INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 13, Number 4, Pages 610–626

A SPLITTING LEAST-SQUARES MIXED FINITE ELEMENT METHOD FOR ELLIPTIC OPTIMAL CONTROL PROBLEMS

HONGFEI FU, HONGXING RUI[1], HUI GUO, JIANSONG ZHANG, AND JIAN HOU

Abstract. In this paper, we propose a splitting least-squares mixed finite element method for the approximation of elliptic optimal control problem with the control constrained by pointwise inequality. By selecting a properly least-squares minimization functional, we derive equivalent two independent, symmetric and positive definite weak formulation for the primal state variable and its flux. Then, using the first order necessary and also sufficient optimality condition, we deduce another two corresponding adjoint state equations, which are both independent, symmetric and positive definite. Also, a variational inequality for the control variable is involved. For the discretization of the state and adjoint state equations, either RT mixed finite element or standard C^0 finite element can be used, which is not necessary subject to the Ladyzhenkaya-Babuska-Brezzi condition. Optimal a priori error estimates in corresponding norms are derived for the control, the states and adjoint states, respectively. Finally, we use some numerical examples to validate the theoretical analysis.

Key words. Optimal control, splitting least-squares, mixed finite element method, positive definite, a priori error estimates.

1. Introduction

Optimal control problems are playing an increasingly important role in modern scientific and engineering numerical simulations. Nowadays, finite element method seems to be the most widely used numerical method in practical computation. The readers are referred to, for example, Refs. [1, 2, 3, 4, 5] for systematic introductions of finite element methods and optimal control problems.

In this paper, we are interested in the following convex quadratic optimal control problem with the control constrained by pointwise inequality:

$$\min_{u\in U_{ad}} \mathcal{J}(y,\sigma,u) = \frac{1}{2}\left(\int_\Omega (y-y_d)^2 + \int_\Omega (\sigma-\sigma_d)^2 + \gamma\int_{\Omega_U} u^2\right) \tag{1}$$

subject to

$$\begin{cases} \mathrm{div}\sigma + cy = f + \mathcal{B}u, & \text{in } \Omega, \\ \sigma + \mathcal{A}\nabla y = 0, & \text{in } \Omega, \\ y = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

and

$$\xi_1 \le u(x) \le \xi_2, \ a.e. \text{ in } \Omega_U. \tag{3}$$

Here $\gamma > 0$ is a constant, Ω and $\Omega_U \subseteq \Omega$ are two bounded domain in $\mathbb{R}^2$, with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega_U$. A precise formulation of this problem including a functional analytic setting is given in the next section.

Received by the editors June 22, 2015 and, in revised form, December 3, 2015.
2000 *Mathematics Subject Classification.* 49K20, 49M25, 65N15, 65N30.

[1]Corresponding author: Hongxing Rui.

This research was supported by the NSF of China (11201485, 91330106, 11571367, 11401588), the NSF of Shandong Province (BS2013NJ001), and the Fundamental Research Funds for the Central Universities (14CX02217, 15CX08004A).

Classical mixed finite element methods (see Refs. [6, 7, 8, 9, 10]) have been proved effectively for solving elliptic equations and fluid problems. They have an advantage of approximating the unknown scalar variable and its diffusive flux simultaneously. Besides, these methods can approximate the unknown variable and its flux to a same order of accuracy. Recently, there are some research articles on these methods for solving optimal control problems, see Refs. [11, 12, 13], for example. However, it is well known that these methods usually produce a symmetric but indefinite system for elliptic equations. Thus the popular conjugate gradient (CG) or algebraic multi-grid (AMG) solvers can not be used for the solution of linear algebraic equation systems.

To conquer these difficulties appeared in using classical mixed finite element methods, least-squares mixed finite element method, for first-order elliptic mixed system in unknown variable y and unknown velocity flux σ, was introduced by Pehlivanov et al. [14]. It is well known that the least-squares mixed finite element method has two typical advantages: First, it is not subjected to the Ladyzhenkaya-Babuska-Brezzi consistency condition, so the choice of finite element spaces becomes flexible; Second, it results in a symmetric and positive definite system, which can be solved using those solvers such as CG and AMG quickly. The idea of splitting least-squares was first proposed by Rui et al. in [15] for a reaction-diffusion equation, where by selecting a properly least-squares functional, the authors derived two independent, symmetric and positive definite equations, respectively, for the unknown state variable y and its flux σ. Then it is applied to solve linear and nonlinear parabolic equations [16], sobolev equations [17], pseudo-parabolic equations [18], and nonlinear convection-diffusion equations [19] and so on.

In this paper, we apply the splitting least-squares mixed finite element method for the discretization of elliptic optimal control problem. Pointwise inequality constraints on the control variable are considered. We derive optimal a priori error estimates, respectively, for the optimal control u^* in $L^2(\Omega_U)$-norm, which is approximated by piecewise constant or piecewise linear discontinuous elements; for the primal state y^* and adjoint state z^* both in $L^2(\Omega)$-norm and $H^1(\Omega)$-norm, which are approximated by standard piecewise linear C^0 finite elements; for the flux state σ^* and adjoint state ω^* in $H(\mathrm{div};\Omega)$-norm, which are approximated by the lowewt-order RT mixed finite elements or standard piecewise linear C^0 finite elements. Here, the Ladyzhenkaya-Babuska-Brezzi consistency condition for the discretization spaces of y^* and σ^* is not needed.

This paper is organized as follows. In Sect. 2, we introduce the optimal control problem and derive the continuous optimality conditions based on the idea of least-squares. In Sect. 3, a splitting least-squares mixed finite element approximation to the continuous optimal control problem is proposed, and then we derive the corresponding discrete optimality conditions. In Sect. 4, some a priori error estimates for the states, adjoint states and control are derived under control constrained by pointwise inequality. In Sect. 5, we conduct some numerical experiments to observe the convergence behavior of the numerical scheme. In the last section, some concluding remarks are given.

In the following, we employ the standard notations $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,p}(\Omega)}$ and seminorm $|\cdot|_{W^{m,p}(\Omega)}$. We set $W_0^{m,p}(\Omega)=\{v\in W^{m,p}(\Omega): v=0 \text{ on } \partial\Omega\}$. For $p=2$, we denote $H^m(\Omega)=W^{m,2}(\Omega)$ and $H_0^m(\Omega)=W_0^{m,2}(\Omega)$. In addition, C denotes a general positive constant which is independent of the spatial mesh parameters.

2. Optimality system based on splitting least-squares

In this section, we first describe the optimal control problem under consideration, and then derive the continuous optimality system based on the idea of splitting least-squares.

Let

$$H(\mathrm{div};\Omega) = \{\tau \in L^2(\Omega)^2 : \mathrm{div}\tau \in L^2(\Omega)\},$$

endowed with the norm given by

$$\|\tau\|_{H(\mathrm{div};\Omega)} = \left(\|\tau\|^2_{L^2(\Omega)^2} + \|\mathrm{div}\tau\|^2_{L^2(\Omega)}\right)^{1/2}.$$

To formulate the optimal control problem, in the rest we shall take the state spaces

$$V = H^1_0(\Omega), \quad W = H(\mathrm{div};\Omega),$$

the control space

$$U = L^2(\Omega_U),$$

and the admissible control set

$$U_{ad} = \{u \in U : \xi_1 \le u \le \xi_2,\ a.e. \text{ in } \Omega_U\}, \tag{4}$$

where the bounds ξ_1, $\xi_2 \in \mathbb{R}$ fulfill $\xi_1 < \xi_2$.

For the optimal control problem (1)-(2), we need the following assumptions.

(**H-1**) The desired states and source function satisfy the following regularity:

$$y_d \in H^1(\Omega),\ \sigma_d \in H^1(\Omega)^2,\ f \in L^2(\Omega).$$

(**H-2**) $\mathcal{A} = \mathcal{A}(x) = (a_{i,j}(x))_{2\times 2}$ is a bounded symmetric and positive definite matrix, i.e., there are positive constants α and β such that

$$\alpha|\mathcal{X}|^2 \le \mathcal{X}^T\mathcal{A}\mathcal{X} \le \beta|\mathcal{X}|^2, \quad \forall \mathcal{X} \in \mathbb{R}^2.$$

(**H-3**) $c = c(x)$ is positive definite and bounded, that is, there exist positive constants c_1 and c_2 such that

$$0 < c_1 \le c \le c_2.$$

(**H-4**) $\mathcal{B}$ is a linear continuous operator from $U = L^2(\Omega_U)$ to $L^2(\Omega)$, such that

$$|(\mathcal{B}u, v)| = |(\mathcal{B}^*v, u)_U| \le C\|u\|_{L^2(\Omega_U)}\|v\|_{L^2(\Omega)}, \quad \forall u \in U,\ v \in V,$$

where $\mathcal{B}^*$ is the adjoint operator of $\mathcal{B}$.

The classical mixed weak formula for problems (1)-(3) is to find $(y^*, \sigma^*, u^*) \in L^2(\Omega) \times H(\mathrm{div};\Omega) \times L^2(\Omega_U)$ such that

$$\mathcal{J}(y^*, \sigma^*, u^*) = \min_{u \in U_{ad}} \mathcal{J}(y, \sigma, u) \tag{5}$$

and

$$\begin{cases} (\mathcal{A}^{-1}\sigma, \tau) - (y, \mathrm{div}\tau) = 0, & \forall \tau \in H(\mathrm{div};\Omega), \\ (\mathrm{div}\sigma, v) + (cy, v) = (f + \mathcal{B}u, v), & \forall v \in L^2(\Omega). \end{cases} \tag{6}$$

It is clear that the two state variables σ and y in problem (6) are coupled each other, and the discretization of (6) will lead to a saddle-point problem, which is symmetric but indefinite.

To derive a symmetric and positive definite weak formulation for problem (2), we define for a given control u the least-squares functional $\mathcal{F}(v, \tau)$ as follows:

$$\mathcal{F}(v, \tau) = \frac{1}{2}\|c^{-1/2}(\mathrm{div}\tau + cv - f - \mathcal{B}u)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\mathcal{A}^{-1/2}(\tau + \mathcal{A}\nabla v)\|^2_{L^2(\Omega)^2}. \tag{7}$$

Then the solution to (2) is corresponding to the least-squares minimization problem:

$$\mathcal{F}(y,\sigma) = \min_{(v,\tau)\in V\times W} \mathcal{F}(v,\tau). \tag{8}$$

Furthermore, it follows from the first order necessary and sufficient minimization condition, problem (8) becomes: find $(y,\sigma) \in V\times W$ such that

$$\begin{cases} (\mathcal{A}\nabla y, \nabla v) + (cy, v) = (f + \mathcal{B}u, v), & \forall v \in V, \\ \left(c^{-1}\mathrm{div}\sigma, \mathrm{div}\tau\right) + (\mathcal{A}^{-1}\sigma, \tau) = \left(c^{-1}(f+\mathcal{B}u), \mathrm{div}\tau\right), & \forall \tau \in W. \end{cases} \tag{9}$$

Remark 2.1. *It is clear that the two equations in problem* (9) *are split, and thus can be solved independently. Besides, it is easy to check that they are also symmetric, coercive and continuous.*

For the given control set U_{ad}, we now restate the optimal control problem (1)-(3) as follows: (OCP)

$$\mathcal{J}(y^*, \sigma^*, u^*) = \min_{u\in U_{ad}} \mathcal{J}(y,\sigma,u) \tag{10}$$

such that $(y,\sigma,u) \in V\times W\times U$ and

$$\begin{cases} (\mathcal{A}\nabla y, \nabla v) + (cy, v) = (f + \mathcal{B}u, v), & \forall v \in V, \\ \left(c^{-1}\mathrm{div}\sigma, \mathrm{div}\tau\right) + (\mathcal{A}^{-1}\sigma, \tau) = \left(c^{-1}(f+\mathcal{B}u), \mathrm{div}\tau\right), & \forall \tau \in W. \end{cases}$$

Here the inner product in $L^2(\Omega)$ or $L^2(\Omega)^2$ is indicated by $(\cdot,\cdot)$.

Since the objective functional $\mathcal{J}(y,\sigma,u)$ is convex, it then follows from Ref. [2] that the optimal control problem (OCP) has a unique solution $(y^*,\sigma^*,u^*) \in V\times W\times U_{ad}$. Furthermore, (y^*,σ^*,u^*) is the solution of (OCP) if and only if there is a pair of adjoint state $(z^*,\omega^*) \in V\times W$, such that $(y^*,\sigma^*,z^*,\omega^*,u^*) \in (V\times W)^2\times U_{ad}$ satisfies the following optimality conditions: (OCP-OPT)

$$\begin{cases} (\mathcal{A}\nabla y^*, \nabla v) + (cy^*, v) = (f + \mathcal{B}u^*, v), & \forall v \in V, \\ \left(c^{-1}\mathrm{div}\sigma^*, \mathrm{div}\tau\right) + (\mathcal{A}^{-1}\sigma^*, \tau) = \left(c^{-1}(f+\mathcal{B}u^*), \mathrm{div}\tau\right), & \forall \tau \in W. \end{cases} \tag{11}$$

$$\begin{cases} (\mathcal{A}\nabla z^*, \nabla v) + (cz^*, v) = -(y^* - y_d, v), & \forall v \in V, \\ \left(c^{-1}\mathrm{div}\omega^*, \mathrm{div}\tau\right) + (\mathcal{A}^{-1}\omega^*, \tau) = -(\sigma^* - \sigma_d, \tau), & \forall \tau \in W. \end{cases} \tag{12}$$

$$\left(\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\mathrm{div}\omega^*, u - u^*\right)_U \geq 0, \quad \forall u \in U_{ad} \subset U, \tag{13}$$

where $(\cdot,\cdot)_U$ denotes the inner product in $U = L^2(\Omega_U)$.

Remark 2.2. *It is clear that the two variables z^* and ω^* in problem* (12) *are also independent, so they can be solved separately, too. Besides, the two equations in problem* (12) *are also symmetric, coercive and continuous.*

Remark 2.3. [2]. *Inequality* (13) *is equivalent to the following:*

$$\begin{cases} \gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\mathrm{div}\omega^* \geq 0, \ u^* = \xi_1; \\ \gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\mathrm{div}\omega^* \leq 0, \ u^* = \xi_2; \\ \gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\mathrm{div}\omega^* = 0, \ \xi_1 < u^* < \xi_2. \end{cases} \tag{14}$$

Define a pointwise projection operator $P_{U_{ad}}$ from U to U_{ad} that

$$P_{U_{ad}}(f(x,t)) = \max\left(\xi_1, \min(\xi_2, f(x,t))\right).$$

Then, the optimality condition (13) *can be expressed as*

$$u^* = P_{U_{ad}}\left(\gamma^{-1}\mathcal{B}^*(z^* + c^{-1}\mathrm{div}\omega^*)\right). \tag{15}$$

3. Splitting least-squares mixed finite element approximation

In this section, we consider the approximation of optimal control problem (OCP) based on the splitting least-squares mixed finite element method.

Let $\mathcal{T}_{h_y}$ and $\mathcal{T}_{h_\sigma}$ be two families of regular triangulations of Ω, which could be identical or not. Here h_y and h_σ denote the largest diameters of elements in $\mathcal{T}_{h_y}$ and $\mathcal{T}_{h_\sigma}$, respectively. Construct two finite element spaces $V^h \times W^h \subset V \times W$ with the following approximate properties (see Refs. [1, 8, 9, 10]): There exist two integers $k_1 \geq k \geq 0$ and $l \geq 1$ such that

$$(16)\qquad \begin{cases} \inf_{v_h \in V^h}\{\|v - v_h\|_{L^2(\Omega)} + h_y\|\nabla(v - v_h)\|_{L^2(\Omega)^2}\} \leq C h_y^{l+1}\|v\|_{H^{l+1}(\Omega)}, \\ \inf_{\tau_h \in W^h}\|\tau - \tau_h\|_{L^2(\Omega)^2} \leq C h_\sigma^{k+1}\|\tau\|_{H^{k+1}(\Omega)^2}, \\ \inf_{\tau_h \in W^h}\|\mathrm{div}(\tau - \tau_h)\|_{L^2(\Omega)} \leq C h_\sigma^{k_1}\|\tau\|_{H^{k_1+1}(\Omega)^2}, \end{cases}$$

for any $v \in V \cap H^{l+1}(\Omega)$ and $\tau \in W \cap (H^{k_1+1}(\Omega))^2$. Here $k_1 = k+1$ when W^h is chosen as RT mixed finite elements in Ref. [8], and $k_1 = k \geq 1$ when W^h is any one of the other mixed elements in Refs. [9, 10] and standard C^0 elements in Ref. [1].

Let $\mathcal{T}_{h_u}$ be a family of regular triangulations of Ω_U, where h_u denotes the largest diameters of elements in $\mathcal{T}_{h_u}$. Associate with $\mathcal{T}_{h_u}$ is another finite element space $U^h \subset U = L^2(\Omega_U)$ consist of piecewise r-order polynomials on each element of $\mathcal{T}_{h_u}$. Here there is no requirement for the continuity, due to the limited regularity of the optimal control. Let U_{ad}^h be a closed convex set in U^h, that is,

$$(17)\qquad U_{ad}^h = \{u_h \in U^h : \xi_1 \leq u_h \leq \xi_2,\ a.e. \text{ in } \Omega_U\}.$$

Then the corresponding splitting least-squares mixed finite element approximation of (OCP), which will be labeled as $(\mathrm{OCP})_h$, is defined as follows:

$$(18)\qquad \mathcal{J}(y_h^*, \sigma_h^*, u_h^*) = \min_{u_h \in U_{ad}^h} \mathcal{J}(y_h, \sigma_h, u_h)$$

where $(y_h, \sigma_h, u_h) \in V^h \times W^h \times U^h$ satisfies

$$\begin{cases} (\mathcal{A}\nabla y_h, \nabla v_h) + (c y_h, v_h) = (f + \mathcal{B}u_h, v_h), & \forall v_h \in V^h, \\ (c^{-1}\mathrm{div}\sigma_h, \mathrm{div}\tau_h) + (\mathcal{A}^{-1}\sigma_h, \tau_h) = \left(c^{-1}(f + \mathcal{B}u_h), \mathrm{div}\tau_h\right), & \forall \tau_h \in W^h. \end{cases}$$

It is again well known (see, e.g., Ref. [2]) that the optimal control problem $(\mathrm{OCP})_h$ has a unique solution $(y_h^*, \sigma_h^*, u_h^*) \in V^h \times W^h \times U_{ad}^h$, and that a triplet $(y_h^*, \sigma_h^*, u_h^*)$ is the solution of $(\mathrm{OCP})_h$ if and only if there is a pair of adjoint state $(z_h^*, \omega_h^*) \in V^h \times W^h$, such that $(y_h^*, \sigma_h^*, z_h^*, \omega_h^*, u_h^*) \in (V^h \times W^h)^2 \times U_{ad}^h$ satisfies the following discrete optimality conditions: $(\mathrm{OCP\text{-}OPT})_h$

$$(19)\qquad \begin{cases} (\mathcal{A}\nabla y_h^*, \nabla v_h) + (c y_h^*, v_h) = (f + \mathcal{B}u_h^*, v_h), & \forall v_h \in V^h, \\ (c^{-1}\mathrm{div}\sigma_h^*, \mathrm{div}\tau_h) + (\mathcal{A}^{-1}\sigma_h^*, \tau_h) = \left(c^{-1}(f + \mathcal{B}u_h^*), \mathrm{div}\tau_h\right), & \forall \tau_h \in W^h. \end{cases}$$

$$(20)\qquad \begin{cases} (\mathcal{A}\nabla z_h^*, \nabla v_h) + (c z_h^*, v_h) = -(y_h^* - y_d, v_h), & \forall v_h \in V^h, \\ (c^{-1}\mathrm{div}\omega_h^*, \mathrm{div}\tau_h) + (\mathcal{A}^{-1}\omega_h^*, \tau_h) = -(\sigma_h^* - \sigma_d, \tau_h), & \forall \tau_h \in W^h. \end{cases}$$

$$(21)\qquad \left(\gamma u_h^* - \mathcal{B}^* z_h^* - \mathcal{B}^* c^{-1}\mathrm{div}\omega_h^*, u_h - u_h^*\right)_U \geq 0, \quad \forall u_h \in U_{ad}^h \subset U^h.$$

To obtain a priori error estimates for the above numerical schemes (19)-(21), in the end of this section, we introduce auxiliary variables $(y_h(u^*), \sigma_h(u^*)) \in V^h \times W^h$

and $(z_h(u^*), \omega_h(u^*)) \in V^h \times W^h$, associated with the optimal control u^* as follows:

$$\begin{cases} (\mathcal{A}\nabla y_h(u^*), \nabla v_h) + (cy_h(u^*), v_h) = (f + \mathcal{B}u^*, v_h), \\ (c^{-1}\mathrm{div}\sigma_h(u^*), \mathrm{div}\tau_h) + (\mathcal{A}^{-1}\sigma_h(u^*), \tau_h) = \left(c^{-1}(f + \mathcal{B}u^*), \mathrm{div}\tau_h\right), \end{cases} \tag{22}$$

and

$$\begin{cases} (\mathcal{A}\nabla z_h(u^*), \nabla v_h) + (cz_h(u^*), v_h) = -(y_h(u^*) - y_d, v_h), \\ (c^{-1}\mathrm{div}\omega_h(u^*), \mathrm{div}\tau_h) + (\mathcal{A}^{-1}\omega_h(u^*), \tau_h) = -(\sigma_h(u^*) - \sigma_d, \tau_h), \end{cases} \tag{23}$$

for $\forall v_h \in V^h$ and $\forall \tau_h \in W^h$.

For simplicity, in the following we denote

$$\widetilde{y}_h := y_h(u^*),\ \widetilde{z}_h := z_h(u^*),\ \widetilde{\sigma}_h := \sigma_h(u^*),\ \widetilde{\omega}_h := \omega_h(u^*).$$

4. A priori error estimates

In this section, we are able to derive some a priori error estimates for the optimal control problem (OCP-OPT) and its splitting least-squares mixed finite element approximation $(\text{OCP-OPT})_h$. Due to the weak regularity of the optimal control problem, we consider piecewise constant elements ($r = 0$) or piecewise linear discontinuous elements ($r = 1$) for the approximation of the optimal control u^*, and piecewise linear C^0 elements ($l = 1$) are adopted for both the approximations of the states y^* and z^*. While for the discretizations of the flux states σ^* and ω^*, we use either the lowest-order RT mixed finite elements ($k_1 = 1, k = 0$) or standard piecewise linear C^0 elements ($k_1 = k = 1$), where in both cases the Ladyzhenkaya-Babuska-Brezzi consistency condition is not satisfied.

Let

$$\begin{aligned} \Omega_U^* &= \{\cup\tau_U : \tau_U \subset \Omega_U, \xi_1 < u|_{\tau_U} < \xi_2\}, \\ \Omega_U^a &= \{\cup\tau_U : \tau_U \subset \Omega_U, u|_{\tau_U} \equiv \xi_1, \text{ or } u|_{\tau_U} \equiv \xi_2\}, \\ \Omega_U^b &= \Omega_U \setminus (\Omega_U^* \cup \Omega_U^a). \end{aligned} \tag{24}$$

It is easy to check that the three parts do not intersect on each other, and $\Omega_U = \Omega_U^* \cup \Omega_U^a \cup \Omega_U^b$. In this paper, we assume that u and $\mathcal{T}_U^h$ are regular such that $\mathrm{meas}(\Omega_U^b) \leq Ch_u$ (see Refs. [21, 22]). Moreover, set

$$\Omega_U^{**} = \{x \in \Omega_U, \xi_1 < u(x) < \xi_2\}.$$

Then it is easy to see that $\Omega_U^* \subset \Omega_U^{**}$.

Before deriving the main error estimates for the splitting least-squares mixed finite element approximation of optimal control problem governed by elliptic equations, some interpolation and projection estimates are prepared without proof.

Lemma 4.1. [1]. *Let $\mathcal{P}_h$ be the L^2-projection from $U = L^2(\Omega_U)$ to U^h such that for any $u \in U$*

$$(u - \mathcal{P}_h u, \phi_h)_U = 0, \quad \forall \phi_h \in U^h. \tag{25}$$

Then we have

$$\|u - \mathcal{P}_h u\|_{L^2(\Omega_U)} \leq Ch_u \|u\|_{H^1(\Omega_U)}$$

for any $u \in H^1(\Omega_U)$.

Lemma 4.2. *Let $\mathcal{I}_h$ be the standard Lagrange interpolation operator defined in Ref. [1]. Then there is a constant $C > 0$ such that*

$$|u - \mathcal{I}_h u|_{W^{m,p}(\Omega)} \leq Ch_u^{2-m} |u|_{W^{2,p}(\Omega)}$$

for $u \in W^{2,p}(\Omega)$, $p > 1$, and $m = 0$ or 1.

Lemma 4.3. *Let $\mathcal{R}_h y \in V^h$ be defined as the elliptic projection of $y \in V$ such that*

$$(\mathcal{A}\nabla(y - \mathcal{R}_h y), v_h) + (c(y - \mathcal{R}_h y), v_h) = 0, \quad \forall w_h \in V^h. \tag{26}$$

It then follows from Ref. [20] *that*

$$\|y - \mathcal{R}_h y\|_{L^2(\Omega)} + h_y\|y - \mathcal{R}_h y\|_{H^1(\Omega)} \le C h_y^2 \|y\|_{H^2(\Omega)}.$$

Lemma 4.4. [15]. *From the approximate property* (16) *of finite element spaces we know that for any $\sigma \in W \cap H^2(\Omega)^2$, there exists a vector-valued function $\mathcal{Q}_h\sigma \subset W^h$ such that*

$$\begin{cases} \|\sigma - \mathcal{Q}_h\sigma\|_{L^2(\Omega)^2} \le C h_\sigma^{k+1}\|\sigma\|_{H^{k+1}(\Omega)^2}, \\ \|\mathrm{div}(\sigma - \mathcal{Q}_h\sigma)\|_{L^2(\Omega)} \le C h_\sigma \|\sigma\|_{H^2(\Omega)^2}, \end{cases}$$

where $k = 0$ for the lowest-order RT mixed finite elements, and $k = 1$ for piecewise linear C^0 elements.

In the following, we wish to demonstrate three main lemmas which can be collected to obtain the main conclusion of this paper.

Lemma 4.5. *Let $(y_h^*, \sigma_h^*, \omega_h^*, z_h^*)$ and $(\widetilde{y}_h, \widetilde{\sigma}_h, \widetilde{\omega}_h, \widetilde{z}_h)$ be the solutions of* (19)-(20) *and* (22)-(23), *respectively. Then the following estimates hold*

$$\|y_h^* - \widetilde{y}_h\|_{H^1(\Omega)} + \|z_h^* - \widetilde{z}_h\|_{H^1(\Omega)} \le C\|u^* - u_h^*\|_{L^2(\Omega_U)}, \tag{27}$$

$$\|\sigma_h^* - \widetilde{\sigma}_h\|_{H(\mathrm{div};\Omega)} + \|\omega_h^* - \widetilde{\omega}_h\|_{H(\mathrm{div};\Omega)} \le C\|u^* - u_h^*\|_{L^2(\Omega_U)}. \tag{28}$$

Proof. It follows from (19) and (22) that

$$\begin{cases} (\mathcal{A}\nabla(y_h^* - \widetilde{y}_h), \nabla v_h) + (c(y_h^* - \widetilde{y}_h), v_h) = (\mathcal{B}(u_h^* - u^*), v_h), \\ (c^{-1}\mathrm{div}(\sigma_h^* - \widetilde{\sigma}_h), \mathrm{div}\tau_h) + (\mathcal{A}^{-1}(\sigma_h^* - \widetilde{\sigma}_h), \tau_h) = \left(c^{-1}\mathcal{B}(u_h^* - u^*), \mathrm{div}\tau_h\right), \end{cases} \tag{29}$$

for $\forall v_h \in V^h$ and $\forall \tau_h \in W^h$.

Note that the two equations in problem (29) are split. By selecting $v_h = y_h^* - \widetilde{y}_h$, $\tau_h = \sigma_h - \widetilde{\sigma}_h$, respectively, and using the assumptions (**H-2**)-(**H-4**), we can easily prove that

$$\|y_h^* - \widetilde{y}_h\|_{H^1(\Omega)} \le C\|u^* - u_h^*\|_{L^2(\Omega_U)}, \tag{30}$$

$$\|\sigma_h^* - \widetilde{\sigma}_h\|_{H(\mathrm{div};\Omega)} \le C\|u^* - u_h^*\|_{L^2(\Omega_U)}. \tag{31}$$

Similarly, one can obtain from (20) and (23) that

$$\begin{cases} (\mathcal{A}\nabla(z_h^* - \widetilde{z}_h), \nabla v_h) + (c(z_h^* - \widetilde{z}_h), v_h) = (\widetilde{y}_h - y_h^*, v_h), \\ (c^{-1}\mathrm{div}(\omega_h^* - \widetilde{\omega}_h), \mathrm{div}\tau_h) + (\mathcal{A}^{-1}(\omega_h^* - \widetilde{\omega}_h), \tau_h) = (\widetilde{\sigma}_h - \sigma_h^*, \tau_h), \end{cases} \tag{32}$$

for $\forall v_h \in V^h$ and $\forall \tau_h \in W^h$.

Then with $v_h = z_h^* - \widetilde{z}_h$ and $\tau_h = \omega_h^* - \widetilde{\omega}_h$ in (32), and again using the assumptions (**H-2**)-(**H-4**), we similarly derive

$$\|z_h^* - \widetilde{z}_h\|_{H^1(\Omega)} \le C\|y_h^* - \widetilde{y}_h\|_{L^2(\Omega)} \le C\|u^* - u_h^*\|_{L^2(\Omega_U)}, \tag{33}$$

$$\|\omega_h^* - \widetilde{\omega}_h\|_{H(\mathrm{div};\Omega)} \le C\|\sigma_h^* - \widetilde{\sigma}_h\|_{L^2(\Omega)^2} \le C\|u^* - u_h^*\|_{L^2(\Omega_U)}, \tag{34}$$

which ends the proof of Lemma 4.5. ∎

Lemma 4.6. *Let $(y^*, \sigma^*, z^*, \omega^*, u^*)$ and $(y_h^*, \sigma_h^*, z_h^*, \omega_h^*, u_h^*)$ be the solutions of* (OCP-OPT) *and* (OCP-OPT)$_h$*, respectively. Let U^h be the piecewise constant element space $(r = 0)$. Assume that $u^* \in H^1(\Omega_U)$, $z^* \in H^1(\Omega)$, and $\mathrm{div}\omega^* \in H^1(\Omega)$. Then we have*

$$\|u^* - u_h^*\|_{L^2(\Omega_U)} \leq C \left(h_u + \|z^* - \widetilde{z}_h\|_{L^2(\Omega)} + \|\mathrm{div}(\omega^* - \widetilde{\omega}_h)\|_{L^2(\Omega)}\right), \tag{35}$$

where $(\widetilde{z}_h, \widetilde{\omega}_h)$ is defined in (23), and the constant C depends on some norms of the solution (z^*, ω^*, u^*), but is independent of the mesh parameters h_u.

Let U^h be the piecewise linear discontinuous element space $(r = 1)$. Furthermore, assume that $u^* \in W^{1,\infty}(\Omega_U) \cap H^2(\Omega_U^{**})$, where Ω_U^{**} is defined above, $z^* \in W^{1,\infty}(\Omega)$, and $\mathrm{div}\omega^* \in W^{1,\infty}(\Omega)$. Then we have

$$\|u^* - u_h^*\|_{L^2(\Omega_U)} \leq C \left(h_u^{\frac{3}{2}} + \|z^* - \widetilde{z}_h\|_{L^2(\Omega)} + \|\mathrm{div}(\omega^* - \widetilde{\omega}_h)\|_{L^2(\Omega)}\right), \tag{36}$$

where the constant C depends on some norms of the solution (z^*, ω^*, u^*), but is independent of the mesh parameter h_u.

Proof. Let $u_I^* \in U_{ad}^h$ be an approximation of u^*. Note that $U^h \subset U$ and $U_{ad}^h \subset U_{ad}$. Recalling the variational inequalities (13) and (21) imply that

$$\begin{aligned} &(\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\mathrm{div}\omega^*, u^* - u_h^*)_U \leq 0, \\ &(\gamma u_h^* - \mathcal{B}^* z_h^* - \mathcal{B}^* c^{-1}\mathrm{div}\omega_h^*, u_h^* - u_I^*)_U \leq 0. \end{aligned} \tag{37}$$

Hence, we can deduce that

$$\begin{aligned} &\gamma\|u^* - u_h^*\|_{L^2(\Omega_U)}^2 = (\gamma u^*, u^* - u_h^*)_U - (\gamma u_h^*, u^* - u_h^*)_U \\ = \;& (\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\mathrm{div}\omega^*, u^* - u_h^*)_U \\ & +(\gamma u_h^* - \mathcal{B}^* z_h^* - \mathcal{B}^* c^{-1}\mathrm{div}\omega_h^*, u_h^* - u_I^*)_U \\ & +(\gamma u_h^* - \mathcal{B}^* z_h^* - \mathcal{B}^* c^{-1}\mathrm{div}\omega_h^*, u_I^* - u^*)_U \\ & +(\mathcal{B}^*(z^* - z_h^*), u^* - u_h^*)_U + \left(\mathcal{B}^* c^{-1}\mathrm{div}(\omega^* - \omega_h^*), u^* - u_h^*\right)_U \\ \leq \;& (\gamma(u_h^* - u^*), u_I^* - u^*)_U + (\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\mathrm{div}\omega^*, u_I^* - u^*)_U \\ & +(\mathcal{B}^*(z^* - \widetilde{z}_h), u_I^* - u^*)_U + \left(\mathcal{B}^* c^{-1}\mathrm{div}(\omega^* - \widetilde{\omega}_h), u_I^* - u^*\right)_U \\ & +(\mathcal{B}^*(\widetilde{z}_h - z_h^*), u_I^* - u^*)_U + \left(\mathcal{B}^* c^{-1}\mathrm{div}(\widetilde{\omega}_h - \omega_h^*), u_I^* - u^*\right)_U \\ & +(\mathcal{B}^*(z^* - \widetilde{z}_h), u^* - u_h^*)_U + \left(\mathcal{B}^* c^{-1}\mathrm{div}(\omega^* - \widetilde{\omega}_h), u^* - u_h^*\right)_U \\ & +(\mathcal{B}^*(\widetilde{z}_h - z_h^*), u^* - u_h^*)_U + \left(\mathcal{B}^* c^{-1}\mathrm{div}(\widetilde{\omega}_h - \omega_h^*), u^* - u_h^*\right)_U. \end{aligned} \tag{38}$$

The right-hand side of (38) can be bounded in a standard way. We here just pay special attentions on the last two terms of the right-hand side of (38). By selecting $v_h = z_h^* - \widetilde{z}_h$, $\tau_h = \omega_h^* - \widetilde{\omega}_h$ in (29) and $v_h = y_h^* - \widetilde{y}_h$, $\tau_h = \sigma_h^* - \widetilde{\sigma}_h$ in (32) we have

$$\begin{aligned} &(\mathcal{B}^*(\widetilde{z}_h - z_h^*), u^* - u_h^*)_U = (\widetilde{z}_h - z_h^*, \mathcal{B}(u^* - u_h^*)) \\ = \;& (\mathcal{A}\nabla(y_h^* - \widetilde{y}_h), \nabla(z_h^* - \widetilde{z}_h)) + (c(y_h^* - \widetilde{y}_h), z_h^* - \widetilde{z}_h) \\ = \;& -(y_h^* - \widetilde{y}_h, y_h^* - \widetilde{y}_h) = -\|y_h^* - \widetilde{y}_h\|_{L^2(\Omega)}^2 \leq 0, \end{aligned} \tag{39}$$

and

$$\begin{aligned} &\left(\mathcal{B}^* c^{-1}\mathrm{div}(\widetilde{\omega}_h - \omega_h^*), u^* - u_h^*\right)_U = \left(c^{-1}\mathrm{div}(\widetilde{\omega}_h - \omega_h^*), \mathcal{B}(u^* - u_h^*)\right) \\ = \;& (c^{-1}\mathrm{div}(\sigma_h^* - \widetilde{\sigma}_h), \mathrm{div}(\omega_h^* - \widetilde{\omega}_h)) + (\mathcal{A}^{-1}(\sigma_h^* - \widetilde{\sigma}_h), \omega_h^* - \widetilde{\omega}_h) \end{aligned} \tag{40}$$

$$= \quad (\widetilde{\sigma}_h - \sigma_h^*, \sigma_h^* - \widetilde{\sigma}_h) = -\|\sigma_h^* - \widetilde{\sigma}_h\|^2_{L^2(\Omega)^2} \leq 0.$$

Thus we obtain from (39)-(40), Lemma 4.5 and Cauchy-Schwarz inequality that

$$\begin{aligned} & \gamma\|u^* - u_h^*\|^2_{L^2(\Omega_U)} \\ \leq\ & (\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\mathrm{div}\omega^*, u_I^* - u^*)_U + C(\delta)\|u^* - u_I^*\|^2_{L^2(\Omega_U)} \\ & + C(\delta)\|z^* - \widetilde{z}_h\|^2_{L^2(\Omega)} + C(\delta)\|\mathrm{div}(\omega^* - \widetilde{\omega}_h)\|^2_{L^2(\Omega)^2} \\ & + C\delta\|\widetilde{z}_h - z_h^*\|^2_{L^2(\Omega)} + C\delta\|\mathrm{div}(\widetilde{\omega}_h - \omega_h^*)\|^2_{L^2(\Omega)} + C\delta\|u^* - u_h^*\|^2_{L^2(\Omega_U)} \\ \leq\ & (\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\mathrm{div}\omega^*, u_I^* - u^*)_U + C\|u^* - u_I^*\|^2_{L^2(\Omega_U)} \\ & + C\|z^* - \widetilde{z}_h\|^2_{L^2(\Omega)} + C\|\mathrm{div}(\omega^* - \widetilde{\omega}_h)\|^2_{L^2(\Omega)} + C\delta\|u^* - u_h^*\|^2_{L^2(\Omega_U)}, \end{aligned} \tag{41}$$

where δ is an arbitrary small positive number.

Thanks to the definition of the control finite element space U^h, in the following we shall discuss two different cases for estimates of the first two terms on the right-hand side of (41). First let us consider the case that U^h is the piecewise constant element space. Let u_I^* be defined as the L^2-projection $\mathcal{P}_h u^*$ (see, Lemma 4.1). It is clear that $\mathcal{P}_h u^* \in U_{ad}^h$, and

$$\|u^* - \mathcal{P}_h u^*\|_{L^2(\Omega_U)} \leq C h_u \|u^*\|_{H^1(\Omega_U)}. \tag{42}$$

Moreover, if $u^* \in H^1(\Omega_U)$, $z^* \in H^1(\Omega)$, and $\mathrm{div}\omega^* \in H^1(\Omega)$, we conclude from the definition of $\mathcal{P}_h$ in (25) that

$$\begin{aligned} & \left(\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\mathrm{div}\omega^*, \mathcal{P}_h u^* - u^*\right)_U \\ =\ & \int_{\Omega_U} (\Theta - \mathcal{P}_h\Theta)(\mathcal{P}_h u^* - u^*) \\ \leq\ & C h_u^2 \left(\|\mathrm{div}\omega^*\|^2_{H^1(\Omega)} + \|z^*\|^2_{H^1(\Omega)} + \|u^*\|^2_{H^1(\Omega_U)}\right), \end{aligned} \tag{43}$$

where $\Theta := \gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\mathrm{div}\omega^*$. Then (35) follows from (41)-(43) with $C\delta = \gamma/2$.

Next we consider the case that U^h is the piecewise linear element space. Let $u_I^* = \mathcal{I}_h u^* \in U^h$ be defined as the standard Lagrange linear interpolation of u^* (see, Lemma 4.2). It is clear that $\mathcal{I}_h u^* \in U_{ad}^h$, and for $u^* \in W^{1,\infty}(\Omega_U) \cap H^2(\Omega_U^{**})$ we have

$$\|u^* - \mathcal{I}_h u^*\|_{L^2(\Omega_U^*)} \leq C h_u^2 \|u^*\|_{H^2(\Omega_U^*)}, \quad \|u^* - \mathcal{I}_h u^*\|_{L^\infty(\Omega_U^b)} \leq C h_u \|u^*\|_{W^{1,\infty}(\Omega_U^b)}.$$

Note that $\mathcal{I}_h u^* = u^*$ on Ω_U^a, then it follows that

$$\begin{aligned} \|u^* - \mathcal{I}_h u^*\|^2_{L^2(\Omega_U)} &= \left(\int_{\Omega_U^*} + \int_{\Omega_U^a} + \int_{\Omega_U^b}\right)(u^* - \mathcal{I}_h u^*)^2 \\ \leq\ & C h_u^4 \|u^*\|^2_{H^2(\Omega_U^*)} + 0 + C h_u^2 \|u^*\|^2_{W^{1,\infty}(\Omega_U^b)} \mathrm{meas}(\Omega_U^b) \\ \leq\ & C h_u^3 \left(\|u\|^2_{H^2(\Omega_U^{**})} + \|u\|^2_{W^{1,\infty}(\Omega_U)}\right). \end{aligned} \tag{44}$$

Moreover, it follows from (14) that $\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\mathrm{div}\omega^* = 0$ on Ω_U^*, and we conclude from the definition of Ω_U^b that for any element $\tau_U \subset \Omega_U^b$, there is a x_0 such that $\xi_1 < u(x_0) < \xi_2$, and hence $(\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\text{div}\omega^*)(x_0) = 0$. Therefore, for any $\tau_U \subset \Omega_U^b$ we have

$$\begin{aligned} & \|\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\text{div}\omega^*\|_{L^\infty(\tau_U)} \\ = \;& \|\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\text{div}\omega^* - (\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\text{div}\omega^*)(x_0)\|_{L^\infty(\tau_U)} \\ \le \;& Ch_u \|\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\text{div}\omega^*\|_{W^{1,\infty}(\tau_U)}. \end{aligned}$$

Thus

$$\begin{aligned} & (\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\text{div}\omega^*, \mathcal{I}_h u^* - u^*)_U \\ = \;& \left(\int_{\Omega_U^*} + \int_{\Omega_U^a} + \int_{\Omega_U^b} \right) (\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\text{div}\omega^*)(\mathcal{I}_h u^* - u^*) \\ = \;& 0 + 0 + \int_{\Omega_U^b} (\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\text{div}\omega^*)(\mathcal{I}_h u^* - u^*) \qquad (45) \\ \le \;& \|\gamma u^* - \mathcal{B}^* z^* - \mathcal{B}^* c^{-1}\text{div}\omega^*\|_{L^\infty(\Omega_U^b)} \|\mathcal{I}_h u^* - u^*\|_{L^\infty(\Omega_U^b)} \text{meas}(\Omega_U^b) \\ \le \;& Ch_u^3 \left(\|u^*\|^2_{W^{1,\infty}(\Omega_U)} + \|z^*\|^2_{W^{1,\infty}(\Omega)} + \|\text{div}\omega^*\|^2_{W^{1,\infty}(\Omega)} \right). \end{aligned}$$

Then (36) is proved by inserting (44)-(45) into (41) with $C\delta = \gamma/2$. ■

Lemma 4.7. *Let $(y^*, \sigma^*, z^*, \omega^*)$ and $(\widetilde{y}_h, \widetilde{\sigma}_h, \widetilde{z}_h, \widetilde{\omega}_h)$ be the solutions of* (11)-(12) *and* (22)-(23), *respectively. Assume that problem* (2) *is H^2-regular, and the solutions y^*, $z^* \in H_0^1(\Omega) \cap H^2(\Omega)$ and σ^*, $\omega^* \in H^2(\Omega)^2$. Then the following estimates hold*

$$\|y^* - \widetilde{y}_h\|_{H^s(\Omega)} + \|z^* - \widetilde{z}_h\|_{H^s(\Omega)} \le Ch_y^{2-s}, \;\; s = 0, 1, \qquad (46)$$

$$\|\sigma^* - \widetilde{\sigma}_h\|_{H(\text{div};\Omega)} + \|\omega^* - \widetilde{\omega}_h\|_{H(\text{div};\Omega)} \le Ch_\sigma, \qquad (47)$$

where the constant C depends on some norms of $(y^*, \sigma^*, z^*, \omega^*)$, but is independent of the mesh parameters h_y and h_σ.

Proof. In the following, we divide the proof into two parts.

Part I. Note that $(\widetilde{\sigma}_h, \widetilde{y}_h)$ is the least-squares mixed finite element solution of (σ^*, y^*). Therefore, it follows immediately from Ref. [15] combined with Lemmas 4.3 and 4.4 that

$$\|y^* - \widetilde{y}_h\|_{H^s(\Omega)} \le Ch_y^{2-s} \|y^*\|_{H^2(\Omega)}, \;\; s = 0, 1, \qquad (48)$$

and

$$\|\sigma^* - \widetilde{\sigma}_h\|_{H(\text{div};\Omega)} \le Ch_\sigma \|\sigma^*\|_{H^2(\Omega)^2}. \qquad (49)$$

Part II. To derive corresponding error estimates for $\|z^* - \widetilde{z}_h\|_{H^s(\Omega)}$ and $\|\omega^* - \widetilde{\omega}_h\|_{H(\text{div};\Omega)}$ in (46)-(47), we write

$$z^* - \widetilde{z}_h = (z^* - \mathcal{R}_h z^*) + (\mathcal{R}_h z^* - \widetilde{z}_h) := \rho + \pi,$$
$$\omega^* - \widetilde{\omega}_h = (\omega^* - \mathcal{Q}_h \omega^*) + (\mathcal{Q}_h \omega^* - \widetilde{\omega}_h) := \zeta + \eta.$$

Here $\mathcal{R}_h z^* \in V^h$ and $\mathcal{Q}_h \omega^* \in W^h$ are defined as in Lemmas 4.3-4.4, and of course ρ and ζ are bounded in the desired way.

In the following, we first estimate $\pi = \mathcal{R}_h z^* - \widetilde{z}_h$. By subtracting the first equation in (23) with that of (12), we obtain for $\forall v_h \in V^h$,

$$(\mathcal{A}\nabla(z^* - \widetilde{z}_h), \nabla v_h) + (c(z^* - \widetilde{z}_h), v_h) = (\widetilde{y}_h - y^*, v_h), \qquad (50)$$

which implies

$$(\mathcal{A}\nabla\pi, \nabla v_h) + (c\pi, v_h) = (\widetilde{y}_h - y^*, v_h), \quad \forall v_h \in V^h. \tag{51}$$

Let $v_h = \pi = \mathcal{R}_h z^* - \widetilde{z}_h$. It then follows from the standard finite element error estimates and (48) that

$$\|\mathcal{R}_h z^* - \widetilde{z}_h\|_{H^1(\Omega)} \le C\|y^* - \widetilde{y}_h\|_{L^2(\Omega)} \le Ch_y^2\|y^*\|_{H^2(\Omega)}. \tag{52}$$

Similarly, we can obtain another error equation

$$\begin{aligned} & (c^{-1}\mathrm{div}(\omega - \widetilde{\omega}_h), \mathrm{div}\tau_h) + (\mathcal{A}^{-1}(\omega - \widetilde{\omega}_h), \tau_h) \\ = \;& (\widetilde{\sigma}_h - \sigma^*, \tau_h), \quad \forall \tau_h \in W^h. \end{aligned} \tag{53}$$

Using the interpolation operator $\mathcal{Q}_h$, equation (53) can also be written as

$$\begin{aligned} & (c^{-1}\mathrm{div}\eta, \mathrm{div}\tau_h) + (\mathcal{A}^{-1}\eta, \tau_h) \\ = \;& (\widetilde{\sigma}_h - \sigma^*, \tau_h) - (c^{-1}\mathrm{div}\zeta, \mathrm{div}\tau_h) - (\mathcal{A}^{-1}\zeta, \tau_h), \quad \forall \tau_h \in W^h. \end{aligned} \tag{54}$$

Let $\tau_h = \eta = \mathcal{Q}_h\omega^* - \widetilde{\omega}_h$ in (54). It is easy to see that

$$\begin{aligned} & c_2^{-1}\|\mathrm{div}(\mathcal{Q}_h\omega^* - \widetilde{\omega}_h)\|_{L^2(\Omega)}^2 + \beta^{-1}\|\mathcal{Q}_h\omega^* - \widetilde{\omega}_h\|_{L^2(\Omega)^2}^2 \\ \le \;& \|\sigma^* - \widetilde{\sigma}_h\|_{L^2(\Omega)^2}\|\mathcal{Q}_h\omega^* - \widetilde{\omega}_h\|_{L^2(\Omega)^2} \\ & + c_1^{-1}\|\mathrm{div}(\omega^* - \mathcal{Q}_h\omega^*)\|_{L^2(\Omega)}\|\mathrm{div}(\mathcal{Q}_h\omega^* - \widetilde{\omega}_h)\|_{L^2(\Omega)} \\ & + \alpha^{-1}\|\omega^* - \mathcal{Q}_h\omega^*\|_{L^2(\Omega)^2}\|\mathcal{Q}_h\omega^* - \widetilde{\omega}_h\|_{L^2(\Omega)^2} \\ \le \;& C\left(\|\sigma^* - \widetilde{\sigma}_h\|_{L^2(\Omega)^2} + \|\mathrm{div}(\omega^* - \mathcal{Q}_h\omega^*)\|_{L^2(\Omega)} + \|\omega^* - \mathcal{Q}_h\omega^*\|_{L^2(\Omega)^2}\right) \\ & \|\mathcal{Q}_h\omega^* - \widetilde{\omega}_h\|_{H(\mathrm{div};\Omega)}. \end{aligned} \tag{55}$$

Following from Lemma 4.4 and the proved result (49), we deduce

$$\begin{aligned} & \|\mathcal{Q}_h\omega^* - \widetilde{\omega}_h\|_{H(\mathrm{div};\Omega)} \\ \le \;& C\left(\|\sigma^* - \widetilde{\sigma}_h\|_{L^2(\Omega)^2} + \|\mathrm{div}(\omega^* - \mathcal{Q}_h\omega^*)\|_{L^2(\Omega)} + \|\omega^* - \mathcal{Q}_h\omega^*\|_{L^2(\Omega)^2}\right) \\ \le \;& Ch_\sigma \sum_{\tau=\sigma,\omega} \|\tau^*\|_{H^2(\Omega)^2}. \end{aligned} \tag{56}$$

Incorporating the above results (52), (56) with the well-known estimates for ρ in Lemma 4.3 and ζ in Lemma 4.4, we derive

$$\|z^* - \widetilde{z}_h\|_{H^s(\Omega)} \le Ch_y^{2-s} \sum_{v=y,z} \|v^*\|_{H^2(\Omega)}, \; s = 0, 1, \tag{57}$$

and

$$\|\omega^* - \widetilde{\omega}_h\|_{H(\mathrm{div};\Omega)} \le Ch_\sigma \sum_{\tau=\sigma,\omega} \|\tau^*\|_{H^2(\Omega)^2}. \tag{58}$$

Thus Lemma 4.7 is proved. ■

Collecting the bounds given by Lemmas 4.5-4.7, we have the following main result.

Theorem 4.8. *Suppose that $(y^*, \sigma^*, z^*, \omega^*, u^*)$ and $(y_h^*, \sigma_h^*, z_h^*, \omega_h^*, u_h^*)$ are the solutions of* (OCP-OPT) *and* (OCP-OPT)$_h$*, respectively. Assume that all conditions of Lemmas 4.5-4.7 are valid. Then for $r, s = 0, 1$, we have*

$$\|u^* - u_h^*\|_{L^2(\Omega_U)} \le C\left(h_u^{1+\frac{r}{2}} + h_y^2 + h_\sigma\right), \tag{59}$$

$$\|y^* - y_h^*\|_{H^s(\Omega)} + \|z^* - z_h^*\|_{H^s(\Omega)} \le C\left(h_u^{1+\frac{r}{2}} + h_y^{2-s} + h_\sigma\right), \tag{60}$$

$$\|\sigma^* - \sigma_h^*\|_{H(\mathrm{div};\Omega)} + \|\omega^* - \omega_h^*\|_{H(\mathrm{div};\Omega)} \le C\left(h_u^{1+\frac{r}{2}} + h_y^2 + h_\sigma\right), \tag{61}$$

where the constant C depends on some norms of $(y^*, \sigma^*, z^*, \omega^*, u^*)$, but is independent of the mesh parameters h_y, h_σ and h_u.

Proof. It follows from Lemmas 4.6-4.7 that

$$\begin{aligned} &\|u^* - u_h^*\|_{L^2(\Omega_U)} \\ \le\ & C\left(h_u^{1+\frac{r}{2}} + \|z^* - \widetilde{z}_h\|_{L^2(\Omega)} + \|\mathrm{div}(\omega^* - \widetilde{\omega}_h)\|_{L^2(\Omega)}\right) \\ \le\ & C\left(h_u^{1+\frac{r}{2}} + h_y^2 + h_\sigma\right). \end{aligned} \tag{62}$$

Moreover, it follows from Lemma 4.5, Lemma 4.7 and (62) that

$$\begin{aligned} &\|y^* - y_h^*\|_{H^s(\Omega)} + \|z^* - z_h^*\|_{H^s(\Omega)} \\ \le\ & \|y_h^* - \widetilde{y}_h\|_{H^s(\Omega)} + \|z_h^* - \widetilde{z}_h\|_{H^s(\Omega)} \\ & + \|y^* - \widetilde{y}_h\|_{H^s(\Omega)} + \|z^* - \widetilde{z}_h\|_{H^s(\Omega)} \\ \le\ & C\left(h_u^{1+\frac{r}{2}} + h_y^{2-s} + h_\sigma\right), \end{aligned} \tag{63}$$

$$\begin{aligned} &\|\sigma^* - \sigma_h^*\|_{H(\mathrm{div};\Omega)} + \|\omega^* - \omega_h^*\|_{H(\mathrm{div};\Omega)} \\ \le\ & \|\sigma_h^* - \widetilde{\sigma}_h\|_{H(\mathrm{div};\Omega)} + \|\omega_h^* - \widetilde{\omega}_h\|_{H(\mathrm{div};\Omega)} \\ & + \|\sigma^* - \widetilde{\sigma}_h\|_{H(\mathrm{div};\Omega)} + \|\omega^* - \widetilde{\omega}_h\|_{H(\mathrm{div};\Omega)} \\ \le\ & C\left(h_u^{1+\frac{r}{2}} + h_y^2 + h_\sigma\right). \end{aligned} \tag{64}$$

Thus, Theorem 4.8 follows immediately from (62)-(64). ■

5. Numerical experiments

In this section, we conduct some numerical examples to check the efficiency of the spitting least-squares mixed finite element scheme (19)-(21). Let the spatial domain $\Omega = (0,1)^2$ and consider the problem:

$$\frac{1}{2}\left(\int_\Omega (y - y_d)^2 + \int_\Omega (\sigma - \sigma_d)^2 + \int_\Omega u^2\right) \tag{65}$$

subject to

$$\begin{cases} -\mathrm{div}(\mathcal{A}\nabla y) + cy = f + u, & \text{in } \Omega, \\ y = 0, & \text{on } \partial\Omega, \end{cases} \tag{66}$$

with $\sigma = -\mathcal{A}\nabla y$. In fact, the adjoint state z satisfies the following equation:

$$\begin{cases} -\mathrm{div}(\mathcal{A}\nabla z) + cz = -(y - y_d), & \text{in } \Omega, \\ z = 0, & \text{on } \partial\Omega. \end{cases} \tag{67}$$

For simplicity we shall adopt the same mesh partitions for the states and control in the numerical tests, i.e., $\mathcal{T}_y^h = \mathcal{T}_\sigma^h = \mathcal{T}_u^h$. To solve the optimal control problem numerically, we use the C++ software package: AFEpack, it is available at http://dsec.pku.edu.cn/~rli/. As for constrained optimal control problems, people pay more attention on the states and the control. Therefore, in the following numerical examples, we mostly center on the primal state variable y, which is approximated by piecewise linear C^0 elements; the flux state variable σ, which is approximated by piecewise linear C^0 elements or the lowest-order RT elements; and the control variable u, which is discretized using piecewise constant elements or piecewise linear discontinuous elements. Furthermore, in the following tests, we use E_u, E_y^0, E_y^1, and E_σ to represent the error norms $\|u-u_h\|_{L^2(\Omega)}$, $\|y-y_h\|_{L^2(\Omega)}$, $\|y-y_h\|_{H^1(\Omega)}$, and $\|\sigma-\sigma_h\|_{H(\mathrm{div};\Omega)}$, respectively.

Example 1. For the first example, the corresponding analytical solutions for problems (65)-(67) with a constant diffusion coefficient matrix $\mathcal{A}=I$ and a constant reaction coefficient $c=1$ are given by

$$
\begin{aligned}
y(x) &= \sin(\pi x_1)\sin(\pi x_2),\\
\sigma(x) &= -\mathcal{A}\nabla y,\\
z(x) &= 2\sin(\pi x_1)\sin(\pi x_2),\\
\omega(x) &= (\sin(\pi x_2), \sin(\pi x_1)),\\
u(x) &= \max\{0, \min(1.0, z + c^{-1}\mathrm{div}\omega)).
\end{aligned}
$$

The source function f, the desired states y_d and σ_d can be determined using the above functions.

In this test, we consider the lowest-order RT elements for the approximation of the flux state σ, and piecewise constant elements for that of the control u. Table 1 displays the errors and convergence orders for the control u, the primal state y, and the flux state σ in corresponding norms. We can see that the convergence orders for the control u in $L^2(\Omega)$-norm, the state y in $H^1(\Omega)$-norm, and σ in $H(\mathrm{div};\Omega)$-norm are almost first-order, while the convergence order for the state y in $L^2(\Omega)$-norm is second-order, which are consistent with Theorem 4.8 very well. Figure 1 shows the approximate profiles of the control u and the primal state y. The two components of the numerical flux state σ are presented in Figure 2. It can be observed that the numerical figures are agreement with the analytical solutions very well.

TABLE 1. σ: the lowest-order RT elements, u: piecewise constant elements.

h	E_u	Order	E_y^0	Order	E_y^1	Order	E_σ	Order
$\frac{1}{10}$	7.7865E-02	—	6.9370E-03	—	2.4620E-01	—	9.2504E-01	—
$\frac{1}{20}$	3.8356E-02	1.02	1.7639E-03	1.98	1.2364E-01	0.99	4.6058E-01	1.01
$\frac{1}{40}$	1.9264E-02	0.99	4.4563E-04	1.98	6.1994E-02	1.00	2.3082E-01	1.00
$\frac{1}{80}$	9.5384E-03	1.01	1.1092E-04	2.01	3.0873E-02	1.01	1.1456E-01	1.01

Example 2. For the second example, the corresponding analytical solutions for problems (65)-(67) are

$$
\begin{aligned}
y(x) &= x_1(1-x_1)x_2(1-x_2),\\
\sigma(x) &= -\mathcal{A}\nabla y,\\
z(x) &= 16x_1(1-x_1)x_2(1-x_2),\\
\omega(x) &= (x_2(1-x_2), x_1(1-x_1)),\\
u(x) &= \max(0, \min(0.5, z + c^{-1}\mathrm{div}\omega)),
\end{aligned}
$$

where $\mathcal{A}=I$ and $c=10$. The source function f and the desired states y_d and σ_d can also be determined using the above functions.

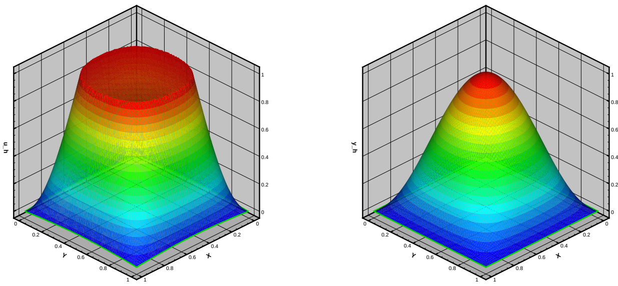

FIGURE 1. The approximate control u_h (*left*) and the approximate state y_h (*right*) for Example 1.

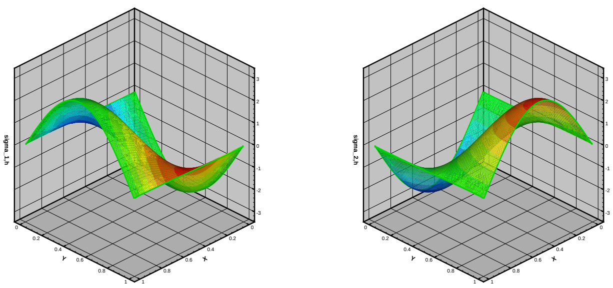

FIGURE 2. The approximate flux state σ_h for Example 1.

In this example, we consider piecewise linear C^0 elements for the approximation of the flux state σ, and piecewise constant elements for that of the control u. We list the corresponding numerical errors and convergence orders for the control u, the primal state y and the flux state σ in Table 2. The convergence orders for the control, the primal state, and the flux state are also well matched with the theoretical analysis in Theorem 4.8. We also show the approximate profiles of the control u, the primal state y and the flux state σ in Figures 3-4, respectively. It can be observed that the numerical results are also agreement with the analytical solutions very well.

TABLE 2. σ: piecewise linear C^0 elements, u: piecewise constant elements.

h	E_u	Order	E_y^0	Order	E_y^1	Order	E_σ	Order
$\frac{1}{10}$	4.2662E-02	—	6.3657E-04	—	1.7283E-02	—	6.1190E-02	—
$\frac{1}{20}$	2.0861E-02	1.03	1.6964E-04	1.91	8.7530E-03	0.98	3.0421E-02	1.01
$\frac{1}{40}$	1.0534E-02	0.99	4.5355E-05	1.90	4.3967E-03	0.99	1.5131E-02	1.01
$\frac{1}{80}$	5.2129E-03	1.01	1.1351E-05	2.00	2.1861E-03	1.01	7.1589E-03	1.08

Example 3. For the third example, we consider a reaction-dominated diffusion problem. In this case, the solution typically has exponential boundary layers on all sides of Ω or internal layers in Ω. We take the analytical solutions for problems

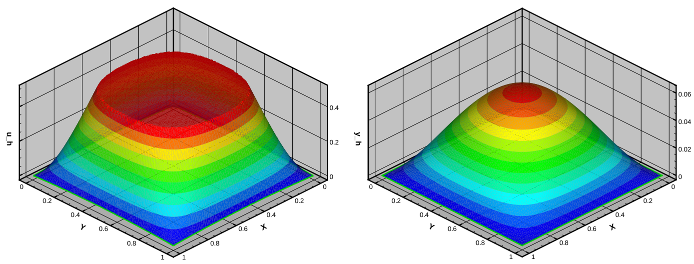

FIGURE 3. The approximate control u_h (*left*) and the approximate state y_h (*right*) for Example 2.

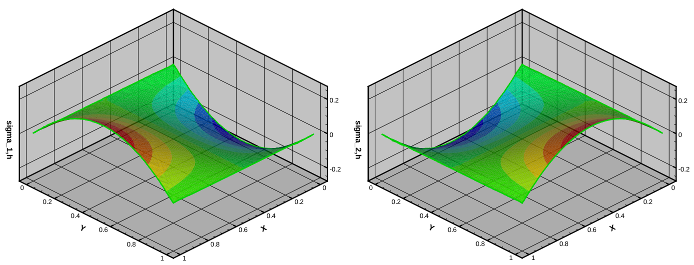

FIGURE 4. The approximate flux state σ_h for Example 2.

(65)-(67) as follows:

$$
\begin{aligned}
y(x) &= \exp(-((x_1-0.5)^2+(x_2-0.5)^2)/\varepsilon),\\
\sigma(x) &= -\varepsilon\nabla y,\\
z(x) &= 2\exp(-((x_1-0.5)^2+(x_2-0.5)^2)/\varepsilon),\\
\omega(x) &= (0,0),\\
u(x) &= \max(0,\min(0.5, z+c^{-1}\mathrm{div}\omega)),
\end{aligned}
$$

where $\varepsilon = 10^{-2}$ and $c = 1$. The source function f and the desired states y_d and σ_d can also be determined using the above functions.

In this example, we consider the lowest-order RT elements for the approximation of the flux state σ, and piecewise linear discontinuous elements for that of the control u. Some corresponding numerical errors and convergence orders for the control u, the primal state y and the flux state σ are presented in Table 3. It can be observed that an nearly $\mathcal{O}(h^{\frac{3}{2}})$-order convergence for the control is derived, which is consistent with Theorem 4.8 for the case $r = 1$. The approximate profiles of the control u, the primal state y and the flux state σ are displayed in Figures 5-6, respectively. All these results also confirms the theoretical analysis.

6. Concluding remarks

In this paper, we derive a priori error estimates for the splitting least-squares mixed finite element discretization of optimal control problem governed by elliptic

TABLE 3. σ: the lowest-order RT elements; u: piecewise linear discontinuous elements.

h	E_u	Order	E_y^0	Order	E_y^1	Order	E_σ	Order
$\frac{1}{10}$	1.8586E-02	—	2.2373E-02	—	8.7984E-01	—	1.9923E-01	—
$\frac{1}{20}$	6.5260E-03	1.51	5.0760E-03	2.14	4.4054E-01	1.00	8.8062E-02	1.18
$\frac{1}{40}$	2.2771E-03	1.52	1.2568E-03	2.01	2.2071E-01	1.00	4.4178E-02	1.00
$\frac{1}{80}$	7.8744E-04	1.53	3.1259E-04	2.01	1.1062E-01	1.00	2.2144E-02	1.00

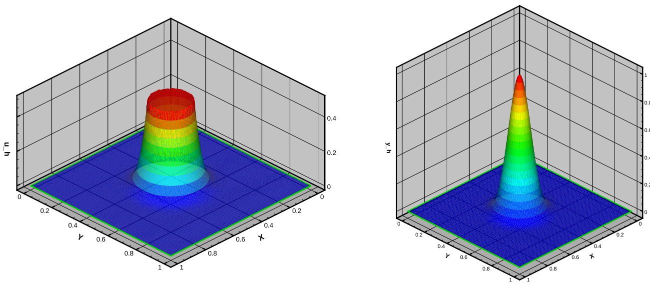

FIGURE 5. The approximate control u_h (*left*) and the approximate state y_h (*right*) for Example 3.

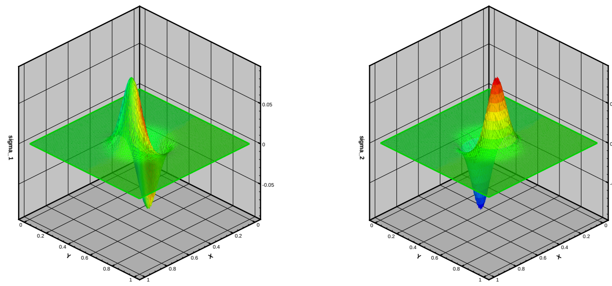

FIGURE 6. The approximate flux state σ_h for Example 3.

equations, where pointwise inequality constraints on the control variable are considered. This splitting mixed method leads to uncoupled positive definite systems for the primal state variable and its flux, also for the adjoint state and flux state variables. Finally, numerical experiments are addressed to verify the theoretical analysis.

Acknowledgments

The authors would like to thank the editor and the anonymous referee for their valuable comments and suggestions on an earlier version of this paper. The first author was partially supported by the China Scholarship Council.

References

[1] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, SIAM, Philadelphia, 2002.
[2] J. L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, Berlin, 1971.

[3] W. Liu and N. Yan, Adaptive Finite Element Methods for Optimal Control Governed by PDEs, Series in Information and Computational Science 41, Science Press, Beijing, 2008.
[4] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich, Optimization with PDE Constraints, Mathematical Modelling: Theory and Applications 23, Springer, Netherlands, 2009.
[5] D. Tiba, Lectures on the Optimal Control of Elliptic Equations, University of Jyvaskyla Press, Finland, 1995.
[6] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, 1991.
[7] J. Douglas Jr. and J. E. Roberts, Global estimates for mixed methods for second order elliptic equations, Math. Comput., 44 (1985) 39-52.
[8] P. A. Raviart and J. M. Thomas, A mixed finite element method for 2nd order elliptic problems, Lecture Notes in Mathematics, vol. 606, Springer, Berlin, 1977, 292-315.
[9] F. Brezzi, J. Douglas Jr., M. Fortin and L. D. Marini, Efficient rectangular mixed finite elements in two and three space variables, RAIRO Model. Math. Numer. Anal., 21 (1987) 581-603.
[10] F. Brezzi, J. Douglas Jr. and L. D. Marini, Two family of mixed finite elements for second order elliptic problems, Numer. Math., 47 (1985) 217-235.
[11] Y. Chen and W. Liu, Error estimates and superconvergence of mixed finite elements for quadratic optimal control, Int. J. Numer. Anal. Model., 3 (2006) 311-321.
[12] Y. Chen and Z. Lu, Error estimates of fully discrete mixed finite element methods for semi-linear quadratic parabolic optimal control problem, Comput. Methods Appl. Mech. Engrg., 199 (2010) 1415-1423.
[13] J. Zhou, Y. Chen and Y. Dai, Superconvergence of triangular mixed finite elements for optimal control problems with an integral constraint, Appl. Math. Comput., 217 (2010) 2057-2066.
[14] A. Pehlivanov, G. F. Carey and D. Lazarov, Least-squares mixed finite elements for second-order elliptic problems, SIAM J. Numer. Anal., 31 (1994) 1368-1377.
[15] H. Rui, S. Kim and S. D. Kim, A remark on least-squares mixed element methods for reaction-diffusion problems, J. Comput. Appl. Math., 202 (2007) 230-236.
[16] H. Rui, S. D. Kim and S. Kim, Split least-squares finite element methods for linear and nonlinear parabolic problems, J. Comput. Appl. Math., 223 (2009) 938-952.
[17] F. Gao and H. Rui, A split least-squares characteristic mixed finite element method for Sobolev equations with convection term, Math. Comput. Simulation, 80 (2009) 341-351.
[18] H. Guo, A remark on split least-squares mixed element procedures for pseudo-parabolic equations, Appl. Math. Comput., 217 (2010) 4682-4690.
[19] J. Zhang and H. Guo, A split least-squares characteristic mixed element method for nonlinear nonstationary convection-diffusion problem, Int. J. Comput. Math., 89 (2012) 932-943.
[20] M. F. Wheeler, A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations, SIAM J. Numer. Anal., 10 (1973) 723-759.
[21] R. Becker and B. Vexler, Optimal control of the convection-diffusion equation using stabilized finite element methods, Numer. Math., 106 (2007) 349-367.
[22] C. Meyer and A. Rösch, Superconvergence properties of optimal control problems, SIAM J. Control Optim., 43 (2004) 970-985.

College of Science, China University of Petroleum, Qingdao 266580, China
E-mail: `hongfeifu@upc.edu.cn`

School of Mathematics, Shandong University, Jinan 250100, China
E-mail: `hxrui@sdu.edu.cn`

College of Science, China University of Petroleum, Qingdao 266580, China
E-mail: `sdugh@163.com`

College of Science, China University of Petroleum, Qingdao 266580, China
E-mail: `jszhang@upc.edu.cn`

College of Petroleum Engineering, China University of Petroleum, Qingdao 266580, China
E-mail: `houjian@upc.edu.cn`