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PARTIALLY OBSERVABLE STOCHASTIC OPTIMAL CONTROL

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Abstract. This paper is a survey on some recent results in optimal control and stochastic filtering. The goal is not to cover all recent developments in control and filtering, instead we focus on maximum principle for optimality of partial information backward or forward-backward stochastic differential equations and branching particle approximation of nonlinear filtering.

Key words. Branching particle system, forward-backward stochastic differential equation, numerical approximation, maximum principle, stochastic filtering.

1. Introduction

Stochastic control is the study of uncertain dynamical systems which can be controlled by decision makers so as to reach the best expected goals. In the real-world, the decision makers are usually only able to observe partially the state by other noisy observations. For example, in financial models, risky asset prices are observable but the appreciation rates of the assets are unavailable. See e.g. Xiong and Zhou [42] and the references therein. See also Huang, Wang and Wu [23] for optimal premium of insurance company with partial information. In these situations, we are facing optimal control problems of partially observable systems.

Such a kind of partially observed optimal control problem is composed of filtering and control. The filtering part is related to two stochastic processes: signal and observation. The signal process is what we want to estimate based on the observation which provides the information we can use. Analytical solutions to the filtering problems are rarely available in general. Thus, we have to resort to numerical schemes. Particle system approximation is an effective class of numerical schemes. The main idea is to represent the solution as a stochastic partial differential equation (SPDE) via a system of weighted particles whose locations and weights obey stochastic differential equations (SDEs) which can be solved numerically. The particle system approximation was studied in heuristic schemes by Gordon, Salmond and Ewing [20], Gordon, Salmond and Smith [21], Kitagawa [25], Carvalho et al. [3], Del Moral, Noyer and Salut [18]. Del Moral [14] considered a particle approximation for a model with independent observation noise that discounted past information. Florchinger and Gland [19] formulated a particle approximation for optimal filter. A rigorous proof of the convergent result for the particle filter is published by Del Moral [15], and independently, by Crisan and Lyons [11]. After that, many improvements were made by various authors. See e.g. Crisan and Lyons [10], Crisan [4], [5], [6], [7], Crisan, Gaines and Lyons [12], Crisan, Del Moral and Lyons [9], Crisan and Doucet [8], Del Moral and Guionnet [16], Del Moral and Miclo [17]. Later, Crisan and Xiong [13] proved a central limit type theorem for a new class of hybrid filters as well as for the original branching particle filters based on Kurtz and Xiong [26].

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In tradition, the partially observable optimal control problem is turned into a full information optimal control problem governed by Zakai equation, which is an SPDE driven by the observation process. However, this leads to an infinite dimensional optimal control problem, which is difficult to solve. See e.g. Bensoussan [2] for a systematic account. Recently, Wang and Wu [32] proposed a backward separation approach in order to study partially observed optimal control. The main idea is to decouple optimal control and state estimate by formally deducing optimal control first and then computing optimal filtering. An advantage of the approach is as follows. We use the original state and observation equation–which are finite dimensional–to calculate the variation, rather than the Zakai equation of the state based on the observation, which is infinite dimensional in general. Making use of this separating technique, lots of complicated stochastic calculus in infinite dimensional spaces are avoided. The approach is applicable to a broad class of control systems, say, backward or forward-backward stochastic differential equation (BSDE or FBSDE) systems. See e.g. Wang and Wu [33], Huang, Wang and Xiong [24], Wu [37], Shi and Wu [30], Xiao and Wang [39, 40], Xiao [38], Wang, Wu and Xiong [34, 35] for more details. See also Tang [31], Hu and Øksendal [22], Øksendal and Sulem [29], Meng [28], where optimal filtering was not studied.

The rest of this paper is organized as follows. The next section establishes several maximum principles for optimality of BSDEs and FBSDEs with partial information. To illustrate the maximum principles, a linear-quadratic (LQ) optimal control problem by means of BSDE is presented. Section 3 gives a brief introduction to the theory of nonlinear filter. A branching particle system is used to approximate the nonlinear filter in Section 4. Some numerical results will be presented in Section 5 to compare the particle filter, the optimal filter and the underlying state process. Finally, Section 6 lists some concluding remarks.

2. Maximum principle

Maximum principle is a set of necessary conditions satisfied by optimal solutions, which offers an approach for solving optimal control problems. This section is concerned with optimal control of BSDEs and FBSDEs with partial information. Two maximum principles for optimality are established, and an LQ example is used to shed light on the application of the maximum principles. These results are taken from the articles of Huang, Wang and Xiong [24], Wang, Wu and Xiong [34, 35].

2.1. The case of controlled BSDEs with partial information. We begin with a complete filtered probability space $(\Omega, \mathcal{F}^{W,Y}, (\mathcal{F}_t^{W,Y})_{0\le t\le 1}, \mathbb{P})$ on which an $\mathbb{R}^{m+d}$-valued standard Brownian motion (W, Y) is defined, and let $(\mathcal{F}_t^{W,Y})_{0\le t\le 1}$ be the natural filtration generated by (W, Y), and $\mathcal{F}^{W,Y} = \mathcal{F}_1^{W,Y}$. If $x : [0, 1] \times \Omega \to S$ is an $\mathcal{F}_t$-adapted and square-integrable process, we write $x \in L^2_{\mathcal{F}^{W,Y}}(0, 1; S)$; if $x : \Omega \to S$ is an $\mathcal{F}_1^{W,Y}$-measurable and square-integrable random variable, we write $x \in L^2_{\mathcal{F}_1^{W,Y}}(\Omega; S)$.

Let U be a non-empty convex subset of $\mathbb{R}^k$. Consider now a BSDE

$$
\begin{cases} -dy_t = f(t, y_t, z_t, \bar{z}_t, v_t)dt - z_t dY_t - \bar{z}_t dW_t, \\ \quad y_1 = \xi, \end{cases} \tag{1}
$$

where $\xi \in L^2_{\mathcal{F}_1^{W,Y}}(\Omega; \mathbb{R}^n)$, $v : [0, 1] \times \Omega \to U$ is a control process, and $f : [0, 1] \times \mathbb{R}^{n+n\times m+n\times d} \times U \to \mathbb{R}^n$ is a continuous mapping and satisfies

(H2.1). *The function f is continuously differentiable with respect to $(y, z, \bar{z}, v)$ and the partial derivatives f_y, f_z, $f_{\bar{z}}$ and f_v are uniformly bounded.*

Definition 2.1. *Let $\mathcal{G}_t$ be a sub-σ-algebra of $\mathcal{F}_t^{W,Y}$, which represents the information available at time t. A control process v is called admissible, if it is a $\mathcal{G}_t$-adapted and square integrable process, i.e., $v \in L^2_{\mathcal{G}}(0, 1; U)$. The collection of all admissible controls is denoted by $\mathcal{U}_{ad}$.*

Under (H2.1) and Definition 2.1, (1) admits a unique solution which is denoted by the triple $(y^v, z^v, \bar{z}^v)$. The associated cost functional is in the form of

$$J[v] = \mathbb{E}\left[\int_0^1 l(t, y_t^v, z_t^v, \bar{z}_t^v, v_t)dt + \phi(y_0^v)\right], \tag{2}$$

where $l : [0, 1] \times \mathbb{R}^{n+n\times m+n\times d} \times U \to \mathbb{R}$ is a continuous mapping, and $\phi : \mathbb{R}^n \to \mathbb{R}$.

(H2.2). *For any $0 \le t \le 1$, there exists a constant $K > 0$ such that*

$$\begin{aligned}
&(1 + |y|^2 + |z|^2 + |\bar{z}|^2 + |v|^2)^{-1}|l(t, y, z, \bar{z}, v)| + \\
&(1 + |y| + |z| + |\bar{z}| + |v|)^{-1}(|l_y(t, y, z, \bar{z}, v)| + \\
&|l_z(t, y, z, \bar{z}, v)| + |l_{\bar{z}}(t, y, z, \bar{z}, v)| + |l_v(t, y, z, \bar{z}, v)|) \le K, \\
&(1 + |y|^2)^{-1}|\phi| + (1 + |y|)^{-1}|\phi_y| \le K.
\end{aligned}$$

We state an optimal control problem of BSDEs with partial information.

Problem 1. Find $u \in \mathcal{U}_{ad}$ such that

$$J[u] = \min_{v\in\mathcal{U}_{ad}} J[v]$$

subject to (1). If such a u exists, we call it an optimal control, and $(y^u, z^u, \bar{z}^u)$ an optimal trajectory. When there is no confusion from the context, we take the shorthand notation $(y, z, \bar{z}) = (y^u, z^u, \bar{z}^u)$.

Let $(y^{u+\varepsilon v}, z^{u+\varepsilon v}, \bar{z}^{u+\varepsilon v})$ be the solution of (1) corresponding to the perturbation $u+\varepsilon v$ of u, where $0 \le \varepsilon \le 1$ and $v \in \mathcal{U}_{ad}$. We first introduce a variational equation

$$\begin{cases}
-dy_t^1 = (f_y(t, y_t, z_t, \bar{z}_t, u_t)y_t^1 + f_z(t, y_t, z_t, \bar{z}_t, u_t)z_t^1 + f_{\bar{z}}(t, y_t, z_t, \bar{z}_t, u_t)\bar{z}_t^1)dt \\
\qquad\qquad - z_t^1 dY_t - \bar{z}_t^1 dW_t, \\
y_1^1 = 0.
\end{cases}$$

Once $(y, z, \bar{z})$ is determined, the variational equation admits a unique solution under (H2.1). By Itô's formula and Gronwall inequality, we get a continuity result.

Lemma 2.1. *If (H2.1) holds, it yields*

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \sup_{0\le t\le 1} \mathbb{E}\left|\frac{y_t^{u+\varepsilon v} - y_t}{\varepsilon} - y_t^1\right|^2 = 0, \\
&\lim_{\varepsilon\to 0} \mathbb{E}\int_0^1 \left|\frac{z_t^{u+\varepsilon v} - z_t}{\varepsilon} - z_t^1\right|^2 dt = 0, \\
&\lim_{\varepsilon\to 0} \mathbb{E}\int_0^1 \left|\frac{\bar{z}_t^{u+\varepsilon v} - \bar{z}_t}{\varepsilon} - \bar{z}_t^1\right|^2 dt = 0.
\end{aligned}$$

Next, we introduce an adjoint system

$$(3)\qquad \begin{cases} dp_t = (f_y^*(t,y_t,z_t,\bar{z}_t,u_t)p_t - l_y(t,y_t,z_t,\bar{z}_t,u_t))dt \\ \qquad\quad + (f_z^*(t,y_t,z_t,\bar{z}_t,u_t)p_t - l_z(t,y_t,z_t,\bar{z}_t,u_t))dY_t \\ \qquad\quad + (f_{\bar{z}}^*(t,y_t,z_t,\bar{z}_t,u_t)p_t - l_{\bar{z}}(t,y_t,z_t,\bar{z}_t,u_t))dW_t, \\ p_0 = \phi_y^*(y_0), \end{cases}$$

which has a unique solution under (H2.1) and (H2.2). Hereinafter, $*$ appearing in superscript denotes the transpose of a matrix and a vector. Then, we define a Hamiltonian function $H : [0,1] \times \mathbb{R}^{n+n\times m+n\times d} \times U \times \mathbb{R}^n \to \mathbb{R}$ by

$$H(t,y,z,\bar{z},v,p) = l(t,y,z,\bar{z},v) - f^*(t,y,z,\bar{z},v)p.$$

(H2.3). *(i) For any t, τ such that $0 \le t+\tau \le 1$, and bounded $\mathcal{G}_t$-measurable random variable ν, we formulate a control process $v_s \in U$, with*

$$v_s = \nu I_{[t,t+\tau]}(s), \quad 0 \le s \le 1,$$

where $I_{[t,t+\tau]}(s)$ is the indicator function on the set $[t,t+\tau]$.
(ii) For any $v_s \in \mathcal{G}_s$ with v_s bounded, $s \in [0,1]$, there is an $\epsilon > 0$ such that $u + \varepsilon v \in \mathcal{U}_{ad}$ for $\varepsilon \in (-\epsilon,\epsilon)$.

Finally, applying the first variation of $J[v]$ with Taylor's expansion and Lemma 2.1, we derive a maximum principle for optimality of Problem 1.

Theorem 2.1. *Under (H2.1), (H2.2) and (H2.3), if u is a local minimum for $J[v]$, we have*

$$\mathbb{E}[H_v(t,y_t,z_t,\bar{z}_t,u_t,p_t)|\mathcal{G}_t] = 0,$$

where p is the unique solution of (3).

To show the application of Theorem 2.1, let us now solve an LQ optimal control problem with partial information. For notational simplicity, we assume that $m = n = k = d = 1$.

Example 2.1. Find a $u \in \mathcal{U}_{ad}$ to minimize

$$J[v] = \frac{1}{2}\mathbb{E}\left\{\int_0^1 \left[O_t(y_t^v)^2 + R_t v_t^2\right] dt + N(y_0^v)^2\right\}$$

subject to the state equation

$$(4)\qquad \begin{cases} -dy_t^v = \left(B_t y_t^v + C_t z_t^v + \bar{C}_t \bar{z}_t^v + D_t v_t\right) dt - z_t^v dY_t - \bar{z}_t^v dW_t, \\ \quad y_T^v = \xi \end{cases}$$

and the observable filtration

$$\mathcal{G}_t = \mathcal{F}_t^Y = \sigma\{Y_s; 0 \le s \le t\}.$$

Here $O_t \ge 0$, $R_t \ge 0$, B_t, C_t, $\bar{C}_t$ and D_t are uniformly bounded and deterministic functions; $1/R_t$ is also bounded; $N \ge 0$ is constant, and $\xi \in L^2_{\mathcal{F}_1}(\Omega;\mathbb{R})$.

It seems that [24] formulated originally the example. However, [24] were only able to solve the case that the drift term of the state equation does not depend on $\bar{C}\bar{z}$ due to the limit of techniques used there. Very recently, Example 2.1 was solved by Wang, Wu and Xiong [35], where some complicated techniques were used, for example, maximum principle, optimal filtering for FBSDEs, existence and uniqueness of FBSDEs.

With the above data, the adjoint equation is

$$\begin{cases} dp_t = (B_t p_t - O_t y_t)dt + C_t p_t dY_t + \bar{C}_t p_t dW_t, \\ p_0 = -Ny_0. \end{cases} \tag{5}$$

Introduce a Riccati differential equation

$$\begin{cases} \dot{\alpha}_t - \left(2B_t + C_t^2 + \bar{C}_t^2\right)\alpha_t - \dfrac{1}{R_t} D_t^2 \alpha_t^2 + O_t = 0, \\ \alpha_0 = -N, \end{cases} \tag{6}$$

and give an additional assumption.

(H2.4). *The solution α of (6) satisfies*

$$\frac{1}{\alpha_t}\bar{C}_t^2 + \frac{1}{R_t} D_t^2 \geq 0.$$

With these preparations, we now obtain

Proposition 2.1. *Under (H2.4),*

$$u_t = \frac{1}{R_t} D_t \alpha_t \hat{y}_t$$

is the unique optimal control of Example 2.1, where α satisfies (6), and the optimal filtering $(\hat{y}, \hat{z}, \hat{p})$ of the solution (y, z, p) to (4) and (5) with respect to $\mathcal{G}_t$ solves

$$\begin{cases} d\hat{p}_t = (B_t \hat{p}_t - O_t \hat{y}_t)dt + C_t \hat{p}_t dY_t, \\ -d\hat{y}_t = \left[\left(\dfrac{1}{\alpha_t}\bar{C}_t^2 + \dfrac{1}{R_t}D_t^2\right)\hat{p}_t + B_t \hat{y}_t + C_t \hat{z}_t\right] dt - \hat{z}_t dY_t, \\ \hat{p}_0 = -N\hat{y}_0, \quad \hat{y}_1 = \mathbb{E}[\xi|\mathcal{G}_1]. \end{cases} \tag{7}$$

Note that (7) is referred to as a kind of forward-backward stochastic differential filtering equation.

2.2. The case of controlled FBSDEs with partial observation.

Let $C(0,1;\mathbb{R})$ be a space of continuous functions from $[0,1]$ to $\mathbb{R}$, let (W, Y) a standard Brownian motion with values in $\mathbb{R}^2$, let $(\mathcal{F}_t^W)_{0\leq t\leq 1}$ and $(\mathcal{F}_t^Y)_{0\leq t\leq 1}$ the natural filtrations generated by W and Y, and let $\mathbb{P}_W$ and $\mathbb{P}_Y$ the probabilities on $C(0,1;\mathbb{R})$, respectively. Set $\Omega = C(0,1;\mathbb{R}) \times C(0,1;\mathbb{R})$, $\mathcal{F}_t = \mathcal{F}_t^W \otimes \mathcal{F}_t^Y$, $\mathcal{F} = \mathcal{F}_1$ and $\mathbb{P} = \mathbb{P}_W \times \mathbb{P}_Y$.

Let U be a non-empty convex subset of $\mathbb{R}$. Consider an FBSDE

$$\begin{cases} dx_t = b(t, x_t, v_t)dt + \sigma(t, x_t, v_t)dW_t + \tilde{\sigma}(t, x_t, v_t)d\tilde{W}_t^v, \\ -dy_t = g(t, x_t, y_t, z_t, \tilde{z}_t, v_t)dt - z_t dW_t - \tilde{z}_t dY_t, \\ x_0 = x^0, \quad y_1 = f(x_1). \end{cases} \tag{8}$$

Here $v : [0,1] \times C([0,1];\mathbb{R}) \to U$ is a control process, and $(x, y, z, \tilde{z})$ is the state process of (8), taking values in $\mathbb{R}^4$, with initial state $x^0 \in \mathbb{R}$. $b, \sigma, \tilde{\sigma} : [0,1]\times\mathbb{R}\times U \to \mathbb{R}$, $g : [0,1] \times \mathbb{R}^4 \times U \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous mappings. $\tilde{W}^v$ is a stochastic process depending on v and takes values in $\mathbb{R}$.

Suppose that $(x, y, z, \tilde{z})$ is observed partially by a noisy process

$$Y_t = \int_0^t h(s, x_s)ds + \tilde{W}_t^v, \tag{9}$$

where $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping.

Definition 2.2. *A control process v is called admissible, if v_t is $\mathcal{F}_t^Y$-adapted with*

$$\sup_{0 \le t \le 1} \mathbb{E} v_t^8 < +\infty.$$

The set of all admissible controls is denoted by $\mathcal{V}_{ad}$.

(H2.5). *The functions b, σ, $\tilde{\sigma}$, f, g and h are continuously differentiable in (x, v), $(x, y, z, \tilde{z}, v)$ and x, respectively. h and the partial derivatives b_x, b_v, σ_x, σ_v, $\tilde{\sigma}_x$, $\tilde{\sigma}_v$, h_x, g_x, g_y, g_z, g_v and f_x are uniformly bounded.*

For any $v \in \mathcal{V}_{ad}$, (8) admits a unique solution $(x^v, y^v, z^v, \tilde{z}^v)$ under (H2.5). Introduce a martingale process

$$Z_t^v = \exp\left\{ \int_0^t h(s, x_s^v) dY_s - \frac{1}{2} \int_0^t h^2(s, x_s^v) ds \right\},$$

whose differential form reads

$$\begin{cases} dZ_t^v = Z_t^v h(t, x_t^v) dY_t, \\ Z_0^v = 1. \end{cases} \tag{10}$$

We are able to define a new probability $\mathbb{P}^v$ by

$$\frac{d\mathbb{P}^v}{d\mathbb{P}} = Z_1^v.$$

Thanks to Girsanov's theorem and (9), $(W, \tilde{W}^v)$ is a 2-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le 1}, \mathbb{P}^v)$.

The cost functional is of the form

$$J[v] = \mathbb{E}^v \left[\int_0^1 l(t, x_t^v, y_t^v, z_t^v, \tilde{z}_t^v, v_t) dt + \phi(x_1^v) + \gamma(y_0^v) \right]. \tag{11}$$

Here $\mathbb{E}^v$ is the expectation with respect to $\mathbb{P}^v$. $l : [0,1] \times \mathbb{R}^4 \times U \to \mathbb{R}$, $\phi : \mathbb{R} \to \mathbb{R}$ and $\gamma : \mathbb{R} \to \mathbb{R}$ are continuous mappings.

(H2.6). *l, ϕ and γ are continuously differentiable with respect to $(x, y, z, \tilde{z}, v)$, x and y, respectively, and there is a constant $K > 0$ such that*

$$\begin{aligned}
&|l(t, x, y, z, \tilde{z}, v)| \le K(1 + |x|^2 + |y|^2 + |z|^2 + |\tilde{z}|^2 + |v|^2), \\
&|l_x(t, x, y, z, \tilde{z}, v)| + |l_y(t, x, y, z, \tilde{z}, v)| + |l_z(t, x, y, z, \tilde{z}, v)| + |l_{\tilde{z}}(t, x, y, z, \tilde{z}, v)| \\
&\quad + |l_v(t, x, y, z, \tilde{z}, v)| \le K(1 + |x| + |y| + |z| + |v|), \\
&(1 + |x|^2)^{-1} |\phi(x)| + (1 + |y|^2)^{-1} |\gamma(y)| \le K, \\
&(1 + |x|)^{-1} |\phi_x(x)| + (1 + |y|)^{-1} |\gamma_y(y)| \le K.
\end{aligned}$$

Problem 2. Find an admissible control u such that

$$J[u] = \min_{v \in \mathcal{V}_{ad}} J[v]$$

subject to (8) and (9).

Any u satisfying the above equality is called an optimal control process of Problem 2, the corresponding state processes, denoted by $(x, y, z, \tilde{z}) = (x^u, y^u, z^u, \tilde{z}^u)$ and $Z = Z^u$, are called the optimal state processes. For simplicity, we also adopt the shorthand notation $\tilde{W} = \tilde{W}^u$. According to Bayes' formula, (11) is rewritten as

$$J[v] = \mathbb{E} \left[\int_0^1 Z_t^v l(t, x_t^v, y_t^v, z_t^v, \tilde{z}_t^v, v_t) dt + Z_1^v \phi(x_1^v) + \gamma(y_0^v) \right]. \tag{12}$$

Thus, Problem 2 is equivalent to minimizing (12) over $\mathcal{V}_{ad}$ subject to (8) and (10).

Let $(x^{u+\varepsilon v}, y^{u+\varepsilon v}, z^{u+\varepsilon v}, \tilde{z}^{u+\varepsilon v})$ and $Y^{u+\varepsilon v}$ be the states of (8) and (10) corresponding to the perturbation $u+\varepsilon v$ of u, where $0 \leq \varepsilon \leq 1$ and $v \in \mathcal{V}_{ad}$. Introduce a variational equation

$$\begin{cases} dZ_t^1 = (Z_t^1 h(t,x_t) + Z_t h_x(t,x_t) x_t^1) dY_t, \\ Z_0^1 = 0 \end{cases} \tag{13}$$

and

$$\begin{cases} dx_t^1 = [(b_x(t,x_t,u_t) - \tilde{\sigma}_x(t,x_t,u_t)h(t,x_t) - \tilde{\sigma}(t,x_t,u_t)h_x(t,x_t))x_t^1 \\ \qquad\quad + (b_v(t,x_t,u_t) - \tilde{\sigma}_v(t,x_t,u_t)h(t,x_t))v_t]dt \\ \qquad\quad + (\sigma_x(t,x_t,u_t)x_t^1 + \sigma_v(t,x_t,u_t)v_t)dW_t \\ \qquad\quad + (\tilde{\sigma}_x(t,x_t,u_t)x_t^1 + \tilde{\sigma}_v(t,x_t,u_t)v_t)dY_t, \\ -dy_t^1 = (g_x(t,x_t,y_t,z_t,\tilde{z}_t,u_t)x_t^1 + g_y(t,x_t,y_t,z_t,\tilde{z}_t,u_t)y_t^1 \\ \qquad\quad + g_z(t,x_t,y_t,z_t,\tilde{z}_t,u_t)z_t^1 + g_{\tilde{z}}(t,x_t,y_t,z_t,\tilde{z}_t,u_t)\tilde{z}_t^1 \\ \qquad\quad + g_v(t,x_t,y_t,z_t,\tilde{z}_t,u_t)v_t)dt - z_t^1 dW_t - \tilde{z}_t^1 dY_t, \\ x_0^1 = 0, \quad y_1^1 = f_x(x_1)x_1^1. \end{cases} \tag{14}$$

For any $v \in \mathcal{V}_{ad}$, it is easy to see that (13) and (14) admit a unique solution under (H2.5), respectively. Thanks to Itô's formula and Gronwall inequality, we derive a continuity result.

Lemma 2.2. *If (H2.5) holds, it yields*

$$\lim_{\varepsilon\to 0} \sup_{0\leq t\leq 1} \mathbb{E}\left|\frac{x_t^{u+\varepsilon v} - x_t}{\varepsilon} - x_t^1\right|^4 = 0,$$
$$\lim_{\varepsilon\to 0} \sup_{0\leq t\leq 1} \mathbb{E}\left|\frac{Z_t^{u+\varepsilon v} - Z_t}{\varepsilon} - Z_t^1\right|^2 = 0,$$
$$\lim_{\varepsilon\to 0} \sup_{0\leq t\leq 1} \mathbb{E}\left|\frac{y_t^{u+\varepsilon v} - y_t}{\varepsilon} - y_t^1\right|^2 = 0,$$
$$\lim_{\varepsilon\to 0} \mathbb{E}\int_0^1 \left|\frac{z_t^{u+\varepsilon v} - z_t}{\varepsilon} - z_t^1\right|^2 dt = 0,$$
$$\lim_{\varepsilon\to 0} \mathbb{E}\int_0^1 \left|\frac{\tilde{z}_t^{u+\varepsilon v} - \tilde{z}_t}{\varepsilon} - \tilde{z}_t^1\right|^2 dt = 0.$$

Next, we introduce a Hamiltonian function

$$\begin{aligned} H(t,x,y,z,\tilde{z},v;p,q,k,\tilde{k},\tilde{Q}) &= b(t,x,v)q + \sigma(t,x,v)k + \tilde{\sigma}(t,x,v)\tilde{k} + h(t,x)\tilde{Q} \\ &\quad - (g(t,x,y,z,\tilde{z},v) - h(t,x)\tilde{z})p + l(t,x,y,z,\tilde{z},v), \end{aligned} \tag{15}$$

and an adjoint system

$$\begin{cases} -dP_t = l(t,x_t,y_t,z_t,\tilde{z}_t,u_t)dt - Q_t dW_t - \tilde{Q}_t d\tilde{W}_t, \\ P_1 = \phi(x_1) \end{cases} \tag{16}$$

and

(17)
$$\begin{cases} dp_t = & (g_y(t,x_t,y_t,z_t,\tilde{z}_t,u_t)p_t - l_y(t,x_t,y_t,z_t,\tilde{z}_t,u_t))\,dt \\ & + (g_z(t,x_t,y_t,z_t,\tilde{z}_t,u_t)p_t - l_z(t,x_t,y_t,z_t,\tilde{z}_t,u_t))\,dW_t, \\ & + [(g_{\tilde{z}}(t,x_t,y_t,z_t,\tilde{z}_t,u_t) - h(t,x_t))p_t - l_{\tilde{z}}(t,x_t,y_t,z_t,\tilde{z}_t,u_t)]d\tilde{W}_t, \\ -dq_t = & [(b_x(t,x_t,u_t) - \tilde{\sigma}(t,x_t,u_t)h_x(t,u_t))q_t + \sigma_x(t,x_t,u_t)k_t + \tilde{\sigma}_x(t,x_t,u_t)\tilde{k}_t \\ & + h_x(t,x_t)\tilde{Q}_t - g_x(t,x_t,y_t,z_t,\tilde{z}_t,u_t)p_t + l_x(t,x_t,y_t,z_t,\tilde{z}_t,u_t)]dt \\ & - k_t dW_t - \tilde{k}_t d\tilde{W}_t, \\ p_0 = & -\gamma_y(y_0), \quad q_1 = -f_x(x_1)p_1 + \phi_x(x_1). \end{cases}$$

It is easy to see that (16) and (17) admit unique solutions under (H2.5) and (H2.6).

Similar to Theorem 2.1, we obtain a maximum principle for optimality of Problem 2 by applying Taylor's expansion and Lemma 2.2.

Theorem 2.2. *Under (H2.3) with $\mathcal{G}_t$ being replaced by $\mathcal{F}_t^Y$, (H2.5) and (H2.6), if u is a local minimum of Problem 2, we have*

$$\mathbb{E}^u\left[H_v(t,x_t,y_t,z_t,\tilde{z}_t,u_t;p_t,q_t,k_t,\tilde{k}_t,\tilde{Q}_t)\middle|\mathcal{F}_t^Y\right] = 0,$$

where H is defined by (15), and $(x,y,z,\tilde{z})$, $\tilde{Q}$ and $(p,q,k,\tilde{k})$ are the solutions of (8), (16) and (17), respectively.

The maximum condition in Theorem 2.2 depends explicitly on the boundedness of the drift term h of (9). It excludes some important applications in practice. Recently, an approximation method and a decomposition method are used to extend Problem 2 under the assumption that h grows linearly in x. We omit the extension for saving space. The interested reader can refer to Wang, Wu and Xiong [34, 35] for more details.

3. Optimal filtering

In this section, we give a brief introduction to the nonlinear filtering theory. We refer the reader to the monographs of Bensoussan [2], Xiong [41], Bain and Crisan [1] for a detailed account.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space. The signal is modeled by a d-dimensional diffusion process x_t governed by an SDE

$$dx_t = b(x_t)dt + c(x_t)dW_t + \sigma(x_t)dB_t,$$

where (W,B) is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion taking values in $\mathbb{R}^{m+d}$. The mappings $b:\mathbb{R}^d\to\mathbb{R}^d$, $c:\mathbb{R}^d\to\mathbb{R}^{d\times m}$ and $\sigma:\mathbb{R}^d\to\mathbb{R}^{d\times d}$ are bounded and Lipschitz continuous. The observation process is an m-dimensional process satisfying

$$Y_t = \int_0^t h(x_s)ds + W_t,$$

where $h:\mathbb{R}^d\to\mathbb{R}^m$ is a bounded and Lipschitz continuous mapping. Let

$$\mathcal{F}_t^Y = \sigma(Y_s : 0\leq s\leq t)$$

be the observable information at time t. The optimal filter π_t is the conditional probability distribution of x_t given $\mathcal{F}_t^Y$, i.e.,

$$\mathbb{E}(f(x_t)|\mathcal{F}_t^Y) = \langle \pi_t, f\rangle\,.$$

Denote by $\hat{\mathbb{E}}$ the expectation with respect to a probability $\hat{\mathbb{P}}$ such that

$$\left.\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(-\int_0^t h^*(x_s)\, dW_s - \frac{1}{2}\int_0^t |h(x_s)|^2\, ds\right).$$

Set

$$M_t = \exp\left(\int_0^t h^*(x_s)\, dY_s - \frac{1}{2}\int_0^t |h(x_s)|^2\, ds\right).$$

It yields from the observation equation that

$$\left.\frac{d\mathbb{P}}{d\hat{\mathbb{P}}}\right|_{\mathcal{F}_t} = M_t.$$

By Girsanov's theorem, Y becomes a Brownian motion under $\hat{\mathbb{P}}$ which is independent of B. The signal is written as

$$dx_t = \tilde{b}(x_t)dt + c(x_t)dY_t + \sigma(x_t)dB_t,$$

where

$$\tilde{b} = b - ch.$$

Theorem 3.1 (Kallianpur-Striebel formula). *The optimal filter π_t is represented by the unnormalized filter V_t, i.e.,*

$$\langle \pi_t, f\rangle = \frac{\langle V_t, f\rangle}{\langle V_t, 1\rangle}, \quad \forall f \in C_b(\mathbb{R}^d), \tag{18}$$

where

$$\langle V_t, f\rangle = \hat{\mathbb{E}}(M_t f(x_t)|\mathcal{F}_t^Y), \tag{19}$$

and $C_b(\mathbb{R}^d)$ is the set of all bounded continuous real-valued functions.

Next, we give a particle system representation for $\langle V_t, f\rangle$ which will be used in the rest of this paper. On the probability space, let B^i $(i = 1, 2, \cdots)$ be independent copies of B, and let them be independent of Y. For $i = 1, 2, \cdots$, consider an interacting particle system

$$\begin{cases} dx_t^i = \tilde{b}(x_t^i)dt + c(x_t^i)dY_t + \sigma(x_t^i)dB_t^i, \\ dM_t^i = M_t^i h(x_t^i)dY_t, \quad M_0^i = 1. \end{cases} \tag{20}$$

By (19) and the conditional law of large numbers, we derive

Theorem 3.2. *If $\{x_0^i, i = 1, 2, \cdots\}$ are i.i.d. random vectors with common distribution π_0 on $\mathbb{R}^d$, then*

$$\langle V_t, f\rangle = \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^k M_t^i f(x_t^i), \tag{21}$$

where $\{(M^i, x^i), i = 1, 2, \cdots\}$ is the unique strong solution to the particle system above.

Applying Itô's formula to (19), (20) and (21), we get

Theorem 3.3 (Zakai's equation). *The unnormalized filter V_t satisfies*

$$\langle V_t, f\rangle = \langle V_0, f\rangle + \int_0^t \langle V_s, Lf\rangle\, ds + \int_0^t \langle V_s, \nabla^* f c + f h^*\rangle\, dY_s, \quad \forall f \in C_b^2(\mathbb{R}^d), \tag{22}$$

where

$$Lf = \nabla^* f b + \frac{1}{2} tr[c\partial^2 f c^* + \sigma \partial^2 f \sigma^*],$$

$tr(A)$ denotes the trace of the square matrix A, and $C_b^2(\mathbb{R}^d)$ is the set of all bounded continuous real-valued functions with their derivatives up to order 2.

Applying Itô's formula to (18) and (22), we obtain

Theorem 3.4 (Kushner-FKK equation). *The optimal filter π_t satisfies*
(23)

$$\langle \pi_t, f\rangle = \langle \pi_0, f\rangle + \int_0^t \langle \pi_s, Lf\rangle\, ds + \int_0^t \left(\langle \pi_s, \nabla^* f c + f h^*\rangle - \langle \pi_s, f\rangle \langle \pi_s, h^*\rangle\right) d\nu_s,$$

where $f \in C_b^2(\mathbb{R}^d)$, and the innovation process μ_t, given by

$$\nu_t = Y_t - \int_0^t \langle \pi_s, h\rangle\, ds$$

is a Brownian motion with respect to the probability $\mathbb{P}$.

We note that the boundedness condition about b, c, σ and h is not necessary for the nonlinear filtering thoery.

4. Branching particle approximation of π_t

This section is concerned with the branching particle approximation to the optimal filter π_t given by (23). These results are based on the recent article of Liu and Xiong [27] and the monograph of Xiong [41].

4.1. Branching particle system. We now introduce the branching particle system in a random environment, which is used to define an approximate filter.

Let π_0 be the initial distribution of the signal. We sample n particles from $\pi_0^{\otimes n}$. Denote by x_0^i $(i = 1, 2, \cdots, n)$ their locations and assign weight $\frac{1}{n}$ to each of them. Let $\delta = n^{-2\alpha}$, $0 < \alpha \le \frac{1}{2}$. Define

$$\delta(t) = j\delta \text{ for } j\delta \le t < (j+1)\delta.$$

During the time interval $[j\delta, (j+1)\delta)$, there are $m_j^n (j = 0, 1, 2, \cdots)$ number of particles alive and they move according to the following diffusions: for $i = 1, 2, \cdots, m_j^n$,

$$x_t^i = x_{j\delta}^i + \tilde{b}(x_{j\delta}^i)(t - j\delta) + c(x_{j\delta}^i)(Y_t - Y_{j\delta}) + \sigma(x_{j\delta}^i)(B_t^i - B_{j\delta}^i).$$

At the end of the interval, the ith particle $(i = 1, 2, \cdots, m_j^n)$ branches (independent of others) into a random number ξ_{j+1}^i of offsprings which is chosen such that the conditional variance (with respect to $\hat{\mathbb{P}}$)

$$Var\left(\xi_{j+1}^i | \mathcal{F}_{(j+1)\delta-}\right)$$

is minimized subject to

$$\hat{\mathbb{E}}(\xi_{j+1}^i | \mathcal{F}_{(j+1)\delta-}) = \tilde{M}_{j+1}^n(x^i),$$

where

$$\tilde{M}_{j+1}^n(x^i) = \frac{M_{j+1}^n(x^i)}{\frac{1}{m_j^n}\sum_{l=1}^{m_j^n} M_{j+1}^n(x^l)}$$

and

$$M_{j+1}^n(x^i) = \exp\left(h^*(x_{j\delta}^i)\left(Y_{(j+1)\delta} - Y_{j\delta}\right) - \frac{1}{2}|h(x_{j\delta}^i)|^2\delta\right).$$

It is clear that

$$\xi^i_{j+1} = \begin{cases} [\tilde{M}^n_{j+1}(x^i)] & \text{with probability } 1 - \{\tilde{M}^n_{j+1}(x^i)\}, \\ [\tilde{M}^n_{j+1}(x^i)] + 1 & \text{with probability } \{\tilde{M}^n_{j+1}(x^i)\}, \end{cases}$$

where $[x]$ is the largest integer which is not greater than x and $\{x\} = x - [x]$ is the fraction of x. Denote the conditional variance of ξ^i_{j+1} by $\gamma^n_{j+1}(x^i)$. We have

$$\gamma^n_{j+1}(x^i) = \{\tilde{M}^n_{j+1}(x^i)\} \left(1 - \{\tilde{M}^n_{j+1}(x^i)\}\right).$$

The approximation to π_t is defined by

$$\pi^n_t = \frac{1}{m^n_j} \sum_{i=1}^{m^n_j} \delta_{x^i_{j\delta}}, \quad j\delta \le t < (j+1)\delta.$$

4.2. Duality representation and some estimates. Let $\mathcal{X}$ be a Hilbert space, and let $C^q_b(\mathbb{R}^d, \mathcal{X})$ be the set of all bounded continuous maps from $\mathbb{R}^d$ to $\mathcal{X}$ with bounded partial derivatives up to order q. We endow $C^q_b(\mathbb{R}^d, \mathcal{X})$ with the norm

$$\|\varphi\|_{q,\infty} = \sum_{|\alpha| \le q} \sup_{x \in \mathbb{R}^d} \|D_\alpha \varphi(x)\|_{\mathcal{X}}, \quad \varphi \in C^q_b(\mathbb{R}^d, \mathcal{X}),$$

where $\alpha = (\alpha^1, \cdots, \alpha^d)$ is a multi-index and $D_\alpha \varphi = \partial_1^{\alpha^1} \partial_2^{\alpha^2} \cdots \partial_d^{\alpha^d} \varphi$. Denote by $W^q_p(\mathbb{R}^d, \mathcal{X})$ the set of all functions with generalized partial derivatives up to order q with both the function and all its partial derivatives being p-integrable. We endow $W^q_p(\mathbb{R}^d, \mathcal{X})$ with the Sobolev norm

$$\|\varphi\|_{q,p} = \left(\sum_{|\alpha| \le q} \int_{\mathbb{R}^d} \|D_\alpha \varphi(x)\|^p \, dx \right)^{\frac{1}{p}}.$$

(H4.1). *The mappings a, b, c, σ, h, ϕ are in $C^q_b(\mathbb{R}^d, \mathcal{X})$ with $q = \left[\frac{d}{2}\right] + 2$ and $\mathcal{X}$ being $\mathcal{S}_d, \mathbb{R}^d, \mathbb{R}^{d\times m}, \mathbb{R}^{d\times d}, \mathbb{R}^m$ and $\mathbb{R}$ respectively. Also, we assume $\phi \in W^q_2(\mathbb{R}^d)$.*

Define the usual distance by

$$d(\mu, \nu) = \sum_{n=0}^{\infty} 2^{-n} \left(|\langle \mu - \nu, f_n \rangle| \wedge 1 \right),$$

where $f_0 = 1$ and for $n \ge 1, f_n \in C^{q+2}_b(\mathbb{R}^d) \cap W^{q+2}_2(\mathbb{R}^d)$ with $\|f_n\|_{q+2,\infty} \le 1$ and $\|f_n\|_{q+2,2} \le 1$, where $q = \left[\frac{d}{2}\right] + 2$ is given in (H4.1).

Consider a backward SPDE

$$\begin{cases} d\psi_s = -L\psi_s ds - (\nabla^* \psi_s c + h^* \psi_s) \hat{d}Y_s, & 0 \le s \le t \\ \psi_t = \phi, \end{cases} \tag{24}$$

where $\hat{d}Y_s$ stands for the backward Itô's integral. The following lemma is taken from Xiong [41].

Lemma 4.1. *Under (H4.1), there is a constant $K > 0$ independent of ϕ and $s \in [0, t]$ such that*

$$\hat{\mathbb{E}}[\|\psi_s\|^2_{q,2}] \le K \|\phi\|^2_{q,2}.$$

As a consequence, $\psi_s \in C^2_b(\mathbb{R}^d)$ a.s. and there is a constant $K > 0$ independent of ϕ and $s \in [0, t]$ such that

$$\hat{\mathbb{E}}[\|\psi_s\|^2_{2,\infty}] \le K \|\phi\|^2_{q,2}.$$

Applying Lemma 6.20 in Xiong [41], we get an identity which plays a key role in proving the convergence of the approximating filter.

Lemma 4.2. *For every $t > 0$, we have*

$$\begin{aligned}\psi_t(x_t)M_t - \psi_0(x_0) = &\int_0^t M_s \nabla^* \psi_s(x_s)\sigma(x_s)dB_s \\ &- \int_0^t M_s \nabla^* \psi_s(x_s)c(x_s)(h(x_s) - h(x_0))^* ds \\ &- \int_0^t M_s \Theta_s(x_s)(h(x_s) - h(x_0))^* dY_s.\end{aligned}$$

Using elementary estimates about stochastic integral, we obtain

Proposition 4.1. *Recall that $\delta(t) = j\delta$ for $j\delta \le t < (j+1)\delta$. Then the locations and weights of the branching particle system satisfy:*

- *For any $0 \le t \le T$, there is a constant $K > 0$ satisfying*
$$\hat{\mathbb{E}}|x_t^i - x_{\delta(t)}^i|^2 \le K\delta \text{ and } \hat{\mathbb{E}}|x_t^i - x_{\delta(t)}^i|^4 \le K\delta^2.$$
- *For any $0 \le j \le [T/\delta]$, $i = 1, 2, \cdots, m_j^n$, we have*
$$\hat{\mathbb{E}}\left((M_{j+1}^n(x^i))^2 | \mathcal{F}_{j\delta}\right) \le e^{K^2\delta} \text{ and } \hat{\mathbb{E}}\left((M_{j+1}^n(x^i))^4 | \mathcal{F}_{j\delta}\right) \le e^{6K^2\delta}.$$
- *For any $0 \le j \le [T/\delta]$, $i = 1, 2, \cdots, m_j^n$, there is a constant $K > 0$ such that*
$$\hat{\mathbb{E}}\left(|M_{j+1}^n(x^i) - 1|^2 | \mathcal{F}_{j\delta}\right) \le K\delta.$$

Similar to Crisan and Xiong [13] or Xiong [41], the following proposition is derived.

Proposition 4.2. *Let*

$$\eta_k^n = \prod_{j=1}^k \frac{1}{m_{j-1}^n} \sum_{l=1}^{m_{j-1}^n} M_j^n(x^l).$$

- *For each $0 \le j \le [T/\delta]$, there is a constant $K > 0$ such that*
$$\hat{\mathbb{E}}\left(m_j^n \left(\eta_j^n\right)^2\right) \le Kn.$$
- *For any $0 \le j \le [T/\delta]$, $i = 1, 2, \cdots, m_j^n$, there is a constant $K > 0$ such that*
$$\hat{\mathbb{E}}\left(\gamma_{j+1}^n \left(X^i\right) \left(\eta_{j+1}^n / \eta_j^n\right)^2 | \mathcal{F}_{j\delta}\right) \le K\sqrt{\delta}.$$
- *For any $1 \le j \le [T/\delta]$, there is a constant $K > 0$ such that*
$$\hat{\mathbb{E}}\left(\left(\eta_j^n / \eta_{j-1}^n\right)^2 | \mathcal{F}_{(j-1)\delta}\right) \le e^{K^2\delta}.$$

The next lemma is obtained by Proposition 4.2.

Lemma 4.3. *For any $0 \le j \le [T/\delta]$, $i = 1, 2, \cdots, m_j^n$, there exists a constant $K > 0$ such that*

$$\hat{\mathbb{E}}\left(\left(\eta_j^n\right)^2 \left(m_j^n\right)^2\right) \le Kn^2.$$

4.3. Branching particle approximation of V_t. In order to study the convergence of π_t^n, we first define the unnormalized filter approximation and prove its convergence. In detail, define the unnormalized filter approximation by

$$V_t^n = \frac{1}{n}\eta_k^n \sum_{i=1}^{m_k^n} M_k^n(x^i,t)\delta_{x_t^i}, \quad k\delta \le t < (k+1)\delta.$$

We first prove the convergence of the approximating unnormalized filter at any fixed time $k\delta$ ($k = 0, 1, \cdots, [T/\delta]$). Next, we study the convergence of the approximating unnormalized filter at any fixed t. Finally, we derive the uniform convergent rate for t in any finite time interval.

Let ψ_s, $0 \le s \le k\delta$ be the solution to (24) with t replaced by $k\delta$. Note that $\langle V_{k\delta}^n, \psi_{k\delta}\rangle - \langle V_0^n, \psi_0\rangle$ can be denoted as a telescopic sum

$$\sum_{j=1}^{k}\left(\left\langle V_{j\delta}^n, \psi_{j\delta}\right\rangle - \left\langle V_{(j-1)\delta}^n, \psi_{(j-1)\delta}\right\rangle\right).$$

As $\psi_{k\delta} = \phi$, we have

$$\begin{aligned}\langle V_{k\delta}^n, \phi\rangle - \langle V_0^n, \psi_0\rangle =& \sum_{j=1}^{k}\left(\left\langle V_{j\delta}^n, \psi_{j\delta}\right\rangle - \hat{\mathbb{E}}\left(\left\langle V_{j\delta}^n, \psi_{j\delta}\right\rangle \middle| \mathcal{F}_{j\delta-} \vee \mathcal{F}_{j\delta,k\delta}^Y\right)\right)\\ &+ \sum_{j=1}^{k}\left(\hat{\mathbb{E}}\left(\left\langle V_{j\delta}^n, \psi_{j\delta}\right\rangle \middle| \mathcal{F}_{j\delta-} \vee \mathcal{F}_{j\delta,k\delta}^Y\right) - \left\langle V_{(j-1)\delta}^n, \psi_{(j-1)\delta}\right\rangle\right)\\ \equiv& I_1^n + I_2^n,\end{aligned}$$

where $\mathcal{F}_{s,t}^Y = \sigma(Y_u - Y_s : s \le u \le t)$,

$$I_1^n = \sum_{j=1}^{k}\eta_j^n \frac{1}{n}\sum_{i=1}^{m_{j-1}^n}\psi_{j\delta}(x_{j\delta}^i)(\xi_j^i - \tilde{M}_j^n(x^i))$$

and

$$I_2^n = \sum_{j=1}^{k}\eta_{j-1}^n \frac{1}{n}\sum_{i=1}^{m_{j-1}^n}\left(\psi_{j\delta}(x_{j\delta}^i)M_j^n(x^i) - \psi_{(j-1)\delta}(x_{(j-1)\delta}^i)\right).$$

Furthermore, it follows from Lemma 4.2 that I_2^n is rewritten as

$$\begin{aligned}I_2^n &\equiv I_{21}^n - I_{22}^n - I_{23}^n.\\ I_{21}^n &= \sum_{j=1}^{k}\eta_{j-1}^n \frac{1}{n}\sum_{i=1}^{m_{j-1}^n}\int_{(j-1)\delta}^{j\delta} M_{j-1}^n(x^i,s)\nabla^*\psi_s(x_s^i)\sigma(x_s^i)dB_s^i,\\ I_{22}^n &= \sum_{j=1}^{k}\eta_{j-1}^n \frac{1}{n}\sum_{i=1}^{m_{j-1}^n}\int_{(j-1)\delta}^{j\delta} M_{j-1}^n(x^i,s)\nabla^*\psi_s(x_s^i)c(x_s^i)\left(h(x_s^i) - h(x_{(j-1)\delta}^i)\right)^* ds,\\ I_{23}^n &= \sum_{j=1}^{k}\eta_{j-1}^n \frac{1}{n}\sum_{i=1}^{m_{j-1}^n}\int_{(j-1)\delta}^{j\delta} M_{j-1}^n(x^i,s)\Theta_s(x_s^i)\left(h(x_s^i) - h(x_{(j-1)\delta}^i)\right)^* dY_s.\end{aligned}$$

Propositions 4.1, 4.2 and Lemma 4.3 imply that

Lemma 4.4. *Suppose that (H4.1) holds, then there exists a constant $K > 0$ such that*

$$\hat{\mathbb{E}}(I_1^n)^2 \le Kn^{-(1-\alpha)}\|\phi\|_{q,2}^2, \quad \hat{\mathbb{E}}(I_{21}^n)^2 \le Kn^{-1}\|\phi\|_{q,2}^2,$$
$$\hat{\mathbb{E}}\left(I_{22}^n\right)^2 \le Kn^{-2\alpha}\|\phi\|_{q,2}^2, \quad \hat{\mathbb{E}}(I_{23}^n)^2 \le Kn^{-2\alpha}\|\phi\|_{q,2}^2.$$

Combining (24) with Lemma 4.4, we get

Theorem 4.1. *Under (H4.1), there is a constant $K > 0$ such that*

$$\hat{\mathbb{E}}\left|\langle V_{k\delta}^n, \phi\rangle - \langle V_{k\delta}, \phi\rangle\right|^2 \le K\left(n^{-(1-\alpha)} \vee n^{-2\alpha}\right)\|\phi\|_{q,2}^2.$$

Now we extend Theorem 4.1 to a general time point t. To do so, we need to estimate the distance between V_t and $V_{k\delta}$, and the distance between V_t^n and $V_{k\delta}^n$.

In terms of the definitions of V_t, V_t^n and martingale properties of stochastic calculus, we have

Lemma 4.5. *For fixed t, there is a constant K such that*

$$\hat{\mathbb{E}}\left|\langle V_t, \phi\rangle - \left\langle V_{\delta(t)}, \phi\right\rangle\right|^2 \le Kn^{-2\alpha}\|\phi\|_{q,2}^2,$$

$$\hat{\mathbb{E}}\left|\langle V_t^n, \phi\rangle - \left\langle V_{\delta(t)}^n, \phi\right\rangle\right|^2 \le Kn^{-2\alpha}\|\phi\|_{q,2}^2.$$

According to Theorem 4.1 and Lemma 4.5, the next theorem is available.

Theorem 4.2. *For fixed t, we have*

$$\hat{\mathbb{E}}\left|\langle V_t^n, \phi\rangle - \langle V_t, \phi\rangle\right| \le K\left(n^{-(1-\alpha)} \vee n^{-2\alpha}\right)\|\phi\|_{q,2}^2.$$

Finally, we discuss the uniform convergence of V_t^n to V_t on any finite time interval $[0, T]$. We first consider the equation satisfied by V_t^n. For any $\delta(t) \le t < \delta(t) + \delta$, it follows from Itô's formula that

$$\begin{aligned}
df(x_t^i) = {} & \nabla^* f(x_t^i)\left(\sigma(x_{\delta(t)}^i)dB_t^i + \tilde{b}(x_{\delta(t)}^i)dt + c(x_{\delta(t)}^i)dY_t\right) \\
& + \frac{1}{2}\sum_{p,q=1}^d a_{pq}(x_{\delta(t)}^i)\partial_{pq}^2 f dt,
\end{aligned}$$

where

$$a = (a_{ij}) = cc^* + \sigma\sigma^*.$$

By Itô's formula, we have

$$\begin{aligned}
& d(M_{\delta(t)/\delta}^n(x^i, t)f(x_t^i)) \\
= & M_{\delta(t)/\delta}^n(x^i, t)h^*(x_{\delta(t)}^i)f(x_t^i)dY_t \\
& + M_{\delta(t)/\delta}^n(x^i, t)\nabla^* f(x_t^i)\left(\sigma(x_{\delta(t)}^i)dB_t^i + \tilde{b}(x_{\delta(t)}^i)dt + c(x_{\delta(t)}^i)dY_t\right) \\
& + M_{\delta(t)/\delta}^n(x^i, t)\frac{1}{2}\sum_{p,q=1}^d a_{pq}(x_{\delta(t)}^i)\partial_{pq}^2 f dt \\
& + M_{\delta(t)/\delta}^n(x^i, t)h^*(x_{\delta(t)}^i)\nabla^* f(x_t^i)c(x_{\delta(t)}^i)dt.
\end{aligned}$$

The jump of V_t^n at $t=(j+1)\delta$ is

$$
\begin{aligned}
&\eta_{j+1}^n \frac{1}{n}\sum_{i=1}^{m_j^n} \xi_{j+1}^i \delta_{x^i_{(j+1)\delta}} - \eta_j^n \frac{1}{n}\sum_{i=1}^{m_j^n} M_{j+1}^n(x^i)\delta_{x^i_{(j+1)\delta}}\\
=&\eta_{j+1}^n \frac{1}{n}\sum_{i=1}^{m_j^n} \left(\xi_{j+1}^i - \tilde{M}_{j+1}^n(x^i)\right)\delta_{x^i_{(j+1)\delta}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\langle V_t^n, f\rangle =& \langle V_0^n, f\rangle + \int_0^t \langle V_s^n, Lf\rangle\, ds + \int_0^t \langle V_s^n, \nabla^* fc + fh^*\rangle\, dY_s\\
&- \sum_{j=0}^{[t/\delta]} \frac{1}{n}\eta_j^n \sum_{i=1}^{m_j^n} \int_{j\delta}^{(j+1)\delta\wedge t} M_j^n(x^i,s)\frac{1}{2}\sum_{p,q=1}^d \left(a_{pq}(x_s^i) - a_{pq}(x^i_{\delta(s)})\right)\partial^2_{pq} f ds\\
&- \sum_{j=0}^{[t/\delta]} \frac{1}{n}\eta_j^n \sum_{i=1}^{m_j^n} \int_{j\delta}^{(j+1)\delta\wedge t} M_j^n(x^i,s)\sum_{p=1}^d \partial_p f(x_s^i)\left(b_p(x_s^i) - b_p(x^i_{\delta(s)})\right) ds\\
&- \sum_{j=0}^{[t/\delta]} \frac{1}{n}\eta_j^n \sum_{i=1}^{m_j^n} \int_{j\delta}^{(j+1)\delta\wedge t} M_j^n(x^i,s)\nabla^* f(x_s^i)\left(c(x_s^i) - c(x^i_{\delta(s)})\right) dY_s\\
&- \sum_{j=0}^{[t/\delta]} \frac{1}{n}\eta_j^n \sum_{i=1}^{m_j^n} \int_{j\delta}^{(j+1)\delta\wedge t} M_j^n(x^i,s) f(x_s^i)\left(h(x_s^i) - h(x^i_{\delta(s)})\right)^* dY_s\\
&+ \sum_{j=0}^{[t/\delta]} \frac{1}{n}\eta_j^n \sum_{i=1}^{m_j^n} \int_{j\delta}^{(j+1)\delta\wedge t} M_j^n(x^i,s)\nabla^* f(x_s^i)\sigma(x^i_{\delta(s)}) dB_s^i\\
&+ \sum_{j=1}^{[t/\delta]} \frac{1}{n}\eta_j^n \sum_{i=1}^{m_{j-1}^n} (\xi_j^i - \tilde{M}_j^n(x^i)) f(x^i_{j\delta}).
\end{aligned}
$$

By Burkholder-Davis-Gundy inequality, Propositions 3.1 and 3.2, Lemma 4.3 and Theorem 4.2, we derive

Theorem 4.3. *Under (H4.1), there is a constant $K>0$, such that*

$$
\hat{\mathbb{E}} \sup_{t\le T} d(V_t^n, V_t)^2 \le K\left(n^{-(1-\alpha)} \vee n^{-2\alpha}\right).
$$

4.4. Convergence of π_t^n. We study the convergence of π_t^n to π_t. For this, define a new filter $\tilde{\pi}_t^n$ by

$$
\tilde{\pi}_t^n = \frac{V_t^n}{\langle V_t^n, 1\rangle}.
$$

Note that $\tilde{\pi}_t^n$ coincides with π_t^n at $t = k\delta (k = 0, 1\cdots, [T/\delta])$.

Applying Cauchy-Schwarz inequality and Theorem 4.3, we get

Theorem 4.4. *Suppose that (H4.1) holds, then there is a constant $K>0$, such that*

$$
\hat{\mathbb{E}} \sup_{0\le t\le T} d(\tilde{\pi}_t^n, \pi_t) \le K\left(n^{-\frac{1-\alpha}{2}} \vee n^{-\alpha}\right).
$$

The next theorem is an immediate conclusion of Theorem 4.4.

Theorem 4.5. *Under (H4.1), there is a constant $K > 0$, such that*

$$\sup_{0 \leq k \leq [T/\delta]} d(\pi^n_{k\delta}, \pi_{k\delta}) \leq K \left(n^{-\frac{1-\alpha}{2}} \vee n^{-\alpha} \right).$$

Applying (23), Jensen's inequality, Burkholder-Davis-Gundy inequality and Theorem 4.5, it follows that

Theorem 4.6. *Suppose that (H4.1) holds. Then, for any $\gamma < \frac{1}{3}$, there is a constant $K > 0$ and α such that*

$$\mathbb{E} \sup_{0 \leq t \leq T} d(\pi^n_t, \pi_t) \leq K n^{-\gamma}.$$

5. Numerical illustrations

To better demonstrate our results in Section 4 that the branching particle indeed approximate the optimal filter, we present a numerical example in this section. Here we consider the model with the state

$$dx_t = b x_t dt + c dW_t + \sigma dB_t,$$

and the observation

$$Y_t = \int_0^t h x_s ds + W_t.$$

We take $b = 1$, $c = 2$, $h = 1$, $\sigma = 1$, $\alpha = 1/3$, T=10.

The following figure represents the distance $\sup_{0 \leq t \leq T} d(\pi^n_t, \pi_t)$ between the particle filter π^n and the optimal filter π, where n is the number of the particles.

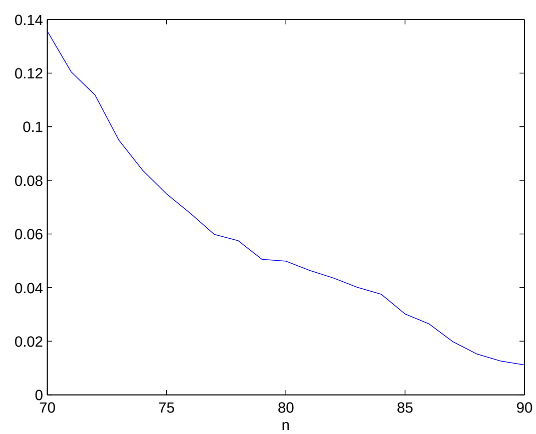

Figure 1. Optimal filter and particle approximation.

Clearly, π^n_t approximate π_t nicely when the number of the particles becomes large.

The next two figures compare the state process x_t with the optimal filter and the particle filter when the testing function f is taken as $f(x) = x$ and the time variable is $t \in [0, 2]$. The Figure 2 is for the smoother case of $\sigma = 1$ and the Figure

3 is for the rougher case of $\sigma = 2$. They show that the particle filter approximates the state process quite nicely (comparable to the optimal filter).

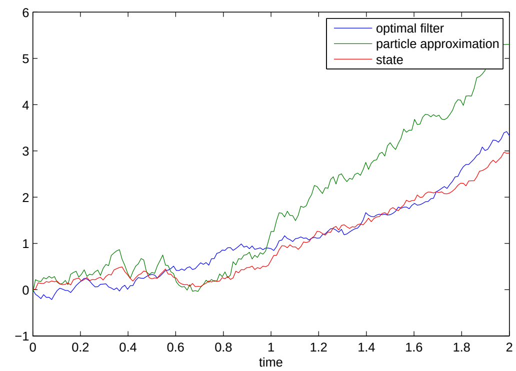

FIGURE 2. Let $n = 88$, $T = 2$, $b = 1$, $c = 2$, $h = 1$, $\sigma = 1$, $\alpha = 1/3$. Comparison among the optimal filter $\langle x, \pi_t \rangle$, the particle filter $\langle x, \pi_t^n \rangle$ and the state $x(t)$.

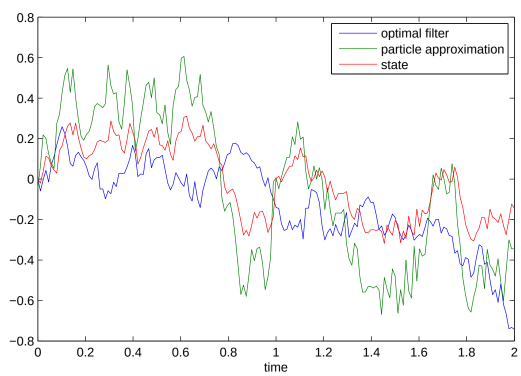

FIGURE 3. Let $n = 88$, $T = 2$, $b = 1$, $c = 2$, $h = 1$, $\sigma = 2$, $\alpha = 1/3$. Comparison among the optimal filter $\langle x, \pi_t \rangle$, the particle filter $\langle x, \pi_t^n \rangle$ and the state process $x(t)$.

6. Conclusion

Many economics, finance and insurance problems can be formulated as partially observable stochastic optimal control models. These models are essentially optimal control problems with infinite dimensional state spaces, which are always hard to solve. Then it is highly desirable to study various numerical methods to approximate these filtering-control problems by problems with finite dimensional states. The branching particle system approximation is an efficient method.

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