$$\mathcal{E}\sigma v = \sigma \mathcal{E}^* v, \quad \forall \sigma \in \mathbb{S}^d, v \in \mathbb{R}^d. \tag{20}$$

In the study of problem (1)-(10) We will need the following hypotheses.

- (h_1) The elasticity operator $\mathfrak{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies $f = (f_{ijkl})$, $f_{ijkl} = f_{jikl} = f_{lkij} \in L^\infty(\Omega)$ and $f_{ijkl}(x)\ \xi_k \xi_l \geq \alpha_a\ \|\xi\|^2 \quad \forall \xi \in \mathbb{S}^d$, $\forall x \in \Omega$ with $\alpha_a > 0$.
- (h_2) The piezoelectric tensor $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies $\mathcal{E} = (e_{ijk})$, $e_{ijk} = e_{ikj} \in L^\infty(\Omega)$.
- (h_3) The electric permittivity tensor $\beta : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies $\beta = (\beta_{ij})$, $\beta_{ij} = \beta_{ji} \in L^\infty(\Omega)$ and $\beta_{ij}\ \xi_i \xi_j \geq \alpha_b\ \|\xi\|^2$ for all $\xi = (\xi_i) \in \mathbb{R}^d$ and $x \in \Omega$, with $\alpha_b > 0$.
- (h_4) The surface electrical conductivity $\psi : \Gamma_C \times \mathbb{R} \to \mathbb{R}_+$ is a bounded function by a constant $M_\psi > 0$, such as, $x \to \psi(x, u)$ is measurable on Γ_C, for all $u \in \mathbb{R}$ and is zero for all $u \leq 0$.
- (h_5) The function $u \to \psi(x, u)$ is a Lipschitz function on $\mathbb{R}$ for all $x \in \Gamma_C$; $|\psi(x, u_1) - \psi(x, u_2)| \leq L_\psi |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R}$, with $L_\psi > 0$.
- (h_6) $f_0 \in L^2(\Omega)^d, \qquad f_2 \in L^2(\Gamma_N)^d$,
- (h_7) $q_0 \in L^2(\Omega), \qquad q_2 \in L^2(\Gamma_b)$.
- (h_8) $\varphi_F \in L^2(\Gamma_C)$.

Next, we use Riesz's representation theorem, consider the elements $f \in V$, and $q \in W$ given by

$$(f, v)_V = \int_\Omega f_0 \cdot v\ dx + \int_{\Gamma_N} f_2 \cdot v\ da, \qquad \forall v \in V, \tag{21}$$

$$(q, \xi)_W = \int_\Omega q_0 \xi\ dx - \int_{\Gamma_b} q_2 \xi\ da, \qquad \forall \xi \in W, \tag{22}$$

and, we define the mapping $\ell : V \times W \times W \to \mathbb{R}$ by

$$\ell(u, \varphi, \xi) = \int_{\Gamma_C} \psi(u_n) \phi_L(\varphi - \varphi_F) \xi\ da, \forall u \in V,\ \forall \varphi, \xi \in W, \tag{23}$$

Keeping in mind assumptions (h_4)-(h_8) it follows that the integrals in (21)-(23) are well-defined. Using Grenn's formula (12), (13) and (19) it is straightforward to see that if $(u,\ \sigma,\ \varphi,\ D)$ are sufficiently regular function which satisfy (3)-(10) then

$$(\sigma(u, \varphi), \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} \geq (f, v - u)_V, \qquad \forall v \in K, \tag{24}$$

$$(D, \nabla \xi)_{L^2(\Omega)^d} = \ell(u, \varphi, \xi) - (q, \xi)_W, \qquad \forall \xi \in W. \tag{25}$$

We plug (1) in (24), (2) in (25) and use the notation $E = -\nabla \varphi$ to obtain the following variational formulation of Problem P, in the terms of displacement field and electric potential.

Problem PV *Find a displacement field* $u \in K$ *and an electric potential* $\varphi \in W$ such that :

$$(\mathfrak{F}\varepsilon(u), \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} + (\mathcal{E}^* \nabla \varphi, \varepsilon(v) - \varepsilon(u))_{L^2(\Omega)^d} \geq (f, v - u)_V,\ \forall v \in K, \tag{26}$$

$$(\beta \nabla \varphi, \nabla \xi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u), \nabla \xi)_{L^2(\Omega)^d} + \ell(u, \varphi, \xi) = (q, \xi)_W,\ \forall \xi \in W. \tag{27}$$

2.3. Existence and uniqueness of the solution. The existence of the solution to Problem PV is given by the following result.

Theorem 2.1. *Assume that* (h_1)*-*(h_4) *and* (h_8) *hold. Then :*

1. *Problem* (PV) *has at least one solution* $(u,\varphi) \in K \times W$*;*
2. *Under the assumption* (h_5)*, there exists* $L^* > 0$ *such that if* $L_\psi L + M_\psi < L^*$*. Then Problem* (PV) *has a unique solution.*

Proof. The proof of Theorem 2.1 will be carried out in several steps. We suppose in the sequel that the assumption of Theorem 2.1 are fulfilled and we consider the product space $X = V \times W$ which is a Hilbert space endowed with the inner product

$$(x,y)_X = (u,v)_V + (\varphi,\xi)_W, \qquad \text{for all } x = (u,\varphi) \text{ and } y = (v,\xi) \in X, \tag{28}$$

The corresponding norm is denoted by $\|\cdot\|_X$. Let $U = K \times W$ be non-empty closed convex subset of X. We also introduce the operator $A : X \to X$, the functions j on X and the element $f^e \in X$ by equalities:

$$\begin{aligned}(Ax,y)_X &= (\mathfrak{F}\varepsilon(u),\varepsilon(v))_{\mathcal{H}} + (\beta\nabla\varphi,\nabla\xi)_{L^2(\Omega)^d} + (\mathcal{E}^*\nabla\varphi,\varepsilon(v))_{L^2(\Omega)^d}\\ &\quad - (\mathcal{E}\varepsilon(u),\nabla\xi)_{L^2(\Omega)^d}, \quad \forall x = (u,\varphi),\ y = (v,\xi) \in X,\end{aligned} \tag{29}$$

$$j(x,y) = \int_{\Gamma_C} \psi(u_n)\phi_L(\varphi - \varphi_F)\xi\, da, \quad \forall x = (u,\varphi),\ y = (v,\xi) \in X, \tag{30}$$

$$f^e = (f,q) \in X. \tag{31}$$

We start by the following equivalence result

Lemma 2.2. *The couple* $x = (u,\varphi)$ *is a solution to problem* (PV) *if and only if:*

$$(Ax, y-x)_X + j(x, y-x) \geq (f^e, y-x)_X, \forall y = (v,\xi) \in U. \tag{32}$$

Proof. Let $x = (u,\varphi) \in U$ be a solution to problem PV and let $y = (v,\xi) \in U$. We use the test function $\xi - \varphi$ in (27), add the corresponding inequality to (26) and use (28) and (29)-(31) to obtain (32). Conversely, let $x = (u,\varphi) \in U$ be a solution to the elliptic variational inequalities (32). We take $y = (v,\varphi)$ in (31) where v is an arbitrary element of K and obtain (26). Then for any $\xi \in W$, we take successively $y = (v,\varphi+\xi)$ and $y = (v,\varphi-\xi)$ in (32) to obtain (27), which concludes the proof of lemma 2.2. □

Let $\eta \in L^2(\Gamma_C)$ be given, and we define the closed convex set

$$\mathcal{K} = \{\eta \in L^2(\Gamma_C)\ /\ \|\eta\|_{L^2(\Gamma_C)} \leq \kappa\},$$

where the constants k is given by

$$\kappa \quad = \quad M_\psi\, L\, meas(\Gamma_C)^{\frac{1}{2}}. \tag{33}$$

In the next step we prove the following existence and uniqueness result

Lemma 2.3. *For any* $\eta \in \mathcal{K}$*, assume that* (h_1)*-*(h_3) *hold. Then, there exists a unique solution* $x_\eta = (u_\eta,\varphi_\eta) \in U$ *such that*

$$(Ax_\eta, y - x_\eta)_X \geq (f^e_\eta, y - x_\eta)_X, \quad \forall\, y = (v,\xi) \in U. \tag{34}$$

Where

$$j_\eta(\xi) = \int_{\Gamma_C} \eta\, \xi da,\ \forall \xi \in W, \tag{35}$$

$$(f^e_\eta, y - x_\eta)_X = (f^e, y - x_\eta)_X - j_\eta(\xi) \quad \forall\, y = (v,\xi) \in U. \tag{36}$$

Proof. Consider two elements $x_1 = (u_1, \varphi_1),\ x_2 = (u_2, \varphi_2) \in X$, using (29), (20), (18), (h_1) and (h_3) there exists $m_A > 0$ which depends only on α_a, β, Ω and Γ_a such that

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geq m_A(\|u_1 - u_2\|_V^2 + \|\varphi_1 - \varphi_2\|_W^2),$$

and, keeping in mind (28), we obtain

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geq m_A\|x_1 - x_2\|_X^2. \tag{37}$$

In the same way, using (h_1)-(h_3), after some algebra it follows that there exists $c_A > 0$ which depends only on $\mathfrak{F}$, β and $\mathcal{E}$ such that

$$\begin{aligned}(Ax_1 - Ax_2, y)_X \quad &\leq \quad c_A\big(\|u_1 - u_2\|_V\|v\|_V + \|\varphi_1 - \varphi_2\|_W\|v\|_V \\ &\qquad + \|u_1 - u_2\|_V\|\xi\|_W + \|\varphi_1 - \varphi_2\|_W\|\xi\|_W\big),\end{aligned}$$

for all $y = (v, \xi) \in X$. We use (28) and the previous inequality to obtain

$$(Ax_1 - Ax_2, y)_X \leq 4c_A(\|x_1 - x_2\|_X\ \|y\|_X),\ \forall\, y \in X,$$

and, taking $y = Ax_1 - Ax_2 \in X$, we find

$$\|Ax_1 - Ax_2\|_X \leq M_A\|x_1 - x_2\|_X, \tag{38}$$

where $M_A = 4c_A$. Moreover, using (35) and (31) it is easy to see that the function f_η^e defined by (36) is an element of X. Lemma 2.3 result now from (54), (55) and standard arguments of elliptic variational inequalities. $\square$

We now consider the operator $\Lambda : L^2(\Gamma_C) \to L^2(\Gamma_C)$ such that for all $\eta \in L^2(\Gamma_C)$, we have

$$\Lambda\eta = \psi(u_{\eta n})\phi_L(\varphi_\eta - \varphi_F),\ \forall\, \eta \in L^2(\Gamma_C), \tag{39}$$

it follows from assumptions (h_4) that the operator Λ is well-defined. In order to prove that Λ has a fixed point, we will need the following result

Lemma 2.4. *The mapping $\eta \to x_\eta$, where x_η is the solution to PV_η, is weakly continuous from $L^2(\Gamma_C)$ to X.*

Proof. Let a sequence (η_k) in $L^2(\Gamma_C)$ converging weakly to η, we denote by $x_{\eta_k} = (u_{\eta_k}, \varphi_{\eta_k}) \in U$ the solution of (34) corresponding to η_k, then we have

$$(Ax_{\eta_k}, y - x_{\eta_k})_X \geq (f_{\eta_k}^e, y - x_{\eta_k})_X, \quad \forall\, y = (v, \xi) \in U, \tag{40}$$

where

$$(f_{\eta_k}^e, y - x_{\eta_k}^n)_X = (f, v - u_{\eta_k})_V + (q, \xi - \varphi_{\eta_k})_W - j_{\eta_k}(\xi - \varphi_{\eta_k}),$$

taking $y = 0$ in (40) and using (54), (18) and $S \geq 0$, we deduce

$$\|x_{\eta_k}\|_X \leq c\left(\|f\|_V + \|q\|_W + \|\eta_k\|_{L^2(\Gamma_C)}\right),$$

that is, the sequence (x_{η_k}) is bounded in X, then, there exists $\widetilde{x} = (\widetilde{u}, \widetilde{\varphi}) \in X$ and a subsequence, denote again (x_{η_k}), such that

$$x_{\eta_k} \rightharpoonup \widetilde{x} \in X, \quad \text{as } k \to +\infty.$$

Moreover, U is closed convex set in a real Hilbert space X, therefor U is weakly closed, then $\widetilde{x} \in U$.

We next prove that $\widetilde{x}$ is solution of (40). First we prove that

$$(f_{\eta_k}^e, y - x_{\epsilon\eta_k})_X \to (f_\eta^e, y - \widetilde{x})_X, \quad \text{as } k \to +\infty. \tag{41}$$

We have

$$\begin{aligned}|j_{\eta_k}(\xi-\widetilde{\varphi})-j_{\eta_k}(\xi-\varphi_{\eta_k})| &\leq \|\eta_k\|_{L^2(\Gamma_C)}\|\widetilde{\varphi}-\varphi_{\eta_k}\|_{L^2(\Gamma_C)}\\ &\leq \|\underbrace{\eta_k}_{bounded}\|_{L^2(\Gamma_C)}\|\widetilde{x}-x_{\eta_k}\|_{L^2(\Gamma_C)\times L^2(\Gamma_C)}.\end{aligned}$$

Since the trace map $\gamma : X \to L^2(\Gamma_C)^d \times L^2(\Gamma_C)$ is compact operator, from the weak convergence $x_{\eta_k} \rightharpoonup \widetilde{x}$ in X, we obtain the convergence $x_{\eta_k} \to \widetilde{x}$ strongly in $L^2(\Gamma_C)^d \times L^2(\Gamma_C)$. So we have (41).

Now, from (40), we have

$$(Ax_{\eta_k}, y - x_{\eta_k})_X \geq (f^e_{\eta_k}, y - x_{\eta_k})_X, \quad \forall y = (v,\xi) \in U.$$

By pseudomonotonicity of A and (40)-(41), we get

$$\left\{\begin{array}{l}\widetilde{x}\in U\\ (A\widetilde{x}, y-\widetilde{x})_X + \widetilde{j}_g(y) - \widetilde{j}_g(\widetilde{x}) \geq (f_\eta, y-\widetilde{x})_X, \quad \forall y=(v,\xi)\in U,\end{array}\right. \tag{42}$$

from (42) we find that $\widetilde{x}$ is a solution of problem PV_η and from the uniqueness of the solution for this variational inequality we obtain $\widetilde{x} = x_\eta$. Since x_η is the unique weak limit of any subsequence of (x_{η_k}), we deduce that the whole sequence (x_{η_k}) is weakly convergent in X to x_η, ensures that the weak continuous mapping $\eta \to x_\eta$, from $L^2(\Gamma_C) \times L^2(\Gamma_C)$ to X. □

Lemma 2.5. *Λ is an operator of $\mathcal{K}$ into itself and has at least one fixed point.*

Proof. Let $\eta \in \mathcal{K}$, i.e.

$$\|\eta\|_{L^2(\Gamma_C)} \leq \kappa.$$

By (39), we have

$$\|\Lambda\eta\|_{L^2(\Gamma_C)} \leq \|\psi(u_{\eta,n})\phi_L(\varphi_\eta - \varphi_F)\|_{L^2(\Gamma_C)},$$

using the properties of ψ and ϕ_L we obtain

$$\|\Lambda\eta\|_{L^2(\Gamma_C)} \leq M_\psi\, L\, meas(\Gamma_C)^{\frac{1}{2}},$$

and keeping in mind (33), we get

$$\|\Lambda\eta\|_{L^2(\Gamma_C)} \leq \kappa,$$

then Λ is an operator of $\mathcal{K}$ into itself, and note that $\mathcal{K}$ is a nonempty, convex and closed subset of $L^2(\Gamma_C)$. Since $L^2(\Gamma_C)$ is a reflexive space, $\mathcal{K}$ is weakly compact. Using continuity of ϕ_L, ψ and lemma 2.4, we deduce that Λ is weakly continuous. Hence, by Schauder's fixed point theorem the operator Λ has at least one fixed point. □

Proof of theorem 2.1 :

1) Existence. Let η^* be the fixed point of operator Λ. We denote by (u^*, φ^*) the solution of the variational problem (34) for $\eta = \eta^*$. Using (34) and (39), it is easy to see that (u^*, φ^*) is a solution of PV. This proves the existence part of theorem 2.1.

2) Uniqueness. We show next that if $L_\psi L + M_\psi < L^*$ the solution is unique.
Let $x_1 = (u_1, \varphi_1)$, $x_2 = (u_2, \varphi_2) \in U$ the solution of problem (32) we have

$$(Ax_1, y - x_1)_X + j(x_1, y - x_1) \geq (f^e, y - x_1)_X, \tag{43}$$

$$(Ax_2, y - x_2)_X + j(x_2, y - x_2) \geq (f^e, y - x_2)_X. \tag{44}$$

We take $y = x_2$ in the first inequality, $y = x_1$ in the second, and add the two inequality to obtain

$$(A_1x_1 - Ax_2, x_1 - x_2)_X \le j(x_1, x_2 - x_1) + j(x_2, x_1 - x_2).$$

From (23), we find

$$(A_1x_1 - Ax_2, x_1 - x_2)_X \le \int_{\Gamma_C} \psi(u_{2,n})\Big(\phi_L(\varphi_2 - \varphi_F) - \phi_L(\varphi_1 - \varphi_F)\Big)\Big(\varphi_1 - \varphi_2\Big)\, da + \int_{\Gamma_C} \phi_L(\varphi_2 - \varphi_F)\Big(\psi(u_{2,n}) - \psi(u_{1,n})\Big)\Big(\varphi_1 - \varphi_2\Big)\, da,$$

thus, by using (h_5), the bounds $|\phi_L(\varphi_2 - \varphi_F)| \le L$, the Lipschitz continuity of the function ϕ_L, (16), (18) and (28) we deduce

$$(A_1x_1 - Ax_2, x_1 - x_2)_X \le (M_\psi\, c_1^2 + L\, L_\psi\, c_0\, c_1)\|x_1 - x_2\|_X^2.$$

Using (54) hence there exists a constant $c_* > 0$ such that

$$\|x_1 - x_2\|_X^2 \le c_*(L_\psi\, L + M_\psi)\, \|x_1 - x_2\|_X^2.$$

Let $L^* = \dfrac{1}{c_*}$, then if $L_\psi\, L + M_\psi < L^*$ therefore $x_1 = x_2$. □

3. The penalty problem PV_ϵ

Let $\epsilon > 0$ be a small parameter, *Find a displacement field* $u_\epsilon : \Omega \to \mathbb{R}^d$, *an electric potentiel* $\varphi_\epsilon : \Omega \to \mathbb{R}$ *such that*

$$(45)\qquad (\mathfrak{F}\varepsilon(u_\epsilon), \varepsilon(v))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_\epsilon, \varepsilon(v))_{\mathcal{H}} + \frac{1}{\epsilon}\langle [u_{\epsilon,n}]_+, v_n\rangle_{\Gamma_C} = (f, v)_V,$$

$$(46)\qquad (\beta\nabla\varphi_\epsilon, \nabla\xi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u_\epsilon), \nabla\xi)_{L^2(\Omega)^d} + \ell(u_\epsilon, \varphi_\epsilon, \xi) = (q, \xi)_W,$$

for all $v \in V$, $\xi \in W$. Where $\langle\cdot\rangle_{\Gamma_C}$ stands for the duality product between W' and W. Note that this formulation is obtained by setting the condition $\sigma_n(u_\epsilon, \varphi_\epsilon) = -\frac{1}{\epsilon}[u_{\epsilon,n}]_+$ with $\forall a \in \mathbb{R}$, $a_+ = a$ if $a \ge 0$ and $a_+ = 0$ if $a \le 0$.
We have the following results

Theorem 3.1. *Assume that* (h_1)-(h_8) *hold. Then there exists* $L^* > 0$ *such that if* $L_\psi L + M_\psi < L^*$, *then The problem* PV_ϵ *has a unique solution such that* $(u_\epsilon, \varphi_\epsilon) \in V \times W$.

Proof. The proof of Theorem 3.1 will be carried out in several steps, based on a fixed point argument. For this purpose, we introduce the operator $A_\epsilon : X \to X$ defined by

$$(47)\qquad (A_\epsilon x, y)_X = (Ax, y)_X + \frac{1}{\epsilon}\langle [u_{\epsilon,n}]_+, v_n\rangle_{\Gamma_C},$$

for all $x = (u, \varphi)$, $y = (v, \xi) \in X$, where A given by (29).

Using (54) and observe that

$$(48)\qquad \langle [u_1]_+, u_1 - u_2\rangle_{\Gamma_C} - \langle [u_2]_+, u_1 - u_2\rangle_{\Gamma_C} \ge 0,\ \forall u_1, u_2 \in V,$$

we find

$$(49)\qquad (A_\epsilon x_1 - A_\epsilon x_2, x_1 - x_2)_X \ge m_A\|x_1 - x_2\|_X^2.$$

We use now (55), (16) and the inequality $|[u_1]_+ - [u_2]_+| \le |u_1 - u_2|$, we obtain

$$(50)\qquad \|A_\epsilon x_1 - A_\epsilon x_2\|_X \le (M_A + \frac{1}{\epsilon}c_0^2)\|x_1 - x_2\|_X.$$

It follows that the operator $A_\epsilon : X \to X$ is strongly monotone and Lipschitz continuous.

Let $\eta \in L^2(\Gamma_3)$ be given, and we construct the following intermediate problem.

Problem PV_ϵ^η. *Let $\eta \in L^2(\Gamma_3)$ be given, find $u_{\epsilon\eta} \in V$ and $\varphi_{\epsilon\eta} \in W$ such that for all $v \in V$, $\xi \in W$*

$$(\mathfrak{F}\varepsilon(u_{\epsilon\eta}), \varepsilon(v))_{\mathcal{H}} - (\mathcal{E}^*\nabla\varphi_{\epsilon\eta}, \varepsilon(v))_{L^2(\Omega)^d} + \frac{1}{\epsilon}\langle [u_{\epsilon,n}]_+, v_n\rangle_{\Gamma_C} = (f, v)_V, \tag{51}$$

$$(\beta\nabla\varphi_{\epsilon\eta}, \nabla\xi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u_{\epsilon\eta}), \nabla\xi)_{L^2(\Omega)^d} = (q, \xi)_W - \ell_\eta(\xi). \tag{52}$$

Using (47), (35) and (36), it is easy to see that $x_{\epsilon\eta} = (u_{\epsilon\eta}, \varphi_{\epsilon\eta})$ is a solution to problem PV_ϵ^η if and only if

$$(A_\epsilon x_{\epsilon\eta}, y)_X = (f_\eta^e, y)_X, \quad \forall y = (v, \xi) \in X. \tag{53}$$

We now use (53) to obtain the following existence and uniqueness result.

Lemma 3.2. *For any $\eta \in L^2(\Gamma_3)$, assume that (h_1)-(h_3) hold. Then, the problem PV_ϵ^η has a unique solution $x_{\epsilon\eta} = (u_{\epsilon\eta}, \varphi_{\epsilon\eta}) \in X$.*

Proof. For all fixed $\epsilon > 0$, it follows from (49), (50), $f_\eta^e \in X$ and a standard result on nonlinear variational equation that there exists a unique element $x_{\epsilon\eta} = (u_{\epsilon\eta}, \varphi_{\epsilon\eta}) \in X$ which satisfies (53). ☐

The rest of the proof follows from arguments of fixed point, similar to those used in the proof of Theorem 2.1. ☐

4. Finite element approximation and error estimates

In this section we introduce the finite element approximation of the variational problem PV_ϵ and derive an error estimate on them. To this end, for any given discretization parameter $h > 0$, let τ^h be a regular family of triangular finite element partitions of $\overline{\Omega}$ that are compatible with the partition of the boundary decompositions $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_C$ and $\Gamma = \Gamma_a \cup \Gamma_b \cup \Gamma_C$, that is, any point when the boundary condition type changes is a vertex of the partitions, then the side lies entirely in $\overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_C$, and $\overline{\Gamma}_a \cup \overline{\Gamma}_b \cup \overline{\Gamma}_C$. Then we consider two finite-dimensional spaces $V^h \subset V$ and $W^h \subset W$, approximating the spaces V and W, respectively, that is

$$V^h = \{v^h \in C(\overline{\Omega})^d,\ v^h_{|T} \in \mathbb{P}_1(T)^d,\ T \in \tau^h,\ v^h = 0 \ on\ \overline{\Gamma}_D\},$$

$$W^h = \{\psi^h \in C(\overline{\Omega}),\ \psi^h_{|T} \in \mathbb{P}_1(T),\ T \in \tau^h,\ \psi^h = 0 \ on\ \overline{\Gamma}_a\}.$$

Here $\mathbb{P}_1(T)$ represents the space of polynomial functions of global degree less or equal to 1 in an element T of the triangulation. We also introduce $\mathcal{X}^h(\Gamma_C)$, the space of normal traces on Γ_C for discrete functions in V^h

$$\mathcal{X}^h(\Gamma_C) = \{\mu_h \in C(\overline{\Gamma_C}) : \quad \exists v^h \in V^h \quad \forall T \in \mathcal{T}^h, v^h.n = \mu_h\}.$$

Thus, the discrete approximation of Problem PV_ϵ is the following.

Problem PV_ϵ^h: *Find $u_\epsilon^h \in V^h$ and $\varphi_\epsilon^h \in W^h$ such that, for all $v^h \in V^h$ and $\xi^h \in W^h$,*

$$(\mathfrak{F}\varepsilon(u_\epsilon^h), \varepsilon(v^h))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_\epsilon^h, \varepsilon(v^h))_{\mathcal{H}} + \frac{1}{\epsilon}\langle [u_{\epsilon,n}^h]_+, v_n^h\rangle_{\Gamma_C} = (f, v^h)_V, \tag{54}$$

$$(\beta\nabla\varphi_\epsilon^h, \nabla\xi^h)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u_\epsilon^h), \nabla\xi^h)_{L^2(\Omega)^d} + \ell(u_\epsilon^h, \varphi_\epsilon^h, \xi^h) = (q, \xi^h)_W. \tag{55}$$

Applying Theorem 3.1, for the case when V and W are replaced by V^h and W^h, respectively, we find that the problem PV_ϵ^h has a unique solution $(u_\epsilon^h, \varphi_\epsilon^h) \in V^h \times W^h$.

We have the following results.

Theorem 4.1. *Suppose that* $u \in \left(H^{\frac{3}{2}+\nu}(\Omega)\right)^d$ *and* $\varphi \in H^{\frac{3}{2}+\nu}(\Omega)$ *with* $\nu \in (0, \frac{1}{2}]$. *Let* $(u_\epsilon^h, \varphi_\epsilon)$ *be the solution of problem* PV_ϵ^h. *We have:*

$$(56)\quad \left\|\sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}^h]_+\right\|_{-\nu,\Gamma_C} \leq C\left(h^\nu \left\|\sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}^h]_+\right\|_{0,\Gamma_C} + h^{\nu-\frac{1}{2}}(\left\|u - u_\epsilon^h\right\|_{1,\Omega} + \left\|\varphi - \varphi_\epsilon^h\right\|_{1,\Omega})\right),$$

$$(57)\quad \left\|\sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}]_+\right\|_{-\nu,\Gamma_C} \leq C\left(\epsilon^\nu \left\|\sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}]_+\right\|_{0,\Gamma_C} + \epsilon^{\nu-\frac{1}{2}}(\|u - u_\epsilon\|_{1,\Omega} + \|\varphi - \varphi_\epsilon\|_{1,\Omega})\right),$$

with C *a constants independent of* u, u_ϵ^h, h *and* ϵ.

Theorem 4.2. *Suppose that* $u \in \left(H^{\frac{3}{2}+\nu}(\Omega)\right)^d$ *and* $\varphi \in H^{\frac{3}{2}+\nu}(\Omega)$ *with* $\nu \in (0, \frac{1}{2}]$. *Let* $(u_\epsilon, \varphi_\epsilon)$ *be the solution of problem* PV_ϵ. *We have:*

$$(58)\quad \|u - u_\epsilon\|_{1,\Omega} \leq C\epsilon^{\nu+\frac{1}{2}}\left(\|u\|_{\frac{3}{2}+\nu,\Omega} + \|\varphi\|_{\frac{3}{2}+\nu,\Omega}\right),$$

$$(59)\quad \|\varphi - \varphi_\epsilon\|_1 \leq C\epsilon^{\nu+\frac{1}{2}}\left(\|u\|_{\frac{3}{2}+\nu,\Omega} + \|\varphi\|_{\frac{3}{2}+\nu,\Omega}\right),$$

$$(60)\quad \left\|\sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}]_+\right\|_{0,\Gamma_C} \leq C\epsilon^\nu\left(\|u\|_{\frac{3}{2}+\nu,\Omega} + \|\varphi\|_{\frac{3}{2}+\nu,\Omega}\right),$$

$$(61)\quad \left\|\sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}]_+\right\|_{-\nu,\Gamma_C} \leq C\epsilon^{2\nu}\left(\|u\|_{\frac{3}{2}+\nu,\Omega} + \|\varphi\|_{\frac{3}{2}+\nu,\Omega}\right),$$

with $C > 0$ *a constant, independent of* ϵ *and* u.

Theorem 4.3. *Suppose that* $u \in \left(H^{\frac{3}{2}+\nu}(\Omega)\right)^d$ *and* $\varphi \in H^{\frac{3}{2}+\nu}(\Omega)$ *with* $\nu \in (0, \frac{1}{2}]$. *Let* $(u_\epsilon^h, \varphi_\epsilon^h)$ *be the solution of problem* PV_ϵ^h *satisfies the following error estimates in two space dimensions:*

$$(62)\quad \|u - u_\epsilon^h\|_{1,\Omega} + \|\varphi - \varphi_\epsilon^h\|_{1,\Omega} + \epsilon^{\frac{1}{2}}\|\sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}^h]_+\|_{0,\Gamma_C} \leq C\begin{cases} \left(h^{\frac{1}{2}+\frac{\nu}{2}+\nu^2} + h^\nu\epsilon^{\frac{1}{2}} + h^{\nu-\frac{1}{2}}\epsilon\right)\left(\|u\|_{\frac{3}{2}+\nu,\Omega} + \|\varphi\|_{\frac{3}{2}+\nu,\Omega}\right) & \text{if } 0 < \nu < \frac{1}{2}, \\ \left(h|\ln h|^{\frac{1}{2}} + (h\epsilon)^{\frac{1}{2}} + \epsilon\right)\left(\|u\|_{2,\Omega} + \|\varphi\|_{2,\Omega}\right) & \text{if } \nu = \frac{1}{2}, \end{cases}$$

$$\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\|_{-\nu,\Gamma_C}$$

(63)

$$\leq C\begin{cases}\Big(h^{\frac12+\frac{3\nu}{2}+\nu^2}\epsilon^{\frac{-1}{2}}+h^{\frac{3\nu}{2}+\nu^2}+h^{2\nu-\frac12}\epsilon^{\frac12}+h^{2\nu-1}\epsilon\Big)\Big(\|u\|_{\frac32+\nu}+\|\varphi\|_{\frac32+\nu}\Big)\\ \text{if } 0<\nu<\frac12,\\ \Big(h^{\frac32}|\ln h|^{\frac12}\epsilon^{\frac{-1}{2}}+h|\ln h|^{\frac12}+(h\epsilon)^{\frac12}+\epsilon\Big)\Big(\|u\|_{2;\Omega}+\|\varphi\|_{2,\Omega}\Big)\\ \text{if } \nu=\frac12,\end{cases}$$

With $C>0$ a constant, independent of ϵ, h and u.
In three space dimensions, the terms $h^{\frac12+\frac{\nu}{2}+\nu^2}$ (resp. $h|\ln h|^{\frac12}$) in (62) have to be replaced with $h^{\frac12+\frac{\nu}{2}}$ (resp.$h^{\frac34}$).

Corollary 4.4. *Suppose that $u\in\Big(H^{\frac32+\nu}(\Omega)\Big)^d$ and $\varphi\in H^{\frac32+\nu}(\Omega)$ with $\nu\in(0,\frac12]$. Suppose also that the parameter is chosen as $\epsilon=h$. The solution u^h_ϵ of the discrete penalty problem PV^h_ϵ satisfies the following error estimates in two space dimensions:*

$$\|u-u^h_\epsilon\|_{1,\Omega}+\|\varphi-\varphi^h_\epsilon\|_{1,\Omega}+h^{\frac12}\big\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\big\|_{0,\Gamma_C}$$
$$+h^{\frac12-\nu}\big\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\big\|_{-\nu,\Gamma_C}$$
$$\leq C\begin{cases}h^{\frac12+\frac{\nu}{2}+\nu^2}\Big(\|u\|_{\frac32+\nu,\Omega}+\|\varphi\|_{\frac32+\nu}\Big) & \text{if } 0<\nu<\frac12,\\ h|\ln h|^{\frac12}\Big(\|u\|_{2,\Omega}+\|\varphi\|_{2,\Omega}\Big) & \text{if } \nu=\frac12,\end{cases}\tag{64}$$

With $C>0$ a constant, independent of h and u.
In three space dimensions, the terms $h^{\frac12+\frac{\nu}{2}+\nu^2}$ (resp. $h|\ln h|^{\frac12}$) in (64) have to be replaced with $h^{\frac12+\frac{\nu}{2}}$ (resp.$h^{\frac34}$).

5. Proof of Theorems 4.1, 4.2 and 4.3

5.1. Proof of theorem 4.1. Let $\mathcal{P}^h:L^2(\Gamma_C)\to\mathcal{X}^h(\Gamma_C)$ denote the $L^2(\Gamma_C)$-projection operator onto $\mathcal{X}^h(\Gamma_C)$. We suppose that the mesh associated to $\mathcal{X}^h(\Gamma_C)$ and the mesh contact boundary are quasi-uniforme.
We recall now some results: there exists a constant $C>0$ such that

- $\forall s\in[0,1],\quad \forall v\in H^s(\Gamma_C)$,

$$\|\mathcal{P}^hv\|_{s,\Gamma_C}\leq C\|v\|_{s,\Gamma_C},\quad \|v-\mathcal{P}^hv\|_{0,\Gamma_C}\leq Ch^s\|v\|_{s,\Gamma_C}.$$

- $\exists\,\mathcal{R}^h:\mathcal{X}^h(\Gamma_C)\to V^h,\quad \forall v^h\in\mathcal{X}^h(\Gamma_C)$,

$$\mathcal{R}^h(v^h)_{|\Gamma_C}.n=v^h,\quad \|\mathcal{R}^h(v^h)\|_{1,\Omega}\leq C\|v^h\|_{\frac12,\Gamma_C}.$$

Keeping in made

$$\Big\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\Big\|_{-\nu,\Gamma_C}=\sup_{v\in H^{-\nu}(\Gamma_C)}\frac{\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+,v\rangle_{\Gamma_C}}{\|v\|_{\nu,\Gamma_C}},$$

we have

$$\begin{aligned}
&\left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\right\|_{-\nu,\Gamma_C}\\
&\le \sup_{v\in H^\nu(\Gamma_C)}\frac{\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+,v-\mathcal{P}^hv\rangle_{\Gamma_C}}{\|v\|_{\nu,\Gamma_C}}+\sup_{v\in H^\nu(\Gamma_C)}\frac{\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+,\mathcal{P}^hv\rangle_{\Gamma_C}}{\|v\|_{\nu,\Gamma_C}}\\
&\le \left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\right\|_{0,\Gamma_C}\sup_{v\in H^\nu(\Gamma_C)}\frac{\|v-\mathcal{P}^hv\|_{0,\Gamma_C}}{\|\mathcal{P}^hv\|_{\nu,\Gamma_C}}\\
&\qquad +C\sup_{v\in H^\nu(\Gamma_C)}\frac{\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+,\mathcal{P}^hv\rangle_{\Gamma_C}}{\|\mathcal{P}^hv\|_{\nu,\Gamma_C}}\\
&\le Ch^\nu\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\|_{0,\Gamma_C}+\sup_{v\in H^\nu(\Gamma_C)}\frac{\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+,\mathcal{P}^hv\rangle_{\Gamma_C}}{\|\mathcal{P}^hv\|_{\nu,\Gamma_C}}.
\end{aligned}$$

On other ship we have:

$$\begin{cases}(\mathfrak{F}\varepsilon(u),\varepsilon(v))_{\mathcal{H}}+(\mathcal{E}^*\nabla\varphi,\varepsilon(v))_{\mathcal{H}}=\langle\sigma_n(u,\varphi),v_n\rangle_{\Gamma_C}+(f,v)_V; \quad v\in V\\ (\mathfrak{F}\varepsilon(u_\epsilon),\varepsilon(v))_{\mathcal{H}}+(\mathcal{E}^*\nabla(\varphi_\epsilon),\varepsilon(v))_{\mathcal{H}}=\langle-\frac{1}{\epsilon}[u_{\epsilon,n}]_+,v_n\rangle_{\Gamma_C}+(f,v)_V; \quad v\in V\end{cases}$$

Hence

$$\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u_{\epsilon,n}]_+,v_n\rangle_{\Gamma_C}=(\mathfrak{F}\varepsilon(u-u_\epsilon),\varepsilon(v))_{\mathcal{H}}+(\mathcal{E}^*\nabla(\varphi-\varphi_\epsilon),\varepsilon(v))_{\mathcal{H}}, \tag{65}$$

for all $v\in V$. When we replace V by V^h we find

$$\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+,v^h_n\rangle_{\Gamma_C}=(\mathfrak{F}\varepsilon(u-u^h_\epsilon),\varepsilon(v^h))_{\mathcal{H}}+(\mathcal{E}^*\nabla(\varphi-\varphi^h_\epsilon),\varepsilon(v^h))_{\mathcal{H}},\ v^h\in V^h.$$

and using the continuity of $(u,v)\to(\mathfrak{F}\varepsilon(u),\varepsilon(v))_{\mathcal{H}}$ and $(\varphi,v)\to(\nabla(\varphi),\mathcal{E}\varepsilon(v))_{\mathcal{H}}$, it results

$$\begin{aligned}
&\sup_{v\in H^\nu(\Gamma_C)}\frac{\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+,\mathcal{P}^hv\rangle_{\Gamma_C}}{\|\mathcal{P}^hv\|_{\nu,\Gamma_C}}\\
&\quad=\sup_{v\in H^\nu(\Gamma_C)}\frac{\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+,\mathcal{R}^h(\mathcal{P}^hv)_{|\Gamma_C}.n\rangle_{\Gamma_C}}{\|\mathcal{P}^hv\|_{\nu,\Gamma_C}}\\
&\quad=\sup_{v\in H^\nu(\Gamma_C)}\frac{(\mathfrak{F}\varepsilon(u-u^h_\epsilon),\mathcal{E}\mathcal{R}^h(\mathcal{P}^hv))_{\mathcal{H}}+(\nabla(\varphi-\varphi^h_\epsilon),\mathcal{E}\mathcal{R}^h(\mathcal{P}^hv))_{\mathcal{H}}}{\|\mathcal{P}^hv\|_{\nu,\Gamma_C}}\\
&\quad\le C(\|u-u^h_\epsilon\|_{1,\Omega}+\|\varphi-\varphi^h_\epsilon\|_{1,\Omega})\sup_{v\in H^\nu(\Gamma_C)}\frac{\|\mathcal{R}^h(\mathcal{P}^hv)\|_{1,\Omega}}{\|\mathcal{P}^hv\|_{\nu,\Gamma_C}}\\
&\quad\le C(\|u-u^h_\epsilon\|_{1,\Omega}+\|\varphi-\varphi^h_\epsilon\|_{1,\Omega})\sup_{v\in H^\nu(\Gamma_C)}\frac{\|\mathcal{P}^hv\|_{\frac{1}{2},\Gamma_C}}{\|\mathcal{P}^hv\|_{\nu,\Gamma_C}}.
\end{aligned}$$

We make use of the inverse inequality

$$\|\mathcal{P}^hv\|_{\frac{1}{2},\Gamma_C}\le Ch^{\nu-\frac{1}{2}}\|\mathcal{P}^hv\|_{\nu,\Gamma_C},$$

we get

$$\sup_{v\in H^\nu(\Gamma_C)}\frac{\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+,\mathcal{P}^hv\rangle_{\Gamma_C}}{\|v\|_{\nu,\Gamma_C}}\le Ch^{\nu-\frac{1}{2}}(\|u-u^h_\epsilon\|_{1,\Omega}+\|\varphi-\varphi^h_\epsilon\|_{1,\Omega}).$$

So we finally obtain:

$$\left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\right\|_{-\nu,\Gamma_C}$$
$$\leq C\left(h^\nu\left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\right\|_{0,\Gamma_C}+h^{\nu-\frac{1}{2}}(\left\|u-u^h_\epsilon\right\|_{1,\Omega}+\left\|\varphi-\varphi^h_\epsilon\right\|_{1,\Omega})\right).$$

Now, we introduce V^ϵ a fictitious finite element space, defined identically as V^h and with the choice of mesh size $h=\epsilon$. We note simply $\mathcal{P}^\epsilon : L^2(\Gamma_C)\to\mathcal{X}^\epsilon(\Gamma_C)$ the $L^2(\Gamma_C)$-projection operator onto $\mathcal{X}^\epsilon(\Gamma_C)$. Therefore, we obtain

$$\left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\right\|_{-\nu,\Gamma_C}$$
$$\leq C\left(\epsilon^\nu\left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\right\|_{0,\Gamma_C}+\epsilon^{\nu-\frac{1}{2}}\big(\left\|u-u^h_\epsilon\right\|_{1,\Omega}+\left\|\varphi-\varphi^h_\epsilon\right\|_{1,\Omega}\big)\right).$$

5.2. Proof of theorem 4.2. From (27) and (46) we find

$$(\beta\nabla(\varphi-\varphi_\epsilon),\nabla\xi)_{L^2(\Omega)^d}-(\mathcal{E}\varepsilon(u-u_\epsilon),\nabla\xi)_{L^2(\Omega)^d}+\ell(u,\varphi,\xi)-\ell(u_\epsilon,\varphi_\epsilon,\xi)=0,$$

this implies

$$\begin{aligned}&(\mathcal{E}\varepsilon(u-u_\epsilon),\nabla(\varphi-\varphi_\epsilon))_{L^2(\Omega)^d}\\&=(\beta\nabla(\varphi-\varphi_\epsilon),\nabla(\varphi-\varphi_\epsilon))_{L^2(\Omega)^d}+\ell(u,\varphi,\varphi-\varphi_\epsilon)-\ell(u_\epsilon,\varphi_\epsilon,\varphi-\varphi_\epsilon).\end{aligned}$$

We know that

$$(\mathfrak{F}\varepsilon(u),\varepsilon(v)-\varepsilon(u))_{\mathcal{H}}+(\mathcal{E}^*\nabla\varphi,\varepsilon(v)-\varepsilon(u))_{\mathcal{H}}=\langle\sigma_n(u,\varphi),v_n-u_n\rangle_{\Gamma_C}+(f,v-u)_V,$$

hence

$$\begin{aligned}&(\mathfrak{F}(\varepsilon(u)-\varepsilon(u_\epsilon)),\varepsilon(u)-\varepsilon(u_\epsilon))_{\mathcal{H}}+(\beta\nabla(\varphi-\varphi_\epsilon),\nabla(\varphi-\varphi_\epsilon))_{L^2(\Omega)^d}\\&=\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u_{\epsilon,n}]_+,u_n-u_{\epsilon,n}\rangle_{\Gamma_C}+\ell(u_\epsilon,\varphi_\epsilon,\varphi-\varphi_\epsilon)-\ell(u,\varphi,\varphi-\varphi_\epsilon).\end{aligned}\tag{66}$$

We denote by $a : V\times V\to\mathbb{R}$ and $b : W\times W\to\mathbb{R}$ the following bilinear and symmetric applications

$$\begin{aligned}a(u,v)&=(\mathfrak{F}(\varepsilon(u)),\varepsilon(v))_{\mathcal{H}},\\b(\varphi,\xi)&=(\beta\nabla(\varphi),\nabla(\xi))_{L^2(\Omega)^d},\end{aligned}\tag{67}$$

By the assumptions (h_1) and (h_3) it is easy to see that a and b are coercive and continuous forms. Moreover,

$$\begin{aligned}&\alpha_a\left\|u-u_\epsilon\right\|^2_{1,\Omega}+\alpha_b\left\|\varphi-\varphi_\epsilon\right\|^2_{1,\Omega}\\&\leq a(u-u_\epsilon,u-u_\epsilon)+b(\varphi-\varphi_\epsilon,\varphi-\varphi_\epsilon)\\&=a(u,u-u_\epsilon)+b(\varphi,\varphi-\varphi_\epsilon)-a(u_\epsilon,u-u_\epsilon)-b(\varphi_\epsilon,\varphi-\varphi_\epsilon)\\&=\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u_{\epsilon,n}]_+,u_n-u_{\epsilon,n}\rangle_{\Gamma_C}+\ell(u_\epsilon,\varphi_\epsilon,\varphi-\varphi_\epsilon)-\ell(u,\varphi,\varphi-\varphi_\epsilon)\\&=\langle\sigma_n(u,\varphi),u_n\rangle_{\Gamma_C}+\langle\frac{1}{\epsilon}[u_{\epsilon,n}]_+,u_n\rangle_{\Gamma_C}-\langle\sigma_n(u,\varphi),u_{\epsilon,n}\rangle_{\Gamma_C}-\langle\frac{1}{\epsilon}[u_{\epsilon,n}]_+,u_{\epsilon,n}\rangle_{\Gamma_C}\\&\quad+\ell(u_\epsilon,\varphi_\epsilon,\varphi-\varphi_\epsilon)-\ell(u,\varphi,\varphi-\varphi_\epsilon),\end{aligned}$$

where $\alpha_a > 0, \alpha_b > 0$ the ellipticy constants.
Due to the contact conditions (7) on Γ_C, we observe that

$$\begin{cases} \langle \sigma_n(u,\varphi), u_n\rangle_{\Gamma_C} = 0, \\ \langle \frac{1}{\epsilon}[u_{\epsilon,n}]_+, u_n\rangle_{\Gamma_C} \leq 0. \end{cases}$$

Then

$$\begin{aligned} -\langle \sigma_n(u,\varphi), u_{\epsilon,n}\rangle_{\Gamma_C} &\leq -\langle \sigma_n(u,\varphi), [u_{\epsilon,n}]_+\rangle_{\Gamma_C} \\ \langle \tfrac{1}{\epsilon}[u_{\epsilon,n}]_+, u_{\epsilon,n}\rangle_{\Gamma_C} &= \langle \tfrac{1}{\epsilon}[u_{\epsilon,n}]_+, [u_{\epsilon,n}]_+\rangle_{\Gamma_C}. \end{aligned} \tag{68}$$

We apply this result to the previous inequality and obtain

$$\alpha_a \|u-u_\epsilon\|^2_{1,\Omega} + \alpha_b \|\varphi - \varphi_\epsilon\|^2_{1,\Omega} \leq -\langle \sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}]_+, [u_{\epsilon,n}]_+\rangle_{\Gamma_C} + R,$$

where

$$R = \ell(u_\epsilon, \varphi_\epsilon, \varphi - \varphi_\epsilon) - \ell(u, \varphi, \varphi - \varphi_\epsilon). \tag{69}$$

We have

$$\begin{aligned} &\alpha_a \|u-u_\epsilon\|^2_{1,\Omega} + \alpha_b \|\varphi - \varphi_\epsilon\|^2_{1,\Omega} \\ &\leq -\langle \epsilon(\sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}]_+), \frac{1}{\epsilon}[u_{\epsilon,n}]_+\rangle_{\Gamma_C} + R \\ &\leq -\langle \epsilon(\sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}]_+), \frac{1}{\epsilon}[u_{\epsilon,n}]_+ + \sigma_n(u,\varphi) - \sigma_n(u,\varphi)\rangle_{\Gamma_C} + R \\ &= -\epsilon \left\| \sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}]_+ \right\|^2_{0,\Gamma_C} + \epsilon\langle \sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}]_+, \sigma_n(u,\varphi)\rangle_{\Gamma_C} + R \end{aligned}$$

From $\sigma_n(u,\varphi) \in H^\nu(\Gamma_C)$, we obtain

$$\langle \sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}]_+, \sigma_n(u,\varphi)\rangle_{\Gamma_C} \leq \left\| \sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}]_+ \right\|_{-\nu,\Gamma_C} \|\sigma_n(u,\varphi)\|_{\nu,\Gamma_C}.$$

With $\delta \in [0,1]$ and $\beta > 0$ we have

$$\begin{aligned} &\alpha_a \|u-u_\epsilon\|^2_{1,\Omega} + \alpha_b \|\varphi - \varphi_\epsilon\|^2_{1,\Omega} \\ &\leq -\epsilon \left\| \sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}]_+ \right\|^2_{0,\Gamma_C} \\ &\quad + \epsilon^\delta \left\| \sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}]_+ \right\|_{-\nu,\Gamma_C} \epsilon^{1-\delta} \|\sigma_n(u,\varphi)\|_{\nu,\Gamma_C} + R \\ &\leq -\epsilon \left\| \sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}]_+ \right\|^2_{0,\Gamma_C} \\ &\quad + \frac{\epsilon^{2\delta}}{2\beta} \left\| \sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}]_+ \right\|^2_{-\nu,\Gamma_C} + \frac{\beta\epsilon^{2-2\delta}}{2} \|\sigma_n(u,\varphi)\|^2_{\nu,\Gamma_C} + R. \end{aligned}$$

From (57) we find

$$\begin{aligned} \alpha_a \|u-u_\epsilon\|^2_{1,\Omega} + \alpha_b \|\varphi - \varphi_\epsilon\|^2_{1,\Omega} &\leq -\epsilon(1 - C\frac{\epsilon^{2(\delta+\nu)-1}}{\beta}) \left\| \sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}]_+ \right\|^2_{0,\Gamma_C} \\ &+ C\frac{\epsilon^{2(\delta+\nu)-1}}{\beta}\big(\|u-u_\epsilon\|^2_{1,\Omega} + \|\varphi - \varphi_\epsilon\|^2_{1,\Omega} \big) + \frac{\beta\epsilon^{2-2\delta}}{2} \|\sigma_n(u,\varphi)\|^2_{\nu,\Gamma_C} + R, \end{aligned}$$

which implies

$$
\begin{aligned}
&(\alpha_a - C\frac{\epsilon^{2(\delta+\nu)-1}}{\beta}) \|u-u_\epsilon\|^2_{1,\Omega} + (\alpha_b - C\frac{\epsilon^{2(\delta+\nu)-1}}{\beta}) \|\varphi-\varphi_\epsilon\|^2_{1,\Omega} \\
&+\epsilon(1-C\frac{\epsilon^{2(\delta+\nu)-1}}{\beta}) \left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u_{\epsilon,n}]_+\right\|^2_{0,\Gamma_C} \qquad (70)\\
\leq &\frac{\beta\epsilon^{2-2\delta}}{2}\|\sigma_n(u,\varphi)\|^2_{\nu,\Gamma_C} + R
\end{aligned}
$$

On the other hand, it follows from (69) and (23) that

$$
\begin{aligned}
R &= \int_{\Gamma_C}\psi(u_{\epsilon,n})\phi_L(\varphi_\epsilon-\varphi_F)(\varphi-\varphi_\epsilon)\,da - \int_{\Gamma_C}\psi(u_n)\phi_L(\varphi-\varphi_F)(\varphi-\varphi_\epsilon)\,da \\
&= \int_{\Gamma_C}\psi(u_{\epsilon,n})(\phi_L(\varphi_\epsilon-\varphi_F)-\phi_L(\varphi-\varphi_F)(\varphi-\varphi_\epsilon)\,da \\
&+\int_{\Gamma_C}(\psi(u_{\epsilon,n})-\psi(u_n))\phi_L(\varphi-\varphi_F)(\varphi-\varphi_\epsilon)\,da. \qquad (71)
\end{aligned}
$$

Then

$$
\begin{aligned}
|R| &\leq M_\psi\|\varphi-\varphi_\epsilon\|^2_{0,\Gamma_C} + L_\psi L\|u_{\epsilon,n}-u_n\|_{0,\Gamma_C}\|\varphi-\varphi_\epsilon\|_{0,\Gamma_C} \\
&\leq (M_\psi+\frac{L_\psi L}{4\gamma})\|\varphi-\varphi_\epsilon\|^2_{0,\Gamma_C} + \gamma L_\psi L\|u_{\epsilon,n}-u_n\|^2_{0,\Gamma_C} \\
&\leq (M_\psi+\frac{C}{4\gamma})\|\varphi-\varphi_\epsilon\|^2_{1,\Omega} + C\gamma\|u_\epsilon-u\|^2_{1,\Omega}, \qquad (72)
\end{aligned}
$$

with $\gamma > 0$. Then

$$
\begin{aligned}
&(\alpha_a-\gamma C - C\frac{\epsilon^{2(\delta+\nu)-1}}{\beta})\|u-u_\epsilon\|^2_{1,\Omega} \\
&+(\alpha_b - C\frac{\epsilon^{2(\delta+\nu)-1}}{\beta} - M_\psi - \frac{C}{4\gamma})\|\varphi-\varphi_\epsilon\|^2_{1,\Omega} \qquad (73)\\
&+\epsilon(1-C\frac{\epsilon^{2(\delta+\nu)-1}}{\beta})\left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u_{\epsilon,n}]_+\right\|^2_{0,\Gamma_C} \\
\leq &\frac{\beta\epsilon^{2-2\delta}}{2}\|\sigma_n(u,\varphi)\|^2_{\nu,\Gamma_C},
\end{aligned}
$$

where $C > L_\psi L$ a constant.
We know that α_a and α_b are sufficiently large then we can find C a constant sufficiently large.

We choose $\delta = \frac{1}{2} - \nu$ and $\begin{cases} \beta = C\left(1+\frac{1}{\alpha_a}+\frac{1}{\alpha_b - M_\psi})\right) \\ \gamma = \frac{1}{C}(\alpha_a - \frac{C}{\beta}) + \frac{C}{2(\alpha_b - M_\psi - \frac{C}{\beta})} \end{cases}$ in order to find

$$
\begin{cases}
\alpha_a - \gamma C - C\frac{\epsilon^{2(\delta+\nu)-1}}{\beta} > 0, \\
\alpha_b - C\frac{\epsilon^{2(\delta+\nu)-1}}{\beta} - M_\psi - \frac{C}{4\gamma} > 0, \\
1 - C\frac{\epsilon^{2(\delta+\nu)-1}}{\beta} > 0,
\end{cases}
$$

and using the estimate $\|\sigma_n(u,\varphi)\|_{\nu,\Gamma_C} \leq C\left(\|u\|_{\nu+\frac{3}{2},\Omega} + \|\varphi\|_{\nu+\frac{3}{2},\Omega}\right)$ proves the bounds (58),(59) and (60) . The bound (61) is a direct consequence of this last result.

5.3. Proof of theorem 4.3. We denote by $\mathcal{L}^h$ (resp.$\mathcal{L'}^h$) the Lagrange interpolation operator mapping onto V^h (resp.W^h). We first use the ellipticity and the continuity of $a(.,.)$ and $b(.,.)$, as well as Youngs inequality, to obtain:

$$\begin{aligned}
&\alpha_a \left\|u - u_\epsilon^h\right\|_{1,\Omega}^2 + \alpha_b \left\|\varphi - \varphi_\epsilon^h\right\|_{1,\Omega}^2 \\
&\leq a(u - u_\epsilon^h, u - u_\epsilon^h) + b(\varphi - \varphi_\epsilon^h, \varphi - \varphi_\epsilon^h) \\
&= a(u - u_\epsilon^h, u - \mathcal{L}^h u) + a(u - u_\epsilon^h, \mathcal{L}^h u - u_\epsilon^h) + b(\varphi - \varphi_\epsilon^h, \varphi - \mathcal{L'}^h \varphi) \\
&\quad + b(\varphi - \varphi_\epsilon^h, \mathcal{L'}^h \varphi - \varphi_\epsilon^h) \\
&\leq C \left\|u - u_\epsilon^h\right\|_{1,\Omega} \left\|u - \mathcal{L}^h u\right\|_{1,\Omega} + a(u - u_\epsilon^h, \mathcal{L}^h u - u_\epsilon^h) \\
&\quad + C \left\|\varphi - \varphi_\epsilon^h\right\|_{1,\Omega} \left\|\varphi - \mathcal{L'}^h \varphi\right\|_{1,\Omega} + b(\varphi - \varphi_\epsilon^h, \mathcal{L'}^h \varphi - \varphi_\epsilon^h) \\
&\leq \frac{\alpha_a}{2} \left\|u - u_\epsilon^h\right\|_{1,\Omega}^2 + \frac{C}{2\alpha_a} \left\|u - \mathcal{L}^h u\right\|_{1,\Omega}^2 + a(u, \mathcal{L}^h u - u_\epsilon^h) \\
&\quad - a(u_\epsilon^h, \mathcal{L}^h u - u_\epsilon^h) + \frac{\alpha_b}{2} \left\|\varphi - \varphi_\epsilon^h\right\|_{1,\Omega}^2 + \frac{C}{2\alpha_b} \left\|\varphi - \mathcal{L'}^h \varphi\right\|_{1,\Omega}^2 \\
&\quad + b(\varphi, \mathcal{L'}^h \varphi - \varphi_\epsilon^h) - b(\varphi_\epsilon^h, \mathcal{L'}^h \varphi - \varphi_\epsilon^h).
\end{aligned} \tag{74}$$

We can transform the term

$$a(u, \mathcal{L}^h u - u_\epsilon^h) - a(u_\epsilon^h, \mathcal{L}^h u - u_\epsilon^h) + b(\varphi, \mathcal{L'}^h \varphi - \varphi_\epsilon^h) - b(\varphi_\epsilon^h, \mathcal{L'}^h \varphi - \varphi_\epsilon^h).$$

So we obtain

$$\begin{aligned}
\frac{\alpha_a}{2} \left\|u - u_\epsilon^h\right\|_{1,\Omega}^2 + \frac{\alpha_b}{2} \left\|\varphi - \varphi_\epsilon^h\right\|_{1,\Omega}^2 &\leq \langle \sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}^h]_+, \mathcal{L}^h u_n - u_{\epsilon,n}^h \rangle_{\Gamma_C} \\
&\quad + \ell(u_\epsilon^h, \varphi_\epsilon^h, \mathcal{L'}^h \varphi - \varphi_\epsilon^h) - \ell(u, \varphi, \mathcal{L'}^h \varphi - \varphi_\epsilon^h).
\end{aligned} \tag{75}$$

Because the $1D-$ Lagrange interpolation with piecewise linear polynomials preserves the positivity, we have that $(\mathcal{L}^h u)_n \leq 0$ on Γ_C. This implies:

$$\langle \frac{1}{\epsilon}[u_{\epsilon,n}^h]_+, (\mathcal{L}^h u)_n \rangle_{\Gamma_C} \leq 0$$

We have again

$$\begin{aligned}
-\langle \sigma_n(u,\varphi), u_{\epsilon,n}^h \rangle_{\Gamma_C} &\leq -\langle \sigma_n(u,\varphi), [u_{\epsilon,n}^h]_+ \rangle_{\Gamma_C} \\
\langle \tfrac{1}{\epsilon}[u_{\epsilon,n}^h]_+, u_{\epsilon,n}^h \rangle_{\Gamma_C} &= \langle \tfrac{1}{\epsilon}[u_{\epsilon,n}^h]_+, [u_{\epsilon,n}^h]_+ \rangle_{\Gamma_C}.
\end{aligned} \tag{76}$$

This results into

$$\begin{aligned}
&\frac{\alpha_a}{2} \left\|u - u_\epsilon^h\right\|_{1,\Omega}^2 + \frac{\alpha_b}{2} \left\|\varphi - \varphi_\epsilon^h\right\|_{1,\Omega}^2 \\
&\leq \frac{C}{2\alpha_a} \left\|u - \mathcal{L}^h u\right\|_{1,\Omega}^2 + \frac{C}{2\alpha_b} \left\|\varphi - \mathcal{L'}^h \varphi\right\|_{1,\Omega}^2 + \langle \sigma_n((u,\varphi), (\mathcal{L}^h u)_n \rangle_{\Gamma_C} \\
&-\langle \sigma_n(u,\varphi) + \frac{1}{\epsilon}[u_{\epsilon,n}^h]_+, u_{\epsilon,n}^h \rangle_{\Gamma_C} + \ell(u_\epsilon^h, \mathcal{L'}^h \varphi_\epsilon^h, \varphi - \varphi_\epsilon^h) - \ell(u, \varphi, \mathcal{L'}^h \varphi - \varphi_\epsilon^h).
\end{aligned} \tag{77}$$

We bound the term: $-\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+,[u^h_{\epsilon,n}]_+\rangle_{\Gamma_C}$ as follows

$$\begin{aligned}
&-\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+,[u^h_{\epsilon,n}]_+\rangle_{\Gamma_C}\\
&=-\epsilon\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+,\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\rangle_{\Gamma_C}\\
&=-\langle\epsilon(\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+),\frac{1}{\epsilon}[u^h_{\epsilon,n}]_++\sigma_n(u,\varphi)-\sigma_n(u,\varphi)\rangle_{\Gamma_C}\\
&=-\epsilon\left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\right\|^2_{0,\Gamma_C}+\epsilon\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+,\sigma_n(u,\varphi)\rangle_{\Gamma_C}\\
&\le-\epsilon\left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\right\|^2_{0,\Gamma_C}+\epsilon\left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\right\|_{-\nu,\Gamma_C}\|\sigma_n(u,\varphi)\|_{\nu,\Gamma_C}.
\end{aligned}$$

Hence

$$\begin{aligned}
&-\langle\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+,[u^h_{\epsilon,n}]_+\rangle_{\Gamma_C}\\
&\le-\epsilon\left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\right\|^2_{0,\Gamma_C}+\frac{\epsilon^2}{2\beta}\left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u^h_{\epsilon,n}]_+\right\|^2_{-\nu,\Gamma_C}\\
&\quad+\frac{\beta}{2}\|\sigma_n(u,\varphi)\|^2_{\nu,\Gamma_C}.
\end{aligned}\tag{78}$$

With $\beta>0$ a constant indepedant of h,ϵ,u and φ.
We bound now the term $\ell(u^h_\epsilon,\varphi^h_\epsilon,\mathcal{L}'^h\varphi-\varphi^h_\epsilon)-\ell(u,\varphi,\mathcal{L}'^h\varphi-\varphi^h_\epsilon)$

$$\begin{aligned}
&\left|\ell(u^h_\epsilon,\varphi^h_\epsilon,\mathcal{L}'^h\varphi-\varphi^h_\epsilon)-\ell(u,\varphi,\mathcal{L}'^h\varphi-\varphi^h_\epsilon)\right|\\
&\le M_\psi\left\|\varphi-\varphi^h_\epsilon\right\|_{0,\Gamma_C}\left\|\mathcal{L}'^h\varphi-\varphi^h_\epsilon\right\|_{0,\Gamma_C}+L_\psi L\left\|u^h_{\epsilon,n}-u_n\right\|_{0,\Gamma_C}\left\|\mathcal{L}'^h\varphi-\varphi^h_\epsilon\right\|_{0,\Gamma_C}\\
&\le M_\psi\left\|\varphi-\varphi^h_\epsilon\right\|^2_{0,\Gamma_C}+M_\psi\left\|\varphi-\varphi^h_\epsilon\right\|_{0,\Gamma_C}\left\|\mathcal{L}'^h\varphi-\varphi\right\|_{0,\Gamma_C}\\
&\quad+L_\psi L\left\|u^h_{\epsilon,n}-u_n\right\|_{0,\Gamma_C}\left(\left\|\mathcal{L}'^h\varphi-\varphi\right\|_{0,\Gamma_C}+\left\|\varphi-\varphi^h_\epsilon\right\|_{0,\Gamma_C}\right)\\
&\le(M_\psi+\frac{L_\psi L+M_\psi}{4\gamma})\|\varphi-\varphi^h_\epsilon\|^2_{0,\Gamma_C}+2\gamma L_\psi L\left\|u^h_{\epsilon,n}-u_n\right\|^2_{0,\Gamma_C}\\
&\quad+(\frac{L_\psi L}{4\gamma}+\gamma M_\psi)\|\mathcal{L}'^h\varphi-\varphi\|^2_{0,\Gamma_C}\\
&\le(M_\psi L+\frac{L_\psi L+M_\psi}{4\gamma})\left\|\varphi-\varphi^h_\epsilon\right\|^2_{1,\Omega}+2\gamma L_\psi L\left\|u^h_\epsilon-u\right\|^2_{1,\Omega}\\
&\quad+(\frac{L_\psi L}{4\gamma}+\gamma M_\psi)\|\mathcal{L}'^h\varphi-\varphi\|^2_{0,\Gamma_C}.
\end{aligned}$$

So we obtain:

$$\begin{aligned}
&\left|\ell(u^h_\epsilon,\varphi^h_\epsilon,\mathcal{L}'^h\varphi-\varphi^h_\epsilon)-\ell(u,\varphi,\mathcal{L}'^h\varphi-\varphi^h_\epsilon)\right|\\
&\le(M_\psi+\frac{L_\psi L+M_\psi}{4\gamma})\left\|\varphi-\varphi^h_\epsilon\right\|^2_{1,\Omega}+2\gamma L_\psi L\left\|u^h_\epsilon-u\right\|^2_{1,\Omega}\\
&\quad+(\gamma M_\psi+\frac{L_\psi L}{4\gamma})\|\mathcal{L}'^h\varphi-\varphi\|^2_{1,\Omega}.
\end{aligned}\tag{79}$$

Now we combine (77), (78) and (79)

$$
\begin{aligned}
&\frac{\alpha_a}{2}\left\|u-u_\epsilon^h\right\|_{1,\Omega}^2+\frac{\alpha_b}{2}\left\|\varphi-\varphi_\epsilon^h\right\|_{1,\Omega}^2\\
&\leq \frac{C}{2\alpha_a}\left\|u-\mathcal{L}^h u\right\|_{1,\Omega}^2+\Big(\frac{C}{2\alpha_b}+\gamma M_\psi+\frac{L_\psi L}{4\gamma}\Big)\left\|\varphi-\mathcal{L'}^h\varphi\right\|_{1,\Omega}^2+\langle\sigma_n(u,\varphi),\mathcal{L}^h u_n\rangle_{\Gamma_C}\\
&\quad-\epsilon\left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u_{\epsilon,n}^h]_+\right\|_{0,\Gamma_C}^2+\frac{\epsilon^2}{2\beta}\left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u_{\epsilon,n}^h]_+\right\|_{-\nu,\Gamma_C}^2+\frac{\beta}{2}\left\|\sigma_n(u,\varphi)\right\|_{\nu,\Gamma_C}^2\\
&\quad+\Big(M_\psi+\frac{L_\psi L+M_\psi}{4\gamma}\Big)\left\|\varphi-\varphi_\epsilon^h\right\|_{1,\Omega}^2+2\gamma L_\psi L\left\|u_\epsilon^h-u\right\|_{1,\Omega}^2\\
&\leq \frac{C}{2\alpha_a}\left\|u-\mathcal{L}^h u\right\|_{1,\Omega}^2+\Big(\frac{C}{2\alpha_b}+\gamma M_\psi+\frac{L_\psi L}{4\gamma}\Big)\left\|\varphi-\mathcal{L'}^h\varphi\right\|_{1,\Omega}^2+\langle\sigma_n(u,\varphi),\mathcal{L}^h u_n\rangle_{\Gamma_C}\\
&\quad-\epsilon\Big(1-C\frac{\epsilon h^{2\nu}}{2\beta}\Big)\left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u_{\epsilon,n}^h]_+\right\|_{0,\Gamma_C}^2+Ch^{2\nu-1}\frac{\epsilon^2}{2\beta}\Big(\left\|u-u_\epsilon^h\right\|_{1,\Omega}^2+\left\|\varphi-\varphi_\epsilon^h\right\|_{1,\Omega}^2\Big)\\
&\quad+\frac{\beta}{2}\left\|\sigma_n(u,\varphi)\right\|_{\nu,\Gamma_C}^2+\Big(M_\psi+\frac{L_\psi L+M_\psi}{4\gamma}\Big)\left\|\varphi-\varphi_\epsilon^h\right\|_{1,\Omega}^2+2\gamma L_\psi L\left\|u_\epsilon^h-u\right\|_{1,\Omega}^2.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\Big(\frac{\alpha_a}{2}-Ch^{2\nu-1}\frac{\epsilon^2}{2\beta}-2\gamma L_\psi L\Big)\left\|u-u_\epsilon^h\right\|_{1,\Omega}^2\\
&+\Big(\frac{\alpha_b}{2}-Ch^{2\nu-1}\frac{\epsilon^2}{2\beta}-M_\psi-\frac{L_\psi L+M_\psi}{4\gamma}\Big)\left\|\varphi-\varphi_\epsilon^h\right\|_{1,\Omega}^2\\
&+\epsilon\Big(1-C\frac{\epsilon h^{2\nu}}{2\beta}\Big)\left\|\sigma_n(u,\varphi)+\frac{1}{\epsilon}[u_{\epsilon,n}^h]_+\right\|_{0,\Gamma_C}^2\\
&\leq \frac{C}{2\alpha_a}\left\|u-\mathcal{L}^h u\right\|_{1,\Omega}^2+\Big(\frac{C}{2\alpha_b}+\gamma M_\psi+\frac{L_\psi L}{4\gamma}\Big)\left\|\varphi-\mathcal{L'}^h\varphi\right\|_{1,\Omega}^2\\
&\quad+\langle\sigma_n(u,\varphi),\mathcal{L}^h u_n\rangle_{\Gamma_C}+\frac{\beta}{2}\left\|\sigma_n(u,\varphi)\right\|_{\nu,\Gamma_C}^2.
\end{aligned}
\tag{80}
$$

We show now that $\beta>0$ and $\gamma>0$ exists such that the terms $\frac{\alpha_a}{2}-Ch^{2\nu-1}\frac{\epsilon^2}{2\beta}-2\gamma L_\psi L$, $\frac{\alpha_b}{2}-Ch^{2\nu-1}\frac{\epsilon^2}{2\beta}-M_\psi-\frac{L_\psi L+M_\psi}{4\gamma}$ and $1-C\frac{\epsilon h^{2\nu}}{2\beta}$ are positives,
it equivalent a

$$
\begin{cases}
\alpha_a-4\gamma L_\psi L>0\\
\frac{\beta}{C}>\frac{h^{2\nu-1}\epsilon^2}{\alpha_a-4\gamma L_\psi L}\\
\alpha_b-2M_\psi>\frac{L_\psi L+M_\psi}{2\gamma}\\
\frac{\beta}{C}>\frac{h^{2\nu-1}\epsilon^2}{\alpha_b-2M_\psi-\frac{L_\psi L+M_\psi}{2\gamma}}\\
\frac{\beta}{C}>\frac{h^{2\nu}\epsilon}{2}
\end{cases}
$$

We know that α_a and α_b are sufficiently large then we can find C a constant sufficiently large and we choose

$$
\begin{cases}
\frac{L_\psi L+M_\psi}{2(\alpha_b-2M_\psi)}<\gamma<\frac{\alpha_a}{4L_\psi L}\\
\beta=C\left(h^{2\nu-1}\epsilon^2\Big(\frac{1}{\alpha_a-2M_\psi}+\frac{1}{\alpha_b-2M_\psi-\frac{L_\psi L+M_\psi}{2\gamma}}\Big)+\frac{h^{2\nu}\epsilon}{2}\right)
\end{cases}
$$

Using $\frac{1}{\alpha_a-2M_\psi}+\frac{1}{\alpha_b-2M_\psi-\frac{L_\psi L+M_\psi}{2\gamma}}\leq 1,$
it results that:

$$\frac{\beta}{2}\left\|\sigma_n(u,\varphi)\right\|^2_{\nu,\Gamma_C}\leq C(h^{2\nu-1}\epsilon^2+h^{2\nu}\epsilon)\left\|\sigma_n(u,\varphi)\right\|^2_{\nu,\Gamma_C}$$

The estimation of the Lagrange interpolations in L^2 and H^1 norms on a domain D is classical (see.e.g. [24]) For all $s\in(1,2]$ we have:

$$\begin{aligned} h^{-1}\left\|u-\mathcal{L}^h u\right\|_{0,D}+\left\|u-\mathcal{L}^h u\right\|_{1,D} &\leq Ch^{s-1}\left\|u\right\|_{s,D}\\ h^{-1}\left\|\varphi-\mathcal{L}'^h\varphi\right\|_{0,D}+\left\|\varphi-\mathcal{L}'^h\varphi\right\|_{1,D} &\leq Ch^{s-1}\left\|\varphi\right\|_{s,D}. \end{aligned} \tag{81}$$

For $s=\frac{3}{2}+\nu$ we find :$\begin{cases}\left\|u-\mathcal{L}^h u\right\|_{1,\Omega} \leq Ch^{\frac{1}{2}+\nu}\left\|u\right\|_{\frac{3}{2}+\nu,\Omega}\\ \left\|\varphi-\mathcal{L}'^h\varphi\right\|_{1,\Omega} \leq Ch^{\frac{1}{2}+\nu}\left\|\varphi\right\|_{\frac{3}{2}+\nu,\Omega}.\end{cases}$

The contact term $\langle\sigma_n(u,\varphi),\mathcal{L}^h u_n\rangle_{\Gamma_C}$ can be estimated in two space dimensions using results from [25] :

(82)
$$\langle\sigma_n(u,\varphi),(\mathcal{L}^h u)_n\rangle_{\Gamma_C}\leq C\begin{cases} h^{1+\nu+2\nu^2}\left(\left\|u\right\|^2_{\frac{3}{2}+\nu,\Omega}+\left\|\varphi\right\|^2_{\frac{3}{2}+\nu,\Omega}\right) & \text{if } 0<\nu<\frac{1}{2}\\ h^2|\ln h|\left(\left\|u\right\|^2_{2,\Omega}+\left\|\varphi\right\|^2_{2,\Omega}\right) & \text{if } \nu=\frac{1}{2}.\end{cases}$$

In three space dimension the bound is obtained in a straightforward way using (81) for any $0<\nu\leq\frac{1}{2}$

$$\langle\sigma_n(u,\varphi),\mathcal{L}^h u_n\rangle_{\Gamma_C}\leq Ch^{1+\nu}\left(\left\|u\right\|^2_{\frac{3}{2}+\nu,\Omega}+\left\|\varphi\right\|^2_{\frac{3}{2}+\nu,\Omega}\right). \tag{83}$$

We combine the last estimations in two space dimension we prove that:

(84)
$$\begin{aligned}&\left\|u-u^h_\epsilon\right\|^2_{1,\Omega}+\left\|\varphi-\varphi^h_\epsilon\right\|^2_{1,\Omega}+\epsilon\left\|\sigma_n(u,\varphi)+\tfrac{1}{\epsilon}[u^h_{\epsilon,n}]_+\right\|^2_{0,\Gamma_C}\\ &\leq C\begin{cases}(h^{1+\nu+2\nu^2}+h^{2\nu-1}\epsilon^2+h^{2\nu}\epsilon)\left(\left\|u\right\|^2_{\frac{3}{2}+\nu,\Omega}+\left\|\varphi\right\|^2_{\frac{3}{2}+\nu,\Omega}\right) & \text{if } 0<\nu<\frac{1}{2}\\ (h^2|\ln h|+\epsilon^2+h\epsilon)\left(\left\|u\right\|^2_{2,\Omega}+\left\|\varphi\right\|^2_{2,\Omega}\right) & \text{if } \nu=\frac{1}{2}.\end{cases}\end{aligned}$$

Finally in two space dimension we prove that:

$$\begin{aligned}&\left\|\sigma_n(u,\varphi)+\tfrac{1}{\epsilon}[u^h_{\epsilon,n}]_+\right\|_{-\nu,\Gamma_C}\\ &\leq C\begin{cases}\left(h^{\frac{1+\nu}{2}+\nu^2}\left(h^\nu\epsilon^{-\frac{1}{2}}+h^{\nu-\frac{1}{2}}\right)+h^{2\nu-\frac{1}{2}}\epsilon^{\frac{1}{2}}+h^{2\nu}+h^{2\nu-1}\epsilon\right)\left(\left\|u\right\|_{\frac{3}{2}+\nu,\Omega}+\left\|\varphi\right\|_{\frac{3}{2}+\nu,\Omega}\right)\\ \quad\text{if } 0<\nu<\frac{1}{2}\\ \left(h^2|\ln h|^{\frac{1}{2}}\left(h^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}+1\right)+h^{\frac{1}{2}}\epsilon^{\frac{1}{2}}+\epsilon+h\right)\left(\left\|u\right\|_{2,\Omega}+\left\|\varphi\right\|_{2,\Omega}\right)\\ \quad\text{if } \nu=\frac{1}{2}\end{cases}\end{aligned}$$

Using $h^{2\nu}\leq h^{3\frac{\nu}{2}+\nu^2}$ and $h\leq h|\ln h|^{\frac{1}{2}}$ ends the proof.

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