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IMPROVED ADI PARALLEL DIFFERENCE METHOD FOR QUANTO OPTIONS PRICING MODEL

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Abstract. The quanto options pricing model is a typical two-dimensional Black-Scholes equation with a mixed derivative term, and it has been increasingly attracting interest over the last decade. A kind of improved alternating direction implicit methods, which is based on the Douglas-Rachford (D-R ADI) and Craig-Sneyd (C-S ADI) split forms, is given in this paper for solving the quanto options pricing model. The improved ADI methods first split the original problem into two separate one-dimensional problems, and then solve the tri-diagonal matrix equations at each time-step. There are several advantages in this method such as: parallel property, unconditional stability, convergency and better accuracy. The numerical experiments show that this kind of methods is very efficient compared to the existent explicit finite difference method. In addition, because of the natural parallel property of the improved ADI methods, the parallel computing is very easy, and about 50% computational cost can been saved. Thus the improved ADI methods can be used to solve the multi-asset option pricing problems effectively.

Key words. Quanto options pricing model, two-dimensional Black-Scholes equation, improved ADI method, parallel computing, numerical experiments.

1. Introduction

In today's financial market, option is one of the most important financial derivatives. With the rapid development of financial market, it is difficult to meet the needs of the financial traders by only using European, American and other single asset options. Therefore, the financial institution designs many multi-asset options.

The name "quanto" is, in fact, derived from the variable notional amount, and is short for "quantity adjusting options". A quanto is a type of derivative in which the underlying is denominated in one currency, but the instrument itself is settled in another currency at some fixed rate. Generally, the value of the quanto options depends not only on the option's intrinsic value in the foreign currency, but also on the foreign currency exchange rate. Therefore, the qunato options pricing is relatively complex. This paper is mainly devoted to the two-dimensional (2D) Black-Scholes equation of quanto options pricing model [1, 2, 3]

$$\frac{\partial V}{\partial t} + F_1(V, S_1, S_2) + G_1(V, S_1, S_2) - r_1 V = 0,$$

$$F_1(V, S_1, S_2) = \frac{1}{2}[\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + 2\rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}],$$
$$G_1(V, S_1, S_2) = (r_1 - \widehat{q_1}) S_1 \frac{\partial V}{\partial S_1} + (r_1 - \widehat{q_2}) S_2 \frac{\partial V}{\partial S_2},$$
$$\widehat{q_1} = r_1 - r_2 + q + \rho\sigma_1\sigma_2,$$
$$\widehat{q_2} = r_2.$$

Here, V, S_1, S_2, r_1, r_2, σ_1^2, σ_2^2, ρ, q and T are the price of the quanto options, price of foreign risk asset, exchange rate of foreign currency against domestic one, domestic risk-free rate, foreign risk-free rate, variance of the rate of return on S_1,

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variance of the rate of return on S_2, correlation coefficient, interest rate and time to expiration, respectively. The quanto options pricing model has the following analytic solution (cf. [1, 2, 3])

$$V(S_1,S_2,t)=\frac{1}{2\pi(T-t)}e^{-r_1(T-t)}\frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}H(S_1,S_2,t), \tag{1}$$

$$H(S_1,S_2,t)=\int_0^\infty\int_0^\infty\frac{(\eta_1-K)^+}{\eta_1}exp[-\frac{\sigma_2^2\alpha_1^2-2\rho\sigma_1\sigma_2\alpha_1\alpha_2+\sigma_1^2\alpha_2^2}{2\sigma_1^2\sigma_2^2(1-\rho^2)(T-t)}]d\eta_1d\eta_2,$$
$$\alpha_1=\ln\frac{S_1}{\eta_1}+(r_2-q-\rho\sigma_1\sigma_2-\frac{\sigma_1^2}{2})(T-t),$$
$$\alpha_2=\ln\frac{S_2}{\eta_2}+(r_1-r_2-\frac{\sigma_2^2}{2})(T-t).$$

Although the quanto options pricing model has an analytic solution (1), it cannot satisfy the application requirements due to the computing complexity of the expression (1). For more general settings, we usually use the numerical method instead, such as the Monte-Carlo method [4] and Binomial Tree method (cf.[5]), but because a large amount of numerical simulations are needed for getting a high accuracy, the calculation efficiency of the above methods are not better than that of the finite difference method, which consists of replacing the partial derivatives by numerical differentiation and then solving the resulting discretized problem.

So far, the finite difference methods used for solving the option pricing problems have got a lot of progresses. M. Gilli *et al.*(2002) [6] investigated an explicit-implicit difference scheme for multi-asset option pricing model, but the solution of a large block triangular linear system was required and the calculation was relatively complex in this method; X.Z. Yang *et al.*(2007) [7] proposed a general difference scheme for solving the one-dimensional Black-Scholes equation, but they did not consider the multi-dimensional ones; A.Q.M. Khaliq *et al.*(2008) [8] put forward a new finite difference method for solving the 2D Black-Scholes equation, but it also needed the using of penalty approach method which was not very convenient to calculate by computers; X.Z. Yang and G.X. Zhou (2011) [9] used the additive operator splitting(AOS) method for solving the quanto options pricing model, but because the approximation to the mixed derivative term was not efficient, the accuracy of this method was not very well. R.Company *et al.* (2008)[10] and D.Y.Tangman (2008) [11] put forward high-order finite difference schemes for solving the nonlinear Black-Scholes equation, but the computational efficiency was not very well.

In the numerical analysis, the alternating direction implicit(ADI) method is mostly notable for solving the partial differential equation in two or more dimension (cf. [12, 13, 14]). I.J.D. Craig and A.D. Sneyd(1988) [15] first put forward an ADI scheme for N-dimensional parabolic equations with a mixed derivative term, but the scheme was conditionally stable and less effective; S. McKee *et al.*(1996) [16] introduced a new ADI scheme which was capable of solving a general parabolic equation in two dimension with mixed derivative and convective terms and was proved to be unconditionally stable, but the equation they considered did not include the one degree term; In addition, D. Jeong, J. Kim(2013) [17] used the ADI difference scheme on multi-dimensional Black-Scholes option pricing models, however the scheme was not unconditionally convergent. Similar researches also have been done for solving the parabolic equations and other types of equations(cf. [22, 23, 24, 25, 32, 33]). For these reasons, this paper gives an improved ADI difference scheme that is capable of solving the quanto options pricing model with unconditional stability and good convergency properties.

2. The 2D Black-Scholes equation of quanto options pricing model

With the usual perfect market and European option assumptions, we consider that each asset price follows the geometric *Brownian* motion and there are no dividends. By the Δ-hedging principle, we can get the 2D Black-Scholes equation of quanto options pricing model [1, 2, 3]

$$\frac{\partial V}{\partial t} + F_1(V, S_1, S_2) + G_1(V, S_1, S_2) - r_1 V = 0, \tag{2}$$

$$\begin{aligned} F_1(V, S_1, S_2) &= \frac{1}{2}[\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + 2\rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}], \\ G_1(V, S_1, S_2) &= (r_1 - \widehat{q_1}) S_1 \frac{\partial V}{\partial S_1} + (r_1 - \widehat{q_2}) S_2 \frac{\partial V}{\partial S_2}, \\ \widehat{q_1} &= r_1 - r_2 + q + \rho\sigma_1\sigma_2, \\ \widehat{q_2} &= r_2. \end{aligned}$$

In theory, the equation (2) is defined on the following domain

$$\Omega = \{(S_1, S_2, t) | 0 < S_1 < \infty, 0 < S_2 < \infty, t \in [0, T]\}.$$

But in the actual transaction, the price of the underlying asset will not always appear to be zero or infinity. Therefore, the financial institution provides a small enough value $S_{min}(S_{min} > 0)$ as the lower boundary and a large enough value $S_{max}(S_{max} > 0)$ as the upper boundary. Then, the pricing problem can be solved on a finite domain

$$\Omega_1 = \{(S_1, S_2, t) | S_{1min} < S_1 < S_{1max},\ S_{2min} < S_2 < S_{2max},\ t \in [0, T]\}.$$

Assume that the underlying asset is a call option, the initial and boundary conditions of equation (2) are constructed under this assumption. As the option pricing is a backward problem, the initial condition is

$$V(S_1, S_2, T) = S_2 \max\{S_1 - K, 0\},$$

and the boundary conditions are

$$\begin{aligned} &V(S_{1min}, S_2, t) = 0, \quad V(S_{1max}, S_2, t) = S_2(S_{1max} - K), \\ &V(S_1, S_{2min}, t) = 0, \quad V(S_1, S_{2max}, t) = S_{2max} \max\{S_1 - K, 0\}. \end{aligned}$$

And if the underlying asset is a put option, the initial conditional will be

$$V(S_1, S_2, T) = S_2 \max\{K - S_1, 0\},$$

and the boundary conditions will be

$$\begin{aligned} &V(S_{1min}, S_2, t) = S_2(K - S_{1min}) \quad , \quad V(S_{1max}, S_2, t) = 0, \\ &V(S_1, S_{2min}, t) = 0 \quad , \quad V(S_1, S_{2max}, t) = S_{2max} \max\{K - S_1, 0\}. \end{aligned}$$

With the following transforms

$$x = \ln S_1,\ y = \ln S_2,\ \tau = T - t,\ U(x, y, \tau) = V(S_1, S_2, t),$$

equation (2) can be transformed into an initial-boundary value problem of partial differential equation with constant coefficients as follows

$$\frac{\partial U}{\partial \tau} - F_2(U, x, y) - G_2(U, x, y) + r_1 U = 0, \tag{3}$$

$$F_2(U,x,y) = \frac{1}{2}[\sigma_1^2 \frac{\partial^2 U}{\partial x^2} + 2\rho\sigma_1\sigma_2 \frac{\partial^2 U}{\partial x \partial y} + \sigma_2^2 \frac{\partial^2 U}{\partial y^2}],$$
$$G_2(U,x,y) = (r_1 - \widehat{q_1} - \frac{\sigma_1^2}{2})\frac{\partial U}{\partial x} + (r_1 - \widehat{q_2} - \frac{\sigma_2^2}{2})\frac{\partial U}{\partial y},$$
$$\widehat{q_1} = r_1 - r_2 + q + \rho\sigma_1\sigma_2,$$
$$\widehat{q_2} = r_2.$$

Then the initial condition can be written as

$$U(x,y,0) = e^y \max\{e^x - K, 0\},$$

with boundary conditions

$$U(x_{min}, y, \tau) = 0, \quad U(x_{max}, y, \tau) = e^y(e^{x_{max}} - K),$$
$$U(x, y_{min}, \tau) = 0, \quad U(x, y_{max}, \tau) = e^{y_{max}} \max\{e^x - K, 0\}.$$

Similarly, we can get the initial and boundary conditions of the quanto options pricing model when the underlying asset is a put option.

3. The D-R ADI method for the quanto options pricing model

3.1. Construction of the D-R ADI difference scheme. The ADI method which proposed by Peaceman and Rachford in 1950's is a common method for solving the 2D parabolic equation [18]. Its main idea is that a 1D implicit discretization at x and y direction is done successively on one time level, then get the numerical solution of the next time level by solving two tri-diagonal sets of equations. The model they studied was the standard parabolic equation $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$. For the quanto options pricing model (3), the improved D-R ADI difference scheme is constructed as follows.

First, for convenience, equation (3) is written as follows

$$\frac{\partial U}{\partial \tau} = LU, \tag{4}$$

where $L \equiv a\frac{\partial^2}{\partial x^2} + b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2} - u\frac{\partial}{\partial x} - v\frac{\partial}{\partial y} - w$, $a = \frac{1}{2}\sigma_1^2$, $b^2 = (\rho\sigma_1\sigma_2)^2$, $c = \frac{1}{2}\sigma_2^2$, $u = \widehat{q_1} + \frac{1}{2}\sigma_1^2 - r_1$, $v = \widehat{q_2} + \frac{1}{2}\sigma_2^2 - r_1$, $w = r_1$. For the coefficients u and v, we can always keep $u > 0$, $v > 0$ if we do some linear transforms on variables x and y. So the coefficients mentioned above satisfy the following equalities:

$$a > 0,\ b^2 < 4ac,\ c > 0,\ u > 0,\ v > 0,\ w > 0.$$

Let us make a mesh partition on the area Ω_1, we consider the function $U(x,y,\tau)$ at the discrete set of points

$$\begin{array}{ll} x_i = \ln S_{1min} + (i-1)h, & i = 1,2,\ldots,m_1+1, \\ y_j = \ln S_{2min} + (j-1)h, & j = 1,2,\ldots,m_2+1, \\ \tau_n = (n-1)k, & n = 1,2,\ldots,N+1, \end{array}$$

where h is the spatial grid step, k is the time step and $m_1+1, m_2+1, N+1$ are the number of grid points in the x, y and τ direction, respectively.

Then for the parabolic equation (4), we propose the improved ADI difference scheme given in Douglas-Rachford split form [19]

$$\left\{\begin{array}{l} Y_0 = U_n + k[A_x(\tau_n, U_n) + A_y(\tau_n, U_n) + A_{xy}(\tau_n, U_n)], \\ Y_1 = Y_0 + \frac{1}{2}k[A_x(\tau_n, Y_1) - A_x(\tau_n, U_n)], \\ Y_2 = Y_1 + \frac{1}{2}k[A_y(\tau_n, Y_2) - A_y(\tau_n, U_n)], \\ U_{n+1} = Y_2, \end{array}\right. \tag{5}$$

here,

$$A_xU = a\frac{\partial^2 U}{\partial x^2} - u\frac{\partial U}{\partial x} - wU,$$

$$A_yU = c\frac{\partial^2 U}{\partial y^2} - v\frac{\partial U}{\partial y} - wU,$$

$$A_{xy}U = b\frac{\partial^2 U}{\partial x \partial y} + wU.$$

And we make the following difference discretization

$$\delta_x^2 U_{i,j}^n = U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n,$$

$$\delta_{x,y} U_{i,j}^n = U_{i+1,j+1}^n - U_{i+1,j-1}^n - U_{i-1,j+1}^n + U_{i-1,j-1}^n,$$

$$\nabla_x U_{i,j}^n = U_{i,j}^n - U_{i-1,j}^n,$$

with corresponding definitions for δ_y^2 and ∇_y. Then the operators can be substituted as

$$A_xU_{i,j} = a\frac{1}{h^2}\delta_x^2 U_{i,j} - u\frac{1}{h}\nabla_x U_{i,j} - wU_{i,j},$$

$$A_yU_{i,j} = c\frac{1}{h^2}\delta_y^2 U_{i,j} - v\frac{1}{h}\nabla_y U_{i,j} - wU_{i,j},$$

$$A_{xy}U_{i,j} = b\frac{1}{4h^2}\delta_{xy} U_{i,j} + wU_{i,j}.$$

The scheme (5) can be written as follows

$$(1 - \frac{1}{2}ar\delta_x^2 + \frac{1}{2}up\nabla_x + \frac{1}{2}wk)U_{i,j}^{n+\frac{1}{2}} = (1 + \frac{1}{2}ar\delta_x^2 + \frac{1}{4}br\delta_{xy} + cr\delta_y^2 - \frac{1}{2}up\nabla_x - vp\nabla_y - \frac{1}{2}wk)U_{i,j}^n, \tag{6}$$

$$(1 - \frac{1}{2}cr\delta_y^2 + \frac{1}{2}vp\nabla_y + \frac{1}{2}wk)U_{i,j}^{n+1} = U_{i,j}^{n+\frac{1}{2}} - (\frac{1}{2}cr\delta_y^2 - \frac{1}{2}vp\nabla_y - \frac{1}{2}wk)U_{i,j}^n, \tag{7}$$

in which $U_{i,j}^n$ is the approximation of $U(x_i, y_j, \tau_n)$, $r = k/h^2, p = k/h$, and $U_{i,j}^{n+\frac{1}{2}}$ are the values of the transition layer. The non-split form is

$$(1 - \frac{1}{2}ar\delta_x^2 + \frac{1}{2}up\nabla_x + \frac{1}{2}wk)(1 - \frac{1}{2}cr\delta_y^2 + \frac{1}{2}vp\nabla_y + \frac{1}{2}wk)U_{i,j}^{n+1} = [(1 + \frac{1}{2}ar\delta_x^2 - \frac{1}{2}up\nabla_x - \frac{1}{2}wk)(1 + \frac{1}{2}cr\delta_y^2 - \frac{1}{2}vp\nabla_y - \frac{1}{2}wk) + \frac{1}{4}br\delta_{xy} + wk]U_{i,j}^n. \tag{8}$$

And the scheme (8) is the improved ADI difference scheme given in the Douglas-Rachford split form (D-R ADI) for the quanto options pricing model (3).

3.2. Parallel realization of the D-R ADI difference scheme. We first introduce the following symbols for convenience:

$$\begin{aligned} &a_1 = \frac{1}{2}ar + \frac{1}{2}up, \quad && b_1 = ar + \frac{1}{2}up + \frac{1}{2}wk, \quad && c_1 = \frac{1}{2}ar, \\ &a_2 = \frac{1}{2}cr + \frac{1}{2}vp, && b_2 = cr + \frac{1}{2}vp + \frac{1}{2}wk, && c_2 = \frac{1}{2}cr, \\ &b_3 = b_1 + 2b_2 - wk, && b_4 = \frac{1}{4}br, \end{aligned}$$

here, a, b, c, u, v, w, r, p and k are the same as mentioned above. Defining the vectors

$$\begin{aligned} A_j^n &= (U_{2,j}^n, U_{3,j}^n, \cdots, U_{m_1,j}^n)\,, j = 1, 2, \cdots, m_2 + 1\,, \\ B_i^n &= (U_{i,2}^n, U_{i,3}^n, \cdots, U_{i,m_2}^n)\,, i = 1, 2, \cdots, m_1 + 1\,, \\ A^n &= (A_2^n, A_3^n, \cdots, A_{m_2}^n)^T, \\ B^n &= (B_2^n, B_3^n, \cdots, B_{m_1}^n)^T\,, \end{aligned}$$

then scheme (6) and (7) can be written as follows

$$(I + H_1)A^{n+\frac{1}{2}} = (I - V_1)A^n + F_1^n\,, \tag{9}$$

$$(I + H_2)B^{n+1} = B^{n+\frac{1}{2}} + V_2B^n + F_2^n\,, \tag{10}$$

where

$$H_1 = \begin{pmatrix} C & & & & \\ & C & & & \\ & & \ddots & & \\ & & & C & \\ & & & & C \end{pmatrix}_{l\times l},$$

$$H_2 = \begin{pmatrix} D & & & & \\ & D & & & \\ & & \ddots & & \\ & & & D & \\ & & & & D \end{pmatrix}_{l\times l},$$

$$V_1 = \begin{pmatrix} C_2 & C_3 & & & \\ C_1 & C_2 & C_3 & & \\ & \ddots & \ddots & \ddots & \\ & & C_1 & C_2 & C_3 \\ & & & C_1 & C_2 \end{pmatrix}_{l\times l},$$

$$C=\begin{pmatrix} b_1 & -c_1 & & & \\ -a_1 & b_1 & -c_1 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_1 & b_1 & -c_1 \\ & & & -a_1 & b_1 \end{pmatrix}_{(m_1-1)\times(m_1-1)},$$

$$D=\begin{pmatrix} b_2 & -c_2 & & & \\ -a_2 & b_2 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_2 & b_2 & -c_2 \\ & & & -a_2 & b_2 \end{pmatrix}_{(m_2-1)\times(m_2-1)},$$

$$C_1=\begin{pmatrix} -2a_2 & b_4 & & & \\ -b_4 & -2a_2 & b_4 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_4 & -2a_2 & b_4 \\ & & & -b_4 & -2a_2 \end{pmatrix}_{(m_1-1)\times(m_1-1)},$$

$$C_2=\begin{pmatrix} b_3 & -c_1 & & & \\ -a_1 & b_3 & -c_1 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_1 & b_3 & -c_1 \\ & & & -a_1 & b_3 \end{pmatrix}_{(m_1-1)\times(m_1-1)},$$

$$C_3=\begin{pmatrix} -2c_2 & -b_4 & & & \\ b_4 & -2c_2 & -b_4 & & \\ & \ddots & \ddots & \ddots & \\ & & b_4 & -2c_2 & -b_4 \\ & & & b_4 & -2c_2 \end{pmatrix}_{(m_1-1)\times(m_1-1)},$$

and $V_2 = H_2$, $l = (m_1 - 1) \times (m_2 - 1)$, $F_1^n = f_1^{n+\frac{1}{2}} + f_1^n + g_1^n - g_2^n$,$F_2^n = f_2^{n+1} - f_2^n$.
$f_1^n = (f_{1,2}^n, f_{1,3}^n, \cdots, f_{1,m_2}^n)^T$, $g_{1,j}^n = (b_4U_{1,j-1}^n + a_1U_{1,j}^n - b_4U_{1,j+1}^n, 0, \cdots, 0,$
$- b_4U_{m_1+1,j-1}^n + c_1U_{m_1+1,j}^n + b_4U_{m_1+1,j+1}^n)_{m_1-1}$,
$f_{1,j}^n = (a_1U_{1,j}^n, 0, \cdots, 0, c_1U_{m_1+1,j}^n)_{m_1-1}$, $j = 2, \cdots, m_2$.
$f_2^n = (f_{2,2}^n, f_{2,3}^n, \cdots, f_{2,m_1}^n)^T$,
$f_{2,i}^n = (a_2U_{i,1}^n, 0, \cdots, 0, c_2U_{i,m_2+1}^n)_{m_2-1}$, $i = 2, \cdots, m_1$.
$g_2^n = (g_{2,2}^n, 0, \cdots, 0, g_{2,m_2}^n)^T$, $g_{2,2}^n = (C_1(A_1^n)^T)^T$, $g_{2,m_2}^n = (C_3(A_{m_2+1}^n)^T)^T$.
And $g_{2,j}^n$, $j = 3, 4, \cdots, m_2 - 1$ are $m_1 - 1$ dimensional zero vectors.

It is obvious that there are actually $m_2 - 1$, $m_1 - 1$ independent systems in scheme (9) and (10) respectively, and they all have tri-diagonal coefficient matrices. Instead of solving $(m_1-1)\times(m_2-1)$ coupled linear equations with a penta-diagonal coefficient matrix, we now only need to solve $m_2 - 1$, $m_1 - 1$ independent systems [20, 21].

Thus a complete D-R ADI method step can now be given as follows:
Step 1. Solve the $m_2 - 1$ independent linear equation systems with tri-diagonal

coefficient matrices in scheme (9) to get $A_i^{n+\frac{1}{2}}$, $i = 2, 3, \cdots, m_2$.

$for \quad i = 2 : m_2$

$\quad for \quad j = 1 : m_1 + 1$

$\quad\quad Set \ f_{1,i,j}^n, \ f_{1,i,j}^{n+\frac{1}{2}}, \ g_{1,i,j}^n \ and \ g_{2,i,j}^n \ defined \ by \ scheme$ (9);

$\quad end$

$$Solve \ \ (I + C)(A_i^{n+\frac{1}{2}})^T = -C_1(A_{i-1}^n)^T + (I - C_2)(A_i^n)^T - C_3(A_{i+1}^n)^T + F_{1,i}^n;$$

end

Step 2. Reorder the grid points.

$for \quad i = 1 : m_1 + 1$

$\quad for \quad j = 1 : m_2 + 1$

$$B_{i,j}^{n+\frac{1}{2}} = A_{j,i}^{n+\frac{1}{2}};$$

$\quad end$

end

Step 3. Solve the $m_1 - 1$ independent linear equation systems with tri-diagonal coefficient matrices in scheme (10) to get B_i^{n+1}, $i = 2, 3, \cdots, m_1$.

$for \quad i = 2 : m_1$

$\quad for \quad j = 1 : m_2 + 1$

$\quad\quad Set \ f_{2,i,j}^n \ and \ f_{2,i,j}^{n+1} \ defined \ by \ scheme$ (10);

$\quad end$

$$Solve \ \ (I + D)(B_i^{n+1})^T = (B_i^{n+\frac{1}{2}})^T + D(B_i^n)^T + F_{2,i}^n;$$

end

Similar to (9) and (10), a complete step of the improved D-R ADI difference scheme needs the solution of two sets of independent equation systems. A straightforward implementation of this scheme on parallel computers or multi-core computers would be, assuming the number of processors is smaller than $m_1 - 1$ and $m_2 - 1$, to assign different tri-diagonal equation systems to different processors, so that $U_{i,j}$ can be solved simultaneously [26, 27, 28]. Thus, compared to the serial difference schemes, the improved ADI difference scheme is easy to realize parallel computing, and it improves the calculation efficiency and reduces the calculation time greatly.

4. Existence and uniqueness of the solution of D-R ADI method for quanto options pricing model

In order to prove the existence and uniqueness of the solution of D-R ADI difference scheme (8), it is sufficient to show that the solution of scheme (9) or (10) is unique. According to the existence and uniqueness theorem for the system of linear equations, we only need to prove that matrix $I+H_1$ and $I+H_2$ are nonsingular.

From the expression of H_1, it is obvious that we now need to prove that $I+C$ is a nonsingular matrix.

$$I+C=\begin{pmatrix} 1+b_1 & -c_1 & & & \\ -a_1 & 1+b_1 & -c_1 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_1 & 1+b_1 & -c_1 \\ & & & -a_1 & 1+b_1 \end{pmatrix}_{(m_1-1)\times(m_1-1)},$$

here, $1+b_1=1+ar+\frac{1}{2}up+\frac{1}{2}wk$, $-a_1=-\frac{1}{2}ar-\frac{1}{2}up$, and $-c_1=-\frac{1}{2}ar$. It is clear that $|1+b_1|>|-a_1|+|-c_1|$, that is $I+C$ is a master diagonal domination matrix. Thus, $I+C$ is nonsingular. And from this, we could get the conclusion that scheme (9) has a unique solution.

In the same way, we could also prove that scheme (10) has a unique solution. Therefore we could get the following theorem.

Theorem 1 The solution of the improved alternating direction implicit difference scheme given in the Douglas-Rachford split form (D-R ADI) (8) of quanto options pricing model (3) is existing and unique.

5. Stability and convergence of the D-R ADI method for quanto options pricing model

We use the traditional von Neumann method to analyze the stability of the improved D-R ADI difference scheme (8). The solution to the difference scheme (8) given by

$$U_{i,j}^n=Ge^{\alpha(n-1)k}e^{\sqrt{-1}\beta(i-1)h}e^{\sqrt{-1}\gamma(j-1)h},$$

where α, β and γ are constant wavenumbers [15]. Then the von Neumann condition requires $|e^{\alpha k}|\le 1$ for all β,γ. It is straightforward to show that

$$\begin{aligned}
&\delta_x^2U_{i,j}^n=2[\cos(\beta h)-1]U_{i,j}^n,\\
&\delta_y^2U_{i,j}^n=2[\cos(\gamma h)-1]U_{i,j}^n,\\
&\delta_{xy}U_{i,j}^n=-4\sin(\beta h)\sin(\gamma h)U_{i,j}^n,\\
&\nabla_xU_{i,j}^n=[1-\cos(\beta h)+\sqrt{-1}\sin(\beta h)]U_{i,j}^n,\\
&\nabla_yU_{i,j}^n=[1-\cos(\gamma h)+\sqrt{-1}\sin(\gamma h)]U_{i,j}^n,
\end{aligned}$$

which, upon substitution into the scheme (8), yield the amplification factor

$$|e^{\alpha k}|=|\frac{z_1z_2+z_0}{z_3z_4}|\,,$$

here,

$$z_0 = 4wk - 4br\sin(\beta h)\sin(\gamma h),$$

$$z_1 = 2 - wk - (2ar + up)[1 - \cos(\beta h)] - \sqrt{-1}up\sin(\beta h),$$

$$z_2 = 2 - wk - (2cr + vp)[1 - \cos(\gamma h)] - \sqrt{-1}vp\sin(\gamma h),$$

$$z_3 = 2 + wk + (2ar + up)[1 - \cos(\beta h)] + \sqrt{-1}up\sin(\beta h),$$

$$z_4 = 2 + wk + (2cr + vp)[1 - \cos(\gamma h)] + \sqrt{-1}vp\sin(\gamma h).$$

Clearly, the von Neumann condition for stability may be equivalently written as $F := |z_3z_4|^2 - |z_1z_2 + z_0|^2 \geq 0$ for all β and γ. Evaluating F gives

$$\begin{aligned}
&F(a,b,c,u,v,w,r,p,k,\theta,\varphi)\\
&= [\frac{1}{2}wk + (2ar + up)\sin^2\frac{\theta}{2}]v^2p^2\sin^2\varphi + [\frac{1}{2}wk + (2cr + vp)\sin^2\frac{\varphi}{2}]u^2p^2\sin^2\theta\\
&+4[wk + (2ar + up)\sin^2\frac{\theta}{2} + (2cr + vp)\sin^2\frac{\varphi}{2}]\{1 + \frac{1}{2}wk[(2ar + vp)\sin^2\frac{\varphi}{2}\\
&+(2cr + vp)\sin^2\frac{\varphi}{2}] + (2ar + up)(2cr + vp)\sin^2\frac{\theta}{2}\sin^2\frac{\varphi}{2} + \frac{1}{4}w^2k^2\}\\
&+(br\sin\theta\sin\varphi - wk)\{2[1 - \frac{1}{2}wk - (2ar + up)\sin^2\frac{\theta}{2}][1 - \frac{1}{2}wk\\
&-(2cr + vp)\sin^2\frac{\varphi}{2}] - \frac{1}{2}uvp^2\sin\theta\sin\varphi - (br\sin\theta\sin\varphi - wk)\},
\end{aligned}$$

where we have introduced

$$\theta = \beta h,\ \varphi = \gamma h\,.$$

Writing

$$\begin{aligned}
&F(a,b,c,u,v,w,r,p,k,\theta,\varphi)\\
&= [F(a,b,c,u,v,w,r,p,k,\theta,\varphi) - F(a,b,c,u,v,0,r,p,k,\theta,\varphi)]\\
&+F(a,b,c,u,v,0,r,p,k,\theta,\varphi)\\
&= D(a,b,c,u,v,w,r,p,k,\theta,\varphi) + F(a,b,c,u,v,0,r,p,k,\theta,\varphi),
\end{aligned}$$

to demonstrate unconditional stability of the scheme (8), it is sufficient to show that both terms of the sum in the expression above are non-negative. However, in the case $w = 0$, the scheme reduces to the Mckee scheme and this scheme is demonstrated to be unconditionally stable [16], or equivalently

$$F(a,b,c,u,v,0,r,p,k,\theta,\varphi) \geq 0\,.$$

Thus we only need to show that

$$\begin{aligned}
&D(a,b,c,u,v,w,r,p,k,\theta,\varphi)\\
&= F(a,b,c,u,v,w,r,p,k,\theta,\varphi) - F(a,b,c,u,v,0,r,p,k,\theta,\varphi) \geq 0\,.
\end{aligned}$$

It is straightforward to show that $D(a,b,c,u,v,w,r,p,k,\theta,\varphi)$ takes the form

$$\begin{aligned}
&D(a,b,c,u,v,w,r,p,k,\theta,\varphi)\\
&=\frac{1}{2}wku^2p^2\sin^2\theta+\frac{1}{2}wkv^2p^2\sin^2\varphi+2wk\{1+\frac{1}{2}wk[(2ar+up)\sin^2\frac{\theta}{2}\\
&+(2cr+vp)\sin^2\frac{\varphi}{2}]+(2ar+up)(2cr+vp)\sin^2\frac{\theta}{2}\sin^2\frac{\varphi}{2}+\frac{1}{4}w^2k^2\}+wk\{wk\\
&+2[(2ar+up)\sin^2\frac{\theta}{2}+(2cr+vp)\sin^2\frac{\varphi}{2}]+\frac{1}{2}uvp^2\sin\theta\sin\varphi+br\sin\theta\sin\varphi\}\\
&+4[(2ar+up)\sin^2\frac{\theta}{2}+(2cr+vp)\sin^2\frac{\varphi}{2}]\{\frac{1}{4}w^2k^2+\frac{1}{2}wk[(2ar+up)\sin^2\frac{\theta}{2}\\
&+(2cr+vp)\sin^2\frac{\varphi}{2}]\}+br\sin\theta\sin\varphi\{-wk+\frac{1}{2}w^2k^2+wk[(2ar+up)\sin^2\frac{\theta}{2}\\
&+(2cr+vp)\sin^2\frac{\varphi}{2}]\},
\end{aligned}$$

In order to demonstrate $D(a,b,c,u,v,w,r,p,k,\theta,\varphi)\geq 0$, we need the following two inequalities:

Lemma 1 $u^2p^2\sin^2\theta+v^2p^2\sin^2\varphi+uvp^2\sin\theta\sin\varphi\geq 0.$

$$\begin{aligned}
&u^2p^2\sin^2\theta+v^2p^2\sin^2\varphi+uvp^2\sin\theta\sin\varphi\\
\geq\;&u^2p^2\sin^2\theta+v^2p^2\sin^2\varphi-2uvp^2|\sin\theta||\sin\varphi|\\
=\;&(up|\sin\theta|-vp|\sin\varphi|)^2\\
\geq\;&0.
\end{aligned}$$

Lemma 2 $4ar\sin^2\frac{\theta}{2}+4cr\sin^2\frac{\varphi}{2}+br\sin\theta\sin\varphi\geq 0.$

$$\begin{aligned}
&4ar\sin^2\frac{\theta}{2}+4cr\sin^2\frac{\varphi}{2}+br\sin\theta\sin\varphi\\
=\;&4ar\sin^2\frac{\theta}{2}+4cr\sin^2\frac{\varphi}{2}+4br\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\\
\geq\;&4[(\sqrt{ar}|\sin\frac{\theta}{2}|)^2-2\sqrt{acr}|\sin\frac{\theta}{2}||\sin\frac{\varphi}{2}|+(\sqrt{cr}|\sin\frac{\varphi}{2}|)^2]\\
=\;&4(\sqrt{ar}|\sin\frac{\theta}{2}|-\sqrt{cr}|\sin\frac{\varphi}{2}|)^2\\
\geq\;&0.
\end{aligned}$$

As the expression of $D(a,b,c,u,v,w,r,p,k,\theta,\varphi)$ may be rearranged as

$$\begin{aligned}
&D(a,b,c,u,v,w,r,p,k,\theta,\varphi)\\
&=\frac{1}{2}wk(u^2p^2\sin^2\theta+v^2p^2\sin^2\varphi+uvp^2\sin\theta\sin\varphi)+\frac{1}{2}w^2k^2(4ar\sin^2\frac{\theta}{2}\\
&+4cr\sin^2\frac{\varphi}{2}+br\sin\theta\sin\varphi)+wk[(2ar+up)\sin^2\frac{\theta}{2}+(2cr+vp)\sin^2\frac{\varphi}{2}]\\
&(br\sin\theta\sin\varphi+4ar\sin^2\frac{\theta}{2}+4cr\sin^2\frac{\varphi}{2})+2wk\{(1+\frac{1}{2}wk+\frac{1}{4}w^2k^2)\\
&+[(2ar+up)\sin^2\frac{\theta}{2}+(2cr+vp)\sin^2\frac{\varphi}{2}](1+up\sin^2\frac{\theta}{2}+vp\sin^2\frac{\varphi}{2})\\
&+(2ar+up)(2cr+vp)\sin^2\frac{\theta}{2}\sin^2\frac{\varphi}{2}\}+w^2k^2[(2ar+2up)\sin^2\frac{\theta}{2}\\
&+(2cr+2vp)\sin^2\frac{\varphi}{2}],
\end{aligned}$$

which, using the results of Lemmas 1 and 2, is clearly non-negative and hence the scheme (8) is unconditionally stable. Therefore we can get the following theorem.

Theorem 2 The improved alternating direction implicit difference scheme given in the Douglas-Rachford split form (D-R ADI) (8) of quanto options pricing model

(3) is unconditionally stable.

In addition, due to the Lax Theorem [29], we can get:

Corollary 1 The improved alternating direction implicit difference scheme based on the Douglas-Rachford split form (D-R ADI) (8) of quanto options pricing model (3) is convergent.

6. Accuracy of the D-R ADI method for quanto options pricing model

Now we will give the error analysis of the scheme (8). In the special case when u, v, w are identically zero, the scheme (8) is consistent to order $O(k+h^2)$. And more generally when only $w=0$, the order is $O(k+h)$ [16]. Now we consider the case when u, v, w are not identically zero. Using the difference of the two sides of the scheme (8), we get the principle part of the truncation error as follows:

$$\begin{aligned} R^n_{i,j} = \; & -\frac{1}{2}uhk\frac{\partial^2 U}{\partial x^2} - \frac{1}{2}vhk\frac{\partial^2 U}{\partial y^2} - bwk^2\frac{\partial^2 U}{\partial x\partial y} + \frac{1}{6}uhk\frac{\partial^3 U}{\partial x^3} + \frac{1}{6}vhk\frac{\partial^3 U}{\partial y^3} \\ & -buk^2\frac{\partial^3 U}{\partial x^2\partial y} - bvk^2\frac{\partial^3 U}{\partial x\partial y^2} - \frac{1}{12}ah^2k\frac{\partial^4 U}{\partial x^4} - \frac{1}{12}ch^2k\frac{\partial^4 U}{\partial y^4} + abk^2\frac{\partial^4 U}{\partial x^3\partial y} \\ & +b^2k^2\frac{\partial^4 U}{\partial x^2\partial y^2} + bck^2\frac{\partial^4 U}{\partial x\partial y^3} - \frac{1}{6}bh^2k\frac{\partial^4 U}{\partial x\partial y^3} - \frac{1}{6}bh^2k\frac{\partial^4 U}{\partial x^3\partial y} + O(h^\alpha k^\beta), \end{aligned}$$

here, $\alpha+\beta=3$. From the expression of the truncation error, we can see that the difference scheme (8) has one order accuracy. So we get the following theorem.

Theorem 3 The improved alternating direction implicit difference scheme given in the Douglas-Rachford split form (D-R ADI) (8) of quanto options pricing model (3) is consistent to order $O(k+h)$, and it is compatible with the quanto options pricing model (3) unconditionally.

7. The improved ADI method given in the Craig-Sneyd split form

Similarly, we can give the improved ADI difference scheme given in the Craig-Rachford split form (C-S ADI) of quanto options pricing model (3). Its construction is based on the improved D-R ADI difference scheme (8). After achieving the first set of correction steps, a new starting solution is used to the same procedure again [14]:

$$(11)\qquad \begin{cases} Y_0 = U_n + k[A_x(\tau_n,U_n) + A_y(\tau_n,U_n) + A_{xy}(\tau_n,U_n)], \\ Y_1 = Y_0 + \theta k[A_x(\tau_n,Y_1) - A_x(\tau_n,U_n)], \\ Y_2 = Y_1 + \theta k[A_y(\tau_n,Y_2) - A_y(\tau_n,U_n)], \\ \widetilde{Y}_0 = Y_0 + \sigma k[A_{xy}(\tau_n,Y_2) - A_{xy}(\tau_n,U_n)], \\ \widetilde{Y}_1 = \widetilde{Y}_0 + \theta k[A_x(\tau_n,\widetilde{Y}_1) - A_x(\tau_n,U_n)], \\ \widetilde{Y}_2 = \widetilde{Y}_1 + \theta k[A_y(\tau_n,\widetilde{Y}_2) - A_y(\tau_n,U_n)], \\ U_{n+1} = \widetilde{Y}_2, \end{cases}$$

here, $0<\theta,\sigma<1$. We take the difference discretization

$$\nabla_x U_{i,j} = U_{i+1,j} - U_{i-1,j},$$

$$\nabla_y U_{i,j} = U_{i,j+1} - U_{i,j-1},$$

and

$$A_xU_{i,j} = a\frac{1}{h^2}\delta_x^2U_{i,j} - u\frac{1}{2h}\nabla_xU_{i,j} - wU_{i,j},$$

$$A_yU_{i,j} = c\frac{1}{h^2}\delta_y^2U_{i,j} - v\frac{1}{2h}\nabla_yU_{i,j} - wU_{i,j},$$

$\delta_x^2U_{i,j}, \delta_y^2U_{i,j}$ and $A_{xy}U_{i,j}$ are defined the same as above. Denote $z_0 = kA_{xy}, z_1 = kA_x, z_2 = kA_y, z = z_1 + z_2, F = (1-\theta z_1)(1-\theta z_2)$, scheme (11) can be written in the non-split form

$$F^2U_{n+1} = [F^2 + F(z_0 + z) + \frac{1}{2}z_0(z_0 + z)]U_n. \tag{12}$$

And the scheme (12) is the improved alternating direction implicit difference scheme given in the Craig-Sneyd split form (C-S ADI) for the quanto options pricing model (3).

It can be verified, through Taylor expansion, the C-S ADI scheme (12) is consistent to one order. However, the precision can be improved to one order in time and two order in space if we take $\theta = \sigma = \frac{1}{2}$. And it can also be proved that the C-S ADI scheme is unconditionally stable and convergent. Thus we can get the following theorem.

Theorem 4 The improved alternating direction implicit difference scheme given in the Craig-Sneyd split form (C-S ADI) (12) of quanto options pricing model (3) is consistent to order $O(k + h)$ and unconditionally stable and convergent; it is consistent to order $O(k + h^2)$ if $\theta = \sigma = \frac{1}{2}$.

8. Numerical experiments

Example 1. We consider an American investor buys a Nikkei index put option. Assuming the current price of Nikkei is 20,000 yen, interest rate is 0.03, volatility of the Nikkei is 0.2, exchange rate of Japanese yen against dollar is 0.01, volatility of the exchange rate is 0.1, correlation coefficient is 0.2, risk-free rates of American and Japan are 0.08 and 0.04, respectively. And the strike price of option is 19,000 yen. Consider the deadline of the option is 3, 6, 9 and 12 months, and the final exchange rate is the spot exchange rate [3, 30, 31].

Here, we use the 4-core CPU for computing, and do numerical experiments under the Matlab R2011 environment. The comparison among the Monto-Carlo solution, which is used to substitute the exact solution similarly, and the numerical solutions, such as the results of the Crank-Nicolson difference scheme and the improved ADI difference scheme is shown as follows.

TABLE 1. Comparison of Monto-Carlo solution and numerical solutions.

Time(mon) / Price($)	3	6	9	12	Relative error
Monto-Carlo solution	3.6449	6.3482	8.4081	10.0941	0
Crank-Nicolson scheme	3.0153	5.4280	7.4140	9.1139	0.1366
D-R ADI scheme	3.3645	5.9680	7.9110	9.3906	0.0679
C-S ADI scheme	3.3600	5.9771	7.9784	9.5669	0.0578

Example 2. Similarly, we consider the case that the underlying asset is a call option, and the values of the variables are the same as that in Example 1. Consider

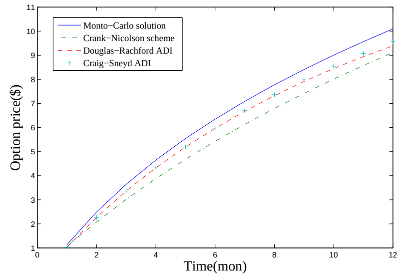

FIGURE 1. Comparison of Monto-Carlo solution and numerical solutions.

the deadline of the option is 3, 6, 9 and 12 months, and the final exchange rate is the spot exchange rate.

First, we give the plots of the option price surface $U(S_1,\, S_2,\, T)$ under the Monto-Carlo solution, Crank-Nicolson difference scheme and improved ADI difference scheme. Here $S_1 \in [19980,\, 20040], S_2 \in [0.010,\, 0.020]$, and the deadline $T = 12$ months.

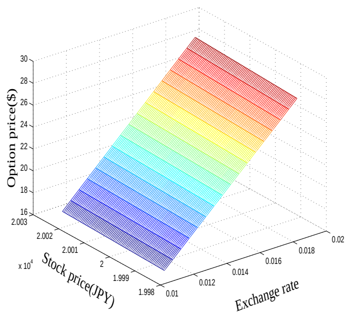

FIGURE 2. The solution of Monto-Carlo method.

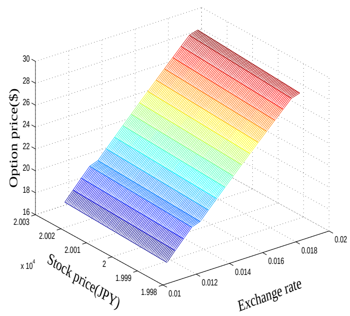

FIGURE 3. Ths solution of Crank-Nicolson scheme.

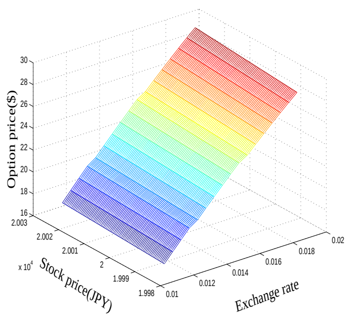

FIGURE 4. The solution of D-R ADI scheme.

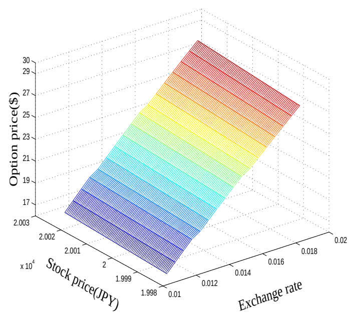

FIGURE 5. The solution of C-S ADI scheme.

Then we give the comparison among the Monto-Carlo solution and numerical solutions, and the result is shown as follows.

TABLE 2. Comparison of Monto-Carlo solution and numerical solutions.

Price($) \ Time(mon)	3	6	9	12	Relative error
Monto-Carlo solution	11.7751	14.8058	17.2481	19.3743	0
Crank-Nicolson scheme	12.1937	15.3263	17.8029	19.9721	0.0318
D-R ADI scheme	11.8990	15.0555	17.4871	19.4817	0.0118
C-S ADI scheme	11.6622	14.7964	17.2051	19.1790	0.0055

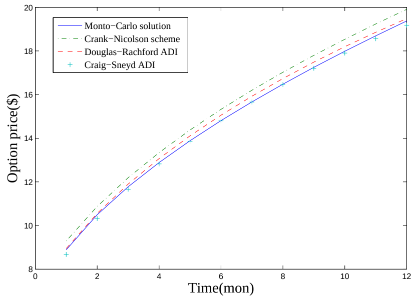

FIGURE 6. Comparison of Monto-Carlo solution and numerical solutions.

From the Table 2 and Figure 6, it is clear to show that the solutions of improved ADI difference schemes are sufficiently close to the Monto-Carlo solution. And we can also see that the improved ADI difference schemes have a higher accuracy than the Crank-Nicolson difference scheme. Thus the schemes given by this paper can be a kind of unconditionally stable finite difference schemes which approximate to equation (3). What's more, as the natural parallel property of improved ADI difference scheme, the calculation time is reduced greatly. The computing time is shown as follows.

TABLE 3. Computing time of the C-N and improved ADI schemes.

Time(min) \ Points' number	1120	1240	1360	1480	1600
Crank-Nicolson scheme	1.7763	2.4331	3.2297	4.0633	4.9922
D-R ADI scheme	0.7243	0.9761	1.2674	1.5814	1.9834
C-S ADI scheme	1.4439	1.9569	2.5305	3.2055	3.9641
D-R ADI time saving	59.22%	59.88%	60.76%	61.08%	60.27%
C-S ADI time saving	18.71%	19.57%	23.25%	21.11%	20.59%

From Table 3 and Figure 7, we can see the speed-up obtained by using the 4-core CPU for computing. And with the amount of calculation data increasing, the improved ADI parallel difference schemes's time-saving property is very apparent. The reason is, for the serial computing, the numerical arrays and the loop bodies are executed in the same Matlab process, there is no data transmission problem. But

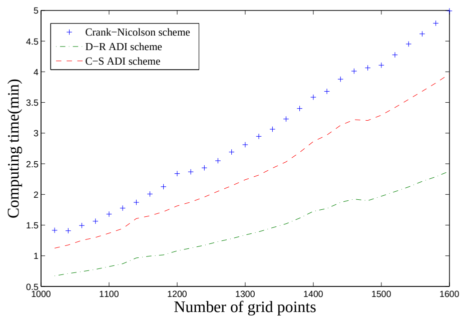

FIGURE 7. Computing time of the C-N and improved ADI schemes.

for the parallel computing, the numerical arrays are executed in the Matlab client while the loop bodies are in the Matlab worker. If the calculation data is fewer, the data transmission will decrease the computing efficiency and thus the parallel computing's time-saving property is not very obvious. However, if the calculation data amount is relatively large, the parallel execution of loop bodies will have much more influence on the computing efficiency than the data transmission does. When we use multi-core CPU for computing, Matlab will assign different loop bodies to different processors and then execute the loops at the same time to get a relatively fast calculation speed. So we could see from the Table 3 and Figure 7, when the amount of calculation data is up to 1500, the computing time of the D-R ADI parallel difference scheme is reduced to almost 40% of the serial scheme's, and the computing time of the C-S ADI parallel difference scheme is reduced to almost 80%. Hence, the improved ADI difference schemes (8) and (12) given by this paper are more efficient in solving the quanto options pricing problem.

9. Conclusion

Based on the Douglas-Rachford (D-R ADI) (8) and the Craig-Sneyd (C-S ADI) (12) splitting forms, a kind of improved alternating direction implicit difference schemes are constructed in this paper for solving the quanto options pricing model. These schemes have been shown to be unconditionally stable, convergent and have better computational accuracy than the existing Crank-Nicolson difference scheme. And it not only avoids the severe restriction for the classical explicit scheme in choosing the time step for ensuring the stability, but also reduces the computation complexity of solving a penta-diagonal equation for the classical implicit scheme.

Moreover, the improved ADI difference schemes are easy to realize parallel computing, so they improve the computational efficiency greatly. Thus compared to the existing explicit, implicit and Crank-Nicolson difference schemes, the improved ADI difference schemes have obvious advantages in solving the multi-asset option pricing problems.

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