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# NEW DEVELOPMENTS ON THE COUPLING OF MIXED-FEM AND BEM FOR THE THREE-DIMENSIONAL EXTERIOR STOKES PROBLEM

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**Abstract.** In this manuscript we consider the three dimensional exterior Stokes problem and study the solvability of the corresponding continuous and discrete formulations that arise from the coupling of a dual-mixed variational formulation (in which the velocity, the pressure and the stress are the original main unknowns) with the boundary integral equation method. The present work is an extended and completed version of the analysis and results provided in our previous paper [ZAMM Z. Angew. Math. Mech. 93 (2013), no. 6-7, 437–445]. More precisely, after employing the incompressibility condition to eliminate the pressure, we consider the resulting velocity-stress-vorticity approach with different kind of boundary conditions on an annular bounded domain, and couple the underlying equations with either one or two boundary integral equations arising from the application of the usual and normal traces to the Green representation formula in the exterior unbounded region. As a result, we obtain saddle point operator equations, which are then analyzed by the well-known Babuška-Brezzi theory. We prove the well-posedness of the continuous formulations, identifying previously the space of solutions of the associated homogeneous problem, and specify explicit hypotheses to be satisfied by the finite element and boundary element subspaces in order to guarantee the stability of the respective Galerkin schemes. In particular, following a similar analysis given recently for the Laplacian, we are able to extend the classical Johnson & Nédélec procedure to the present case, without assuming any restrictive smoothness requirement on the coupling boundary, but only Lipschitz-continuity. In addition, and differently from known approaches for the elasticity problem, we are also able to extend the Costabel & Han coupling procedure to the 3D Stokes problem by providing a direct proof of the required coerciveness property, that is without argueing by contradiction, and by using the natural norm of each space instead of mesh-dependent norms. Finally, we briefly describe concrete examples of discrete spaces satisfying the aforementioned hypotheses.

**Key words.** Mixed-FEM, BEM, 3D Stokes problem, Johnson & Nédélec's coupling, Costabel & Han's coupling.

## 1. Introduction

The classical approach combining finite element (FEM) with boundary element methods (BEM) for solving exterior boundary value problems in continuum mechanics, usually known as the coupling of FEM and BEM, has been extensively employed since its creation during the second half of the seventies up to nowadays. The usual procedure is as follows. The underlying domain is first divided into two subregions by introducing an auxiliary boundary $\Gamma$, if necessary, so that the original exterior problem can be reformulated as a transmission problem through $\Gamma$. Next, the latter is reduced to an equivalent problem in the bounded inner region by imposing nonlocal boundary conditions on $\Gamma$ that are derived by employing boundary integral equation methods in the unbounded outer domain. The resulting nonlocal

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boundary value problem is then solved by a conventional Galerkin method, in which the boundary integral operators involved are discretized using finite element spaces on $\Gamma$.

While detailed surveys on most of the different ways of coupling BEM and FEM can be seen in [43] and [27, Chapter I], we simply recall here that the most popular ones correspond to the *Johnson & Nédélec* (J & N) and *Costabel & Han* (C & H) procedures (cf. [12], [13], [20], [38], [45], and [58]), which employ the Green representation of the solution in the unbounded region. The success of the J & N method, being based on a single boundary integral equation on $\Gamma$ and the Fredholm theory, hinged on the fact that certain boundary integral operators are compact, which usually requires $\Gamma$ to be smooth enough. According to it, it was not possible, at least from a theoretical point of view, to employ this approach when the coupling boundary was non–smooth, say for instance polygonal, which left out the possibility of utilizing classical finite element discretizations. Moreover, the J & N idea seemed to be applicable only to the Laplace operator since for other elliptic systems, such as the elasticity one, and irrespective of the smoothness of the boundaries, the aforementioned compactness did not hold. One attempt to overcome this was suggested in [9] where the underlying transmission problem was replaced by one employing the pseudostress instead of the usual stress. As a consequence, the foregoing mapping property was achieved, but the coupling boundary was still required to be smooth enough. One has to admit, however, that the above described drawbacks were mainly theoretical since no failure of the corresponding discrete schemes was ever reported by users of the method in problems where those hypotheses were not met. Any way, in order to circumvent these apparent difficulties, suitable modifications of the original J & N method, in which neither the compactness nor the smoothness play any role, were proposed by *Costabel* and *Han* in [20] and [38], respectively. Both techniques are based on the addition of a boundary integral equation for the normal derivative (resp. traction in the case of elasticity). The former leads to a symmetric and non-positive definite scheme, while the latter, on the contrary, yields a positive definite and non-symmetric scheme. Nevertheless, and since the only difference between these formulations lies on the sign of an integral identity, from now on we simply refer to either one of them as the C & H approach. Further and later contributions in this direction, including applications to nonlinear problems and coupling with mixed-FEM, non–conforming FEM, local discontinuous Galerkin, and hybridizable discontinuous Galerkin methods, can be found in [8], [14], [15], [16], [17], [18], [19], [21], [24], [25], [26], [34], [35], [36], [49], and the references therein.

The whole picture on the coupling of FEM and BEM, and particularly the widely accepted fact since the eighties concerning the lack of further applicability and usefulness of the J & N method, changed dramatically with [52]. More precisely, it was proved in this paper, without any need of applying Fredholm theory nor assuming smooth domains, that all Galerkin methods for this approach are actually stable, thus allowing the coupling boundary $\Gamma$ to be polygonal/polyhedral. As a consequence, the classical J & N method was begun to be considered as a real competitor of the C & H approach. In other words, the appearing of [52] gave rise to several new contributions within this and related topics. Indeed, we first refer to [48] where the corresponding extension to the combination of mixed-FEM and BEM on any Lipschitz-continuous interface $\Gamma$ was successfully developed. Furthermore, the analysis of the quasi–symmetric procedure from [9] was improved in [30] by showing that the interface $\Gamma$ can also be taken polygonal/polyhedral, and that in

the case of the elasticity problem, the coupling can be performed by employing the usual stress instead of the pseudostress. In addition, a new and extremely simplified proof of the main result in [52], by showing directly ellipticity of the operator equation, was provided in [30]. An alternative proof of this ellipticity result has been recently given in [56], using a particular expression of the Steklov–Poincaré operator, which is based on a Schur complement of a perturbation of the Calderón projector. Some comments on the consequences of the new theory of non-symmetric coupling of BEM and FEM can be found in the republishing of [52] as [53]. Nevertheless, the utilization of mixed-FEM instead of the usual FEM, and the application of the J & N coupling procedure to the 3D Stokes and similar elliptic systems such as Lamé, is still missing. Moreover, most of the related works available in the literature involve either 2D problems or just the coupling of BEM and the usual FEM (see, e.g. [37], [50], [51], and [55]). In addition, the analysis of the C & H approach for the coupling of mixed-FEM and BEM has not yielded too satisfactory results when it has been applied to the elasticity problem (see, e.g. [14]). Other contributions dealing with the coupling of BEM and FEM for exterior boundary value problems in fluid mechanics are available in [39], [40], [41], and [42].

According to the above bibliographic discussion, and specially motivated by the recent results from [52], [48], and [30], we now aim to analyze the coupling of mixed-FEM and BEM, as applied to the 3D exterior Stokes problem, by utilizing both the J & N and the C & H approaches. More precisely, we extend the first method to the present case, without assuming any smoothness requirement on the interface, but only Lipschitz-continuity. Furthermore, and differently from the analysis in [14] for the elasticity problem, we are also able to extend the second coupling procedure to the 3D Stokes problem by providing a direct proof of the required coerciveness property, that is without argueing by contradiction, and by using the natural norm of each space instead of mesh-dependent norms. Our results here can be easily extended to the 2D and 3D Lamé systems. We also remark that a preliminary version of a small part of the contributions to be described in the present work were already published in [29] as part of a special issue honoring Professor Wolfgang L. Wendland. In fact, whereas just a homogeneous Neumann boundary condition and the continuous analysis of the J & N approach were considered in [29], we now address not only the continuous analysis of the C & H method as well, but also several other kind of boundary conditions, and the discrete analysis of each one of the resulting coupled formulations.

The rest of this paper is organized as follows. In Section 2 we introduce the exterior boundary value problem of interest by describing it as the transmission problem between the non-homogeneous Stokes equation (holding in a bounded annular domain $\Omega^-$) and the homogeneous Stokes equation (holding in an unbounded exterior region $\Omega^+$). The incompressibility condition is employed here to eliminate the pressure so that the stress tensor and the velocity vector become the main unknowns of the resulting transformed problem. The dual-mixed formulations in $\Omega^-$ for different boundary conditions on the interior boundary of this region are derived in Section 3. Then, in Section 4 we recall the main aspects and properties of the boundary integral equation approach as applied to the homogeneous Stokes equation in $\Omega^+$. Next, in Section 5 we derive and analyze the coupled variational formulations that arise from the combination of the dual-mixed approach in $\Omega^-$ with the boundary integral equation method in $\Omega^+$. We first identify the solutions of the associated homogeneous problems and then establish the well-posedness of the continuous formulations. In particular, the classical Johnson & Nedelec procedure, which employs

a single boundary integral equation and yields a non-symmetric scheme, is extended to the present case by requiring only a Lipschitz-continuous coupling boundary. In addition, the Costabel & Han approach, which makes use of two bounday integral equations and leads to a symmetric formulation, is also successfully analyzed with the natural norms of the spaces involved and through direct proofs of the required continuous and discrete coerciveness properties. Finally, in Section 6 we consider the Galerkin schemes arising from the coupled formulations studied in Section 5, and provide explicit hypotheses to be satisfied by the respective discrete spaces in order to guarantee their corresponding solvability and stability. Moreover, concrete examples of finite element and boundary element subspaces verifying those conditions are also identified here.

We end this section with some notations to be used below. Given any Hilbert space $U$, we denote by $U^3$ and $U^{3\times 3}$, respectively, the space of vectors and square matrices of order 3 with entries in $U$. In particular, the identity matrix of $\mathrm{R}^{3\times 3}$ is $\mathbf{I}$, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathrm{R}^{3\times 3}$, we write as usual

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji}), \quad \mathrm{tr}\,\boldsymbol{\tau} := \sum_{i=1}^{3} \tau_{ii}, \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{3}\,\mathrm{tr}\,(\boldsymbol{\tau})\,\mathbf{I}, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{3} \tau_{ij}\,\zeta_{ij}.$$

Also, in what follows we utilize the standard terminology for Sobolev spaces and norms. However, given a domain $\mathcal{O}$, a closed manifold $\Gamma$, and $r \in \mathrm{R}$, we simplify notations and define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^3, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{3\times 3}, \quad \text{and} \quad \mathbf{H}^r(\Gamma) := [H^r(\Gamma)]^3.$$

In the special case $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\Gamma)$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\Gamma)$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\Gamma}$ (for $H^r(\Gamma)$ and $\mathbf{H}^r(\Gamma)$). In addition, denoting by $\mathbf{div}$ the usual divergence operator div acting on the rows of a tensor, we define the Hilbert space

$$\mathbb{H}(\mathbf{div}\,;\mathcal{O}) := \left\{\boldsymbol{\tau} \in \mathbb{L}^2(\mathcal{O}) : \quad \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\mathcal{O})\right\},$$

and the subspace

$$\widetilde{\mathbb{H}}(\mathbf{div}\,;\mathcal{O}) := \left\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\mathcal{O}) : \quad \int_{\mathcal{O}} \mathrm{tr}\,\boldsymbol{\tau} = 0\right\} \tag{1}$$

which are both endowed with the norm

$$\|\boldsymbol{\tau}\|_{\mathbf{div}\,;\mathcal{O}} := \left\{\|\boldsymbol{\tau}\|_{0,\mathcal{O}}^2 + \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\mathcal{O}}^2\right\}^{1/2} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\mathcal{O}).$$

Note that there holds the decomposition:

$$\mathbb{H}(\mathbf{div}\,;\mathcal{O}) = \widetilde{\mathbb{H}}(\mathbf{div}\,;\mathcal{O}) \oplus P_0(\mathcal{O})\,\mathbf{I}, \tag{2}$$

where $P_0(\mathcal{O})$ is the space of constant polynomials on $\mathcal{O}$.

Finally, throughout the paper we employ $\mathbf{0}$ to denote a generic null vector, and use $C$ and $c$, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

## 2. The boundary value problem

We now describe the boundary value problem of interest by following basically the same notations and grammar style from [29]. Let $\Omega_0$ be a bounded Lipschitz-continuous domain in $\mathrm{R}^3$ with boundary $\Gamma_0$, let $\Omega^-$ be the annular region bounded by $\Gamma_0$ and another Lipschitz-continuous surface $\Gamma$ whose interior contains $\bar{\Omega}_0$, and let $\Omega^+ := \mathrm{R}^3 \backslash \big(\bar{\Omega}_0 \cup \bar{\Omega}^-\big)$ (see Figure 2.1 below). We consider a steady incompressible flow in the region $\mathrm{R}^3 \backslash \Omega_0$, under the action of external forces on $\bar{\Omega}^-$, and are interested in determining the velocity, the pressure, and the stress of the corresponding fluid. More precisely, given $\mathbf{f} \in \mathbf{L}^2(\Omega^-)$, we seek a vector field $\mathbf{u}$, a scalar field $p$, and a tensor field $\boldsymbol{\sigma}$ such that:

$$
\begin{gathered}
\boldsymbol{\sigma} = \Xi[\mathbf{u}, p] := 2\,\mu\,\mathbf{e}(\mathbf{u}) - p\,\mathbf{I} \quad \text{and} \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \Omega^- \cup \Omega^+ , \\
\mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f} \quad \text{in} \quad \Omega^- , \quad \mathbf{div}\,\boldsymbol{\sigma} = \mathbf{0} \quad \text{in} \quad \Omega^+ , \quad \text{BC} \quad \text{on} \quad \Gamma_0 , \\
[\mathbf{u}] := \mathbf{u}^- - \mathbf{u}^+ = \mathbf{0} \quad \text{and} \quad [\boldsymbol{\sigma}\boldsymbol{\nu}] := (\boldsymbol{\sigma}\boldsymbol{\nu})^- - (\boldsymbol{\sigma}\boldsymbol{\nu})^+ = \mathbf{0} \quad \text{on} \quad \Gamma , \\
\mathbf{u}(\mathbf{x}) = O(\|\mathbf{x}\|^{-1}) \quad \text{and} \quad p(\mathbf{x}) = O(\|\mathbf{x}\|^{-2}) \quad \text{as} \quad \|\mathbf{x}\| \to +\infty ,
\end{gathered}
\tag{3}
$$

where BC stands for a suitable boundary condition on $\Gamma_0$, which will be specified later on. Hereafter, $\Xi$ is the stress operator acting on the velocity/pressure pair, $\mu$ is the kinematic viscosity of the fluid, $\mathbf{e}(\mathbf{u}) := \frac{1}{2}\big(\nabla \mathbf{u} + (\nabla \mathbf{u})^t\big)$ is the strain tensor (or symmetric part of the velocity gradient), $\boldsymbol{\nu}$ is the unit normal on $\Gamma_0$ and $\Gamma$ pointing inside $\Omega^-$ and $\Omega^+$, respectively,

$$
\mathbf{u}^{\pm}(\mathbf{x}) := \lim_{\substack{\tilde{\mathbf{x}} \to \mathbf{x} \\ \tilde{\mathbf{x}} \in \Omega^{\pm}}} \mathbf{u}(\tilde{\mathbf{x}}) \qquad \forall\, \mathbf{x} \in \Gamma ,
$$

and

$$
(\boldsymbol{\sigma}\boldsymbol{\nu})^{\pm}(\mathbf{x}) := \lim_{\substack{\tilde{\mathbf{x}} \to \mathbf{x} \\ \tilde{\mathbf{x}} \in \Omega^{\pm}}} \boldsymbol{\sigma}(\tilde{\mathbf{x}})\,\boldsymbol{\nu}(\mathbf{x}) \qquad \forall\, \mathbf{x} \in \Gamma .
$$

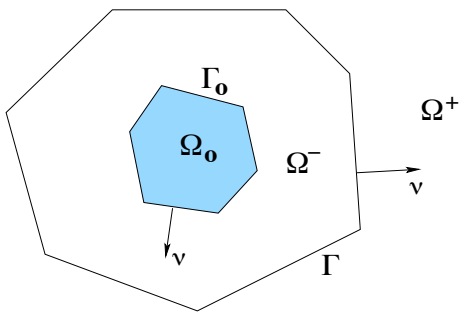


FIGURE 1. Geometry of the problem.

Note that, thanks to the incompressibility condition given by $\operatorname{div}\,\mathbf{u} = 0 \quad \text{in} \quad \Omega^- \cup \Omega^+$, there holds $\mathbf{div}\,\boldsymbol{\sigma} = \mu\,\Delta\,\mathbf{u} - \nabla\,p \quad \text{in} \quad \Omega^- \cup \Omega^+$, which means that the second row of (3) becomes the nonhomogeneous and homogeneous Stokes equations in $\Omega^-$ and $\Omega^+$, respectively. However, since we are going to apply a mixed variational formulation in $\Omega^-$ and the associated boundary integral equation approach in $\Omega^+$, we need to keep $\boldsymbol{\sigma}$ as an independent unknown.

Throughout the rest of the paper, and without loss of generality, we assume that $\mu = 1/2$. Otherwise, we just redefine $p$ as $p/2\mu$ and let $\boldsymbol{\sigma} = \Xi[\mathbf{u}, p] := \mathbf{e}(\mathbf{u}) - p\,\mathbf{I}$ in $\Omega^- \cup \Omega^+$, which yields the datum $\mathbf{f}$ to be replaced by $\mathbf{f}/2\mu$. In addition, it is easy to see, using that $\operatorname{tr}\mathbf{e}(\mathbf{u}) = \operatorname{div}\mathbf{u}$, that the pair of equations

$$\boldsymbol{\sigma} = \Xi[\mathbf{u}, p] := \mathbf{e}(\mathbf{u}) - p\,\mathbf{I} \quad \text{and} \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega^- \cup \Omega^+ ,$$

is equivalent to

$$\boldsymbol{\sigma} = \Xi[\mathbf{u}, p] := \mathbf{e}(\mathbf{u}) - p\,\mathbf{I} \quad \text{and} \quad p + \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} = 0 \quad \text{in} \quad \Omega^- \cup \Omega^+ , \tag{4}$$

which can be rewritten as

$$\boldsymbol{\sigma}^{\mathtt{d}} = \mathbf{e}(\mathbf{u}) \quad \text{and} \quad p + \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} = 0 \quad \text{in} \quad \Omega^- \cup \Omega^+ . \tag{5}$$

Consequently, from now on we replace our transmission problem (3) by the following:

$$\begin{gathered}
\boldsymbol{\sigma}^{\mathtt{d}} = \mathbf{e}(\mathbf{u}) \quad \text{and} \quad p + \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} = 0 \quad \text{in} \quad \Omega^- \cup \Omega^+ , \\
\mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f} \quad \text{in} \quad \Omega^- , \quad \mathbf{div}\,\boldsymbol{\sigma} = \mathbf{0} \quad \text{in} \quad \Omega^+ , \quad \text{BC} \quad \text{on} \quad \Gamma_0 , \\
[\mathbf{u}] := \mathbf{u}^- - \mathbf{u}^+ = \mathbf{0} \quad \text{and} \quad [\boldsymbol{\sigma}\boldsymbol{\nu}] := (\boldsymbol{\sigma}\boldsymbol{\nu})^- - (\boldsymbol{\sigma}\boldsymbol{\nu})^+ = \mathbf{0} \quad \text{on} \quad \Gamma , \\
\mathbf{u}(\mathbf{x}) = O(\|\mathbf{x}\|^{-1}) \quad \text{and} \quad p(\mathbf{x}) = O(\|\mathbf{x}\|^{-2}) \quad \text{as} \quad \|\mathbf{x}\| \to +\infty .
\end{gathered} \tag{6}$$

Our aim throughout the following sections is to introduce and analyze several weak formulations of (6), employing either the Johnson & Nédélec (see [45]) or the Costabel & Han (see [20], [38]) coupling procedures, and taking into account the specific boundary condition on $\Gamma_0$. Since the pressure can be computed in terms of the stress, we focus mainly on the approaches that do not include $p$ as an explicit unknown but only as part of $\boldsymbol{\sigma}$. In what follows we let $\gamma^- : \mathbf{H}^1(\Omega^-) \to \mathbf{H}^{1/2}(\partial\Omega^-)$ and $\gamma_{\boldsymbol{\nu}}^- : \mathbb{H}(\mathbf{div}\,;\Omega^-) \to \mathbf{H}^{-1/2}(\partial\Omega^-)$ be the usual trace and normal trace operators, respectively, on $\partial\Omega^- := \Gamma_0 \cup \Gamma$. Similarly, given a fixed Lipschitz-continuous surface $\Gamma^+$ whose interior region contains $\bar{\Omega}_0 \cup \bar{\Omega}^-$, we let $\Omega^{++}$ be the annular domain bounded by $\Gamma$ and $\Gamma^+$, and let $\gamma^+ : \mathbf{H}^1(\Omega^{++}) \to \mathbf{H}^{1/2}(\Gamma)$ and $\gamma_{\boldsymbol{\nu}}^+ : \mathbb{H}(\mathbf{div}\,;\Omega^{++}) \to \mathbf{H}^{-1/2}(\Gamma)$ be the usual trace and normal trace operators, respectively, on $\Gamma$. In this way, the transmission conditions on $\Gamma$ can be rewritten in (6) as:

$$\gamma^-(\mathbf{u}) = \gamma^+(\mathbf{u}) \quad \text{and} \quad \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\sigma}) = \gamma_{\boldsymbol{\nu}}^+(\boldsymbol{\sigma}) \quad \text{on} \quad \Gamma . \tag{7}$$

## 3. The dual-mixed formulations in $\Omega^-$

We first proceed similarly as for the linear elasticity problem (see, e.g. [2], [22], [57]) and introduce in the bounded domain $\Omega^-$ the vorticity

$$\boldsymbol{\chi} := \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}\right) \in \mathbb{L}^2_{\mathtt{skew}}(\Omega^-) \tag{8}$$

as an auxiliary unknown, where

$$\mathbb{L}^2_{\mathtt{skew}}(\Omega^-) := \left\{ \boldsymbol{\eta} \in \mathbb{L}^2(\Omega^-) : \quad \boldsymbol{\eta}^{\mathtt{t}} = -\boldsymbol{\eta} \right\} .$$

In this way, the constitutive equation relating $\mathbf{u}$ and $\boldsymbol{\sigma}$ in $\Omega^-$ becomes

$$\boldsymbol{\sigma}^{\mathtt{d}} = \nabla\mathbf{u} - \boldsymbol{\chi} \quad \text{in} \quad \Omega^- ,$$

which, multiplying (tensor product :) by $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\Omega^-)$ and integrating by parts, yields

(9)
$$\int_{\Omega^-} \boldsymbol{\sigma}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} - \langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}), \boldsymbol{\varphi}\rangle_\Gamma + \int_{\Omega^-} \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_{\Omega^-} \boldsymbol{\chi} : \boldsymbol{\tau} + \langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}), \gamma^-(\mathbf{u})\rangle_{\Gamma_0} = 0\,,$$

where

$$\boldsymbol{\varphi} := \gamma^-(\mathbf{u}) = \gamma^+(\mathbf{u}) \in \mathbf{H}^{1/2}(\Gamma) \tag{10}$$

is an additional unknown, and, given $S \in \{\Gamma, \Gamma_0\}$, $\langle\cdot,\cdot\rangle_S$ denotes the duality pairing between $\mathbf{H}^{-1/2}(S)$ and $\mathbf{H}^{1/2}(S)$ with respect to the $\mathbf{L}^2(S)$-inner product. On the other hand, incorporating the equilibrium equation in $\Omega^-$ and the symmetry of the stress tensor $\boldsymbol{\sigma}$ in a weak sense, we arrive at

$$\int_{\Omega^-} \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} + \int_{\Omega^-} \boldsymbol{\eta} : \boldsymbol{\sigma} = -\int_{\Omega^-} \mathbf{f}\cdot\mathbf{v} \qquad \forall\,(\mathbf{v},\boldsymbol{\eta}) \in \mathbf{L}^2(\Omega^-)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega^-)\,. \tag{11}$$

**3.1. Dirichlet boundary condition on $\Gamma_0$.** We assume here that BC on $\Gamma_0$ is given by the natural boundary condition

$$\gamma^-(\mathbf{u}) = \mathbf{g}_D \in \mathbf{H}^{1/2}(\Gamma_0)\,, \tag{12}$$

whence (9) becomes

$$\int_{\Omega^-} \boldsymbol{\sigma}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} - \langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}), \boldsymbol{\varphi}\rangle_\Gamma + \int_{\Omega^-} \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_{\Omega^-} \boldsymbol{\chi} : \boldsymbol{\tau} = -\langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}), \mathbf{g}_D\rangle_{\Gamma_0} \tag{13}$$

for each $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\Omega^-)$.

Then, we introduce the spaces

$$\mathbf{X}_D := \mathbb{H}(\mathbf{div}\,;\Omega^-)\times\mathbf{H}^{1/2}(\Gamma)\,, \quad \mathbf{Y}_D := \mathbf{L}^2(\Omega^-)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega^-)\,, \tag{14}$$

endowed with the product norms, and let $\mathbf{a}_D : \mathbf{X}_D\times\mathbf{X}_D \to \mathrm{R}$ and $\mathbf{b}_D : \mathbf{X}_D \times \mathbf{Y}_D \to \mathrm{R}$ be the bilinear forms given by

(15)
$$\mathbf{a}_D((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\tau},\boldsymbol{\psi})) := \int_{\Omega^-} \boldsymbol{\sigma}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} - \langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}), \boldsymbol{\varphi}\rangle_\Gamma \quad \forall\,((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\tau},\boldsymbol{\psi})) \in \mathbf{X}_D\times\mathbf{X}_D\,,$$

and

(16)
$$\mathbf{b}_D((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{v},\boldsymbol{\eta})) := \int_{\Omega^-} \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_{\Omega^-} \boldsymbol{\eta} : \boldsymbol{\tau} \qquad \forall\,((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{v},\boldsymbol{\eta})) \in \mathbf{X}_D\times\mathbf{Y}_D\,.$$

Also, we let $\mathbf{F}_D : \mathbf{X}_D \to \mathrm{R}$ and $\mathbf{G}_D : \mathbf{Y}'_D \to \mathrm{R}$ be the linear functionals given by the right hand side of (13) and (11), respectively, that is

$$\mathbf{F}_D(\boldsymbol{\tau},\boldsymbol{\psi}) := -\langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}), \mathbf{g}_D\rangle_{\Gamma_0} \qquad \text{and} \qquad \mathbf{G}_D(\mathbf{v},\boldsymbol{\eta}) := -\int_{\Omega^-} \mathbf{f}\cdot\mathbf{v}. \tag{17}$$

Then, collecting (13) and (11), we find that the dual-mixed formulation in $\Omega^-$ can be stated as: Find $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})) \in \mathbf{X}_D\times\mathbf{Y}_D$ such that

(18)
$$\begin{aligned} \mathbf{a}_D((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\tau},\boldsymbol{\psi})) + \mathbf{b}_D((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{u},\boldsymbol{\chi})) &= \mathbf{F}_D(\boldsymbol{\tau},\boldsymbol{\psi}) \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{X}_D\,,\\ \mathbf{b}_D((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{v},\boldsymbol{\eta})) &= \mathbf{G}_D(\mathbf{v},\boldsymbol{\eta}) \qquad \forall\,(\mathbf{v},\boldsymbol{\eta}) \in \mathbf{Y}_D\,. \end{aligned}$$

Note, however, that the above is clearly an incomplete variational formulation since it actually concerns four unknowns satisfying only three independent equations. In other words, though the bilinear forms $\mathbf{a}_D$ and $\mathbf{b}_D$ are originally defined in $\mathbf{X}_D\times\mathbf{X}_D$ (cf. (15)) and $\mathbf{X}_D\times\mathbf{Y}_D$ (cf. (16)), respectively, the first equation in (18) does not

really involve the test function $\boldsymbol{\psi}$. In Sections 4 and 5 below we complete this formulation through the application of the boundary integral equation method in the unbounded exterior domain $\Omega^+$.

**3.2. Non-homogeneous Neumann boundary condition on $\Gamma_0$.** We assume now that BC on $\Gamma_0$ is given by the essential boundary condition

$$\gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\sigma}) \;=\; \mathbf{g}_N \;\in\; \mathbf{H}^{-1/2}(\Gamma_0)\,, \tag{19}$$

which is imposed weakly as

$$\langle \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\sigma}), \boldsymbol{\xi}\rangle_{\Gamma_0} \;=\; \langle \mathbf{g}_N, \boldsymbol{\xi}\rangle_{\Gamma_0} \qquad \forall\, \boldsymbol{\xi} \;\in\; \mathbf{H}^{1/2}(\Gamma_0)\,. \tag{20}$$

Then, introducing the further unknown

$$\boldsymbol{\lambda} \;:=\; \gamma^-(\mathbf{u}) \;\in\; \mathbf{H}^{1/2}(\Gamma_0)\,, \tag{21}$$

we find that (9) becomes

$$\int_{\Omega^-} \boldsymbol{\sigma}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} - \langle \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}), \boldsymbol{\varphi}\rangle_{\Gamma} + \int_{\Omega^-} \mathbf{u}\cdot \mathbf{div}\, \boldsymbol{\tau} + \int_{\Omega^-} \boldsymbol{\chi} : \boldsymbol{\tau} + \langle \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}), \boldsymbol{\lambda}\rangle_{\Gamma_0} \;=\; 0 \tag{22}$$

for each $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\Omega^-)$, whence $\boldsymbol{\lambda}$ constitutes the Lagrange multiplier associated with (20).

Next, we let $\mathbf{X}_N = \mathbf{X}_D$ (cf. (14)), $\mathbf{a}_N = \mathbf{a}_D$ (cf. (15)), define the space

$$\mathbf{Y}_N \;:=\; \mathbf{L}^2(\Omega^-) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega^-) \times \mathbf{H}^{1/2}(\Gamma_0)\,, \tag{23}$$

endowed with the product norm, and introduce the bilinear form $\mathbf{b}_N \,:\, \mathbf{X}_N \times \mathbf{Y}_N \to \mathrm{R}$ given by

$$\mathbf{b}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi})) \;:=\; \int_{\Omega^-} \mathbf{v}\cdot \mathbf{div}\, \boldsymbol{\tau} \;+\; \int_{\Omega^-} \boldsymbol{\eta} : \boldsymbol{\tau} \;+\; \langle \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}), \boldsymbol{\xi}\rangle_{\Gamma_0} \tag{24}$$

for each $((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi})) \in \mathbf{X}_N \times \mathbf{Y}_N$. Also, we let $\mathbf{F}_N : \mathbf{X}_N \to \mathrm{R}$ and $\mathbf{G}_N : \mathbf{Y}_N \to \mathrm{R}$ be the linear functionals given by

$$\mathbf{F}_N(\boldsymbol{\tau},\boldsymbol{\psi}) \;:=\; 0 \qquad \text{and} \qquad \mathbf{G}_N(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi}) \;:=\; -\int_{\Omega^-} \mathbf{f}\cdot\mathbf{v} \;+\; \langle \mathbf{g}_N, \boldsymbol{\xi}\rangle_{\Gamma_0}. \tag{25}$$

Then, collecting (22), (20), and (11), we find that the dual-mixed formulation in $\Omega^-$ can be stated as: Find $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi},\boldsymbol{\lambda})) \in \mathbf{X}_N \times \mathbf{Y}_N$ such that

$$\begin{aligned} \mathbf{a}_N((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\tau},\boldsymbol{\psi})) \;+\; \mathbf{b}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{u},\boldsymbol{\chi},\boldsymbol{\lambda})) \;&=\; \mathbf{F}_N(\boldsymbol{\tau},\boldsymbol{\psi}) \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{X}_N\,, \\ \mathbf{b}_N((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi})) \;&=\; \mathbf{G}_N(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi}) \quad \forall\,(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi}) \in \mathbf{Y}_N\,. \end{aligned} \tag{26}$$

**3.3. Homogeneous Neumann boundary condition on $\Gamma_0$.** When the Neumann boundary condition (19) is homogeneous, that is if $\mathbf{g}_N = \mathbf{0}$, then there is no need of including the additional unknown $\boldsymbol{\lambda}$ (cf. (21)). In fact, we just introduce the space

$$\mathbb{H}_0(\mathbf{div}\,;\Omega^-) \;:=\; \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\Omega^-) : \quad \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}) \;=\; \mathbf{0} \quad \text{on} \quad \Gamma_0 \Big\}, \tag{27}$$

and redefine $\mathbf{X}_N$, $\mathbf{Y}_N$, and $\mathbf{b}_N$, respectively, as

$$\mathbf{X}_N \;:=\; \mathbb{H}_0(\mathbf{div}\,;\Omega^-) \times \mathbf{H}^{1/2}(\Gamma)\,, \quad \mathbf{Y}_N \;:=\; \mathbf{L}^2(\Omega^-) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega^-)\,, \tag{28}$$

and

$$\mathbf{b}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{v},\boldsymbol{\eta})) \;:=\; \int_{\Omega^-} \mathbf{v}\cdot\mathbf{div}\, \boldsymbol{\tau} + \int_{\Omega^-} \boldsymbol{\eta} : \boldsymbol{\tau} \qquad \forall\,((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{v},\boldsymbol{\eta})) \;\in\; \mathbf{X}_N \times \mathbf{Y}_N\,. \tag{29}$$

Then, letting $\mathbf{F}_N : \mathbf{X}_N \to \mathrm{R}$ and $\mathbf{G}_N : \mathbf{Y}_N \to \mathrm{R}$ be the linear functionals given by

$$\mathbf{F}_N(\boldsymbol{\tau},\boldsymbol{\psi}) := 0 \qquad \text{and} \qquad \mathbf{G}_N(\mathbf{v},\boldsymbol{\eta}) := -\int_{\Omega^-} \mathbf{f}\cdot\mathbf{v}, \tag{30}$$

we find that the dual-mixed formulation in $\Omega^-$ becomes: Find $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})) \in \mathbf{X}_N \times \mathbf{Y}_N$ such that

(31)
$$\begin{aligned} \mathbf{a}_N((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\tau},\boldsymbol{\psi})) + \mathbf{b}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{u},\boldsymbol{\chi})) &= \mathbf{F}_N(\boldsymbol{\tau},\boldsymbol{\psi}) \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{X}_N\,, \\ \mathbf{b}_N((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{v},\boldsymbol{\eta})) &= \mathbf{G}_N(\mathbf{v},\boldsymbol{\eta}) \qquad \forall\,(\mathbf{v},\boldsymbol{\eta}) \in \mathbf{Y}_N\,. \end{aligned}$$

An analogue remark to the one given at the end of Section 3.1 is valid here. In fact, it is clear that (26) and (31) constitute incomplete variational formulations since they concern five and four unknowns satisfying only four and three independent equations, respectively. Hence, similarly as we did for (18), we now announce that (26) and (31) will also be completed in Sections 4 and 5 by applying the boundary integral equation method in the unbounded exterior domain $\Omega^+$.

## 4. The boundary integral equation approach in $\Omega^+$

The contents of this section, being mainly a compilation of classical known results, are presented following the same notations and grammar style from [29]. We begin by recalling from (6) that in $\Omega^+$ there hold the homogeneous Stokes equations with decay conditions at infinity given by

$$\mathbf{u}(\mathbf{x}) = O(\|\mathbf{x}\|^{-1}) \quad \text{and} \quad p(\mathbf{x}) = O(\|\mathbf{x}\|^{-2}) \quad \text{as} \quad \|\mathbf{x}\| \to +\infty\,.$$

Hence, following [44, Chapter 2], our aim in this section is to apply the Green's representation formulae to express the velocity $\mathbf{u}$ and pressure $p$ of the fluid in $\Omega^+$ in terms of the Cauchy data on $\Gamma$. For this purpose, we first let $E$ and $Q$ be the fundamental velocity tensor and its associated pressure vector, respectively, which, using that $\mu = 1/2$, become (see [44, eq. (2.3.10)]):

$$E(\mathbf{x},\mathbf{y}) := \frac{1}{4\,\pi}\left\{\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\,\mathbf{I} + \frac{(\mathbf{x}-\mathbf{y})\,(\mathbf{x}-\mathbf{y})^{\mathtt{t}}}{\|\mathbf{x}-\mathbf{y}\|^3}\right\} \qquad \forall\,\mathbf{x}\neq\mathbf{y}\,, \tag{32}$$

and

$$Q(\mathbf{x},\mathbf{y}) := \frac{1}{4\,\pi}\,\nabla_{\mathbf{y}}\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right) \qquad \forall\,\mathbf{x}\neq\mathbf{y}\,. \tag{33}$$

In addition, we let $(\mathbf{S},\mathbf{D})$ and $(\Phi,\Pi)$ be the pairs of simple and double layer hydrodynamic potentials for the velocity and the pressure, respectively, that is

$$\mathbf{S}\,\boldsymbol{\rho}(\mathbf{x}) := \int_\Gamma E(\mathbf{x},\mathbf{y})\,\boldsymbol{\rho}(\mathbf{y})\,d\mathbf{s}_{\mathbf{y}} \qquad \forall\,\mathbf{x}\,\notin\,\Gamma\,, \quad \forall\,\boldsymbol{\rho}\,\in\,\mathbf{H}^{-1/2}(\Gamma)\,, \tag{34}$$

(35)
$$\mathbf{D}\,\boldsymbol{\psi} := \begin{pmatrix} \mathbf{D}_1\,\boldsymbol{\psi} \\ \mathbf{D}_2\,\boldsymbol{\psi} \\ \mathbf{D}_3\,\boldsymbol{\psi} \end{pmatrix}, \ \mathbf{D}_i\,\boldsymbol{\psi}(\mathbf{x}) := \int_\Gamma \Big\{\Xi[E_i(\mathbf{x},\cdot),-Q_i(\mathbf{x},\cdot)](\mathbf{y})\,\boldsymbol{\nu}(\mathbf{y})\Big\}^{\mathtt{t}}\,\boldsymbol{\psi}(\mathbf{y})\,d\mathbf{s}_{\mathbf{y}}$$
$$\forall\,\mathbf{x}\,\notin\,\Gamma\,, \qquad \forall\,\boldsymbol{\psi}\,\in\,\mathbf{H}^{1/2}(\Gamma)\,,$$

where $E_i(\mathbf{x},\mathbf{y})$ is the $i$-th column of $E(\mathbf{x},\mathbf{y})$ and $Q_i(\mathbf{x},\mathbf{y})$ is the $i$-th component of $Q(\mathbf{x},\mathbf{y})$,

$$\Phi\,\boldsymbol{\rho}(\mathbf{x}) := \int_\Gamma Q(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\rho}(\mathbf{y})\,d\mathbf{s}_{\mathbf{y}} \qquad \forall\,\mathbf{x}\,\notin\,\Gamma\,, \quad \forall\,\boldsymbol{\rho}\,\in\,\mathbf{H}^{-1/2}(\Gamma)\,, \tag{36}$$

and

$$\Pi\,\boldsymbol{\psi}(\mathbf{x}) := \int_\Gamma \nabla_{\mathbf{y}}\, Q(\mathbf{x},\mathbf{y})\,\boldsymbol{\nu}(\mathbf{y})\cdot\boldsymbol{\psi}(\mathbf{y})\,d\mathbf{s}_{\mathbf{y}} \qquad \forall\,\mathbf{x}\,\notin\,\Gamma\,, \qquad \forall\,\boldsymbol{\psi}\,\in\,\mathbf{H}^{1/2}(\Gamma)\,. \tag{37}$$

It is important to recall here that, for each $\boldsymbol{\rho}\,\in\,\mathbf{H}^{-1/2}(\Gamma)$ and for each $\boldsymbol{\psi}\,\in\,\mathbf{H}^{1/2}(\Gamma)$, the velocity/pressure pairs $(\mathbf{S}\,\boldsymbol{\rho},\Phi\,\boldsymbol{\rho})$ and $(\mathbf{D}\,\boldsymbol{\psi},\Pi\,\boldsymbol{\psi})$ satisfy the homogeneous Stokes equations in $\mathrm{R}^3\backslash\Gamma$. In addition, the main continuity properties of $\mathbf{S}$, $\mathbf{D}$, $\Phi$, and $\Pi$ are summarized in the following lemma.

LEMMA 4.1. *The hydrodynamic potentials define the following bounded linear operators:*

$$\begin{aligned}
\mathbf{S}\,:\,\mathbf{H}^{-1/2}(\Gamma) &\to \mathbf{H}^1_{\mathrm{div}}(\Omega^-;\Delta)\times\mathbf{H}^1_{\mathrm{div,loc}}(\Omega^+,\Delta)\,,\\
\mathbf{D}\,:\,\mathbf{H}^{1/2}(\Gamma) &\to \mathbf{H}^1_{\mathrm{div}}(\Omega^-;\Delta)\times\mathbf{H}^1_{\mathrm{div,loc}}(\Omega^+,\Delta)\,,\\
\Phi\,:\,\mathbf{H}^{-1/2}(\Gamma) &\to L^2(\Omega^-)\times L^2(\Omega^+)\,,\\
\Pi\,:\,\mathbf{H}^{1/2}(\Gamma) &\to L^2(\Omega^-)\times L^2(\Omega^+)\,,
\end{aligned}$$

*where*

$$\mathbf{H}^1_{\mathrm{div}}(\Omega^-;\Delta) := \left\{\mathbf{v}\in\mathbf{H}^1(\Omega^-):\ \mathrm{div}\,\mathbf{v}=0 \text{ in } \Omega^- \text{ and } \Delta\,\mathbf{v}\,\in\,\tilde{\mathbf{H}}^{-1}_0(\Omega^-)\right\},$$

$$\mathbf{H}^1_{\mathrm{div,loc}}(\Omega^+,\Delta) := \left\{\mathbf{v}\in\mathbf{H}^1_{\mathrm{loc}}(\Omega^+):\ \mathrm{div}\,\mathbf{v}=0 \text{ in } \Omega^+ \text{ and } \Delta\,\mathbf{v}\in\left(\mathbf{H}^1_{\mathrm{loc}}(\Omega^+)\right)'\right\},$$

*and* $\tilde{\mathbf{H}}^{-1}_0(\Omega^-)$ *is the orthogonal complement in* $\left(\mathbf{H}^1(\Omega^-)\right)'$ *of the space defined as* $\left\{\mathbf{v}\,\in\,\left(\mathbf{H}^1(\Omega^-)\right)':\quad \mathrm{supp}\,\mathbf{v}\,\subseteq\,\Gamma\right\}$.

*Proof.* It follows by combining analogue continuity properties for the Lamé system and the Laplacian. We refer to [44, Lemmas 5.6.4 and 5.6.6] and [46, Theorem 3.3] for details. Alternatively, [54] contains similar boundedness statements using weighted Sobolev spaces. □

We now let $\mathbf{V}$, $\mathbf{K}$, $\mathbf{K}^{\mathrm{t}}$, and $\mathbf{W}$ be the boundary integral operators of the simple, double, adjoint of the double, and hypersingular layer hydrodynamic potentials, respectively. These operators can be defined using lateral traces of the single and double layer hydrodynamic potentials (see [44, Lemma 5.6.5] and [54, Sections 5 and 6]:

$$\gamma^+(\mathbf{S}\,\boldsymbol{\rho}) \,=\, \gamma^-(\mathbf{S}\,\boldsymbol{\rho}) \,=\, \mathbf{V}\,\boldsymbol{\rho} \qquad \forall\,\boldsymbol{\rho}\,\in\,\mathbf{H}^{-1/2}(\Gamma)\,, \tag{38}$$

$$\gamma^\pm(\mathbf{D}\,\boldsymbol{\psi}) \,=\, \left(\pm\,\frac{1}{2}\,\mathbf{I}\,+\,\mathbf{K}\right)\,\boldsymbol{\psi} \qquad \forall\,\boldsymbol{\psi}\,\in\,\mathbf{H}^{1/2}(\Gamma)\,, \tag{39}$$

$$\gamma^\pm_{\boldsymbol{\nu}}\big(\Xi[\mathbf{S}\,\boldsymbol{\rho},\Phi\,\boldsymbol{\rho}]\big) \,=\, \left(\mp\,\frac{1}{2}\,\mathbf{I}\,+\,\mathbf{K}^{\mathrm{t}}\right)\,\boldsymbol{\rho} \qquad \forall\,\boldsymbol{\rho}\,\in\,\mathbf{H}^{-1/2}(\Gamma)\,, \tag{40}$$

$$\gamma^+_{\boldsymbol{\nu}}\big(\Xi[\mathbf{D}\,\boldsymbol{\psi},\Pi\,\boldsymbol{\psi}]\big) \,=\, \gamma^-_{\boldsymbol{\nu}}\big(\Xi[\mathbf{D}\,\boldsymbol{\psi},\Pi\,\boldsymbol{\psi}]\big) \,=\, -\,\mathbf{W}\,\boldsymbol{\psi} \qquad \forall\,\boldsymbol{\psi}\,\in\,\mathbf{H}^{1/2}(\Gamma)\,. \tag{41}$$

As a consequence of (38) - (41) the following jump conditions hold:

$$[\gamma(\mathbf{S}\,\boldsymbol{\rho})] \,:=\, \gamma^-(\mathbf{S}\,\boldsymbol{\rho}) \,-\, \gamma^+(\mathbf{S}\,\boldsymbol{\rho}) \,=\, \mathbf{0} \qquad \forall\,\boldsymbol{\rho}\,\in\,\mathbf{H}^{-1/2}(\Gamma)\,, \tag{42}$$

$$[\gamma(\mathbf{D}\,\boldsymbol{\psi})] \,:=\, \gamma^-(\mathbf{D}\,\boldsymbol{\psi}) \,-\, \gamma^+(\mathbf{D}\,\boldsymbol{\psi}) \,=\, -\,\boldsymbol{\psi} \qquad \forall\,\boldsymbol{\psi}\,\in\,\mathbf{H}^{1/2}(\Gamma)\,, \tag{43}$$

(44)
$$[\gamma_{\boldsymbol{\nu}}\big(\Xi[\mathbf{S}\,\boldsymbol{\rho},\Phi\,\boldsymbol{\rho}]\big) \;:=\; \gamma_{\boldsymbol{\nu}}^{-}\big(\Xi[\mathbf{S}\,\boldsymbol{\rho},\Phi\,\boldsymbol{\rho}]\big) \;-\; \gamma_{\boldsymbol{\nu}}^{+}\big(\Xi[\mathbf{S}\,\boldsymbol{\rho},\Phi\,\boldsymbol{\rho}]\big) \;=\; \boldsymbol{\rho} \quad \forall\,\boldsymbol{\rho}\,\in\,\mathbf{H}^{-1/2}(\Gamma)\,,$$

(45)
$$[\gamma_{\boldsymbol{\nu}}\big(\Xi[\mathbf{D}\,\boldsymbol{\psi},\Pi\,\boldsymbol{\psi}]\big) \;:=\; \gamma_{\boldsymbol{\nu}}^{-}\big(\Xi[\mathbf{D}\,\boldsymbol{\psi},\Pi\,\boldsymbol{\psi}]\big) \;-\; \gamma_{\boldsymbol{\nu}}^{+}\big(\Xi[\mathbf{D}\,\boldsymbol{\psi},\Pi\,\boldsymbol{\psi}]\big) \;=\; \mathbf{0} \quad \forall\,\boldsymbol{\psi}\,\in\,\mathbf{H}^{1/2}(\Gamma)\,.$$

Integral expressions for the above boundary integral operators can be found in the literature. For smooth enough densities $\boldsymbol{\rho}$ and almost everywhere on $\Gamma$ we can write [44, (2.3.15)]

$$\mathbf{V}\,\boldsymbol{\rho}(\mathbf{x}) \;=\; \mathbf{S}\,\boldsymbol{\rho}(\mathbf{x}) \;:=\; \int_{\Gamma} E(\mathbf{x},\mathbf{y})\,\boldsymbol{\rho}(\mathbf{y})\,d\mathbf{s}_{\mathbf{y}}. \tag{46}$$

An integral expression for $\mathbf{K}$ can be found in [44, (2.3.30)]

$$\mathbf{K}\,\boldsymbol{\psi}(\mathbf{x}) \;:=\; \frac{3}{4\,\pi}\,\text{p.v.}\int_{\Gamma} \frac{\big((\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})\big)\,\big((\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\psi}(\mathbf{y})\big)}{\|\mathbf{x}-\mathbf{y}\|^5}\,(\mathbf{x}-\mathbf{y})\,d\mathbf{s}_{\mathbf{y}}. \tag{47}$$

This integral formula involves a Cauchy principal value. For the explicit integral form of $\mathbf{W}$ we refer to [44, eqs. (2.3.30) and (2.3.31)].

Furthermore, some of the main properties of $\mathbf{V}$, $\mathbf{K}$, $\mathbf{K}^{\mathbf{t}}$, and $\mathbf{W}$ are collected next. To this end, given $\mathcal{O}\,\subseteq\,\mathrm{R}^3$ and $\ell\in\mathbb{N}\cup\{0\}$, we now let $P_\ell(\mathcal{O})$ be the space of polynomials of degree $\leq\ell$ on $\mathcal{O}$, and let $\mathbf{RM}(\mathcal{O})$ be the space of rigid motions in $\mathcal{O}$, that is

$$\mathbf{RM}(\mathcal{O}) \;:=\; \big\{\,\mathbf{z}:\quad \mathbf{z}(\mathbf{x}) \;=\; \mathbf{c}\,+\,\mathbf{d}\times\mathbf{x}\quad \forall\,\mathbf{x}\in\mathcal{O}\,;\quad \mathbf{c},\,\mathbf{d}\in\mathrm{R}^3\big\}\,.$$

We also introduce the vector field $\mathbf{m}:\Gamma\to\mathrm{R}^3$ given by

$$\mathbf{m}(\mathbf{x}) := \mathbf{x} - \frac{1}{|\Gamma|}\int_{\Gamma}\mathbf{x}\,d\mathbf{s}_{\mathbf{x}},$$

and the spaces

$$\mathbf{H}_0^{-1/2}(\Gamma) \;:=\; \Big\{\,\boldsymbol{\rho}\,\in\,\mathbf{H}^{-1/2}(\Gamma):\quad \langle\mathbf{m},\boldsymbol{\rho}\rangle_\Gamma \;=\; 0\Big\}$$

and

$$\mathbf{H}_0^{1/2}(\Gamma) \;:=\; \Big\{\,\boldsymbol{\psi}\,\in\,\mathbf{H}^{1/2}(\Gamma):\quad \langle\mathbf{r},\boldsymbol{\psi}\rangle_\Gamma \;=\; 0\quad \forall\,\mathbf{r}\,\in\,\mathbf{RM}(\Gamma)\Big\}.$$

LEMMA 4.2. *The following boundary integral operators are linear and bounded:*

$$\begin{aligned}
\mathbf{V}\;:\;\mathbf{H}^{-1/2}(\Gamma) \quad&\to\quad \mathbf{H}^{1/2}(\Gamma)\,,\\
\mathbf{K}\;:\;\mathbf{H}^{1/2}(\Gamma) \quad&\to\quad \mathbf{H}^{1/2}(\Gamma)\,,\\
\mathbf{K}^{\mathbf{t}}\;:\;\mathbf{H}^{-1/2}(\Gamma) \quad&\to\quad \mathbf{H}^{-1/2}(\Gamma)\,,\\
\mathbf{W}\;:\;\mathbf{H}^{1/2}(\Gamma) \quad&\to\quad \mathbf{H}^{-1/2}(\Gamma)\,.
\end{aligned} \tag{48}$$

*In addition,* $\mathbf{V}$ *and* $\mathbf{W}$ *are selfadjoint,*

(49)
$$\ker(\mathbf{V}) = \ker\left(\frac{1}{2}\,\mathbf{I}-\mathbf{K}^{\mathbf{t}}\right) = P_0(\Gamma)\,\boldsymbol{\nu}\,,\quad \ker(\mathbf{W}) = \ker\left(\frac{1}{2}\,\mathbf{I}+\mathbf{K}\right) = \mathbf{RM}(\Gamma)\,,$$

*and there exist* $\alpha_1\,,\alpha_2\,>\,0$ *such that*

$$\langle\boldsymbol{\rho},\mathbf{V}\,\boldsymbol{\rho}\rangle_\Gamma \;\geq\; \alpha_1\,\|\boldsymbol{\rho}\|^2_{-1/2,\Gamma} \qquad \forall\,\boldsymbol{\rho}\,\in\,\mathbf{H}_0^{-1/2}(\Gamma) \tag{50}$$

*and*

$$\langle\mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma \;\geq\; \alpha_2\,\|\boldsymbol{\psi}\|^2_{1/2,\Gamma} \qquad \forall\,\boldsymbol{\psi}\,\in\,\mathbf{H}_0^{1/2}(\Gamma)\,. \tag{51}$$

*Proof.* The proofs of these results appear in [54, Sections 5 and 6]. □

Note that the decompositions

$$\mathbf{H}^{-1/2}(\Gamma) \,=\, \mathbf{H}_0^{-1/2}(\Gamma) \,\oplus\, P_0(\Gamma)\,\boldsymbol{\nu} \quad \text{and} \quad \mathbf{H}^{1/2}(\Gamma) \,=\, \mathbf{H}_0^{1/2}(\Gamma) \,\oplus\, \mathbf{RM}(\Gamma)\,, \tag{52}$$

are stable and have associated oblique projectors $\boldsymbol{\pi}_{\nu} : \mathbf{H}_0^{-1/2}(\Gamma) \to P_0(\Gamma)\boldsymbol{\nu}$ and $\boldsymbol{\pi}_{RM} : \mathbf{H}^{1/2}(\Gamma) \to \mathbf{RM}(\Gamma)$. Therefore, the inequalities (50) and (51) are equivalent to

$$\langle \boldsymbol{\rho}, \mathbf{V}\,\boldsymbol{\rho}\rangle_\Gamma \,\geq\, \tilde{\alpha}_1\, \|\boldsymbol{\rho} - \boldsymbol{\pi}_{\nu}\boldsymbol{\rho}\|^2_{-1/2,\Gamma} \qquad \forall\, \boldsymbol{\rho} \,\in\, \mathbf{H}^{-1/2}(\Gamma) \tag{53}$$

and

$$\langle \mathbf{W}\,\boldsymbol{\psi}, \boldsymbol{\psi}\rangle_\Gamma \,\geq\, \tilde{\alpha}_2\, \|\boldsymbol{\psi} - \boldsymbol{\pi}_{RM}\boldsymbol{\psi}\|^2_{1/2,\Gamma} \qquad \forall\, \boldsymbol{\psi} \,\in\, \mathbf{H}^{1/2}(\Gamma). \tag{54}$$

As a simple consequence of (49) we can prove that

$$\langle \boldsymbol{\nu}, \mathbf{r}\rangle_\Gamma = \left\langle \left(\frac{1}{2}\mathbf{I} + \mathbf{K}^{\mathrm{t}}\right)\boldsymbol{\nu}, \mathbf{r}\right\rangle = \left\langle \boldsymbol{\nu}, \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)\mathbf{r}\right\rangle = 0 \qquad \forall \mathbf{r} \in \mathbf{RM}(\Gamma). \tag{55}$$

On the other hand, we have the following technical result.

Lemma 4.3. *There holds*

$$\langle \mathbf{W}\,\boldsymbol{\psi}, \boldsymbol{\psi}\rangle_\Gamma \,=\, \|\mathbf{e}(\mathbf{D}\,\boldsymbol{\psi})\|^2_{0,\mathrm{R}^3\setminus\Gamma} \qquad \forall\, \boldsymbol{\psi} \,\in\, \mathbf{H}^{1/2}(\Gamma)\,. \tag{56}$$

*Proof.* Given $\boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Gamma)$, it follows from (41) and (43) that

$$\begin{aligned}\langle \mathbf{W}\,\boldsymbol{\psi}, \boldsymbol{\psi}\rangle_\Gamma \,&=\, \langle -\,\gamma^{\pm}_{\boldsymbol{\nu}}\big(\Xi[\mathbf{D}\,\boldsymbol{\psi}, \Pi\,\boldsymbol{\psi}]\big), \gamma^+(\mathbf{D}\,\boldsymbol{\psi}) \,-\, \gamma^-(\mathbf{D}\,\boldsymbol{\psi})\rangle_\Gamma \\ &=\, \langle \gamma^-_{\boldsymbol{\nu}}\big(\Xi[\mathbf{D}\,\boldsymbol{\psi}, \Pi\,\boldsymbol{\psi}]\big), \gamma^-(\mathbf{D}\,\boldsymbol{\psi})\rangle_\Gamma \,-\, \langle \gamma^+_{\boldsymbol{\nu}}\big(\Xi[\mathbf{D}\,\boldsymbol{\psi}, \Pi\,\boldsymbol{\psi}]\big), \gamma^+(\mathbf{D}\,\boldsymbol{\psi})\rangle_\Gamma\,.\end{aligned}$$

Next, integrating by parts in $\Omega := \Omega_0 \cup \Omega^-$ and recalling that the velocity/pressure pair $(\mathbf{D}\,\boldsymbol{\psi}, \Pi\,\boldsymbol{\psi})$ satisfies the homogeneous Stokes equations, we find that

$$\begin{aligned}&\langle \gamma^-_{\boldsymbol{\nu}}\big(\Xi[\mathbf{D}\,\boldsymbol{\psi}, \Pi\,\boldsymbol{\psi}]\big), \gamma^-(\mathbf{D}\,\boldsymbol{\psi})\rangle_\Gamma \\ &\quad=\, \int_\Omega \nabla\mathbf{D}\,\boldsymbol{\psi} : \Xi[\mathbf{D}\,\boldsymbol{\psi}, \Pi\,\boldsymbol{\psi}] \,+\, \int_\Omega \mathbf{D}\,\boldsymbol{\psi}\cdot\mathbf{div}\;\Xi[\mathbf{D}\,\boldsymbol{\psi}, \Pi\,\boldsymbol{\psi}] \\ &\quad=\, \int_\Omega \nabla\mathbf{D}\,\boldsymbol{\psi} : \Big\{\,\mathbf{e}(\mathbf{D}\,\boldsymbol{\psi}) \,-\, \Pi\,\boldsymbol{\psi}\,\mathbf{I}\,\Big\} \,=\, \int_\Omega \nabla\mathbf{D}\,\boldsymbol{\psi} : \mathbf{e}(\mathbf{D}\,\boldsymbol{\psi}) \,=\, \|\mathbf{e}(\mathbf{D}\,\boldsymbol{\psi})\|^2_{0,\Omega}\,.\end{aligned}$$

Similarly, integrating by parts in $\Omega^+$, noting that $\boldsymbol{\nu}$ points inward $\Omega^+$, and using additionally the conditions at infinity, we deduce that

$$-\,\langle \gamma^+_{\boldsymbol{\nu}}\big(\Xi[\mathbf{D}\,\boldsymbol{\psi}, \Pi\,\boldsymbol{\psi}]\big), \gamma^+(\mathbf{D}\,\boldsymbol{\psi})\rangle_\Gamma \,=\, \|\mathbf{e}(\mathbf{D}\,\boldsymbol{\psi})\|^2_{0,\Omega^+}\,,$$

which, together with the previous identity, yields (56) and ends the proof. Alternatively, this proof can also be found in [54, Proposition 6.4].

□

We now go back to the homogeneous Stokes equations in $\Omega^+$. In fact, according to the Green's formulae provided in [44, Section 2.3.1], we have the representations

$$\mathbf{u} \,=\, -\,\mathbf{S}\,\gamma^+_{\boldsymbol{\nu}}(\boldsymbol{\sigma}) \,+\, \mathbf{D}\,\gamma^+(\mathbf{u}) \quad \text{in} \quad \Omega^+\,, \tag{57}$$

and

$$p \,=\, -\,\Phi\,\gamma^+_{\boldsymbol{\nu}}(\boldsymbol{\sigma}) \,+\, \Pi\,\gamma^+(\mathbf{u}) \quad \text{in} \quad \Omega^+\,. \tag{58}$$

Therefore, evaluating the operators $\gamma^+$ and $\gamma^+_{\boldsymbol{\nu}}$ in $\mathbf{u}$ and $\boldsymbol{\sigma} = \Xi[\mathbf{u}, p]$, respectively, with $\mathbf{u}$ and $p$ given by (57) and (58), and applying the trace properties, we arrive at the following boundary integral equations:

$$\gamma^+(\mathbf{u}) = -\mathbf{V}\,\gamma^+_{\boldsymbol{\nu}}(\boldsymbol{\sigma}) + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)\gamma^+(\mathbf{u}) \quad \text{on} \quad \Gamma\,, \tag{59}$$

and

$$\gamma^+_{\boldsymbol{\nu}}(\boldsymbol{\sigma}) = \left(\frac{1}{2}\mathbf{I} - \mathbf{K}^{\mathtt{t}}\right)\gamma^+_{\boldsymbol{\nu}}(\boldsymbol{\sigma}) - \mathbf{W}\,\gamma^+(\mathbf{u}) \quad \text{on} \quad \Gamma\,. \tag{60}$$

Moreover, thanks to the transmission conditions (7) and the introduction of the additional unknown $\boldsymbol{\varphi}$ (cf. (10)), the above equations become:

$$\boldsymbol{\varphi} = -\mathbf{V}\,\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\sigma}) + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)\boldsymbol{\varphi} \quad \text{on} \quad \Gamma\,, \tag{61}$$

and

$$\mathbf{W}\,\boldsymbol{\varphi} + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}^{\mathtt{t}}\right)\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\sigma}) = \mathbf{0} \quad \text{on} \quad \Gamma\,. \tag{62}$$

## 5. The coupled variational formulations

In this section we combine the dual-mixed approach in $\Omega^-$ (cf. Section 3) with the boundary integral equation method in $\Omega^+$ (cf. Section 4) to derive and analyze coupled variational formulations for the transmission problem (6). We remark in advance that just the contents of Section 5.1 below constitute the contributions described in [29]. However, while we could have simply referred to the results in [29], for sake of completeness we prefer to provide here most of the corresponding details. The remaining approaches and cases analyzed in the present Section 5 were not considered in [29].

**5.1. J & N coupling with homogeneous Neumann boundary conditions on $\Gamma_0$.** Here we follow the Johnson-Nédélec coupling method (see [12], [45]) and incorporate the single boundary integral equation (62) into the dual-mixed variational formulation in $\Omega^-$ given by (31), which considers the homogeneous Neumann boundary condition $\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\sigma}) = \mathbf{0}$ on $\Gamma_0$. More precisely, we test (62) against $\boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Gamma)$ and add the resulting equation to the first equation of (31) thus yielding a redefinition of the bilinear form $\mathbf{a}_N = \mathbf{a}_D$ (cf. (15)). In this way, our coupled variational formulation reads as follows: Find $((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\mathbf{u}, \boldsymbol{\chi})) \in \mathbf{X}_N \times \mathbf{Y}_N$ such that

$$\begin{aligned} \mathbf{a}_N((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\boldsymbol{\tau}, \boldsymbol{\psi})) + \mathbf{b}_N((\boldsymbol{\tau}, \boldsymbol{\psi}), (\mathbf{u}, \boldsymbol{\chi})) &= \mathbf{F}_N(\boldsymbol{\tau}, \boldsymbol{\psi}) \qquad \forall\,(\boldsymbol{\tau}, \boldsymbol{\psi}) \in \mathbf{X}_N\,, \\ \mathbf{b}_N((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\mathbf{v}, \boldsymbol{\eta})) &= \mathbf{G}_N(\mathbf{v}, \boldsymbol{\eta}) \qquad \forall\,(\mathbf{v}, \boldsymbol{\eta}) \in \mathbf{Y}_N\,, \end{aligned} \tag{63}$$

where

$$\mathbf{X}_N := \mathbb{H}_0(\mathbf{div}\,; \Omega^-) \times \mathbf{H}^{1/2}(\Gamma)\,, \quad \mathbf{Y}_N := \mathbf{L}^2(\Omega^-) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega^-)\,, \tag{64}$$

$\mathbf{a}_N : \mathbf{X}_N \times \mathbf{X}_N \to \mathrm{R}$ and $\mathbf{b}_N : \mathbf{X}_N \times \mathbf{Y}_N \to \mathrm{R}$ are the bounded bilinear forms defined by

$$\begin{aligned} \mathbf{a}_N((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\boldsymbol{\tau}, \boldsymbol{\psi})) := \int_{\Omega^-} \boldsymbol{\sigma}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} + \langle \mathbf{W}\,\boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_\Gamma + \left\langle \left(\frac{1}{2}\mathbf{I} + \mathbf{K}^{\mathtt{t}}\right)\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\sigma}), \boldsymbol{\psi}\right\rangle_\Gamma \\ - \langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}), \boldsymbol{\varphi}\rangle_\Gamma \end{aligned} \tag{65}$$

and

$$\mathbf{b}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{v},\boldsymbol{\eta})) := \int_{\Omega^-} \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_{\Omega^-} \boldsymbol{\eta}:\boldsymbol{\tau}\,, \tag{66}$$

and $\mathbf{F}_N : \mathbf{X}_N \to \mathrm{R}$ and $\mathbf{G}_N : \mathbf{Y}_N \to \mathrm{R}$ are the bounded linear functionals given by

$$\mathbf{F}_N(\boldsymbol{\tau},\boldsymbol{\psi}) := 0 \qquad \text{and} \qquad \mathbf{G}_N(\mathbf{v},\boldsymbol{\eta}) := -\int_{\Omega^-} \mathbf{f}\cdot\mathbf{v}. \tag{67}$$

We now observe from (66) that the bounded linear operator induced by $\mathbf{b}_N$, say $\mathbf{B}_N : \mathbf{X}_N \to \mathbf{Y}_N$, is given by $\mathbf{B}_N((\boldsymbol{\tau},\boldsymbol{\psi})) := \big(\mathbf{div}\,\boldsymbol{\tau}, \frac{1}{2}(\boldsymbol{\tau}-\boldsymbol{\tau}^{\mathtt{t}})\big)$ for any $(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{X}_N$. If follows easily that $\mathbf{V}_N$, the kernel of $\mathbf{B}_N$, reduces to

$$\mathbf{V}_N := \Big\{ (\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{X}_N : \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}} \quad \text{and} \quad \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \quad \text{in} \quad \Omega^- \Big\}.$$

The following lemmas, which establish a positiveness property of $\mathbf{a}_N$ on $\mathbf{V}_N$ and an inf-sup condition for $\mathbf{b}_N$, are crucial for the forthcoming analysis (see, also [29, Lemmas 1, 2, and 3]).

Lemma 5.1. *There holds*

$$\mathbf{a}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\boldsymbol{\tau},\boldsymbol{\psi})) \geq \frac{1}{2}\Big\{ \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma \Big\} \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{V}_N\,. \tag{68}$$

*Proof.* Given $(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{V}_N$ we have from (65) and (39)

$$\begin{aligned} \mathbf{a}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\boldsymbol{\tau},\boldsymbol{\psi})) &= \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma + \Big\langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}), \Big(-\frac{1}{2}\mathbf{I} + \mathbf{K}\Big)\,\boldsymbol{\psi}\Big\rangle_\Gamma\,, \\ &= \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma + \big\langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}), \gamma^-(\mathbf{D}\,\boldsymbol{\psi})\big\rangle_\Gamma\,. \end{aligned} \tag{69}$$

Hence, integrating by parts in $\Omega^-$ and using that $\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}) = \mathbf{0}$ on $\Gamma_0$, we find that

$$\begin{aligned} \langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}), \gamma^-(\mathbf{D}\,\boldsymbol{\psi})\rangle_\Gamma &= \int_{\Omega^-} \Big\{ \nabla\mathbf{D}\,\boldsymbol{\psi}:\boldsymbol{\tau} + \mathbf{D}\,\boldsymbol{\psi}\cdot\mathbf{div}\,\boldsymbol{\tau} \Big\} \\ &= \int_{\Omega^-} \mathbf{e}\big(\mathbf{D}\,\boldsymbol{\psi}\big):\boldsymbol{\tau} = \int_{\Omega^-} \mathbf{e}\big(\mathbf{D}\,\boldsymbol{\psi}\big):\boldsymbol{\tau}^{\mathtt{d}}\,, \end{aligned} \tag{70}$$

where the free-divergence and symmetry properties of $\boldsymbol{\tau}$, together with the incompressibility condition satisfied by $\mathbf{D}\,\boldsymbol{\psi}$, have been utilized in the last two equalities. In this way, replacing (70) into (69), and then applying Cauchy-Schwarz's inequality and the identity (56), we deduce that

$$\begin{aligned} a_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\boldsymbol{\tau},\boldsymbol{\psi})) &= \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma + \int_{\Omega^-} \mathbf{e}\big(\mathbf{D}\,\boldsymbol{\psi}\big):\boldsymbol{\tau}^{\mathtt{d}} \\ &\geq \frac{1}{2}\|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma - \frac{1}{2}\|\mathbf{e}(\mathbf{D}\,\boldsymbol{\psi})\|^2_{0,\Omega^-} \\ &\geq \frac{1}{2}\|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma - \frac{1}{2}\|\mathbf{e}(\mathbf{D}\,\boldsymbol{\psi})\|^2_{0,\mathrm{R}^3\setminus\Gamma} \\ &= \frac{1}{2}\|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \frac{1}{2}\langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma\,, \end{aligned}$$

which finishes the proof. □

Lemma 5.2. *There exists $\beta > 0$ such that for any $(\mathbf{v},\boldsymbol{\eta}) \in \mathbf{Y}_N$ there holds*

$$\sup_{(\boldsymbol{\tau},\boldsymbol{\psi})\in\mathbf{X}_N\setminus\{\mathbf{0}\}} \frac{\mathbf{b}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{v},\boldsymbol{\eta}))}{\|(\boldsymbol{\tau},\boldsymbol{\psi})\|_{\mathbf{X}_N}} \geq \beta\,\|(\mathbf{v},\boldsymbol{\eta})\|_{\mathbf{Y}_N}\,. \tag{71}$$

*Proof.* It reduces to show that the operator $\mathbf{B}_N$ is surjective. In fact, given $(\mathbf{v},\boldsymbol{\eta}) \in \mathbf{Y}_N$, we let $\mathbf{z}$ be the unique element in $\mathbf{H}^1_\Gamma(\Omega^-) := \Big\{ \mathbf{w} \in \mathbf{H}^1(\Omega^-) : \quad \mathbf{w} = \mathbf{0} \quad \text{on} \quad \Gamma \Big\}$, whose existence is guaranteed by the second Korn inequality, such that

$$\int_{\Omega^-} \mathbf{e}(\mathbf{z}) : \mathbf{e}(\mathbf{w}) \,=\, -\int_{\Omega^-} \mathbf{v}\cdot\mathbf{w} \,-\, \int_{\Omega^-} \boldsymbol{\eta} : \nabla\mathbf{w} \qquad \forall\, \mathbf{w} \,\in\, \mathbf{H}^1_\Gamma(\Omega^-)\,.$$

Hence, defining $\widehat{\boldsymbol{\tau}} := \mathbf{e}(\mathbf{z}) + \boldsymbol{\eta} \in \mathbb{L}^2(\Omega^-)$, we deduce from the above formulation that $\mathbf{div}\,\widehat{\boldsymbol{\tau}} = \mathbf{v}$ in $\Omega^-$, which shows that $\widehat{\boldsymbol{\tau}} \in \mathbb{H}(\mathbf{div}\,;\Omega^-)$, and then that $\gamma^-_{\boldsymbol{\nu}}(\widehat{\boldsymbol{\tau}}) = \mathbf{0}$ on $\Gamma_0$. In this way, $\widehat{\boldsymbol{\tau}} \in \mathbb{H}_0(\mathbf{div}\,;\Omega^-)$ and it is easy to see that $\mathbf{B}_N((\widehat{\boldsymbol{\tau}},\mathbf{0})) = (\mathbf{v},\boldsymbol{\eta})$, which ends the proof.

□

Note that the fact that $\mathbf{b}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{v},\boldsymbol{\eta}))$ does not depend on $\boldsymbol{\psi}$ guarantees that the inf-sup condition (71) can also be rewritten as

$$\sup_{\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}\,;\Omega^-)\setminus\{\mathbf{0}\}} \frac{\mathbf{b}_N((\boldsymbol{\tau},\mathbf{0}),(\mathbf{v},\boldsymbol{\eta}))}{\|(\boldsymbol{\tau},\mathbf{0})\|_{\mathbf{X}_N}} \,\geq\, \beta\,\|(\mathbf{v},\boldsymbol{\eta})\|_{\mathbf{Y}_N} \qquad \forall\,(\mathbf{v},\boldsymbol{\eta}) \in \mathbf{Y}_N\,. \tag{72}$$

We now begin the solvability analysis of (63) by identifying previously the solutions of the associated homogeneous problem.

Lemma 5.3. *The set of solutions of the homogeneous version of* (63) *is given by*

$$\Big\{ \big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})\big) := \big((\mathbf{0},\mathbf{z}|_\Gamma),(\mathbf{z},\nabla\mathbf{z})\big) : \qquad \mathbf{z} \,\in\, \mathbf{RM}(\Omega^-) \Big\}.$$

*Proof.* Let $\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})\big) \in \mathbf{X}_N \times \mathbf{Y}_N$ be a solution of (63) with $\mathbf{f} = \mathbf{0}$. It is clear from the second equation that $(\boldsymbol{\sigma},\boldsymbol{\varphi}) \in \mathbf{V}_N$, that is $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}}$ and $\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{0}$ in $\Omega^-$. Then, taking in particular $(\boldsymbol{\tau},\boldsymbol{\psi}) = (\boldsymbol{\sigma},\boldsymbol{\varphi})$ in the first equation, and then applying the inequalities (68) (cf. Lemma 5.1) and (54), we find that

$$\begin{aligned} 0 \,=\, \mathbf{a}_N\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\sigma},\boldsymbol{\varphi})\big) \,&\geq\, \frac{1}{2}\Big\{ \|\boldsymbol{\sigma}^{\mathrm{d}}\|^2_{0,\Omega^-} \,+\, \langle \mathbf{W}\,\boldsymbol{\varphi},\boldsymbol{\varphi}\rangle_\Gamma \Big\} \\ &\geq\, \frac{1}{2}\,\|\boldsymbol{\sigma}^{\mathrm{d}}\|^2_{0,\Omega^-} \,+\, \frac{\tilde{\alpha}_2}{2}\,\|\boldsymbol{\varphi} - \boldsymbol{\pi}_{RM}\boldsymbol{\varphi}\|^2_{1/2,\Gamma}\,, \end{aligned}$$

which gives $\boldsymbol{\sigma}^{\mathrm{d}} = \mathbf{0}$ in $\Omega^-$ and $\boldsymbol{\varphi} = \mathbf{z}|_\Gamma$, with $\mathbf{z} \in \mathbf{RM}(\Omega^-)$. In turn, the conditions satisfied by $\boldsymbol{\sigma}$, namely $\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{0}$ and $\boldsymbol{\sigma}^{\mathrm{d}} = \mathbf{0}$ in $\Omega^-$, together with the fact that $\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\sigma}) = \mathbf{0}$ on $\Gamma_0$ imply that $\boldsymbol{\sigma} = \mathbf{0}$. Next, taking $\boldsymbol{\psi} = \mathbf{0}$ in the first equation of our homogeneous problem, and then integrating by parts in $\Omega^-$, we obtain that for any $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega^-)$ there holds

$$\begin{aligned} 0 \,&=\, \mathbf{b}_N((\boldsymbol{\tau},\mathbf{0}),(\mathbf{u},\boldsymbol{\chi})) \,-\, \langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}),\boldsymbol{\varphi}\rangle_\Gamma \,=\, \mathbf{b}_N((\boldsymbol{\tau},\mathbf{0}),(\mathbf{u},\boldsymbol{\chi})) \,-\, \langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}),\mathbf{z}\rangle_\Gamma \\ &=\, \mathbf{b}_N((\boldsymbol{\tau},\mathbf{0}),(\mathbf{u},\boldsymbol{\chi})) \,-\, \int_{\Omega^-} \mathbf{z}\cdot\mathbf{div}\,\boldsymbol{\tau} \,-\, \int_{\Omega^-} \nabla\mathbf{z} : \boldsymbol{\tau} \\ &=\, \mathbf{b}_N((\boldsymbol{\tau},\mathbf{0}),(\mathbf{u}-\mathbf{z},\boldsymbol{\chi}-\nabla\mathbf{z}))\,, \end{aligned}$$

which, thanks to the inf-sup condition (72), gives $(\mathbf{u},\boldsymbol{\chi}) = (\mathbf{z},\nabla\mathbf{z})$. Conversely, it is easy to see, in particular using that $\ker(\mathbf{W}) = \mathbf{RM}(\Gamma)$ (cf. (49)), that for any $\mathbf{z} \in \mathbf{RM}(\Omega^-)$ the element $\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})\big) := \big((\mathbf{0},\mathbf{z}|_\Gamma),(\mathbf{z},\nabla\mathbf{z})\big)$ solves the homogeneous version of (63).

□

According to the above lemma and the decomposition $\mathbf{H}^{1/2}(\Gamma) = \mathbf{H}_0^{1/2}(\Gamma) \oplus \mathbf{RM}(\Gamma)$ (cf. (52)), and in order to guarantee the unique solvability of the coupled problem (63), we now look for the solution $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi}))$ in the space $\widetilde{\mathbf{X}}_N \times \mathbf{Y}_N$, where

$$\widetilde{\mathbf{X}}_N := \mathbb{H}_0(\mathbf{div}\,;\Omega^-) \times \mathbf{H}_0^{1/2}(\Gamma)\,. \tag{73}$$

In turn, it is easy to see, using that $\langle \mathbf{W}\,\boldsymbol{\varphi},\boldsymbol{\psi}\rangle_\Gamma = \langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\varphi}\rangle_\Gamma$ and that $\ker(\mathbf{W}) = \ker\left(\frac{1}{2}\mathbf{I}+\mathbf{K}\right) = \mathbf{RM}(\Gamma)$ (cf. (49)), that the ocurrence of the first equation of (63) can be equivalently established for any $(\boldsymbol{\tau},\boldsymbol{\psi}) \in \widetilde{\mathbf{X}}_N$. As a consequence, and instead of (63), we now seek $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})) \in \widetilde{\mathbf{X}}_N \times \mathbf{Y}_N$ such that

$$\begin{aligned} \mathbf{a}_N((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\tau},\boldsymbol{\psi})) + \mathbf{b}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{u},\boldsymbol{\chi})) &= \mathbf{F}_N(\boldsymbol{\tau},\boldsymbol{\psi}) \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\psi}) \in \widetilde{\mathbf{X}}_N\,, \\ \mathbf{b}_N((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{v},\boldsymbol{\eta})) &= \mathbf{G}_N(\mathbf{v},\boldsymbol{\eta}) \qquad \forall\,(\mathbf{v},\boldsymbol{\eta}) \in \mathbf{Y}_N\,, \end{aligned} \tag{74}$$

The following two lemmas are needed to show the well-posedness of (74). They make use of the decomposition defined by (1) and (2), which says in this case that each $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\Omega^-)$ can be written in a unique way as $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d\,\mathbf{I}$, with $\boldsymbol{\tau}_0 \in \widetilde{\mathbb{H}}(\mathbf{div}\,;\Omega^-)$ and $d \in \mathrm{R}$. The associated projector will be denoted $\boldsymbol{\pi}_I : \mathbb{H}(\mathbf{div}\,;\Omega^-) \to P_0(\Omega^-)\,I$.

Lemma 5.4. *There exists $c_1 > 0$, depending only on $\Omega^-$, such that*

$$\|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega^-} \;\geq\; c_1\,\|\boldsymbol{\tau} - \boldsymbol{\pi}_I\boldsymbol{\tau}\|^2_{0,\Omega^-} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\Omega^-)\,. \tag{75}$$

*Proof.* See [3, Lemma 3.1] or [11, Proposition 3.1, Chapter IV]. □

Lemma 5.5. *There exists $c_2 > 0$, depending only on $\Omega^-$, such that*

$$\|\boldsymbol{\tau} - \boldsymbol{\pi}_I\boldsymbol{\tau}\|^2_{\mathbf{div}\,;\Omega^-} \;\geq\; c_2\,\|\boldsymbol{\tau}\|^2_{\mathbf{div}\,;\Omega^-} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega^-)\,. \tag{76}$$

*Proof.* See [31, Lemma 4.5]. □

We are now in a position to establish the main result of this section.

Theorem 5.1. *Given $\mathbf{f} \in \mathbf{L}^2(\Omega^-)$, there exists a unique $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})) \in \widetilde{\mathbf{X}}_N \times \mathbf{Y}_N$ solution to (74). In addition, there exists $C > 0$ such that*

$$\|((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi}))\|_{\mathbf{X}_N\times\mathbf{Y}_N} \;\leq\; C\,\|\mathbf{f}\|_{0,\Omega^-}\,.$$

*Proof.* It reduces to verify the hypotheses of the classical Babuška-Brezzi theory. The boundedness of $\mathbf{a}_N$ and $\mathbf{b}_N$ was already noticed at the beginning of this section. Also, we observe that Lemma 5.2 establishes the required inf-sup condition for $\mathbf{b}_N$. Next, because of the replacement of the space $\mathbf{X}_N$ by $\widetilde{\mathbf{X}}_N$ (cf. (73)), the kernel of the operator induced by $\mathbf{b}_N : \widetilde{\mathbf{X}}_N \times \mathbf{Y}_N \to \mathrm{R}$ becomes now

$$\widetilde{\mathbf{V}}_N := \left\{(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbb{H}_0(\mathbf{div}\,;\Omega^-) \times \mathbf{H}_0^{1/2}(\Gamma) :\; \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}} \text{ and } \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \text{ in } \Omega^-\right\}.$$

Hence, applying (68) (cf. Lemmas 5.1), (75) (cf. Lemma 5.4), (76) (cf. Lemma 5.5), and (51) (cf. Lemma 4.2), we deduce that for any $(\boldsymbol{\tau},\boldsymbol{\psi}) \in \widetilde{\mathbf{V}}_N$ there holds

$$\begin{aligned}
\mathbf{a}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\boldsymbol{\tau},\boldsymbol{\psi})) &\geq \frac{1}{2}\,\|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \frac{1}{2}\,\langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma \\
&\geq \frac{c_1}{2}\,\|\boldsymbol{\tau}-\boldsymbol{\pi}_I\boldsymbol{\tau}\|^2_{0,\Omega^-} + \frac{1}{2}\,\langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma \\
&= \frac{c_1}{2}\,\|\boldsymbol{\tau}-\boldsymbol{\pi}_I\boldsymbol{\tau}\|^2_{\mathbf{div};\Omega^-} + \frac{1}{2}\,\langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma \\
&\geq \frac{c_1\,c_2}{2}\,\|\boldsymbol{\tau}\|^2_{\mathbf{div};\Omega^-} + \frac{\alpha_2}{2}\,\|\boldsymbol{\psi}\|^2_{1/2,\Gamma}\,,
\end{aligned}$$

which proves that $\mathbf{a}_N$ is $\widetilde{\mathbf{V}}_N$-elliptic. In this way, the proof is completed by applying the corresponding result from the above mentioned theory (see, e.g. [11, Theorem 1.1, Chapter II]).

□

Notice that the result provided by the previous theorem constitutes the natural extension of the continuous analysis developed in [48], which in turn adapts and modifies the main ideas from [52], to the present mixed formulation of the three-dimensional exterior Stokes problem. Furthermore, it is important to remark at this point, as shown in the proof of Lemma 5.1, that the $\widetilde{\mathbf{V}}_N$-ellipticity of $\mathbf{a}_N$, which is certainly needed for the well-posedness of (74), does require that the component $\boldsymbol{\tau}$ of each pair $(\boldsymbol{\tau},\boldsymbol{\psi})$ in $\widetilde{\mathbf{V}}_N$ be free-divergence and symmetric. In particular, recall that the symmetry of $\boldsymbol{\tau}$ is employed to replace $\displaystyle\int_{\Omega^-} \nabla\mathbf{D}\,\boldsymbol{\psi}:\boldsymbol{\tau}$ by $\displaystyle\int_{\Omega^-} \mathbf{e}(\mathbf{D}\,\boldsymbol{\psi}):\boldsymbol{\tau}$ in equation (70), which constitutes a crucial identity for the remaining part of the proof. Analogously, for the analysis of an associated Galerkin scheme, one would need to show that $\mathbf{a}_N$ is elliptic at the discrete kernel of $\mathbf{b}_N$, which is given by

$$\begin{aligned}
\widetilde{\mathbf{V}}_{N,h} := \Bigg\{ (\boldsymbol{\tau}_h,\boldsymbol{\psi}_h) \in \widetilde{\mathbf{X}}_{N,h} := \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \mathbf{H}^{\boldsymbol{\varphi}}_{h,0} : \int_{\Omega^-} \mathbf{v}_h\cdot\mathbf{div}\,\boldsymbol{\tau}_h = 0 \quad \forall\,\mathbf{v}_h \in \mathbf{L}^{\mathbf{u}}_h \\
\text{and} \quad \int_{\Omega^-} \boldsymbol{\tau}_h:\boldsymbol{\eta}_h = 0 \quad \forall\,\boldsymbol{\eta}_h \in \mathbb{L}^{\boldsymbol{\chi}}_h \Bigg\},
\end{aligned}$$

where $\mathbb{H}^{\boldsymbol{\sigma}}_{h,0}$, $\mathbf{H}^{\boldsymbol{\varphi}}_{h,0}$, $\mathbf{L}^{\mathbf{u}}_h$, and $\mathbb{L}^{\boldsymbol{\chi}}_h$ are finite dimensional subspaces of $\mathbb{H}_0(\mathbf{div};\Omega^-)$, $\mathbf{H}^{1/2}_0(\Gamma)$, $\mathbf{L}^2(\Omega^-)$, and $\mathbb{L}^2_{\mathtt{skew}}(\Omega^-)$, respectively. Nevertheless, while it is possible to choose these subspaces so that the discrete inf-sup condition for $\mathbf{b}_N$ is satisfied and the first equation defining $\widetilde{\mathbf{V}}_{N,h}$ yields the components $\boldsymbol{\tau}_h$ of the pairs $(\boldsymbol{\tau}_h,\boldsymbol{\psi}_h) \in \widetilde{\mathbf{V}}_{N,h}$ to be free-divergence, no subspaces implying additionally the symmetry of these components from the second equation defining $\widetilde{\mathbf{V}}_{N,h}$ are known (at least, up to the authors' knowledge). In order to overcome this difficulty, one could consider Galerkin schemes for the simplified continuous formulation that arises from (74) after eliminating the vorticity unknown $\boldsymbol{\chi}$, which means that one looks, from the beginning, for a symmetric stress tensor $\boldsymbol{\sigma}$. The recent availability of new stable mixed finite element methods for linear elasticity with strong symmetry allows for the choice of concrete finite element subspaces towards this purpose (see, e.g. [5], [1]). However, due to the high number of local degrees of freedom involved, this procedure is still a bit prohibitive. Alternatively, instead of proving the $\widetilde{\mathbf{V}}_{N,h}$-ellipticity of $\mathbf{a}_N$, one could try to show that this bilinear form satisfies the discrete inf-sup condition on $\widetilde{\mathbf{V}}_{N,h}$, hoping that the symmetry property in question is not

needed along the way. However, this idea is rather an open question that needs to be further investigated. In the present paper we suggest a different approach which makes no use of any strong symmetry property of the discrete tensors. More precisely, we show below in Section 6.1 that, under a suitable assumption on the mesh sizes involved, $\mathbf{a}_N$ does become uniformly strongly coercive on the discrete kernels of $\mathbf{b}_N$.

On the other hand, another procedure that certainly avoids the need of any symmetry condition, neither for the continuous nor for the discrete kernels of $\mathbf{b}_N$, is based on the incorporation of both integral equations (61) and (62) into the respective variational formulation. This coupling method, known as the Costabel & Han procedure, which has been denoted C & H in Section 1, is analyzed with Dirichlet and Neumann boundary conditions on $\Gamma_0$ in the forthcoming sections.

**5.2. C & H coupling with Dirichlet boundary conditions on $\Gamma_0$.** We now follow the Costabel & Han coupling method (see [20], [38]) and incorporate the boundary integral equations (61) and (62) into the dual-mixed variational formulation in $\Omega^-$ given by (18), which assumes the Dirichlet boundary condition $\gamma^-(\mathbf{u}) = \mathbf{g}_D \in \mathbf{H}^{1/2}(\Gamma_0)$ on $\Gamma_0$. More precisely, we replace $\boldsymbol{\varphi}$ in the first equation of (18) by the right hand side of (61), and simultaneously add (62) tested against $\boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Gamma)$ to the same equation, thus yielding a redefinition of the bilinear form $\mathbf{a}_D$ (cf. (15)). In this way, our coupled variational formulation reads as follows: Find $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})) \in \mathbf{X}_D \times \mathbf{Y}_D$ such that

(77)
$$\begin{aligned} \mathbf{a}_D((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\tau},\boldsymbol{\psi})) + \mathbf{b}_D((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{u},\boldsymbol{\chi})) &= \mathbf{F}_D(\boldsymbol{\tau},\boldsymbol{\psi}) \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{X}_D\,, \\ \mathbf{b}_D((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{v},\boldsymbol{\eta})) &= \mathbf{G}_D(\mathbf{v},\boldsymbol{\eta}) \qquad \forall\,(\mathbf{v},\boldsymbol{\eta}) \in \mathbf{Y}_D\,, \end{aligned}$$

where

$$\mathbf{X}_D := \mathbb{H}(\mathbf{div}\,;\Omega^-) \times \mathbf{H}^{1/2}(\Gamma)\,, \quad \mathbf{Y}_D := \mathbf{L}^2(\Omega^-) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega^-)\,,$$

$\mathbf{a}_D : \mathbf{X}_D \times \mathbf{X}_D \to \mathrm{R}$ and $\mathbf{b}_D : \mathbf{X}_D \times \mathbf{Y}_D \to \mathrm{R}$ are the bounded bilinear forms defined by

(78)
$$\begin{aligned} \mathbf{a}_D((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\tau},\boldsymbol{\psi})) := \int_{\Omega^-} \boldsymbol{\sigma}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} + \langle \mathbf{W}\,\boldsymbol{\varphi},\boldsymbol{\psi}\rangle_\Gamma + \left\langle \left(\frac{1}{2}\mathbf{I} + \mathbf{K}^{\mathtt{t}}\right)\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\sigma}),\boldsymbol{\psi}\right\rangle_\Gamma \\ + \langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}),\mathbf{V}\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\sigma})\rangle_\Gamma - \left\langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}),\left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)\boldsymbol{\varphi}\right\rangle_\Gamma \end{aligned}$$

and

$$\mathbf{b}_D((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{v},\boldsymbol{\eta})) := \int_{\Omega^-} \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_{\Omega^-} \boldsymbol{\eta} : \boldsymbol{\tau}\,, \tag{79}$$

and $\mathbf{F}_D : \mathbf{X}_D \to \mathrm{R}$ and $\mathbf{G}_D : \mathbf{Y}_D \to \mathrm{R}$ are the bounded linear functionals given by

$$\mathbf{F}_D(\boldsymbol{\tau},\boldsymbol{\psi}) := -\langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}),\mathbf{g}_D\rangle_{\Gamma_0} \qquad \text{and} \qquad \mathbf{G}_D(\mathbf{v},\boldsymbol{\eta}) := -\int_{\Omega^-} \mathbf{f}\cdot\mathbf{v}\,. \tag{80}$$

We first let $\mathbf{V}_D$ be the kernel of the bounded linear operator induced by $\mathbf{b}_D$ (cf. (79)), that is

$$\mathbf{V}_D := \left\{ (\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{X}_D : \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}} \quad \text{and} \quad \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \quad \text{in} \quad \Omega^- \right\},$$

and identify the solutions of the homogeneous problem associated with (77).

LEMMA 5.6. *The set of solutions of the homogeneous version of* (77) *is given by*

$$\Big\{ \big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})\big) := \big((\mathbf{0},\mathbf{z}),(\mathbf{0},\mathbf{0})\big) : \qquad \mathbf{z} \in \mathbf{RM}(\Gamma) \Big\}.$$

*Proof.* Let $\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})\big) \in \mathbf{X}_D \times \mathbf{Y}_D$ be a solution of (77) with $\mathbf{g}_D = \mathbf{0}$ on $\Gamma_0$ and $\mathbf{f} = \mathbf{0}$ in $\Omega^-$. It is clear from the second equation that $(\boldsymbol{\sigma},\boldsymbol{\varphi}) \in \mathbf{V}_D$, that is $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathtt{t}}$ and $\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{0}$ in $\Omega^-$. Then, taking in particular $(\boldsymbol{\tau},\boldsymbol{\psi}) = (\boldsymbol{\sigma},\boldsymbol{\varphi})$ in the first equation, recalling that $\mathbf{K}^{\mathtt{t}}$ is the adjoint of $\mathbf{K}$, and then applying the inequalities (75) (cf. Lemma 5.4), (54), and (53), we find that

(81)

$$\begin{aligned} 0 = \mathbf{a}_D\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\sigma},\boldsymbol{\varphi})\big) &= \|\boldsymbol{\sigma}^{\mathtt{d}}\|^2_{0,\Omega^-} + \langle \mathbf{W}\,\boldsymbol{\varphi},\boldsymbol{\varphi}\rangle_\Gamma + \langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\sigma}), \mathbf{V}\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\sigma})\rangle_\Gamma \\ &\geq c_1\|\boldsymbol{\sigma} - \boldsymbol{\pi}_{\scriptscriptstyle I}\boldsymbol{\sigma}\|^2_{0,\Omega^-} + \tilde{\alpha}_2\,\|\boldsymbol{\varphi} - \boldsymbol{\pi}_{\scriptscriptstyle RM}\boldsymbol{\varphi}\|^2_{1/2,\Gamma} + \tilde{\alpha}_1\,\|\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\sigma}) - \boldsymbol{\pi}_{\scriptscriptstyle \nu}\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\sigma})\|^2_{-1/2,\Gamma}\,. \end{aligned}$$

Therefore $\boldsymbol{\sigma} = c\,\mathbf{I}$ and $\boldsymbol{\varphi} = \mathbf{z} \in \mathbf{RM}(\Gamma)$. As a consequence, and using the characterization of the kernels of $\mathbf{V}$ and $\mathbf{W}$ given by (49), we find that the first equation of the homogeneous (77) becomes

$$c\,\langle \boldsymbol{\nu},\boldsymbol{\psi}\rangle_\Gamma + \mathbf{b}_D((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{u},\boldsymbol{\chi})) = 0 \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{X}_D\,.$$

Then, taking $\boldsymbol{\psi} = \mathbf{0}$ in the above equation, and using the inf-sup condition (72), which is possible in this case thanks to the inclusion $\mathbb{H}_0(\mathbf{div}\,;\Omega^-) \subseteq \mathbb{H}(\mathbf{div}\,;\Omega^-)$ and the fact that the expressions defining $\mathbf{b}_N$ and $\mathbf{b}_D$ coincide, we deduce that $(\mathbf{u},\boldsymbol{\chi}) = (\mathbf{0},\mathbf{0})$. In this way, we obtain that for any $\boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Gamma)$ there holds $c\,\langle \boldsymbol{\nu},\boldsymbol{\psi}\rangle_\Gamma = 0$, which necessarily implies that $c = 0$, and thus $\boldsymbol{\sigma} = \mathbf{0}$. Conversely, it is not difficult to see, using again the characterization of $\ker(\mathbf{W})$ (cf. (49)), that for any $\mathbf{z} \in \mathbf{RM}(\Gamma)$, $\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})\big) := \big((\mathbf{0},\mathbf{z}),(\mathbf{0},\mathbf{0})\big)$ solves the homogeneous version of (77).

□

Similarly to the analysis in Section 5.1, and in order to guarantee the unique solvability of the coupled problem (77), we now look for the solution $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi}))$ in the space $\widetilde{\mathbf{X}}_D \times \mathbf{Y}_D$, where

$$\widetilde{\mathbf{X}}_D := \mathbb{H}(\mathbf{div}\,;\Omega^-) \times \mathbf{H}^{1/2}_0(\Gamma)\,, \tag{82}$$

which yields the kernel of the operator defined by $\mathbf{b}_D : \widetilde{\mathbf{X}}_D \times \mathbf{Y}_D \to \mathrm{R}$ to become

$$\widetilde{\mathbf{V}}_D := \Big\{ (\boldsymbol{\tau},\boldsymbol{\psi}) \in \widetilde{\mathbf{X}}_D : \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}} \quad \text{and} \quad \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \quad \text{in} \quad \Omega^- \Big\}.$$

In turn, thanks again to the characterization of the kernels (cf. (49)) and the symmetry-type property of $\mathbf{W}$, we deduce that it suffices to require the first equation of (77) for any $(\boldsymbol{\tau},\boldsymbol{\psi}) \in \widetilde{\mathbf{X}}_D$. Therefore, instead of (77), we now look for $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})) \in \widetilde{\mathbf{X}}_D \times \mathbf{Y}_D$ such that

(83)

$$\begin{aligned} \mathbf{a}_D((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\tau},\boldsymbol{\psi})) + \mathbf{b}_D((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{u},\boldsymbol{\chi})) &= \mathbf{F}_D(\boldsymbol{\tau},\boldsymbol{\psi}) \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\psi}) \in \widetilde{\mathbf{X}}_D\,, \\ \mathbf{b}_D((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{v},\boldsymbol{\eta})) &= \mathbf{G}_D(\mathbf{v},\boldsymbol{\eta}) \qquad \forall\,(\mathbf{v},\boldsymbol{\eta}) \in \mathbf{Y}_D\,, \end{aligned}$$

Note, according to the definition of $\mathbf{a}_D$ (cf. (78)) and the identity $\mathbf{V}(\boldsymbol{\nu}) = \mathbf{0}$ (cf. (49)), that for any $c \in \mathrm{R}$ there holds $\mathbf{a}_D((c\,\mathbf{I},\mathbf{0}),(c\,\mathbf{I},\mathbf{0})) = 0$, which proves that $\mathbf{a}_D$ is not $\widetilde{\mathbf{V}}_D$-elliptic. However, the following lemma establishes the weak-coerciveness of $\mathbf{a}_D$ on this kernel.

LEMMA 5.7. *The bilinear form* $\mathbf{a}_D : \widetilde{\mathbf{V}}_D \times \widetilde{\mathbf{V}}_D \to \mathrm{R}$ *defines an invertible operator* $\mathbf{A}_D : \widetilde{\mathbf{V}}_D \to \widetilde{\mathbf{V}}_D$.

*Proof.* Using that $\mathbf{K}^{\mathtt{t}}$ is the adjoint of $\mathbf{K}$ and the inequalities (75), (51), and (53), it is easy to show that

(84)
$$\mathbf{a}_D\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\sigma},\boldsymbol{\varphi})\big)+\|\boldsymbol{\pi}_I\boldsymbol{\sigma}\|^2_{0,\Omega^-} \;\geq\; C\left\{\|\boldsymbol{\sigma}\|^2_{\mathbf{div}\,;\Omega^-}+\|\boldsymbol{\varphi}\|^2_{1/2,\Gamma}\right\} \qquad \forall\,(\boldsymbol{\sigma},\boldsymbol{\varphi})\;\in\;\widetilde{\mathbf{V}}_D.$$

Therefore, the operator $\mathbf{A}_D$ is Fredholm of index zero. If $\mathbf{A}_D((\boldsymbol{\sigma},\boldsymbol{\varphi}))=0$, then (by the same arguments)

$$0=\mathbf{a}_D\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\sigma},\boldsymbol{\varphi})\big)\geq C\left\{\|\boldsymbol{\sigma}-\boldsymbol{\pi}_I\boldsymbol{\sigma}\|^2_{\mathbf{div}\,;\Omega^-}\;+\;\|\boldsymbol{\varphi}\|^2_{1/2,\Gamma}\right\}, \tag{85}$$

which implies that $\boldsymbol{\varphi}=0$ and $\boldsymbol{\sigma}=c\mathbf{I}$ for some $c\in\mathrm{R}$. Therefore $\boldsymbol{\sigma}^{\mathtt{d}}=0$, $\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\sigma})=c\boldsymbol{\nu}$ and, using (49)

$$0=\mathbf{a}_D\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{0},\boldsymbol{\psi})\big)\;=\;c\left\langle\left(\frac{1}{2}\,\mathbf{I}\;+\;\mathbf{K}^{\mathtt{t}}\right)\boldsymbol{\nu},\boldsymbol{\psi}\right\rangle_\Gamma=\langle\boldsymbol{\nu},\boldsymbol{\psi}\rangle_\Gamma \qquad \forall\boldsymbol{\psi}\in\mathbf{H}^{1/2}_0(\Gamma).$$

This identity, (55), and (52) imply that $c=0$. We have thus proved that $\mathbf{A}_D$ is injective, and therefore invertible. □

The well-posedness of (83) can now be established.

THEOREM 5.2. *Given* $\mathbf{g}_D\;\in\;\mathbf{H}^{1/2}(\Gamma_0)$ *and* $\mathbf{f}\;\in\;\mathbf{L}^2(\Omega^-)$*, there exists a unique* $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi}))\;\in\;\widetilde{\mathbf{X}}_D\times\mathbf{Y}_D$ *solution to* (83)*. In addition, there exists* $C>0$ *such that*

$$\|((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi}))\|_{\mathbf{X}_D\times\mathbf{Y}_D}\;\leq\;C\left\{\|\mathbf{g}_D\|_{1/2,\Gamma_0}\;+\;\|\mathbf{f}\|_{0,\Omega^-}\right\}.$$

*Proof.* It is clear from the beginning of the section that $\mathbf{a}_D$ and $\mathbf{b}_D$ are bounded bilinear forms. In addition, the continuous inf-sup condition for $\mathbf{b}_D:\widetilde{\mathbf{X}}_D\times\mathbf{Y}_D\to\mathrm{R}$ follows straightforwardly from (72) by noting that the expressions defining $\mathbf{b}_D$ (cf. (79)) and $\mathbf{b}_N$ (cf. (66)) coincide and that certainly $\mathbb{H}_0(\mathbf{div}\,;\Omega^-)\;\subseteq\;\mathbb{H}(\mathbf{div}\,;\Omega^-)$. Consequently, the proof is completed by applying the corresponding result from the Babuška-Brezzi theory (see, e.g. [11, Theorem 1.1, Chapter II]).

□

The proof of weak coerciveness of $\mathbf{a}_D$ (cf. Lemma 5.7) uses a compactness argument starting from a Gårding inequality. At the discrete level this would impose a certain restriction on the approximation properties of the space to be fine enough. Note that an even more indirect argument (by contradiction) was used in [14, Lemma 4.3] for the analysis of the coupling of mixed-FEM and BEM as applied to the elasticity problem. However, a mesh-dependent norm had to be employed there instead of the usual norm of $\mathbb{H}(\mathbf{div}\,;\Omega^-)$, and it is not clear from the proof whether the constants involved depend or not on $h$. In the next Lemma we are going to give a refined version of the weak-coercivity that can be inherited at the discrete level.

LEMMA 5.8. *Let* $\widetilde{\boldsymbol{\psi}}$ *be an arbitrary but fixed element in* $\mathbf{H}^{1/2}(\Gamma)$ *such that* $\langle\boldsymbol{\nu},\widetilde{\boldsymbol{\psi}}\rangle_\Gamma\;>\;0$*. Consider the functional* $c:\mathbb{H}(\mathbf{div}\,,\Omega^-)\to\mathrm{R}$ *given by*

$$c(\boldsymbol{\sigma}):=\frac{1}{|\Omega^-|}\int_{\Omega^-}\operatorname{tr}\boldsymbol{\sigma}.$$

*Then there exist positive* $C$ *and* $\delta$*, depending on* $\widetilde{\boldsymbol{\psi}}$*, such that*

$$\mathbf{a}_D\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\sigma},\boldsymbol{\varphi}+\delta c(\boldsymbol{\sigma})\widetilde{\boldsymbol{\psi}})\big)\geq C\|(\boldsymbol{\sigma},\boldsymbol{\varphi})\|^2_{\mathbf{X}_D} \qquad \forall(\boldsymbol{\sigma},\boldsymbol{\varphi})\in\widetilde{\mathbf{V}}_D.$$

*Proof.* Because of (55) we can assume that $\widetilde{\boldsymbol{\psi}} \in \mathbf{H}_0^{1/2}(\Gamma)$ (just write $\widetilde{\boldsymbol{\psi}} - \boldsymbol{\pi}_{RM}\widetilde{\boldsymbol{\psi}}$ instead of $\widetilde{\boldsymbol{\psi}}$). Let us next notice that $\boldsymbol{\pi}_I\boldsymbol{\sigma} = c(\boldsymbol{\sigma})\mathbf{I}$ and therefore

$$\mathbf{a}_D\big(\boldsymbol{\pi}_I\boldsymbol{\sigma}, \mathbf{0}), (\mathbf{0}, c(\boldsymbol{\sigma})\widetilde{\boldsymbol{\psi}})\big) = c(\boldsymbol{\sigma})^2\langle \boldsymbol{\nu}, \widetilde{\boldsymbol{\psi}}\rangle_\Gamma.$$

This inequality, combined with the bound

$$|\mathbf{a}_D\big((\boldsymbol{\sigma} - \boldsymbol{\pi}_I\boldsymbol{\sigma}, \boldsymbol{\varphi}), (0, c(\boldsymbol{\sigma})\widetilde{\boldsymbol{\psi}})\big)| \le \frac{1}{2}c(\boldsymbol{\sigma})^2\langle \boldsymbol{\nu}, \widetilde{\boldsymbol{\psi}}\rangle_\Gamma + \frac{1}{2}\frac{\|\mathbf{a}_D\|^2\|\widetilde{\boldsymbol{\psi}}\|^2_{1/2,\Gamma}}{\langle \boldsymbol{\nu}, \widetilde{\boldsymbol{\psi}}\rangle_\Gamma}\|\boldsymbol{\sigma} - \boldsymbol{\pi}_I\boldsymbol{\sigma}\|^2_{\mathbf{X}_D}$$

prove

$$\mathbf{a}_D\big((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (0, c(\boldsymbol{\sigma})\widetilde{\boldsymbol{\psi}})\big) \ge \frac{1}{2}c(\boldsymbol{\sigma})^2\langle \boldsymbol{\nu}, \widetilde{\boldsymbol{\psi}}\rangle_\Gamma - \frac{1}{2}\frac{\|\mathbf{a}_D\|^2\|\widetilde{\boldsymbol{\psi}}\|^2_{1/2,\Gamma}}{\langle \boldsymbol{\nu}, \widetilde{\boldsymbol{\psi}}\rangle_\Gamma}\|\boldsymbol{\sigma} - \boldsymbol{\pi}_I\boldsymbol{\sigma}\|^2_{\mathbf{X}_D}. \tag{86}$$

At the same time, we know (see the proof of Lemma 5.7) that there exists $\widetilde{C} > 0$ such that

$$\mathbf{a}_D\big((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\boldsymbol{\sigma}, \boldsymbol{\varphi})\big) \ge \widetilde{C}\|(\boldsymbol{\sigma} - \boldsymbol{\pi}_I\boldsymbol{\sigma}, \boldsymbol{\varphi})\|^2_{\mathbf{X}_D} \qquad \forall(\boldsymbol{\sigma}, \boldsymbol{\varphi}) \in \widetilde{\mathbf{V}}_D. \tag{87}$$

Taking

$$\delta := \frac{\widetilde{C}\langle \boldsymbol{\nu}, \widetilde{\boldsymbol{\psi}}\rangle_\Gamma}{\|\mathbf{a}_D\|^2\|\widetilde{\boldsymbol{\psi}}\|^2_{1/2,\Gamma}}$$

and combining (86)-(87), we easily prove that

$$\mathbf{a}_D\big((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\boldsymbol{\sigma}, \boldsymbol{\varphi}+\delta c(\boldsymbol{\sigma})\widetilde{\boldsymbol{\psi}})\big) \ge \frac{\widetilde{C}\langle \boldsymbol{\nu}, \widetilde{\boldsymbol{\psi}}\rangle^2_\Gamma}{2\|\mathbf{a}_D\|^2\|\widetilde{\boldsymbol{\psi}}\|^2_{1/2,\Gamma}}c(\boldsymbol{\sigma})^2 + \frac{\widetilde{C}}{2}\|(\boldsymbol{\sigma} - \boldsymbol{\pi}_I\boldsymbol{\sigma}, \boldsymbol{\varphi})\|^2_{\mathbf{X}_D}. \tag{88}$$

This finishes the proof. □

We note that the result of Lemma 5.8 fits into what is called a T-coercivity result. In this case, the bounded isomorphism $T(\boldsymbol{\sigma}, \boldsymbol{\varphi}) := (\boldsymbol{\sigma}, \boldsymbol{\varphi} + c(\boldsymbol{\sigma})\delta\widetilde{\boldsymbol{\psi}})$ applied to the second component of the bilinear form makes it coercive. This gives an alternative proof of Lemma 5.7 providing additionally an estimate of the norm of the inverse of the operator $\mathbf{A}_D$, which depends exclusively on $\langle \boldsymbol{\nu}, \widetilde{\boldsymbol{\psi}}\rangle_\Gamma$, $\|\widetilde{\boldsymbol{\psi}} - \boldsymbol{\pi}_{RM}\widetilde{\boldsymbol{\psi}}\|_{1/2,\Gamma}$, the norm of $\mathbf{a}_D$, and the coercivity constant of (87).

**5.3. C & H coupling with non-homogeneous Neumann boundary conditions on $\Gamma_0$.** Similarly as in the previous section, we now apply again the Costabel & Han coupling method (see [20], [38]) and incorporate the boundary integral equations (61) and (62) into the dual-mixed variational formulation in $\Omega^-$ given by (26), which considers the non-homogeneous Neumann boundary condition $\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\sigma}) = \mathbf{g}_N \in \mathbf{H}^{-1/2}(\Gamma_0)$ on $\Gamma_0$. In this way, our coupled variational formulation reads as follows: Find $((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\mathbf{u}, \boldsymbol{\chi}, \boldsymbol{\lambda})) \in \mathbf{X}_N \times \mathbf{Y}_N$ such that

$$\begin{aligned} \mathbf{a}_N((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\boldsymbol{\tau}, \boldsymbol{\psi})) + \mathbf{b}_N((\boldsymbol{\tau}, \boldsymbol{\psi}), (\mathbf{u}, \boldsymbol{\chi}, \boldsymbol{\lambda})) &= \mathbf{F}_N(\boldsymbol{\tau}, \boldsymbol{\psi}) \qquad \forall\,(\boldsymbol{\tau}, \boldsymbol{\psi}) \in \mathbf{X}_N\,, \\ \mathbf{b}_N((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\mathbf{v}, \boldsymbol{\eta}, \boldsymbol{\xi})) &= \mathbf{G}_N(\mathbf{v}, \boldsymbol{\eta}, \boldsymbol{\xi}) \quad \forall\,(\mathbf{v}, \boldsymbol{\eta}, \boldsymbol{\xi}) \in \mathbf{Y}_N\,, \end{aligned} \tag{89}$$

where

$$\mathbf{X}_N := \mathbb{H}(\mathbf{div}\,;\Omega^-) \times \mathbf{H}^{1/2}(\Gamma)\,, \quad \mathbf{Y}_N := \mathbf{L}^2(\Omega^-) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega^-) \times \mathbf{H}^{1/2}(\Gamma_0)\,,$$

$\mathbf{a}_N : \mathbf{X}_N \times \mathbf{X}_N \to \mathrm{R}$ and $\mathbf{b}_N : \mathbf{X}_N \times \mathbf{Y}_N \to \mathrm{R}$ are the bounded bilinear forms defined by

(90)
$$\mathbf{a}_N((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\tau},\boldsymbol{\psi})) := \int_{\Omega^-} \boldsymbol{\sigma}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} + \langle \mathbf{W}\,\boldsymbol{\varphi},\boldsymbol{\psi}\rangle_\Gamma + \left\langle \left(\frac{1}{2}\mathbf{I} + \mathbf{K}^{\mathtt{t}}\right)\gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\sigma}),\boldsymbol{\psi}\right\rangle_\Gamma + \langle \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}),\mathbf{V}\gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\sigma})\rangle_\Gamma - \left\langle \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}),\left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)\boldsymbol{\varphi}\right\rangle_\Gamma$$

and

$$\mathbf{b}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi})) := \int_{\Omega^-} \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_{\Omega^-} \boldsymbol{\eta} : \boldsymbol{\tau} + \langle \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}),\boldsymbol{\xi}\rangle_{\Gamma_0}\,, \tag{91}$$

and $\mathbf{F}_N : \mathbf{X}_N \to \mathrm{R}$ and $\mathbf{G}_N : \mathbf{Y}_N \to \mathrm{R}$ are the bounded linear functionals given by

$$\mathbf{F}_N(\boldsymbol{\tau},\boldsymbol{\psi}) := 0\,, \qquad \text{and} \qquad \mathbf{G}_N(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi}) := -\int_{\Omega^-} \mathbf{f}\cdot\mathbf{v} + \langle \mathbf{g}_N,\boldsymbol{\xi}\rangle_{\Gamma_0}\,. \tag{92}$$

We now observe from (91) that $\mathbf{B}_N : \mathbf{X}_N \to \mathbf{Y}_N$, the bounded linear operator induced by $\mathbf{b}_N$, is given by

$$\mathbf{B}_N((\boldsymbol{\tau},\boldsymbol{\psi})) := \left(\mathbf{div}\,\boldsymbol{\tau},\frac{1}{2}(\boldsymbol{\tau}-\boldsymbol{\tau}^{\mathtt{t}}),\mathcal{R}_0\,\gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau})\right) \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{X}_N\,,$$

where $\mathcal{R}_0 : \mathbf{H}^{-1/2}(\Gamma_0) \to \mathbf{H}^{1/2}(\Gamma_0)$ is the respective Riesz operator. Then, we begin the analysis of solvability of (89) by proving next that $\mathbf{b}_N$ satisfies the continuous inf-sup condition, which is equivalent to the surjectivity of $\mathbf{B}_N$.

LEMMA 5.9. *There exists $\beta > 0$ such that for any $(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi}) \in \mathbf{Y}_N$ there holds*

$$\sup_{(\boldsymbol{\tau},\boldsymbol{\psi})\in\mathbf{X}_N\setminus\{\mathbf{0}\}} \frac{\mathbf{b}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi}))}{\|(\boldsymbol{\tau},\boldsymbol{\psi})\|_{\mathbf{X}_N}} \geq \beta\,\|(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi})\|_{\mathbf{Y}_N}\,. \tag{93}$$

*Proof.* We proceed as in the proof of Lemma 5.2. In fact, given $(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi}) \in \mathbf{Y}_N$, we let $\mathbf{z}$ be the unique element in $\mathbf{H}^1_\Gamma(\Omega^-) := \left\{\mathbf{w} \in \mathbf{H}^1(\Omega^-) : \quad \mathbf{w} = \mathbf{0} \quad \text{on} \quad \Gamma\right\}$, whose existence is guaranteed by the second Korn inequality, such that

$$\int_{\Omega^-} \mathbf{e}(\mathbf{z}) : \mathbf{e}(\mathbf{w}) = -\int_{\Omega^-} \mathbf{v}\cdot\mathbf{w} - \int_{\Omega^-} \boldsymbol{\eta} : \nabla\mathbf{w} + \langle \mathcal{R}_0^{-1}(\boldsymbol{\xi}),\gamma^-(\mathbf{w})\rangle_{\Gamma_0} \quad \forall\,\mathbf{w} \in \mathbf{H}^1_\Gamma(\Omega^-)\,.$$

Hence, defining $\widehat{\boldsymbol{\tau}} := \mathbf{e}(\mathbf{z}) + \boldsymbol{\eta} \in \mathbb{L}^2(\Omega^-)$, we deduce from the above formulation that $\mathbf{div}\,\widehat{\boldsymbol{\tau}} = \mathbf{v}$ in $\Omega^-$, which shows that $\widehat{\boldsymbol{\tau}} \in \mathbb{H}(\mathbf{div}\,;\Omega^-)$, and then that $\gamma_{\boldsymbol{\nu}}^-(\widehat{\boldsymbol{\tau}}) = \mathcal{R}_0^{-1}(\boldsymbol{\xi})$ on $\Gamma_0$. In this way, it is easy to see that $\mathbf{B}_N((\widehat{\boldsymbol{\tau}},\mathbf{0})) = (\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi})$, which ends the proof.

□

In what follows we let $\mathbf{V}_N$ be the kernel of $\mathbf{B}_N$, that is

$$\mathbf{V}_N := \left\{(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{X}_N : \ \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}} \text{ and } \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \text{ in } \Omega^-\,, \text{ and } \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}) = \mathbf{0} \text{ on } \Gamma_0\right\},$$

and establish a positiveness property of $\mathbf{a}_N$ on $\mathbf{V}_N$.

LEMMA 5.10. *There exists $\widetilde{\alpha} > 0$ such that*

$$\mathbf{a}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\boldsymbol{\tau},\boldsymbol{\psi})) \geq \widetilde{\alpha}\left\{\|\boldsymbol{\tau}\|^2_{\mathbf{div}\,;\Omega^-} + \|\boldsymbol{\psi} - \boldsymbol{\pi}_{RM}\,\boldsymbol{\psi}\|^2_{1/2,\Gamma}\right\} \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{V}_N., \tag{94}$$

*Proof.* Let $(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{V}_N$. Then, recalling that $\mathbf{K}^{\mathtt{t}}$ is the adjoint of $\mathbf{K}$, noting that $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega^-)$, and then applying the inequalities (75) (cf. Lemma 5.4), (76) (cf. Lemma 5.5), (54), and (53), we find that

(95)
$$\begin{aligned}\mathbf{a}_N\big((\boldsymbol{\tau},\boldsymbol{\psi}),(\boldsymbol{\tau},\boldsymbol{\psi})\big) &= \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma + \langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}),\mathbf{V}\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau})\rangle_\Gamma \\ &\geq c_1\|\boldsymbol{\tau}-\boldsymbol{\pi}_I\boldsymbol{\tau}\|^2_{\mathbf{div}\,;\Omega^-} + \tilde{\alpha}_2\,\|\boldsymbol{\psi}-\boldsymbol{\pi}_{RM}\boldsymbol{\psi}\|^2_{1/2,\Gamma} + \tilde{\alpha}_1\,\|\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau})-\boldsymbol{\pi}_\nu\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau})\|^2_{-1/2,\Gamma} \\ &\geq c_1\,c_2\,\|\boldsymbol{\tau}\|^2_{\mathbf{div}\,;\Omega^-} + \tilde{\alpha}_2\,\|\boldsymbol{\psi}-\boldsymbol{\pi}_{RM}\|^2_{1/2,\Gamma}\,,\end{aligned}$$

which yields the required inequality (94).

□

LEMMA 5.11. *The set of solutions of the homogeneous version of* (89) *is given by*

$$\Big\{\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi},\boldsymbol{\lambda})\big) := \big((\mathbf{0},\mathbf{z}),(\mathbf{0},\mathbf{0},\mathbf{0})\big): \qquad \mathbf{z} \in \mathbf{RM}(\Gamma)\Big\}.$$

*Proof.* It follows similarly to the proof of Lemma 5.6. Indeed, let $\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi},\boldsymbol{\lambda})\big) \in \mathbf{X}_N \times \mathbf{Y}_N$ be a solution of (77) with $\mathbf{g}_N = \mathbf{0}$ on $\Gamma_0$ and $\mathbf{f} = \mathbf{0}$ in $\Omega^-$. It is clear from the second equation that $(\boldsymbol{\sigma},\boldsymbol{\varphi}) \in \mathbf{V}_N$. Then, taking in particular $(\boldsymbol{\tau},\boldsymbol{\psi}) = (\boldsymbol{\sigma},\boldsymbol{\varphi})$ in the first equation, and using the inequality (94) (cf. Lemma 5.10), we find that

$$0 = \mathbf{a}_N\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\sigma},\boldsymbol{\varphi})\big) \geq \widetilde{\alpha}\Big\{\|\boldsymbol{\sigma}\|^2_{\mathbf{div}\,;\Omega^-} + \|\boldsymbol{\varphi}-\boldsymbol{\pi}_{RM}\boldsymbol{\varphi}\|^2_{1/2,\Gamma}\Big\}, \tag{96}$$

from where it follows that $\boldsymbol{\sigma} = \mathbf{0}$ in $\Omega^-$ and $\boldsymbol{\varphi} = \mathbf{z}$ for $\mathbf{z} \in \mathbf{RM}(\Gamma)$. As a consequence, and using the characterization of the kernel $\mathbf{W}$ given by (49), we find that the first equation of the homogeneous (89) becomes

$$\mathbf{b}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{u},\boldsymbol{\chi},\boldsymbol{\lambda})) = 0 \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{X}_N\,,$$

which, thanks to the inf-sup condition (93) (cf. Lemma 5.9), yields $(\mathbf{u},\boldsymbol{\chi},\boldsymbol{\lambda}) = (\mathbf{0},\mathbf{0},\mathbf{0})$. Conversely, it is not difficult to see, using again the characterization of $\ker(\mathbf{W})$ (cf. (49)), that for any $\mathbf{z} \in \mathbf{RM}(\Gamma)$, $\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi},\boldsymbol{\lambda})\big) := \big((\mathbf{0},\mathbf{z}),(\mathbf{0},\mathbf{0},\mathbf{0})\big)$ solves the homogeneous version of (89).

□

Therefore, similarly as for the analysis in Sections 5.1 and 5.2, and in order to guarantee the unique solvability of the coupled problem (89), we now look for the solution $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi},\boldsymbol{\lambda}))$ in the space $\widetilde{\mathbf{X}}_N \times \mathbf{Y}_N$, where

$$\widetilde{\mathbf{X}}_N := \mathbb{H}(\mathbf{div}\,;\Omega^-) \times \mathbf{H}^{1/2}_0(\Gamma)\,, \tag{97}$$

which yields the kernel of the operator defined by $\mathbf{b}_N : \widetilde{\mathbf{X}}_N \times \mathbf{Y}_N \to \mathrm{R}$ to become

$$\widetilde{\mathbf{V}}_N := \Big\{(\boldsymbol{\tau},\boldsymbol{\psi}) \in \widetilde{\mathbf{X}}_N :\ \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}} \text{ and } \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \text{ in } \Omega^-\,,\ \text{and } \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}) = \mathbf{0} \text{ on } \Gamma_0\Big\}.$$

In addition, applying again the characterization of the kernels (cf. (49)) and the symmetry-type property of $\mathbf{W}$, we find that it suffices to require the first equation of (89) for any $(\boldsymbol{\tau},\boldsymbol{\psi}) \in \widetilde{\mathbf{X}}_N$. As a consequence, instead of (89), we now look for $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi},\boldsymbol{\lambda})) \in \widetilde{\mathbf{X}}_N \times \mathbf{Y}_N$ such that

(98)
$$\begin{aligned}\mathbf{a}_N((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\tau},\boldsymbol{\psi})) + \mathbf{b}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{u},\boldsymbol{\chi},\boldsymbol{\lambda})) &= \mathbf{F}_N(\boldsymbol{\tau},\boldsymbol{\psi}) \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\psi}) \in \widetilde{\mathbf{X}}_N\,, \\ \mathbf{b}_N((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi})) &= \mathbf{G}_N(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi}) \quad \forall\,(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi}) \in \mathbf{Y}_N\,,\end{aligned}$$

Next, it follows from Lemma 5.10 and the above characterization of $\widetilde{\mathbf{V}}_N$ that the bilinear form $\mathbf{a}_N$ is strongly coercive on $\widetilde{\mathbf{V}}_N$. In addition, since actually $\mathbf{b}_N$ does not depend on the component $\boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Gamma)$, it is quite clear from Lemma 5.9 that $\mathbf{b}_N$ satisfies the continuous inf-sup condition on $\widetilde{\mathbf{X}}_N \times \mathbf{Y}_N$ as well. Hence, the well-posedness of (98) is readily established as follows.

THEOREM 5.3. *Given* $\mathbf{g}_N \in \mathbf{H}^{-1/2}(\Gamma_0)$ *and* $\mathbf{f} \in \mathbf{L}^2(\Omega^-)$*, there exists a unique* $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi},\boldsymbol{\lambda})) \in \widetilde{\mathbf{X}}_N \times \mathbf{Y}_N$ *solution to* (98)*. In addition, there exists* $C > 0$ *such that*

$$\|((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi},\boldsymbol{\lambda}))\|_{\mathbf{X}_N\times\mathbf{Y}_N} \le C\left\{ \|\mathbf{g}_N\|_{-1/2,\Gamma_0} + \|\mathbf{f}\|_{0,\Omega^-} \right\}.$$

*Proof.* According to the previous discussion, the proof follows by applying once again the usual result from the Babuška-Brezzi theory (see, e.g. [11, Theorem 1.1, Chapter II]).

□

**5.4. C & H coupling with homogeneous Neumann boundary conditions on $\Gamma_0$.** In what follows we proceed similarly as in the previous section and apply the Costabel & Han coupling method to the case of homogeneous Neumann boundary conditions on $\Gamma_0$. This means that we now incorporate the boundary integral equations (61) and (62) into the dual-mixed variational formulation in $\Omega^-$ given by (31). In this way, as in Sections 3.3 and 5.1, there is no need of introducing the additional unknown $\lambda \in \mathbf{H}^{1/2}(\Gamma_0)$, and hence our coupled variational formulation simply reads as follows: Find $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})) \in \mathbf{X}_N \times \mathbf{Y}_N$ such that

(99)
$$\begin{aligned} \mathbf{a}_N((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\tau},\boldsymbol{\psi})) + \mathbf{b}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{u},\boldsymbol{\chi})) &= \mathbf{F}_N(\boldsymbol{\tau},\boldsymbol{\psi}) \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{X}_N\,, \\ \mathbf{b}_N((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{v},\boldsymbol{\eta})) &= \mathbf{G}_N(\mathbf{v},\boldsymbol{\eta}) \qquad \forall\,(\mathbf{v},\boldsymbol{\eta}) \in \mathbf{Y}_N\,, \end{aligned}$$

where

$$\mathbf{X}_N := \mathbb{H}_0(\mathbf{div}\,;\Omega^-) \times \mathbf{H}^{1/2}(\Gamma)\,, \quad \mathbf{Y}_N := \mathbf{L}^2(\Omega^-) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega^-)\,,$$

$\mathbf{a}_N : \mathbf{X}_N \times \mathbf{X}_N \to \mathrm{R}$ and $\mathbf{b}_N : \mathbf{X}_N \times \mathbf{Y}_N \to \mathrm{R}$ are the bounded bilinear forms defined by

(100)
$$\begin{aligned} \mathbf{a}_N((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\tau},\boldsymbol{\psi})) := \int_{\Omega^-} \boldsymbol{\sigma}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} + \langle \mathbf{W}\,\boldsymbol{\varphi},\boldsymbol{\psi}\rangle_\Gamma + \left\langle \left(\frac{1}{2}\mathbf{I} + \mathbf{K}^{\mathtt{t}}\right)\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\sigma}),\boldsymbol{\psi}\right\rangle_\Gamma \\ + \langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}),\mathbf{V}\gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\sigma})\rangle_\Gamma - \left\langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}),\left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)\boldsymbol{\varphi}\right\rangle_\Gamma \end{aligned}$$

and

$$\mathbf{b}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{v},\boldsymbol{\eta})) := \int_{\Omega^-} \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_{\Omega^-} \boldsymbol{\eta} : \boldsymbol{\tau}\,, \tag{101}$$

and $\mathbf{F}_N : \mathbf{X}_N \to \mathrm{R}$ and $\mathbf{G}_N : \mathbf{Y}_N \to \mathrm{R}$ are the bounded linear functionals given by

$$\mathbf{F}_N(\boldsymbol{\tau},\boldsymbol{\psi}) := 0\,, \qquad \text{and} \qquad \mathbf{G}_N(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi}) := -\int_{\Omega^-} \mathbf{f}\cdot\mathbf{v}\,. \tag{102}$$

Concerning the solvability analysis of (99), we first observe that the continuous inf-sup condition for $\mathbf{b}_N$ was already proved by Lemma 5.2. In addition, it is clear that the kernel of $\mathbf{b}_N$ is given by

$$\mathbf{V}_N := \left\{ (\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{X}_N : \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}} \quad \text{and} \quad \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \quad \text{in} \quad \Omega^- \right\},$$

and that $\mathbf{a}_N$ satisfies the same positiveness property from Lemma 5.10, that is
(103)
$$\mathbf{a}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\boldsymbol{\tau},\boldsymbol{\psi})) \,\geq\, \widetilde{\alpha}\Big\{\,\|\boldsymbol{\tau}\|^2_{\mathbf{div}\,;\Omega^-} \,+\, \|\boldsymbol{\psi}-\boldsymbol{\pi}_{_{RM}}\boldsymbol{\psi}\|^2_{1/2,\Gamma}\Big\} \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\psi}) \,\in\, \mathbf{V}_N\,.$$

Moreover, basically the same proof of Lemma 5.11 shows that the set of solutions of the homogeneous version of (99) is given by

$$\Big\{\,\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})\big) \,:=\, \big((\mathbf{0},\mathbf{z}),(\mathbf{0},\mathbf{0})\big) : \qquad \mathbf{z} \,\in\, \mathbf{RM}(\Gamma)\,\Big\}\,.$$

Consequently, following a similar analysis to the one from the previous section, and in order to guarantee unique solvability of the resulting problem, (99) is reformulated as: Find $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})) \,\in\, \widetilde{\mathbf{X}}_N \times \mathbf{Y}_N$ such that
(104)
$$\begin{aligned}\mathbf{a}_N((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\tau},\boldsymbol{\psi})) \,+\, \mathbf{b}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{u},\boldsymbol{\chi})) \;&=\; \mathbf{F}_N(\boldsymbol{\tau},\boldsymbol{\psi}) \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\psi}) \in \widetilde{\mathbf{X}}_N\,,\\ \mathbf{b}_N((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{v},\boldsymbol{\eta})) \;&=\; \mathbf{G}_N(\mathbf{v},\boldsymbol{\eta}) \qquad \forall\,(\mathbf{v},\boldsymbol{\eta}) \in \mathbf{Y}_N\,,\end{aligned}$$

where

$$\widetilde{\mathbf{X}}_N \,:=\, \mathbb{H}_0(\mathbf{div}\,;\Omega^-) \times \mathbf{H}_0^{1/2}(\Gamma)\,.$$

Then, the kernel of $\mathbf{b}_N : \widetilde{\mathbf{X}}_N \times \mathbf{Y}_N \to \mathrm{R}$ becomes now

$$\widetilde{\mathbf{V}}_N \,:=\, \Big\{\,(\boldsymbol{\tau},\boldsymbol{\psi}) \in \widetilde{\mathbf{X}}_N : \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}} \quad \text{and} \quad \mathbf{div}\,\boldsymbol{\tau} \,=\, \mathbf{0} \quad \text{in} \quad \Omega^-\,\Big\}\,,$$

whence (103) yields the strong coerciveness of $\mathbf{a}_N$ on $\widetilde{\mathbf{V}}_N$. In turn, since $\mathbf{b}_N$ does not depend on the component $\boldsymbol{\psi} \in \mathbf{H}_0^{1/2}(\Gamma)$, it is also clear from Lemma 5.2 that $\mathbf{b}_N$ satisfies the continuous inf-sup condition on $\widetilde{\mathbf{X}}_N \times \mathbf{Y}_N$ as well. These remarks and [11, Theorem 1.1, Chapter II] imply the well-posedness of (104).

THEOREM 5.4. *Given* $\mathbf{f} \,\in\, \mathbf{L}^2(\Omega^-)$*, there exists a unique* $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})) \,\in\, \widetilde{\mathbf{X}}_N \times \mathbf{Y}_N$ *solution to* (104)*. In addition, there exists* $C > 0$ *such that*

$$\|((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi}))\|_{\mathbf{X}_N\times\mathbf{Y}_N} \,\leq\, C\,\|\mathbf{f}\|_{0,\Omega^-}\,.$$

## 6. Galerkin schemes of the coupled formulations

In this section we study the well-posedness of the Galerkin schemes associated with each one of the coupled variational formulations analyzed in Section 5. Needless to say, no discrete analysis was reported in [29].

**6.1. J & N coupling with homogeneous Neumann on $\Gamma_0$.** We first let $\mathbb{H}^{\boldsymbol{\sigma}}_{h,0}$, $\mathbf{H}^{\boldsymbol{\varphi}}_{\tilde{h}}$, $\mathbf{L}^{\mathbf{u}}_h$, and $\mathbb{L}^{\boldsymbol{\chi}}_h$ be finite dimensional subspaces of $\mathbb{H}_0(\mathbf{div}\,;\Omega^-)$, $\mathbf{H}^{1/2}(\Gamma)$, $\mathbf{L}^2(\Omega^-)$, and $\mathbb{L}^2_{\mathtt{skew}}(\Omega^-)$, respectively, and define

$$\mathbf{H}^{\boldsymbol{\varphi}}_{\tilde{h},0} \,:=\, \Big\{\boldsymbol{\psi} \in \mathbf{H}^{\boldsymbol{\varphi}}_{\tilde{h}} : \quad \langle \mathbf{r},\boldsymbol{\psi}\rangle_{1/2,\Gamma} \,=\, 0 \quad \forall\,\mathbf{r} \in \mathbf{RM}(\Gamma)\Big\}\,, \tag{105}$$

which is clearly a subspace of $\mathbf{H}_0^{1/2}(\Gamma)$. Note that, because of reasons that will become clear below, we take a different meshsize for defining $\mathbf{H}^{\boldsymbol{\varphi}}_{\tilde{h}}$ (and hence $\mathbf{H}^{\boldsymbol{\varphi}}_{\tilde{h},0}$). Then we introduce the product spaces

$$\widetilde{\mathbf{X}}_{N,h} \,:=\, \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \,\times\, \mathbf{H}^{\boldsymbol{\varphi}}_{\tilde{h},0} \quad \text{and} \quad \mathbf{Y}_{N,h} \,:=\, \mathbf{L}^{\mathbf{u}}_h \,\times\, \mathbb{L}^{\boldsymbol{\chi}}_h\,,$$

and define the Galerkin scheme associated with (74) as: Find $((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_{\tilde{h}}), (\mathbf{u}_h, \boldsymbol{\chi}_h)) \in \widetilde{\mathbf{X}}_{N,h} \times \mathbf{Y}_{N,h}$ such that

(106)
$$\begin{aligned} \mathbf{a}_N((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_{\tilde{h}}), (\boldsymbol{\tau}, \boldsymbol{\psi})) + \mathbf{b}_N((\boldsymbol{\tau}, \boldsymbol{\psi}), (\mathbf{u}_h, \boldsymbol{\chi}_h)) &= \mathbf{F}_N(\boldsymbol{\tau}, \boldsymbol{\psi}) \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{\psi}) \in \widetilde{\mathbf{X}}_{N,h}\,, \\ \mathbf{b}_N((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_{\tilde{h}}), (\mathbf{v}, \boldsymbol{\eta})) &= \mathbf{G}_N(\mathbf{v}, \boldsymbol{\eta}) \quad \forall\, (\mathbf{v}, \boldsymbol{\eta}) \in \mathbf{Y}_{N,h}\,, \end{aligned}$$

where $\mathbf{a}_N$, $\mathbf{b}_N$, $\mathbf{F}_N$, and $\mathbf{G}_N$ are the bilinear forms and functionals defined in Section 5.1.

In order to prove the unique solvability, stability, and convergence of (106), we have in mind the discrete Babuška-Brezzi theory and consider in what follows the following assumptions:

**(H.1)** the bilinear form $\mathbf{b}_N$ satisfies the discrete inf-sup condition uniformly on $\widetilde{\mathbf{X}}_{N,h} \times \mathbf{Y}_{N,h}$, that is there exists $\tilde{\beta} > 0$, independent of $h$ and $\tilde{h}$, such that

(107)
$$\sup_{(\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}}) \in \widetilde{\mathbf{X}}_{N,h} \setminus \{\mathbf{0}\}} \frac{\mathbf{b}_N((\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}}), (\mathbf{v}, \boldsymbol{\eta}))}{\|(\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}})\|_{\mathbf{X}_N}} \geq \tilde{\beta}\, \|(\mathbf{v}, \boldsymbol{\eta})\|_{\mathbf{Y}_N} \qquad \forall\, (\mathbf{v}, \boldsymbol{\eta}) \in \mathbf{Y}_{N,h}\,.$$

**(H.2)** $\mathbf{div}\, \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \subseteq \mathbf{L}^{\mathbf{u}}_h$.

**(H.3)** there exists $\epsilon \in (0, 1/2)$ such that $\mathbf{H}^{\boldsymbol{\varphi}}_{\tilde{h},0} \subseteq \mathbf{H}^{1/2+\epsilon}(\Gamma)$ for each $\tilde{h} > 0$.

**(H.4)** the finite element subspace $\mathbf{H}^{\boldsymbol{\varphi}}_{\tilde{h},0}$ satisfies the inverse inequality, that is there exists $C > 0$, independent of $\tilde{h}$, such that

$$\|\boldsymbol{\varphi}_{\tilde{h}}\|_{1/2+\delta,\Gamma} \leq C\, \tilde{h}^{-\delta}\, \|\boldsymbol{\varphi}_{\tilde{h}}\|_{1/2,\Gamma} \qquad \forall\, \boldsymbol{\varphi}_{\tilde{h}} \in \mathbf{H}^{\boldsymbol{\varphi}}_{\tilde{h},0}\,, \quad \forall\, \delta \in [0, \epsilon]\,. \tag{108}$$

**(H.5)** the orthogonal projector $\Pi^{\boldsymbol{\chi}}_h : \mathbb{L}^2_{\mathtt{skew}}(\Omega^-) \to \mathbb{L}^{\boldsymbol{\chi}}_h$ satisfies the approximation property

(109)
$$\|\boldsymbol{\eta} - \Pi^{\boldsymbol{\chi}}_h(\boldsymbol{\eta})\|_{0,\Omega^-} \leq C\, h^{\delta}\, \|\boldsymbol{\eta}\|_{\delta,\Omega^-} \qquad \forall\, \boldsymbol{\eta} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega^-) \cap \mathbb{H}^{\delta}(\Omega^-)\,, \quad \forall\, \delta \in [0, 1]\,.$$

We notice that, being the bilinear form $\mathbf{b}_N$ independent of the $\boldsymbol{\psi}$-component, its discrete inf-sup condition (cf. (107) in **(H.1)**) involves only the subspaces $\mathbb{H}^{\boldsymbol{\sigma}}_{h,0}$, $\mathbf{L}^{\mathbf{u}}_h$, and $\mathbb{L}^{\boldsymbol{\chi}}_h$. In addition, since the discrete kernel of $\mathbf{b}_N$ becomes

(110)
$$\widetilde{\mathbf{V}}_{N,h} := \left\{ (\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}}) \in \widetilde{\mathbf{X}}_{N,h} := \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \mathbf{H}^{\boldsymbol{\varphi}}_{\tilde{h},0} : \int_{\Omega^-} \mathbf{v}_h \cdot \mathbf{div}\, \boldsymbol{\tau}_h = 0 \;\forall\, \mathbf{v}_h \in \mathbf{L}^{\mathbf{u}}_h \right.$$
$$\left. \text{and} \quad \int_{\Omega^-} \boldsymbol{\tau}_h : \boldsymbol{\eta}_h = 0 \quad \forall\, \boldsymbol{\eta}_h \in \mathbb{L}^{\boldsymbol{\chi}}_h \right\},$$

it is straightforward to see that hypothesis **(H.2)** yields

$$\mathbf{div}\, \boldsymbol{\tau}_h = \mathbf{0} \quad \text{in} \quad \Omega^- \qquad \forall\, (\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}}) \in \widetilde{\mathbf{V}}_{N,h}\,. \tag{111}$$

The statement (111) and hypotheses **(H.3)** - **(H.5)** are utilized next to prove that $\mathbf{a}_N$ is uniformly strongly coercive on $\widetilde{\mathbf{V}}_{N,h}$. The fact that $\mathbf{D} : \mathbf{H}^{1/2+\delta}(\Gamma) \to \mathbf{H}^{1+\delta}(\Omega^-)$ is a bounded linear operator for each $\delta \in [0, 1/2)$, which follows from similar arguments to those mentioned in the proof of Lemma 4.1, and which certainly extends the corresponding result for $\mathbf{D}$, will also be employed.

LEMMA 6.1. *There exist* $\tilde{\alpha}$, $c_0 > 0$, *independent of* $h$ *and* $\tilde{h}$, *such that whenever* $h \leq c_0\,\tilde{h}$, *there holds*

$$\mathbf{a}_N\big((\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}}), (\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}})\big) \;\geq\; \tilde{\alpha}\,\|(\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}})\|^2_{\mathbf{X}_N} \qquad \forall\,(\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}}) \in \widetilde{\mathbf{V}}_{N,h}\,. \tag{112}$$

*Proof.* Given $(\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}}) \in \widetilde{\mathbf{V}}_{N,h}$, we first proceed as in the proof of Lemma 5.1 and obtain

$$\mathbf{a}_N\big((\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}}), (\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}})\big) \;=\; \|\boldsymbol{\tau}_h^{\mathtt{d}}\|^2_{0,\Omega^-} \;+\; \langle \mathbf{W}\boldsymbol{\psi}_{\tilde{h}}, \boldsymbol{\psi}_{\tilde{h}}\rangle_\Gamma \;+\; \int_{\Omega^-} \nabla\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big) : \boldsymbol{\tau}_h^{\mathtt{d}}\,.$$

Note that, while $\boldsymbol{\tau}_h$ is divergence free (according to (111)), its lack of strong symmetry stops us of replacing $\displaystyle\int_{\Omega^-} \nabla\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big) : \boldsymbol{\tau}_h^{\mathtt{d}}$ exactly by $\displaystyle\int_{\Omega^-} \mathbf{e}\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big) : \boldsymbol{\tau}_h^{\mathtt{d}}$, as we did in that proof. However, what we can certainly do in the present discrete case is to write

$$\nabla\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big) \;=\; \mathbf{e}\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big) \;+\; \boldsymbol{\eta}\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big)\,,$$

where

$$\boldsymbol{\eta}\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big) \;=\; \frac{1}{2}\Big\{\,\nabla\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big) \;+\; \Big(\nabla\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big)\Big)^{\mathtt{t}}\,\Big\}\,.$$

In this way, we have that

$$\begin{aligned}
&\mathbf{a}_N\big((\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}}), (\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}})\big)\\
&\quad= \|\boldsymbol{\tau}_h^{\mathtt{d}}\|^2_{0,\Omega^-} \;+\; \langle \mathbf{W}\boldsymbol{\psi}_{\tilde{h}}, \boldsymbol{\psi}_{\tilde{h}}\rangle_\Gamma \;+\; \int_{\Omega^-} \mathbf{e}\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big) : \boldsymbol{\tau}_h^{\mathtt{d}} \;+\; \int_{\Omega^-} \boldsymbol{\eta}\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big) : \boldsymbol{\tau}_h^{\mathtt{d}}\,,
\end{aligned}$$

which, following the last part of the proof of Lemma 5.1, yields

$$\mathbf{a}_N\big((\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}}), (\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}})\big) \;\geq\; \frac{1}{2}\Big\{\|\boldsymbol{\tau}_h^{\mathtt{d}}\|^2_{0,\Omega^-} + \langle \mathbf{W}\boldsymbol{\psi}_{\tilde{h}}, \boldsymbol{\psi}_{\tilde{h}}\rangle_\Gamma\Big\} \;-\; \Big|\int_{\Omega^-} \boldsymbol{\eta}\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big) : \boldsymbol{\tau}_h^{\mathtt{d}}\Big|\,. \tag{113}$$

Next, using from (110) that $\displaystyle\int_{\Omega^-} \boldsymbol{\tau}_h^{\mathtt{d}} : \boldsymbol{\eta}_h \;=\; \int_{\Omega^-} \boldsymbol{\tau}_h : \boldsymbol{\eta}_h \;=\; 0 \quad \forall\,\boldsymbol{\eta}_h \in \mathbb{L}_h^{\boldsymbol{\chi}}$, we find that

$$\int_{\Omega^-} \boldsymbol{\eta}\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big) : \boldsymbol{\tau}_h^{\mathtt{d}} \;=\; \int_{\Omega^-} \Big\{\,\boldsymbol{\eta}\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big) \;-\; \Pi_h^{\boldsymbol{\chi}}\big(\boldsymbol{\eta}\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big)\big)\,\Big\} : \boldsymbol{\tau}_h^{\mathtt{d}}\,,$$

from which, applying the approximation property of $\mathbb{L}_h^{\boldsymbol{\chi}}$ (cf. (109) in **(H.5)**), **(H.3)**, the boundedness of $\mathbf{D} : \mathbf{H}^{1/2+\epsilon}(\Gamma) \to \mathbf{H}^{1+\epsilon}(\Omega^-)$, and the inverse inequality satisfied by $\mathbf{H}_{\tilde{h},0}^{\boldsymbol{\varphi}}$ (cf. (108) in **(H.4)**), we deduce that

$$\begin{aligned}
&\Big|\int_{\Omega^-} \boldsymbol{\eta}\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big) : \boldsymbol{\tau}_h^{\mathtt{d}}\Big| \;\leq\; \big\|\boldsymbol{\eta}\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big) \;-\; \Pi_h^{\boldsymbol{\chi}}\big(\boldsymbol{\eta}\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big)\big)\big\|_{0,\Omega^-}\;\|\boldsymbol{\tau}_h^{\mathtt{d}}\|_{0,\Omega^-}\\
&\quad\leq\; C\,h^\epsilon\,\big\|\boldsymbol{\eta}\big(\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big)\big\|_{\epsilon,\Omega^-}\;\|\boldsymbol{\tau}_h^{\mathtt{d}}\|_{0,\Omega^-} \;\leq\; C\,h^\epsilon\,\big\|\nabla\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big\|_{\epsilon,\Omega^-}\;\|\boldsymbol{\tau}_h^{\mathtt{d}}\|_{0,\Omega^-}\\
&\quad\leq\; C\,h^\epsilon\,\big\|\mathbf{D}\,\boldsymbol{\psi}_{\tilde{h}}\big\|_{1+\epsilon,\Omega^-}\;\|\boldsymbol{\tau}_h^{\mathtt{d}}\|_{0,\Omega^-} \;\leq\; C\,h^\epsilon\,\big\|\boldsymbol{\psi}_{\tilde{h}}\big\|_{1/2+\epsilon,\Gamma}\;\|\boldsymbol{\tau}_h^{\mathtt{d}}\|_{0,\Omega^-}\\
&\quad\leq\; C\,\Big\{\frac{h}{\tilde{h}}\Big\}^\epsilon\;\big\|\boldsymbol{\psi}_{\tilde{h}}\big\|_{1/2,\Gamma}\,\|\boldsymbol{\tau}_h^{\mathtt{d}}\|_{0,\Omega^-} \;\leq\; C\,\Big\{\frac{h}{\tilde{h}}\Big\}^\epsilon\,\Big\{\,\frac{1}{2}\,\big\|\boldsymbol{\psi}_{\tilde{h}}\big\|^2_{1/2,\Gamma} + \frac{1}{2}\,\|\boldsymbol{\tau}_h^{\mathtt{d}}\|^2_{0,\Omega^-}\,\Big\}\,.
\end{aligned} \tag{114}$$

Therefore, using that $\langle \mathbf{W}\,\boldsymbol{\psi}, \boldsymbol{\psi}\rangle_\Gamma \;\geq\; \alpha_2\,\|\boldsymbol{\psi}\|^2_{1/2,\Gamma} \quad \forall\,\boldsymbol{\psi} \in \mathbf{H}_0^{1/2}(\Gamma)$ (cf. (51)), it follows from (113) and (114) that

$$\begin{aligned}
&\mathbf{a}_N\big((\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}}), (\boldsymbol{\tau}_h, \boldsymbol{\psi}_{\tilde{h}})\big)\\
&\quad\geq\; \frac{1}{2}\,\Big\{1 \;-\; C\,\Big\{\frac{h}{\tilde{h}}\Big\}^\epsilon\Big\}\,\|\boldsymbol{\tau}_h^{\mathtt{d}}\|^2_{0,\Omega^-} \;+\; \frac{1}{2}\,\Big\{\alpha_2 \;-\; C\,\Big\{\frac{h}{\tilde{h}}\Big\}^\epsilon\Big\}\,\|\boldsymbol{\psi}_{\tilde{h}}\|^2_{1/2,\Gamma}\,,
\end{aligned}$$

which yields the existence of a sufficiently small constant $c_0 > 0$ such that for each $h \leq c_0\,\tilde{h}$, $\mathbf{a}_N$ is uniformly strongly coercive on $\widetilde{\mathbf{V}}_{N,h}$.

□

The well-posedness and convergence of (106) can now be established.

THEOREM 6.1. *Assume that the hypotheses* **(H.1)** *up to* **(H.5)** *are satisfied, and let $c_0$ be the positive constant provided by Lemma* 6.1. *Then, for each $h \leq c_0\,\tilde{h}$ there exists a unique $((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_{\tilde{h}}), (\mathbf{u}_h, \boldsymbol{\chi}_h)) \in \widetilde{\mathbf{X}}_{N,h} \times \mathbf{Y}_{N,h}$ solution of* (106). *In addition, there exist $C_1$, $C_2 > 0$, independent of $h$ and $\tilde{h}$, such that*

$$\|((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_{\tilde{h}}), (\mathbf{u}_h, \boldsymbol{\chi}_h))\|_{\mathbf{X}_N \times \mathbf{Y}_N} \leq C_1\, \|\mathbf{f}\|_{0,\Omega^-}\,,$$

*and*

$$\begin{aligned} &\|((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\mathbf{u}, \boldsymbol{\chi})) - ((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_{\tilde{h}}), (\mathbf{u}_h, \boldsymbol{\chi}_h))\|_{\mathbf{X}_N \times \mathbf{Y}_N} \\ &\qquad \leq C_2\, \mathrm{dist}\Big(((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\mathbf{u}, \boldsymbol{\chi})), \widetilde{\mathbf{X}}_{N,h} \times \mathbf{Y}_{N,h}\Big)\,. \end{aligned} \tag{115}$$

**6.2. C & H coupling with non-homogeneous Dirichlet on $\Gamma_0$.** We now let $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\boldsymbol{\varphi}}$, $\mathbf{L}_h^{\mathbf{u}}$, and $\mathbb{L}_h^{\boldsymbol{\chi}}$ be finite dimensional subspaces of $\mathbb{H}(\mathbf{div}\,;\Omega^-)$, $\mathbf{H}^{1/2}(\Gamma)$, $\mathbf{L}^2(\Omega^-)$, and $\mathbb{L}^2_{\mathtt{skew}}(\Omega^-)$, respectively, and define

$$\mathbf{H}_{h,0}^{\boldsymbol{\varphi}} := \Big\{\boldsymbol{\psi} \in \mathbf{H}_h^{\boldsymbol{\varphi}} : \quad \langle \mathbf{r}, \boldsymbol{\psi}\rangle_\Gamma = 0 \quad \forall\, \mathbf{r} \in \mathbf{RM}(\Gamma)\Big\} = \mathbf{H}_h^{\boldsymbol{\varphi}} \cap \mathbf{H}_0^{1/2}(\Gamma). \tag{116}$$

Note that, differently from the previous section, in this case we do not need to take any different meshsize for $\mathbf{H}_{h,0}^{\boldsymbol{\varphi}}$. Then we introduce the product spaces

$$\widetilde{\mathbf{X}}_{D,h} := \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_{h,0}^{\boldsymbol{\varphi}} \quad \text{and} \quad \mathbf{Y}_{D,h} := \mathbf{L}_h^{\mathbf{u}} \times \mathbb{L}_h^{\boldsymbol{\chi}}\,,$$

and define the Galerkin scheme associated with (83) as: Find $((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\mathbf{u}_h, \boldsymbol{\chi}_h)) \in \widetilde{\mathbf{X}}_{D,h} \times \mathbf{Y}_{D,h}$ such that

$$\begin{aligned} \mathbf{a}_D((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_)), (\boldsymbol{\tau}, \boldsymbol{\psi})) + \mathbf{b}_D((\boldsymbol{\tau}, \boldsymbol{\psi}), (\mathbf{u}_h, \boldsymbol{\chi}_h)) &= \mathbf{F}_D(\boldsymbol{\tau}, \boldsymbol{\psi}) \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{\psi}) \in \widetilde{\mathbf{X}}_{D,h}\,, \\ \mathbf{b}_D((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\mathbf{v}, \boldsymbol{\eta})) &= \mathbf{G}_D(\mathbf{v}, \boldsymbol{\eta}) \quad \forall\, (\mathbf{v}, \boldsymbol{\eta}) \in \mathbf{Y}_{D,h}\,, \end{aligned} \tag{117}$$

where $\mathbf{a}_D$, $\mathbf{b}_D$, $\mathbf{F}_D$, and $\mathbf{G}_D$ are the bilinear forms and functionals defined in Section 5.2.

Next, we apply again the discrete Babuška-Brezzi theory to prove the unique solvability, stability, and convergence of (117). To this end, in what follows we assume the following hypotheses:

$\widetilde{\textbf{(H.1)}}$ the bilinear form $\mathbf{b}_D$ satisfies the discrete inf-sup condition uniformly on $\widetilde{\mathbf{X}}_{D,h} \times \mathbf{Y}_{D,h}$, that is there exists $\tilde{\beta} > 0$, independent of $h$, such that

$$\sup_{(\boldsymbol{\tau}_h, \boldsymbol{\psi}_h) \in \widetilde{\mathbf{X}}_{D,h} \setminus \{\mathbf{0}\}} \frac{\mathbf{b}_D((\boldsymbol{\tau}_h, \boldsymbol{\psi}_h), (\mathbf{v}, \boldsymbol{\eta}))}{\|(\boldsymbol{\tau}_h, \boldsymbol{\psi}_h)\|_{\mathbf{X}_D}} \geq \tilde{\beta}\, \|(\mathbf{v}, \boldsymbol{\eta})\|_{\mathbf{Y}_D} \qquad \forall\, (\mathbf{v}, \boldsymbol{\eta}) \in \mathbf{Y}_{D,h}\,. \tag{118}$$

$\widetilde{\textbf{(H.2)}}$ $\mathbf{div}\, \mathbb{H}_h^{\boldsymbol{\sigma}} \subseteq \mathbf{L}_h^{\mathbf{u}}$.

$\widetilde{\textbf{(H.3)}}$ there holds $P_0(\Omega^-)\,\mathbf{I} \subseteq \mathbb{H}_h^{\boldsymbol{\sigma}}$ and $\mathbf{RM}(\Gamma) \subseteq \mathbf{H}_h^{\boldsymbol{\varphi}}$.

$\widetilde{\textbf{(H.4)}}$ there exists $\widetilde{\boldsymbol{\psi}} \in \mathbf{H}^{1/2}(\Gamma)$ such that $\langle \boldsymbol{\nu}, \widetilde{\boldsymbol{\psi}}\rangle_\Gamma > 0$ and $\widetilde{\boldsymbol{\psi}} \in \mathbf{H}_h^{\boldsymbol{\varphi}}$ for each $h > 0$.

Similarly as in the previous section, the bilinear form $\mathbf{b}_D$ is independent of the $\boldsymbol{\psi}$-component, and hence its discrete inf-sup condition (cf. (118) in $\widetilde{(\mathbf{H}.\mathbf{1})}$) involves only the subspaces $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{L}_h^{\mathbf{u}}$, and $\mathbb{L}_h^{\boldsymbol{\chi}}$. In addition, hypothesis $\widetilde{(\mathbf{H}.\mathbf{2})}$ implies now that the discrete kernel of $\mathbf{b}_D$ reduces to

(119)
$$\widetilde{\mathbf{V}}_{D,h} := \left\{ (\boldsymbol{\tau}_h, \boldsymbol{\psi}_h) \in \widetilde{\mathbf{X}}_{D,h} : \ \mathbf{div}\, \boldsymbol{\tau}_h = \mathbf{0} \text{ in } \Omega^- \text{ and } \int_{\Omega^-} \boldsymbol{\tau}_h : \boldsymbol{\eta}_h = 0 \ \forall\, \boldsymbol{\eta}_h \in \mathbb{L}_h^{\boldsymbol{\chi}} \right\}.$$

As announced right before the statement of Lemma 5.8, we now establish the discrete weak-coerciveness of $\mathbf{a}_D$.

LEMMA 6.2. *There exists $\widehat{\alpha} > 0$, independent of $h$ but depending explicitly on $\widetilde{\boldsymbol{\psi}}$, such that*

(120)
$$\sup_{(\boldsymbol{\tau},\boldsymbol{\psi}) \in \widetilde{\mathbf{V}}_{D,h} \setminus \{\mathbf{0}\}} \frac{|\, \mathbf{a}_D((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\boldsymbol{\tau}, \boldsymbol{\psi}))\,|}{\|(\boldsymbol{\tau}, \boldsymbol{\psi})\|_{\mathbf{X}_D}} \ \geq\ \widehat{\alpha}\, \|(\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h)\|_{\mathbf{X}_D} \qquad \forall\, (\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h) \ \in\ \widetilde{\mathbf{V}}_{D,h}\,.$$

*Proof.* This is a direct consequence of Lemma 5.8 using hypotheses $\widetilde{(\mathbf{H}.\mathbf{3})}$ and $\widetilde{(\mathbf{H}.\mathbf{4})}$. □

As a consequence of the foregoing analysis, we can provide now the well-posedness and convergence of (117).

THEOREM 6.2. *Assume that the hypotheses $\widetilde{(\mathbf{H}.\mathbf{1})}$ up to $\widetilde{(\mathbf{H}.\mathbf{4})}$ are satisfied. Then, there exists a unique $((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\mathbf{u}_h, \boldsymbol{\chi}_h)) \in \widetilde{\mathbf{X}}_{D,h} \times \mathbf{Y}_{D,h}$ solution of* (117). *In addition, there exist $C_1$, $C_2 > 0$, independent of $h$, such that*

$$\|((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\mathbf{u}_h, \boldsymbol{\chi}_h))\|_{\mathbf{X}_D \times \mathbf{Y}_D} \ \leq\ C_1 \left\{ \|\mathbf{g}_D\|_{1/2,\Gamma_0} \ +\ \|\mathbf{f}\|_{0,\Omega^-} \right\},$$

*and*

(121)
$$\begin{aligned} &\|((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\mathbf{u}, \boldsymbol{\chi})) \ -\ ((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\mathbf{u}_h, \boldsymbol{\chi}_h))\|_{\mathbf{X}_D \times \mathbf{Y}_D} \\ &\qquad \leq\ C_2\, \mathrm{dist}\Big( ((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\mathbf{u}, \boldsymbol{\chi})), \widetilde{\mathbf{X}}_{D,h} \times \mathbf{Y}_{D,h} \Big)\,. \end{aligned}$$

**6.3. C & H coupling with non-homogeneous Neumann on $\Gamma_0$.** We let $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\boldsymbol{\varphi}}$, $\mathbf{L}_h^{\mathbf{u}}$, $\mathbb{L}_h^{\boldsymbol{\chi}}$, and $\mathbf{H}_{\tilde{h}}^{\lambda}$ be finite dimensional subspaces of $\mathbb{H}(\mathbf{div}\,;\Omega^-)$, $\mathbf{H}^{1/2}(\Gamma)$, $\mathbf{L}^2(\Omega^-)$, $\mathbb{L}^2_{\mathtt{skew}}(\Omega^-)$, and $\mathbf{H}^{1/2}(\Gamma_0)$, respectively, and define, as in (116), the subspace

$$\mathbf{H}_{h,0}^{\boldsymbol{\varphi}} \ :=\ \mathbf{H}_h^{\boldsymbol{\varphi}} \cap \mathbf{H}_0^{1/2}(\Gamma)\,. \tag{122}$$

Note that, in order to be able to define specific discrete subspaces satisfying the assumptions to be specified below, we need to take a different meshsize for the finite element subspace of $\mathbf{H}^{1/2}(\Gamma_0)$. However, as in the previous section, this is not required for $\mathbf{H}_{h,0}^{\boldsymbol{\varphi}}$. Then we introduce the product spaces

$$\widetilde{\mathbf{X}}_{N,h} \ :=\ \mathbb{H}_h^{\boldsymbol{\sigma}} \ \times\ \mathbf{H}_{h,0}^{\boldsymbol{\varphi}} \quad \text{and} \quad \mathbf{Y}_{N,h,\tilde{h}} \ :=\ \mathbf{L}_h^{\mathbf{u}} \ \times\ \mathbb{L}_h^{\boldsymbol{\chi}} \times \mathbf{H}_{\tilde{h}}^{\lambda}\,,$$

and define the Galerkin scheme for (98) as: Find $((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\mathbf{u}_h, \boldsymbol{\chi}_h, \boldsymbol{\lambda}_{\tilde{h}})) \in \widetilde{\mathbf{X}}_{N,h} \times \mathbf{Y}_{N,h,\tilde{h}}$ such that

(123)
$$\begin{aligned} \mathbf{a}_N((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_)), (\boldsymbol{\tau}, \boldsymbol{\psi})) + \mathbf{b}_N((\boldsymbol{\tau}, \boldsymbol{\psi}), (\mathbf{u}_h, \boldsymbol{\chi}_h, \boldsymbol{\lambda}_{\tilde{h}})) \ &=\ \mathbf{F}_N(\boldsymbol{\tau}, \boldsymbol{\psi})\,, \\ \mathbf{b}_N((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\mathbf{v}, \boldsymbol{\eta}, \boldsymbol{\xi})) \ &=\ \mathbf{G}_N(\mathbf{v}, \boldsymbol{\eta}, \boldsymbol{\xi})\,, \end{aligned}$$

for all $(\boldsymbol{\tau},\boldsymbol{\psi}) \in \widetilde{\mathbf{X}}_{N,h}$ and $(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi}) \in \mathbf{Y}_{N,h,\tilde{h}}$, where $\mathbf{a}_N$, $\mathbf{b}_N$, $\mathbf{F}_N$, and $\mathbf{G}_N$ are those bilinear forms and functionals defined in Section 5.3.

Proceeding as before, in what follows we apply the discrete Babuška-Brezzi theory to establish the well-posedness and convergence of (123). For this purpose, we now assume the following hypotheses:

$\widehat{(\mathbf{H}.\mathbf{1})}$ the bilinear form $\mathbf{b}_N$ satisfies the discrete inf-sup condition uniformly on $\widetilde{\mathbf{X}}_{N,h} \times \mathbf{Y}_{N,h,\tilde{h}}$, that is there exists $\tilde{\beta} > 0$, independent of $h$ and $\tilde{h}$, such that

(124)
$$\sup_{(\boldsymbol{\tau}_h,\boldsymbol{\psi}_h)\in\widetilde{\mathbf{X}}_{N,h}\setminus\{\mathbf{0}\}} \frac{\mathbf{b}_N((\boldsymbol{\tau}_h,\boldsymbol{\psi}_h),(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi}))}{\|(\boldsymbol{\tau}_h,\boldsymbol{\psi}_h)\|_{\mathbf{X}_N}} \;\geq\; \tilde{\beta}\,\|(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi})\|_{\mathbf{Y}_N} \quad \forall\,(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi}) \in \mathbf{Y}_{N,h,\tilde{h}}\,.$$

$\widehat{(\mathbf{H}.\mathbf{2})}$ $\mathbf{div}\,\mathbb{H}_h^{\boldsymbol{\sigma}} \;\subseteq\; \mathbf{L}_h^{\mathbf{u}}$.

$\widehat{(\mathbf{H}.\mathbf{3})}$ there holds $P_0(\Omega^-)\,\mathbf{I} \subseteq \mathbb{H}_h^{\boldsymbol{\sigma}}$.

$\widehat{(\mathbf{H}.\mathbf{4})}$ there exists $\widetilde{\boldsymbol{\xi}} \in \mathbf{H}^{1/2}(\Gamma_0)$ such that $\langle \boldsymbol{\nu},\widetilde{\boldsymbol{\xi}}\rangle_{\Gamma_0} \;>\; 0$ and $\widetilde{\boldsymbol{\xi}} \in \mathbf{H}_{\tilde{h}}^{\lambda}$ for each $h > 0$.

Note, as in both previous sections, that the bilinear form $\mathbf{b}_N$ is also independent of the $\boldsymbol{\psi}$-component, and hence its discrete inf-sup condition (cf. (124) in $\widehat{(\mathbf{H}.\mathbf{1})}$) involves only the subspaces $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{L}_h^{\mathbf{u}}$, and $\mathbb{L}_h^{\boldsymbol{\chi}}$. In addition, hypothesis $\widehat{(\mathbf{H}.\mathbf{2})}$ implies now that the discrete kernel of $\mathbf{b}_N$ reduces to

(125)
$$\widetilde{\mathbf{V}}_{N,h} := \Big\{(\boldsymbol{\tau}_h,\boldsymbol{\psi}_h) \in \widetilde{\mathbf{X}}_{N,h} :\; \mathbf{div}\,\boldsymbol{\tau}_h = \mathbf{0} \text{ in } \Omega^-\,,\; \int_{\Omega^-} \boldsymbol{\tau}_h : \boldsymbol{\eta}_h = 0\;\forall\,\boldsymbol{\eta}_h \in \mathbb{L}_h^{\boldsymbol{\chi}}\,,$$
$$\text{and} \qquad \langle \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}_h),\boldsymbol{\xi}\rangle_{\Gamma_0} \;=\; 0 \quad \forall\,\boldsymbol{\xi} \in \mathbf{H}_{\tilde{h}}^{\lambda}\Big\}\,.$$

In turn, $\widehat{(\mathbf{H}.\mathbf{3})}$ guarantees that there holds the decomposition (cf. (1) - (2))
$$\mathbb{H}_h^{\boldsymbol{\sigma}} \;=\; \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \;\oplus\; P_0(\Omega^-)\,\mathbf{I}\,,$$
where
$$\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \;:=\; \Big\{\boldsymbol{\tau}_h \;\in\; \mathbb{H}_h^{\boldsymbol{\sigma}} :\quad \int_{\Omega^-} \operatorname{tr}\boldsymbol{\tau}_h \;=\; 0\Big\}\,.$$
In other words, $\boldsymbol{\tau}_h - \boldsymbol{\pi}_I\boldsymbol{\tau}_h \in \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ for each $\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}$. Hence, the discrete analogue of Lemma 5.5 is established now as a consequence of $\widehat{(\mathbf{H}.\mathbf{4})}$.

Lemma 6.3. *Let* $\mathbb{H}_{h,\tilde{h}} \;:=\; \Big\{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} :\quad \langle \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}_h),\boldsymbol{\xi}\rangle_{\Gamma_0} \;=\; 0 \quad \forall\,\boldsymbol{\xi} \;\in\; \mathbf{H}_{\tilde{h}}^{\lambda}\Big\}$. *Then there exists* $\widetilde{c}_2 > 0$, *independent of* $h$ *and* $\tilde{h}$, *such that*

(126)
$$\|\boldsymbol{\tau}_h\|^2_{\mathbf{div}\,;\Omega^-} \;\leq\; \widetilde{c}_2\,\|\boldsymbol{\tau}_h - \boldsymbol{\pi}_I\boldsymbol{\tau}_h\|^2_{\mathbf{div}\,;\Omega^-} \qquad \forall\,\boldsymbol{\tau}_h \;\in\; \mathbb{H}_{h,\tilde{h}}\,.$$

*Proof.* Let $\boldsymbol{\tau}_h \in \mathbb{H}_{h,\tilde{h}}$ and write $\boldsymbol{\pi}_I\boldsymbol{\tau}_h = d_h\mathbf{I}$. Then
$$0 \;=\; \langle \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}_h),\widetilde{\boldsymbol{\xi}}\rangle_{\Gamma_0} \;=\; \langle \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}_h - \boldsymbol{\pi}_I\boldsymbol{\tau}_h),\widetilde{\boldsymbol{\xi}}\rangle_{\Gamma_0} \;+\; d_h\,\langle \boldsymbol{\nu},\widetilde{\boldsymbol{\xi}}\rangle_{\Gamma_0}\,,$$
which gives
$$|d_h| \;\leq\; C\,\frac{\|\widetilde{\boldsymbol{\xi}}\|_{1/2,\Gamma_0}}{|\,\langle \boldsymbol{\nu},\widetilde{\boldsymbol{\xi}}\rangle_{\Gamma_0}\,|}\;\|\boldsymbol{\tau}_h - \boldsymbol{\pi}_I\boldsymbol{\tau}_h\|_{\mathbf{div}\,;\Omega^-}\,.$$
This inequality and the fact that $\|\boldsymbol{\tau}_h\|^2_{\mathbf{div}\,;\Omega^-} \;=\; \|\boldsymbol{\tau}_h - \boldsymbol{\pi}_I\boldsymbol{\tau}_h\|^2_{\mathbf{div}\,;\Omega^-} \;+\; 2\,d_h^2\,|\Omega^-|$ imply (126). $\square$

In this way, noting that $\widetilde{\mathbf{V}}_{N,h} \subseteq \mathbb{H}_{h,\tilde{h}} \times \mathbf{H}^{\boldsymbol{\varphi}}_{h,0}$, and using now Lemma 6.3 (instead of Lemma 5.5) together with (75) (cf. Lemma 5.4), (54), and (53), we conclude the strong coerciveness of $\mathbf{a}_N$ on $\widetilde{\mathbf{V}}_{N,h}$. Therefore, we summarize the foregoing analysis in the following main result.

THEOREM 6.3. *Assume that the hypotheses* $\widehat{(\mathbf{H.1})}$ *up to* $\widehat{(\mathbf{H.4})}$ *are satisfied. Then, there exists a unique* $((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\mathbf{u}_h, \boldsymbol{\chi}_h, \boldsymbol{\lambda}_{\tilde{h}})) \in \widetilde{\mathbf{X}}_{N,h} \times \mathbf{Y}_{N,h,\tilde{h}}$ *solution of* (123). *In addition, there exist* $C_1$, $C_2 > 0$, *independent of* $h$, *such that*

$$\|((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\mathbf{u}_h, \boldsymbol{\chi}_h, \boldsymbol{\lambda}_{\tilde{h}}))\|_{\mathbf{X}_N \times \mathbf{Y}_N} \leq C_1 \left\{ \|\mathbf{g}_N\|_{-1/2,\Gamma_0} + \|\mathbf{f}\|_{0,\Omega^-} \right\},$$

*and*

$$\begin{aligned} &\|((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\mathbf{u}, \boldsymbol{\chi}, \boldsymbol{\lambda})) - ((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\mathbf{u}_h, \boldsymbol{\chi}_h, \boldsymbol{\lambda}_{\tilde{h}}))\|_{\mathbf{X}_N \times \mathbf{Y}_N} \\ &\qquad \leq C_2 \operatorname{dist}\Big( ((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\mathbf{u}, \boldsymbol{\chi}, \boldsymbol{\lambda})), \widetilde{\mathbf{X}}_{N,h} \times \mathbf{Y}_{N,h,\tilde{h}} \Big). \end{aligned} \tag{127}$$

**6.4. C & H coupling with homogeneous Neumann on $\Gamma_0$.** We now let $\mathbb{H}^{\boldsymbol{\sigma}}_{h,0}$, $\mathbf{H}^{\boldsymbol{\varphi}}_h$, $\mathbf{L}^{\mathbf{u}}_h$, and $\mathbb{L}^{\boldsymbol{\chi}}_h$ be finite dimensional subspaces of $\mathbb{H}_0(\mathbf{div}\,;\Omega^-)$, $\mathbf{H}^{1/2}(\Gamma)$, $\mathbf{L}^2(\Omega^-)$, and $\mathbb{L}^2_{\mathtt{skew}}(\Omega^-)$, respectively, and define, as in previous occasions,

$$\mathbf{H}^{\boldsymbol{\varphi}}_{h,0} = \mathbf{H}^{\boldsymbol{\varphi}}_h \cap \mathbf{H}^{1/2}_0(\Gamma). \tag{128}$$

Then we introduce the product spaces

$$\widetilde{\mathbf{X}}_{N,h} := \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \mathbf{H}^{\boldsymbol{\varphi}}_{h,0} \quad \text{and} \quad \mathbf{Y}_{N,h} := \mathbf{L}^{\mathbf{u}}_h \times \mathbb{L}^{\boldsymbol{\chi}}_h,$$

and define the Galerkin scheme associated with (104) as: Find $((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\mathbf{u}, \boldsymbol{\chi})) \in \widetilde{\mathbf{X}}_{N,h} \times \mathbf{Y}_{N,h}$ such that

$$\begin{aligned} \mathbf{a}_N((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\boldsymbol{\tau}, \boldsymbol{\psi})) + \mathbf{b}_N((\boldsymbol{\tau}, \boldsymbol{\psi}), (\mathbf{u}_h, \boldsymbol{\chi}_h)) &= \mathbf{F}_N(\boldsymbol{\tau}, \boldsymbol{\psi}) \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{\psi}) \in \widetilde{\mathbf{X}}_{N,h}, \\ \mathbf{b}_N((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\mathbf{v}, \boldsymbol{\eta})) &= \mathbf{G}_N(\mathbf{v}, \boldsymbol{\eta}) \quad \forall\, (\mathbf{v}, \boldsymbol{\eta}) \in \mathbf{Y}_{N,h}, \end{aligned} \tag{129}$$

where $\mathbf{a}_N$, $\mathbf{b}_N$, $\mathbf{F}_N$, and $\mathbf{G}_N$ are those bilinear forms and functionals defined in Section 5.4.

We assume the following hypotheses:

$\overline{(\mathbf{H.1})}$ the bilinear form $\mathbf{b}_N$ satisfies the discrete inf-sup condition uniformly on $\widetilde{\mathbf{X}}_{N,h} \times \mathbf{Y}_{N,h}$, that is there exists $\tilde{\beta} > 0$, independent of $h$, such that

$$\sup_{(\boldsymbol{\tau}_h, \boldsymbol{\psi}_h) \in \widetilde{\mathbf{X}}_{N,h} \setminus \{\mathbf{0}\}} \frac{\mathbf{b}_N((\boldsymbol{\tau}_h, \boldsymbol{\psi}_h), (\mathbf{v}, \boldsymbol{\eta}))}{\|(\boldsymbol{\tau}_h, \boldsymbol{\psi}_h)\|_{\mathbf{X}_N}} \geq \tilde{\beta}\, \|(\mathbf{v}, \boldsymbol{\eta})\|_{\mathbf{Y}_N} \qquad \forall\, (\mathbf{v}, \boldsymbol{\eta}) \in \mathbf{Y}_{N,h}. \tag{130}$$

$\overline{(\mathbf{H.2})}$ $\mathbf{div}\, \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \subseteq \mathbf{L}^{\mathbf{u}}_h$.

$\overline{(\mathbf{H.3})}$ there holds $\mathbf{RM}(\Gamma) \subseteq \mathbf{H}^{\boldsymbol{\varphi}}_h$.

As in all the previous subsections, we note once again that the bilinear form $\mathbf{b}_N$ does not depend on the $\boldsymbol{\psi}$-component, and hence the eventual verification of its discrete inf-sup condition (cf. (130) in $\overline{(\mathbf{H.1})}$) is determined only by the subspaces $\mathbb{H}^{\boldsymbol{\sigma}}_{h,0}$, $\mathbf{L}^{\mathbf{u}}_h$, and $L^{\boldsymbol{\chi}}_h$. In turn, it is also quite obvious from $\overline{(\mathbf{H.2})}$ that the discrete kernel of $\mathbf{b}_N$ reduces to

$$\widetilde{\mathbf{V}}_{N,h} := \left\{ (\boldsymbol{\tau}_h, \boldsymbol{\psi}_h) \in \widetilde{\mathbf{X}}_{N,h} : \ \mathbf{div}\, \boldsymbol{\tau}_h = \mathbf{0} \text{ in } \Omega^-, \ \int_{\Omega^-} \boldsymbol{\tau}_h : \boldsymbol{\eta}_h = 0 \ \forall\, \boldsymbol{\eta}_h \in \mathbb{L}^{\boldsymbol{\chi}}_h \right\}. \tag{131}$$

Furthermore, it is clear that $\overline{(\mathbf{H.3})}$ yields the decomposition $\mathbf{H}_h^{\boldsymbol{\varphi}} = \mathbf{H}_{h,0}^{\boldsymbol{\varphi}} \oplus \mathbf{RM}(\Gamma)$, and $\boldsymbol{\psi}_h - \boldsymbol{\pi}_{RM}\boldsymbol{\psi}_h \in \mathbf{H}_{h,0}^{\boldsymbol{\varphi}}$ for all $\boldsymbol{\psi}_h \in \mathbf{H}_h^{\boldsymbol{\varphi}}$. Consequently, (131) and the application of (103) to the above discrete context confirm that the bilinear form $\mathbf{a}_N$ is strongly coercive on $\widetilde{\mathbf{V}}_{N,h}$.

The well-posedness and convergence of (129) is established next as a straightforward consequence of the foregoing analysis.

THEOREM 6.4. *Assume that the hypotheses* $\overline{(\mathbf{H.1})}$ *up to* $\overline{(\mathbf{H.3})}$ *are satisfied. Then, there exists a unique* $((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\mathbf{u}_h, \boldsymbol{\chi}_h)) \in \widetilde{\mathbf{X}}_{N,h} \times \mathbf{Y}_{N,h}$ *solution of* (129). *In addition, there exist* $C_1$, $C_2 > 0$, *independent of* $h$, *such that*

$$\|((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\mathbf{u}_h, \boldsymbol{\chi}_h))\|_{\mathbf{X}_N \times \mathbf{Y}_N} \le C_1 \|\mathbf{f}\|_{0,\Omega^-} ,$$

*and*

$$\begin{aligned} &\|((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\mathbf{u}, \boldsymbol{\chi})) - ((\boldsymbol{\sigma}_h, \boldsymbol{\varphi}_h), (\mathbf{u}_h, \boldsymbol{\chi}_h))\|_{\mathbf{X}_N \times \mathbf{Y}_N} \\ &\qquad \le C_2 \operatorname{dist}\Big( ((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\mathbf{u}, \boldsymbol{\chi})), \widetilde{\mathbf{X}}_{N,h} \times \mathbf{Y}_{N,h} \Big) . \end{aligned} \tag{132}$$

We end this paper by remarking that specific subspaces satisfying the hypotheses described in each one of the subsections of the present section can be easily found in the existing literature. Indeed, we first observe that the expressions defining the bilinear forms $\mathbf{b}_N$, $\mathbf{b}_D$, and $\mathbf{b}_N$ involved in $(\mathbf{H.1})$, $\widetilde{(\mathbf{H.1})}$, and $\overline{(\mathbf{H.1})}$, respectively, are the same as the one arising from the mixed formulation of the linear elasticity problem in $\Omega^-$ with Dirichlet or homogeneous Neumann boundary conditions on $\partial\Omega^-$. Hence, any triple of finite element subspaces $(\mathbb{H}_h^{\boldsymbol{\sigma}}, \mathbf{L}_h^{\mathbf{u}}, \mathbb{L}_h^{\boldsymbol{\chi}})$ (or $(\mathbb{H}_{h,0}^{\boldsymbol{\sigma}}, \mathbf{L}_h^{\mathbf{u}}, \mathbb{L}_h^{\boldsymbol{\chi}})$) yielding the discrete stability of that problem will satisfy our aforementioned hypotheses. In particular, besides the classical PEERS (cf. [2]), we can consider for any integer $k \ge 1$ the PEERS$_k$ and the BDMS$_k$ elements, whose definitions and corresponding proofs of stability are provided in [47]. In addition, new stable mixed finite element methods for 3D linear elasticity with a weak symmetry condition for the stresses have been constructed recently in [4] and [6] by using the finite element exterior calculus. This is a quite abstract framework involving several sophisticated mathematical tools (see also [5] and [7] for further details), which has been simplified in some particular cases by employing more elementary and classical techniques (see, e.g. [10]). The resulting Arnold-Falk-Winther (AFW) element with the lowest polynomial degrees, which is referred to as of order 1, and which certainly constitutes another feasible choice verifying $(\mathbf{H.1})$, $\widetilde{(\mathbf{H.1})}$, and $\overline{(\mathbf{H.1})}$, consists of piecewise linear approximations for the stress $\boldsymbol{\sigma}$ and piecewise constants functions for both the velocity $\mathbf{u}$ and vorticity $\boldsymbol{\chi}$ unknowns.

In turn, the bilinear form $\mathbf{b}_N$ involved in $\widehat{(\mathbf{H.1})}$ corresponds also to the one arising from the mixed formulation of the linear elasticity problem in $\Omega^-$, but now with Dirichlet boundary condition on $\Gamma$ and non-homogeneous Neumann boundary condition on $\Gamma_0$. As stated at the beginning of Section 6.3, we recall here that a different meshsize $\tilde{h}$ is assumed to define the finite element subspace $\mathbf{H}_{\tilde{h}}^{\boldsymbol{\lambda}}$ of $\mathbf{H}^{1/2}(\Gamma_0)$. In other words, we consider a partition $\Gamma_{0,\tilde{h}}$ of $\Gamma_0$ that is independent of the partition $\Gamma_{0,h}$ of $\Gamma_0$ inherited from a regular triangulation $\mathcal{T}_h$ of $\Omega^-$. In this case, given an integer $k \ge 0$, and denoting the classical PEERS from [2] by PEERS$_0$, we first take either PEERS$_k$ or BDMS$_k$ (when $k \ge 1$) to define the triple $(\mathbb{H}_h^{\boldsymbol{\sigma}}, \mathbf{L}_h^{\mathbf{u}}, \mathbb{L}_h^{\boldsymbol{\chi}})$, and then let $\mathbf{H}_{\tilde{h}}^{\boldsymbol{\lambda}}$ be the space of continuous piecewise polynomials on $\Gamma_{0,\tilde{h}}$ of degree $\le k+1$. Similarly, one could also take the above described AFW element of order 1

and set $\mathbf{H}_{\tilde{h}}^{\boldsymbol{\lambda}}$ as the space of continuous piecewise polynomials on $\Gamma_{0,\tilde{h}}$ of degree $\leq 1$. Hence, proceeding analogously as in the proofs of [32, Lemmata 4.1, 4.2, and 4.3], one can easily show that for each one of the foregoing choices there exist $C_0$, $\tilde{\beta} > 0$, independent of $h$ and $\tilde{h}$, such that whenever $h \leq C_0\,\tilde{h}$, the hypothesis $\widehat{(\mathbf{H}.\mathbf{1})}$ is satisfied with the constant $\tilde{\beta}$. We can also refer to [28, Lemma 7.5] for a closely related argument.

Furthermore, it is easy to see that the above described pairs of finite element subspaces $\big(\mathbb{H}_h^{\boldsymbol{\sigma}}, \mathbf{L}_h^{\mathbf{u}}\big)$ (or $\big(\mathbb{H}_{h,0}^{\boldsymbol{\sigma}}, \mathbf{L}_h^{\mathbf{u}}\big)$) also verify the respective hypotheses $(\mathbf{H}.\mathbf{2})$, $\widetilde{(\mathbf{H}.\mathbf{2})}$, $\widehat{(\mathbf{H}.\mathbf{2})}$, and $\overline{(\mathbf{H}.\mathbf{2})}$, which imply that the first components of the discrete kernels of $\mathbf{b}_N$ and $\mathbf{b}_D$ become all free divergent. In addition, it is clear that the corresponding subspaces $\mathbf{L}_h^{\boldsymbol{\chi}}$ satisfy the approximation property required by $(\mathbf{H}.\mathbf{5})$. Moreover, it is quite straightforward to notice that the first condition in $\widetilde{(\mathbf{H}.\mathbf{3})}$, and $\widehat{(\mathbf{H}.\mathbf{3})}$, are both satisfied by any of the previously mentioned choices of $\mathbb{H}_h^{\boldsymbol{\sigma}}$.

Next, with respect to the boundary element subspaces of $\mathbf{H}^{1/2}(\Gamma)$ and $\mathbf{H}_0^{1/2}(\Gamma)$, we first let $\Gamma_h$ and $\Gamma_{\tilde{h}}$ be independent partitions of $\Gamma$, with $\Gamma_h$ being the one inherited from a regular triangulation $\mathcal{T}_h$ of $\Omega^-$. Hence, in order to satisfy the regularity assumption $(\mathbf{H}.\mathbf{3})$ and the inverse inequality $(\mathbf{H}.\mathbf{4})$, it suffices to consider an integer $k \geq 0$, set $\mathbf{H}_{\tilde{h}}^{\boldsymbol{\varphi}}$ as the space of continuous piecewise polynomials on $\Gamma_{\tilde{h}}$ of degree $\leq k+1$, and define $\mathbf{H}_{\tilde{h},0}^{\boldsymbol{\varphi}}$ as the intersection of $\mathbf{H}_0^{1/2}(\Gamma)$ with $\mathbf{H}_{\tilde{h}}^{\boldsymbol{\varphi}}$. Similarly, defining $\mathbf{H}_h^{\boldsymbol{\varphi}}$ as the space of continuous piecewise polynomials on $\Gamma_h$ of degree $\leq k+1$, we find that the second condition in $\widetilde{(\mathbf{H}.\mathbf{3})}$, and $\overline{(\mathbf{H}.\mathbf{3})}$, follow straightforwardly from the fact that $\mathbf{RM}(\Gamma)$ is certainly contained in the space of continuous piecewise polynomials on $\Gamma_h$ of degree $\leq 1$. On the other hand, concerning $\widetilde{(\mathbf{H}.\mathbf{4})}$ and $\widehat{(\mathbf{H}.\mathbf{4})}$, we just remark that the existence of $\widetilde{\boldsymbol{\psi}} \in \mathbf{H}^{1/2}(\Gamma)$ and $\widetilde{\boldsymbol{\xi}} \in \mathbf{H}^{1/2}(\Gamma_0)$ satisfying the required conditions in those hypotheses, follows by simply adapting the procedure suggested in [23, paragraph right before Lemma 8] to the present 3D case (see also [33, Section 3.2]). To this end, we just need to define $\mathbf{H}_h^{\boldsymbol{\varphi}}$ (resp. $\mathbf{H}_{\tilde{h}}^{\boldsymbol{\lambda}}$) as any subspace of $\mathbf{H}^{1/2}(\Gamma)$ (resp. $\mathbf{H}^{1/2}(\Gamma_0)$) containing the space of continuous piecewise polynomials on $\Gamma_h$ (resp. $\Gamma_{0,h}$) of degree $\leq 1$.

Finally, while the definitions of $\mathbf{H}_{\tilde{h},0}^{\boldsymbol{\varphi}}$ and $\mathbf{H}_{h,0}^{\boldsymbol{\varphi}}$ (cf. (105), (116), (122), and (128)) are theoretically correct, we remark that for the sake of the computational implementation of the corresponding Galerkin schemes, it will be better off to introduce Lagrange multipliers handling the orthogonality conditions defining these boundary element subspaces. We omit further details and leave this issue and other related matters, including numerical essays and the analysis of the associated experimental rates of convergence, for a separate work.

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