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A CONVERGENCE ANALYSIS OF ORTHOGONAL SPLINE COLLOCATION FOR SOLVING TWO-POINT BOUNDARY VALUE PROBLEMS WITHOUT THE BOUNDARY SUBINTERVALS

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Abstract. We consider a new Hermite cubic orthogonal spline collocation (OSC) scheme to solve a two-point boundary value problem (TPBVP) with boundary subintervals excluded from the given interval. Such TPBVPs arise, for example, in the alternating direction implicit OSC solution of parabolic problems on arbitrary domains. The scheme involves transfer of the given Dirichlet boundary values to the end points of the interior interval. The convergence analysis shows that the scheme is of optimal fourth order accuracy in the maximum norm. Numerical results confirm the theoretical results.

Key words. Two-point boundary value problem, orthogonal spline collocation, optimal order of accuracy.

1. Introduction

The orthogonal spline collocation (OSC) technique is an efficient way to solve a wide variety of problems that are modeled by ordinary and partial differential equations, see [10] and references therein. OSC for solving a two-point boundary value problem (TPBVP) has been introduced and analyzed in [5]. We consider a new Hermite cubic OSC scheme to solve a TPBVP with boundary subintervals excluded from the given interval. Such TPBVPs arise, for example, in the alternating direction implicit (ADI) OSC solution of parabolic problems on some non-rectangular domains [3] with non-uniform consistent partitions. We expect to use the idea of transfer of Dirichlet boundary values presented in this paper to generalize the ADI OSC method of [3] to the solution of parabolic problems on arbitrary domains with uniform non-consistent partitions [4]. Figure 1 shows collocation points and horizontal line segments, without the boundary subintervals, on each of which a TPBVP is solved in the x-direction when the ADI OSC method is used to discretize a parabolic problem on an arbitrary domain with a uniform partition. Figure 2 shows the corresponding collocations points and vertical line segments, without the boundary subintervals, on each of which a TPBVP is solved in the y-direction. We believe that a theoretical convergence analysis of OSC for solving TPBVPs without the boundary subintervals is a first important and necessary step in justifying the ADI OSC method of [3] for solving parabolic problems on some non-rectangular domains as well as its generalization to arbitrary domains [4].

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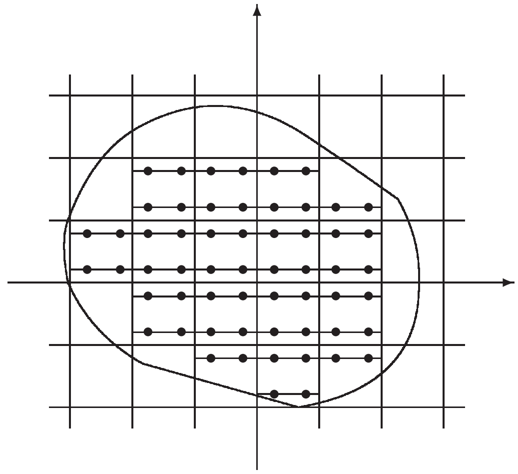


Figure 1. Horizontal line segments.

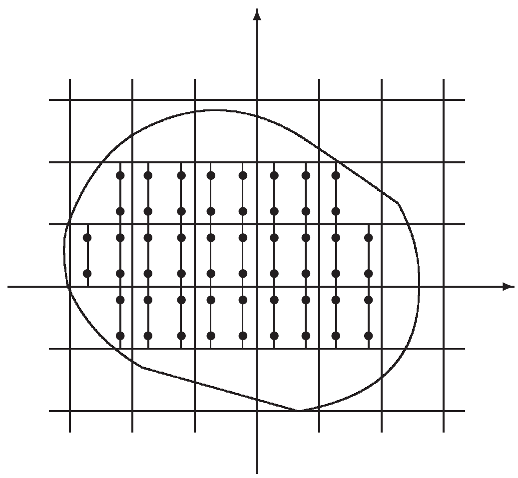


Figure 2. Vertical line segments.

A brief outline of the paper is as follows. In section 2 we present the formulation and implementation of the new Hermite cubic OSC method for the solution of a TPBVP with both end subintervals removed from the original interval. In particular, we explain the transfer of the Dirichlet boundary values on which our approach for higher dimensional problems will depend. In section 3, by converting the resulting linear system of equations to a tridiagonal one, we prove, using the discrete maximum principle, the optimal fourth order accuracy of our scheme. (For completeness we present the discrete maximum principle in the Appendix.) Numerical results presented in section 4 confirm our theoretical results. Concluding remarks are given in section 5.

2. OSC for two-point BVP without end subintervals

Consider the two-point BVP on $[a, b]$ with Dirichlet boundary conditions

$$Lu = f(x), \quad x \in (a, b), \quad u(a) = u_a, \quad u(b) = u_b, \tag{1}$$

where a, b, u_a, u_b are given numbers, $a < b$, f is a given function on (a, b), and, with r a given nonnegative function on (a, b),

$$Lu = -u'' + r(x)u. \tag{2}$$

Assume that a given stepsize $h > 0$, $Z = \{0, \pm 1, \pm 2, \ldots\}$, and $x_i = ih$, $i \in Z$. Let i_* and i^* in Z be such that

$$i_* = \min\{i \in Z : x_i \geq a\}, \quad i^* = \max\{i \in Z : x_i \leq b\}. \tag{3}$$

Then (see Figure 3)

$$x_{i_*-1} < a \leq x_{i_*} < \cdots < x_{i^*} \leq b < x_{i^*+1}$$

and hence for

$$h_l = x_{i_*} - a, \quad h_r = b - x_{i^*},$$

we have

$$0 \leq h_l < h, \quad 0 \leq h_r < h. \tag{4}$$

First we want to approximate u of (1)–(2) on $[x_{i_*}, x_{i^*}]$ (see Figure 3).

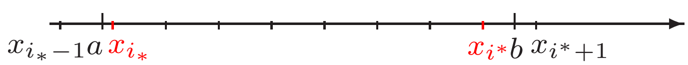


Figure 3. Partition points.

Let V be the space of piecewise Hermite cubics on $[x_{i_*}, x_{i^*}]$ defined by

$$V = \{v \in C^1[x_{i_*}, x_{i^*}] : v|_{[x_i, x_{i+1}]} \in P_3, i = i_*, \ldots, i^* - 1\},$$

where P_3 is the set of polynomials of degree ≤ 3. Let the Gauss points be given by

$$\xi_{i,1} = x_i + \frac{3 - \sqrt{3}}{6}h, \quad \xi_{i,2} = x_i + \frac{3 + \sqrt{3}}{6}h, \quad i \in Z. \tag{5}$$

Note that there are two Gauss (collocation) points in each subinterval $[x_i, x_{i+1}]$, $i = i_*, \ldots, i^* - 1$, (see black dots in Figure 4).

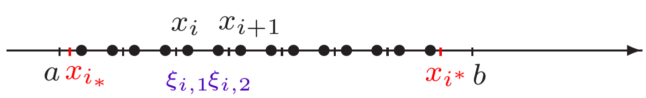


Figure 4. Collocation points.

We look for the approximate solution $U \in V$ such that

$$LU(\xi_{i,k}) = f(\xi_{i,k}), \quad i = i_*, \ldots, i^* - 1, \quad k = 1, 2. \tag{6}$$

Since dim $V = 2(i^* - i_*) + 2$ and the number of equations in (6) is $2(i^* - i_*)$, we require two additional equations. If $a = x_{i_*}$ and $x_{i^*} = b$, then the two additional equations, complementing (6), are

$$U(a) = u_a, \quad U(b) = u_b. \tag{7}$$

If $a < x_{i_*}$ and $x_{i^*} < b$, then the two additional equations, complementing (6), are

$$U'(x_{i_*}) = p'(x_{i_*}), \quad U'(x_{i^*}) = q'(x_{i^*}), \tag{8}$$

where p and q in P_3 satisfy respectively the following interpolation conditions

$$p(a) = u_a, \quad p(x_{i_*}) = U(x_{i_*}), \quad p(x_{i_*+1}) = U(x_{i_*+1}), \quad p'(x_{i_*+1}) = U'(x_{i_*+1}),$$
$$q(x_{i^*-1}) = U(x_{i^*-1}), \quad q'(x_{i^*-1}) = U'(x_{i^*-1}), \quad q(x_{i^*}) = U(x_{i^*}), \quad q(b) = u_b.$$

The equations in (8) are not the only choice of two additional equations complementing (6). It is possible, for example, to use, in place of (8), the following two equations

$$U(x_{i_*}) = p(x_{i_*}), \quad U(x_{i^*}) = q(x_{i^*}),$$

where p and q in P_3 satisfy respectively the following interpolation conditions

$$\begin{aligned}
&p(a) = u_a, \quad p(\xi_{i_*,1}) = U(\xi_{i_*,1}), \quad p(\xi_{i_*,2}) = U(\xi_{i_*,2}),\\
&p(x_{i_*+1}) = U(x_{i_*+1}), \quad q(x_{i^*-1}) = U(x_{i^*-1}),\\
&q(\xi_{i^*-1,1}) = U(\xi_{i^*-1,1}), \quad q(\xi_{i^*-1,2}) = U(\xi_{i^*-1,2}), \quad q(b) = u_b.
\end{aligned}$$

To describe the algebraic problem corresponding to (6) and (8), we note that

$$p(x) = u_a p_1(x) + U(x_{i_*})p_2(x) + U(x_{i_*+1})p_3(x) + U'(x_{i_*+1})p_4(x), \tag{9}$$

$$q(x) = U(x_{i^*-1})q_1(x) + U'(x_{i^*-1})q_2(x) + U(x_{i^*})q_3(x) + u_b q_4(x), \tag{10}$$

where the p_j in P_3 are defined by

$$\begin{aligned}
&p_1(a) = 1, \quad p_1(x_{i_*}) = 0, \quad p_1(x_{i_*+1}) = 0, \quad p_1'(x_{i_*+1}) = 0,\\
&p_2(a) = 0, \quad p_2(x_{i_*}) = 1, \quad p_2(x_{i_*+1}) = 0, \quad p_2'(x_{i_*+1}) = 0,\\
&p_3(a) = 0, \quad p_3(x_{i_*}) = 0, \quad p_3(x_{i_*+1}) = 1, \quad p_3'(x_{i_*+1}) = 0,\\
&p_4(a) = 0, \quad p_4(x_{i_*}) = 0, \quad p_4(x_{i_*+1}) = 0, \quad p_4'(x_{i_*+1}) = 1,
\end{aligned}$$

and the q_j in P_3 associated with x_{i^*-1} (twice through $q_j(x_{i^*-1})$ and $q_j'(x_{i^*-1})$), x_{i^*}, and b are defined similarly. Substituting (9) and (10) into (8), we obtain

$$\begin{aligned}
&-p_2'(x_{i_*})U(x_{i_*}) + U'(x_{i_*}) - p_3'(x_{i_*})U(x_{i_*+1}) - p_4'(x_{i_*})U'(x_{i_*+1})\\
&= u_a p_1'(x_{i_*}),
\end{aligned} \tag{11}$$

$$\begin{aligned}
&-q_1'(x_{i^*})U(x_{i^*-1}) - q_2'(x_{i^*})U'(x_{i^*-1}) - q_3'(x_{i^*})U(x_{i^*}) + U'(x_{i^*})\\
&= u_b q_4'(x_{i^*}).
\end{aligned} \tag{12}$$

It can be verified that

$$p_1(x) = \frac{x_{i_*} - x}{h_l}\left(\frac{x - x_{i_*+1}}{h + h_l}\right)^2, \quad p_2(x) = \frac{x - a}{h_l}\left(\frac{x - x_{i_*+1}}{h}\right)^2,$$

$$p_3(x) = \frac{x-a}{h+h_l}\frac{x - x_{i_*}}{h} - \frac{2h + h_l}{h^2(h+h_l)^2}(x-a)(x - x_{i_*})(x - x_{i_*+1}),$$

$$p_4(x) = \frac{(x-a)(x-x_{i_*})(x-x_{i_*+1})}{h(h+h_l)},$$

satisfy all conditions defining the p_j. Evaluating directly, we have

$$p_1'(x_{i_*}) = -\frac{h^2}{h_l(h+h_l)^2}, \quad p_2'(x_{i_*}) = \frac{h - 2h_l}{hh_l},$$

$$p_3'(x_{i_*}) = h_l\frac{3h + 2h_l}{h(h+h_l)^2}, \quad p_4'(x_{i_*}) = -\frac{h_l}{h+h_l}.$$

Substituting these values into (11) and multiplying through by $-h_l$, we obtain

$$\begin{aligned}
&\left(1 - 2\frac{h_l}{h}\right)U(x_{i_*}) - h_l U'(x_{i_*}) + h_l^2\frac{3h + 2h_l}{h(h+h_l)^2}U(x_{i_*+1})\\
&- \frac{h_l^2}{h+h_l}U'(x_{i_*+1}) = \frac{h^2}{(h+h_l)^2}u_a.
\end{aligned} \tag{13}$$

If $a = x_{i_*}$ ($h_l = 0$), then the last equation reduces to the first equation in (7). In a similar way, (12), multiplied through by h_r, yields

$$h_r^2 \frac{3h + 2h_r}{h(h+h_r)^2} U(x_{i^*-1}) + \frac{h_r^2}{h+h_r} U'(x_{i^*-1}) + \left(1 - 2\frac{h_r}{h}\right) U(x_{i^*}) + h_r U'(x_{i^*}) = \frac{h^2}{(h+h_r)^2} u_b. \tag{14}$$

If $x_{i^*} = b$ ($h_r = 0$), then the last equation reduces to the second equation in (7). We write the approximate solution U in the form

$$U(x) = \sum_{i=i_*}^{i^*} \left[\alpha_i \phi_i(x) + \beta_i \psi_i(x)\right], \quad x \in [x_{i_*}, x_{i^*}], \tag{15}$$

where the ϕ_i and ψ_i in V are respectively the value and scaled slope basis functions associated with the partition point x_i. These functions are given by (see, e.g., (5.1) and (5.3) in [2])

$$\phi_i(x) = h^{-3} \begin{cases} [h - 2(x - x_i)]\,(x - x_{i-1})^2, & x \in [x_{i-1}, x_i], \\ [h + 2(x - x_i)]\,(x - x_{i+1})^2, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise}, \end{cases} \tag{16}$$

$$\psi_i(x) = h^{-3} \begin{cases} (x - x_i)(x - x_{i-1})^2, & x \in [x_{i-1}, x_i], \\ (x - x_i)(x - x_{i+1})^2, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise}, \end{cases} \tag{17}$$

with an obvious modifications for $i = i_*$ and $i = i^*$. For $i, j = i_*, \dots, i^*$, we have

$$\phi_i(x_i) = 1, \quad \phi_i'(x_i) = 0, \quad \phi_i(x_j) = \phi_i'(x_j) = 0, \quad j \neq i,$$

$$\psi_i(x_i) = 0, \quad \psi_i'(x_i) = h^{-1}, \quad \psi_i(x_j) = \psi_i'(x_j) = 0, \quad j \neq i.$$

Hence (15) implies

$$U(x_i) = \alpha_i, \quad U'(x_i) = h^{-1}\beta_i, \quad i = i_*, \dots, i^*. \tag{18}$$

Substituting (15) into (6) and using (13), (14), and (18), we obtain the $[2(i^* - i_*) + 2] \times [2(i^* - i_*) + 2]$ linear system

$$A\mathbf{c} = \mathbf{f} \tag{19}$$

with

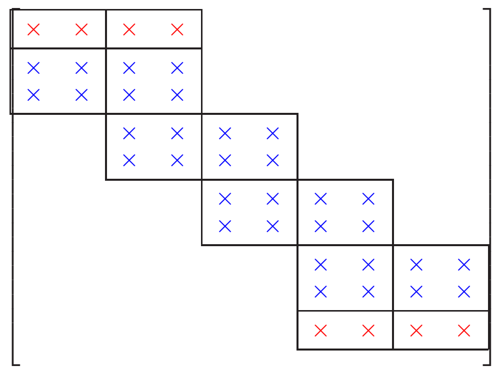
$$\mathbf{c} = [\alpha_{i_*}, \beta_{i_*}, \dots, \alpha_{i^*}, \beta_{i^*}]^T, \quad \mathbf{f} = [f_{i_*}, f_{i_*,1}, f_{i_*,2}, \dots, f_{i^*-1,1}, f_{i^*-1,2}, f_{i^*}]^T, \tag{20}$$

where

$$f_{i_*} = \frac{h^2}{(h+h_l)^2} u_a, \; f_{i,k} = f(\xi_{i,k}), \quad i = i_*, \dots, i^* - 1, \quad k = 1, 2, \; f_{i^*} = \frac{h^2}{(h+h_r)^2} u_b.$$

Note that ϕ_{i_*} and ψ_{i_*} are 0 outside the interval $[x_{i_*}, x_{i_*+1}]$, for $i = i_*+1, \dots, i^*-1$, ϕ_i and ψ_i are 0 outside the interval $[x_{i-1}, x_{i+1}]$, and ϕ_{i^*} and ψ_{i^*} are 0 outside the interval $[x_{i^*-1}, x_{i^*}]$. Therefore, the matrix A has the following structure, displayed here for $i^* - i_* = 4$,

$A =$



.

We assume that A is nonsingular. Then the system (19) can be solved at a cost $O(i^* - i_*)$ using the capacitance matrix method [6] and the package COLROW [7, 8]. To describe this computation, we consider the $[2(i^* - i_*) + 2] \times [2(i^* - i_*) + 2]$ matrix B whose first and last rows are

$$(21) \qquad [1, 0, 0, 0 \ldots, 0, 0, 0, 0], \quad [0, 0, 0, 0 \ldots, 0, 0, 1, 0],$$

respectively, and whose remaining rows are equal to the corresponding rows of A, that is,

$$(22) \qquad A(i, :) = B(i, :), \quad i = 2, \ldots, 2(i^* - i_*) + 1.$$

(The rows in (21) correspond to specifying U of (15) at x_{i_*} and x_{i^*}, respectively.) The matrix B is almost block diagonal (ABD) [7] and nonsingular [9] since the function r in (2) is nonnegative. A linear system with B can be solved at a cost $O(i^* - i_*)$ using the package COLROW of [7, 8] for solving ABD linear systems. In the capacitance matrix approach we look for the solution $\mathbf{c}$ of (19) in the form

$$(23) \qquad \mathbf{c} = \mathbf{d} + \gamma_1 \mathbf{d}_1 + \gamma_2 \mathbf{d}_2,$$

where the numbers γ_1 and γ_2 are to be determined and the vectors $\mathbf{d}$, $\mathbf{d}_1$, $\mathbf{d}_2$ are solutions of the linear systems

$$(24) \quad B\mathbf{d} = [0, f_{i_*,1}, \ldots, f_{i^*-1,2}, 0]^T, \quad B\mathbf{d}_1 = [1, 0, \ldots, 0, 0]^T, \quad B\mathbf{d}_2 = [0, 0, \ldots, 0, 1]^T.$$

Using (22) and (24) it is easy to verify that, for arbitrary γ_1 and γ_2,

$$A(2 : 2(i^* - i_*) + 1, :)(\mathbf{d} + \gamma_1 \mathbf{d}_1 + \gamma_2 \mathbf{d}_2) = [f_{i_*,1}, \ldots, f_{i^*-1,2}]^T.$$

Moreover, for $i = 1$ and $i = 2(i^* - i_*) + 2$, we have

$$A(i, :)(\mathbf{d} + \gamma_1 \mathbf{d}_1 + \gamma_2 \mathbf{d}_2) = A(i, :)\mathbf{d} + \gamma_1 A(i, :)\mathbf{d}_1 + \gamma_2 A(i, :)\mathbf{d}_2.$$

Hence $\mathbf{c}$ given by the right-hand side of (23) solves (19) if and only if γ_1 and γ_2 solve the 2×2 linear system

(25)
$$\begin{bmatrix} A(1, :)\mathbf{d}_1 & A(1, :)\mathbf{d}_2 \\ A(2(i^* - i_*) + 2, :)\mathbf{d}_1 & A(2(i^* - i_*) + 2, :)\mathbf{d}_2 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} f_{i_*} \\ f_{i^*} \end{bmatrix} - \begin{bmatrix} A(1, :)\mathbf{d} \\ A(2(i^* - i_*) + 2, :)\mathbf{d} \end{bmatrix}.$$

Since A and B are nonsingular, it follows from Theorem 1 in [6] that the 2×2 matrix in (25) is also nonsingular. Thus we obtain solution $\mathbf{c}$ of the system (19) by first computing, with the use of COLROW, the vectors $\mathbf{d}$, $\mathbf{d}_1$, and $\mathbf{d}_2$ of (24). Then we set up and solve the system (25) and finally we form $\mathbf{c}$ using (23). The cost of this computation is $O(i^* - i_*)$.

If $a < x_{i_*}$ and $x_{i^*} < b$, then the approximate solution U on $[x_{i_*}, x_{i^*}]$ is extended onto $[a, b]$ in the following way. For $x \in [a, x_{i_*}]$, we define

$$(26) \qquad U(x) = p(x),$$

where p in P_3 is such that

(27)
$$p(x_{i_*}) = U(x_{i_*}), \quad p'(x_{i_*}) = U'(x_{i_*}), \quad p(x_{i_*+1}) = U(x_{i_*+1}), \quad p'(x_{i_*+1}) = U'(x_{i_*+1}).$$

Similarly,

$$U(x) = q(x), \quad x \in [x_{i^*}, b], \tag{28}$$

where q in P_3 is such that

$$q(x_{i^*-1}) = U(x_{i^*-1}), \quad q'(x_{i^*-1}) = U'(x_{i^*-1}), \quad q(x_{i^*}) = U(x_{i^*}), \quad q'(x_{i^*}) = U'(x_{i^*}). \tag{29}$$

An alternative to (27) is $p(a) = u_a$ and the first three equations of (27) and an alternative to (29) is the equations one, three, four of (29) and $q(b) = u_b$.

Remark 2.1. *We refer to the two additional equations in (8), involving respectively the interpolation nodes a, x_{i_*}, x_{i_*+1} and x_{i^*-1}, x_{i^*}, b, as transferring of the Dirichlet boundary values at a and b (see green dots in Figure 5) to x_{i_*} and x_{i^*} (see red dots in Figure 5).*

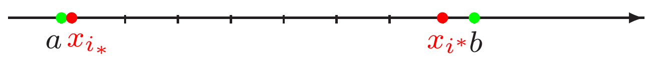


Figure 5. *Transfer of Dirichlet boundary values.*

For finite differences [11], *a similar linear polynomial transfer of order two to x_{i_*} and x_{i^*} involves the interpolation nodes a, x_{i_*+1} and x_{i^*-1}, b, respectively. Higher order finite difference transfer is possible* [12] *but using additional interpolation nodes. In comparison, for OSC with Hermite cubics or higher degree splines, only three interpolation nodes near the boundary points are involved in transferring of the Dirichlet boundary values.*

3. Convergence Analysis

For the convergence analysis, we first express the OSC scheme as a tridiagonal linear system. We bound the truncation error for this system and use the discrete maximum principle, provided for completeness in the Appendix, to establish its stability. Then, we show that the scheme is fourth order accurate in the discrete and global maximum norms. In the remainder of this section, for simplicity, we assume that $r(x) = 0$ in (2). In what follows, C denotes a generic positive constant that is independent of h, h_l, h_r but which, in general, depends on the exact solution u. Also, $O(h^p)$ denotes a quantity whose absolute value is $\le Ch^p$.

Lemma 3.1. *Assume that U of (15) is the solution of the OSC scheme defined by (6), (13) and (14). Then α_i, β_i, $i_* \le i \le i^*$, satisfy*

$$2\frac{h+h_l}{h}\alpha_{i_*} - 2\frac{h_l}{h}\alpha_{i_*+1} = 2u_a + \frac{h_l(h+h_l)}{2\sqrt{3}h}\left\{\left[\left(\sqrt{3}+1\right)h + 2h_l\right] f(\xi_{i_*,1}) + \left[\left(\sqrt{3}-1\right)h - 2h_l\right] f(\xi_{i_*,2})\right\}, \tag{30}$$

$$\frac{-\alpha_{i-1} + 2\alpha_i - \alpha_{i+1}}{h^2} = \frac{\sqrt{3}-1}{4\sqrt{3}}\left[f(\xi_{i-1,1}) + f(\xi_{i,2})\right] + \frac{\sqrt{3}+1}{4\sqrt{3}}\left[f(\xi_{i-1,2}) + f(\xi_{i,1})\right], \quad i = i_*+1, \dots, i^*-1, \tag{31}$$

$$-2\frac{h_r}{h}\alpha_{i^*-1} + 2\frac{h+h_r}{h}\alpha_{i^*} = 2u_b + \frac{h_r(h+h_r)}{2\sqrt{3}h}\left\{\left[\left(\sqrt{3}-1\right)h - 2h_r\right] f(\xi_{i^*-1,1}) + \left[\left(\sqrt{3}+1\right)h + 2h_r\right] f(\xi_{i^*-1,2})\right\}, \tag{32}$$

$$\beta_{i+1} = \alpha_{i+1} - \alpha_i - \frac{h^2}{4\sqrt{3}}\left[(\sqrt{3}-1)f(\xi_{i,1}) + (\sqrt{3}+1)f(\xi_{i,2})\right], \quad i = i_*, \dots, i^*-1, \tag{33}$$

(34) $$\beta_i = \alpha_{i+1} - \alpha_i + \frac{h^2}{4\sqrt{3}}\left[(\sqrt{3}+1)f(\xi_{i,1}) + (\sqrt{3}-1)f(\xi_{i,2})\right], \quad i = i_*, \dots, i^*-1.$$

Proof. Using (18) in (13) and (14), we obtain

(35) $$\left(1 - 2\frac{h_l}{h}\right)\alpha_{i_*} - \frac{h_l}{h}\beta_{i_*} + h_l^2\frac{3h+2h_l}{h(h+h_l)^2}\alpha_{i_*+1} - \frac{h_l^2}{h(h+h_l)}\beta_{i_*+1} = \frac{h^2}{(h+h_l)^2}u_a,$$

(36) $$h_r^2\frac{3h+2h_r}{h(h+h_r)^2}\alpha_{i^*-1} + \frac{h_r^2}{h(h+h_r)}\beta_{i^*-1} + \left(1 - 2\frac{h_r}{h}\right)\alpha_{i^*} + \frac{h_r}{h}\beta_{i^*} = \frac{h^2}{(h+h_r)^2}u_b.$$

Substituting (15) into (6) and using (16), (17), and (5), we obtain

(37) $$\frac{1}{h^2}\left[2\sqrt{3}\alpha_i + (\sqrt{3}+1)\beta_i - 2\sqrt{3}\alpha_{i+1} + (\sqrt{3}-1)\beta_{i+1}\right] = f(\xi_{i,1}), \quad i = i_*, \dots, i^*-1.$$

(38) $$\frac{1}{h^2}\left[-2\sqrt{3}\alpha_i - (\sqrt{3}-1)\beta_i + 2\sqrt{3}\alpha_{i+1} - (\sqrt{3}+1)\beta_{i+1}\right] = f(\xi_{i,2}), \quad i = i_*, \dots, i^*-1.$$

To eliminate β_i, we add (37) multiplied by $\sqrt{3}-1$ to (38) multiplied by $\sqrt{3}+1$ to obtain

(39) $$\begin{aligned}&\frac{1}{h^2}\left[-4\sqrt{3}\alpha_i + 4\sqrt{3}\alpha_{i+1} - 4\sqrt{3}\beta_{i+1}\right]\\ =&(\sqrt{3}-1)f(\xi_{i,1}) + (\sqrt{3}+1)f(\xi_{i,2}),\ i = i_*, \dots, i^*-1.\end{aligned}$$

To eliminate β_{i+1}, we add (37) multiplied by $\sqrt{3}+1$ to (38) multiplied by $\sqrt{3}-1$ to obtain

(40) $$\begin{aligned}&\frac{1}{h^2}\left[4\sqrt{3}\alpha_i + 4\sqrt{3}\beta_i - 4\sqrt{3}\alpha_{i+1}\right]\\ =&(\sqrt{3}+1)f(\xi_{i,1}) + (\sqrt{3}-1)f(\xi_{i,2}), \quad i = i_*, \dots, i^*-1.\end{aligned}$$

Adding (39), with i replaced by $i-1$, to (40) and dividing by $4\sqrt{3}$, we obtain (31). Equation (33) and (34) follow from (39) and (40), respectively. Substituting β_{i_*+1} from (33) and β_{i_*} from (34) into (35), simplifying and multiplying by $2\frac{(h+h_l)^2}{h^2}$, we obtain (30). Also, substituting β_{i^*} from (33) and β_{i^*-1} from (34) into (36), simplifying and multiplying by $2\frac{(h+h_r)^2}{h^2}$, we obtain (32). □

Remark 3.1. *Equations (30)–(32) form a tridiagonal system in the unknowns* $\alpha_{i_*}, \dots, \alpha_{i^*}$.

Let u be the exact solution of (1)–(2). Then the truncation errors of the system (30)–(32) are defined by

(41) $$\begin{aligned}\epsilon_{i_*} =& 2\frac{h+h_l}{h}u(x_{i_*}) - 2\frac{h_l}{h}u(x_{i_*+1}) - 2u_a\\ &-\frac{h_l(h+h_l)}{2\sqrt{3}h}\left\{\left[\left(\sqrt{3}+1\right)h + 2h_l\right]f(\xi_{i_*,1}) + \left[\left(\sqrt{3}-1\right)h - 2h_l\right]f(\xi_{i_*,2})\right\},\end{aligned}$$

(42) $$\begin{aligned}\epsilon_i =& \frac{-u(x_{i-1}) + 2u(x_i) - u(x_{i+1})}{h^2} - \frac{\sqrt{3}-1}{4\sqrt{3}}\left[f(\xi_{i-1,1}) + f(\xi_{i,2})\right]\\ &-\frac{\sqrt{3}+1}{4\sqrt{3}}\left[f(\xi_{i-1,2}) + f(\xi_{i,1})\right], \quad i = i_*+1, \dots, i^*-1,\end{aligned}$$

(43) $$\begin{aligned}\epsilon_{i^*} =& -2\frac{h_r}{h}u(x_{i^*-1}) + 2\frac{h+h_r}{h}u(x_{i^*}) - 2u_b\\ &-\frac{h_r(h+h_r)}{2\sqrt{3}h}\left\{\left[\left(\sqrt{3}-1\right)h - 2h_r\right]f(\xi_{i^*-1,1}) + \left[\left(\sqrt{3}+1\right)h + 2h_r\right]f(\xi_{i^*-1,2})\right\}.\end{aligned}$$

The truncation errors (41)–(43) indicate by how much u fails to satisfy (30)–(32).

Lemma 3.2. *If u is sufficiently smooth, then the truncation errors (41)–(43) satisfy*

$$\epsilon_{i_*} = -\frac{h_l^2(h+h_l)^2}{12}u^{(4)}(a) + O(h^5), \tag{44}$$

$$\epsilon_i = O(h^4), \quad i = i_*+1,\dots,i^*-1, \tag{45}$$

$$\epsilon_{i^*} = -\frac{h_r^2(h+h_r)^2}{12}u^{(4)}(b) + O(h^5). \tag{46}$$

Proof. With u sufficiently smooth, Taylor's theorem, (1), (2) with $r(x) = 0$, (5), and (4) give

$$u(x_{i_*}) = u_a + h_l u'(a) + \frac{h_l^2}{2}u''(a) + \frac{h_l^3}{6}u^{(3)}(a) + \frac{h_l^4}{24}u^{(4)}(a) + O(h^5), \tag{47}$$

(48)
$$u(x_{i_*+1}) = u_a+(h+h_l)u'(a)+\frac{(h+h_l)^2}{2}u''(a)+\frac{(h+h_l)^3}{6}u^{(3)}(a)+\frac{(h+h_l)^4}{24}u^{(4)}(a)+O(h^5),$$

$$\begin{aligned} -f(\xi_{i_*,1}) &= u''(\xi_{i_*,1}) \\ &= u''(a) + \left(h_l + \frac{3-\sqrt{3}}{6}h\right)u^{(3)}(a) + \frac{1}{2}\left(h_l + \frac{3-\sqrt{3}}{6}h\right)^2 u^{(4)}(a) + O(h^3), \end{aligned} \tag{49}$$

$$\begin{aligned} -f(\xi_{i_*,2}) &= u''(\xi_{i_*,2}) \\ &= u''(a) + \left(h_l + \frac{3+\sqrt{3}}{6}h\right)u^{(3)}(a) + \frac{1}{2}\left(h_l + \frac{3+\sqrt{3}}{6}h\right)^2 u^{(4)}(a) + O(h^3). \end{aligned} \tag{50}$$

Hence, substituting (47)–(50) into (41), we see that the coefficients of u_a, $u'(a)$, $u''(a)$, $u^{(3)}(a)$, and $u^{(4)}(a)$ are respectively equal to

$$2\frac{h+h_l}{h} - 2\frac{h_l}{h} - 2 = \frac{2}{h}[h+h_l-h_l-h] = 0, \tag{51}$$

$$2\frac{h+h_l}{h}h_l - 2\frac{h_l}{h}(h+h_l) = 0, \tag{52}$$

$$2\frac{h+h_l}{h}\frac{h_l^2}{2} - 2\frac{h_l}{h}\frac{(h+h_l)^2}{2} + \frac{h_l(h+h_l)}{2\sqrt{3}h}2\sqrt{3}h = (h+h_l)\frac{h_l}{h}[h_l-h-h_l+h] = 0, \tag{53}$$

$$\begin{aligned} &2\frac{h+h_l}{h}\frac{h_l^3}{6} - 2\frac{h_l}{h}\frac{(h+h_l)^3}{6} + \frac{h_l(h+h_l)}{2\sqrt{3}h}\left\{\left[(\sqrt{3}+1)h+2h_l\right]\left(h_l + \frac{3-\sqrt{3}}{6}h\right)\right. \\ &\qquad\qquad \left. + \left[(\sqrt{3}-1)h-2h_l\right]\left(h_l+\frac{3+\sqrt{3}}{6}h\right)\right\} \\ &= \frac{h_l(h+h_l)}{6h}\left[2h_l^2 - 2(h+h_l)^2 + \sqrt{3}\left\{(\sqrt{3}+1)hh_l + 2h_l^2 + \frac{\sqrt{3}}{3}h^2 + \left(1-\frac{1}{\sqrt{3}}\right)hh_l\right.\right. \\ &\qquad\qquad \left.\left. +(\sqrt{3}-1)hh_l - 2h_l^2 + \frac{\sqrt{3}}{3}h^2 - \left(1+\frac{1}{\sqrt{3}}\right)hh_l\right\}\right] \\ &= \frac{h_l(h+h_l)}{6h}\left[-2h^2 - 4hh_l + 6hh_l + 2h^2 - 2hh_l\right] = 0, \end{aligned} \tag{54}$$

$$(55)\quad 2\frac{h+h_l}{h}\frac{h_l^4}{24}-2\frac{h_l}{h}\frac{(h+h_l)^4}{24}+\frac{h_l(h+h_l)}{2\sqrt{3}h}\left\{\frac{1}{2}\left[(\sqrt{3}+1)h+2h_l\right]\left(h_l+\frac{3-\sqrt{3}}{6}h\right)^2\right.$$
$$\left.+\frac{1}{2}\left[(\sqrt{3}-1)h-2h_l\right]\left(h_l+\frac{3+\sqrt{3}}{6}h\right)^2\right\}$$
$$=\frac{h_l(h+h_l)}{12h}[h_l^3-(h+h_l)^3+\sqrt{3}\{\sqrt{3}h\left[2h_l^2+2hh_l+\frac{2}{3}h^2\right]$$
$$+(2h_l+h)\left(-\frac{h}{\sqrt{3}}\right)(2h_l+h)\}]$$
$$=\frac{h_l(h+h_l)}{12h}\left[-h(3h_l^2+3hh_l+h^2)+h(6h_l^2+6hh_l+2h^2)-h(4h_l^2+4hh_l+h^2)\right]$$
$$=-\frac{h_l^2(h+h_l)^2}{12}.$$

Equality (44) follows from (41), (47)–(55), and (4).

In a similar way we obtain (46). To obtain (45) from (42), we first note the well-known finite difference formula

(56)
$$\frac{-u(x_{i-1})+2u(x_i)-u(x_{i+1})}{h^2}=-u''(x_i)-\frac{h^2}{12}u^{(4)}(x_i)+O(h^4),\quad i=i_*+1,\dots,i^*-1.$$

Using (1), (2) with $r(x)=0$, and (5), we also have

$$(57)\quad -f(\xi_{i,1})=u''(\xi_{i,1})=u''(x_i)+\frac{3-\sqrt{3}}{6}hu^{(3)}(x_i)+\frac{1}{2}\left(\frac{3-\sqrt{3}}{6}\right)^2h^2u^{(4)}(x_i)$$
$$+\frac{1}{6}\left(\frac{3-\sqrt{3}}{6}\right)^3h^3u^{(5)}(x_i)+O(h^4),\quad i=i_*,\dots,i^*-1,$$

$$(58)\quad -f(\xi_{i,2})=u''(\xi_{i,2})=u''(x_i)+\frac{3+\sqrt{3}}{6}hu^{(3)}(x_i)+\frac{1}{2}\left(\frac{3+\sqrt{3}}{6}\right)^2h^2u^{(4)}(x_i)$$
$$+\frac{1}{6}\left(\frac{3+\sqrt{3}}{6}\right)^3h^3u^{(5)}(x_i)+O(h^4),\quad i=i_*,\dots,i^*-1,$$

$$(59)\quad -f(\xi_{i-1,1})=u''(\xi_{i-1,1})=u''(x_i)-\frac{3+\sqrt{3}}{6}hu^{(3)}(x_i)+\frac{1}{2}\left(\frac{3+\sqrt{3}}{6}\right)^2h^2u^{(4)}(x_i)$$
$$-\frac{1}{6}\left(\frac{3+\sqrt{3}}{6}\right)^3h^3u^{(5)}(x_i)+O(h^4),\quad i=i_*+1,\dots,i^*,$$

$$(60)\quad -f(\xi_{i-1,2})=u''(\xi_{i-1,2})=u''(x_i)-\frac{3-\sqrt{3}}{6}hu^{(3)}(x_i)+\frac{1}{2}\left(\frac{3-\sqrt{3}}{6}\right)^2h^2u^{(4)}(x_i)$$
$$-\frac{1}{6}\left(\frac{3-\sqrt{3}}{6}\right)^3h^3u^{(5)}(x_i)+O(h^4),\quad i=i_*+1,\dots,i^*.$$

Hence (59), (58), (60), (57) give

$$f(\xi_{i-1,1})+f(\xi_{i,2})=-2u''(x_i)-\left(\frac{3+\sqrt{3}}{6}\right)^2h^2u^{(4)}(x_i)+O(h^4),$$

$$f(\xi_{i-1,2})+f(\xi_{i,1})=-2u''(x_i)-\left(\frac{3-\sqrt{3}}{6}\right)^2h^2u^{(4)}(x_i)+O(h^4),$$

which, in turn, yield

$$(61)\quad -\frac{\sqrt{3}-1}{4\sqrt{3}}\left[f(\xi_{i-1,1})+f(\xi_{i,2})\right]-\frac{\sqrt{3}+1}{4\sqrt{3}}\left[f(\xi_{i-1,2})+f(\xi_{i,1})\right]$$

$$= -2\left(\frac{1-\sqrt{3}}{4\sqrt{3}} - \frac{\sqrt{3}+1}{4\sqrt{3}}\right)u''(x_i)$$

$$- \left[\frac{1-\sqrt{3}}{4\sqrt{3}}\left(\frac{3+\sqrt{3}}{6}\right)^2 - \frac{\sqrt{3}+1}{4\sqrt{3}}\left(\frac{3-\sqrt{3}}{6}\right)^2\right] h^2 u^{(4)}(x_i) + O(h^4)$$

$$= u''(x_i) + \frac{h^2}{12}u^{(4)}(x_i) + O(h^4).$$

Using (42), (56), and (61), we obtain (45). □

Remark 3.2. *Equations (44), (46), and (4) imply, of course, that*

$$|\epsilon_{i_*}|, |\epsilon_{i^*}| \leq Ch^4.$$

However, (44) and (46) also show that ϵ_{i_*} and ϵ_{i^*} are not, in general, proportional to h^4 since $h_l \neq h$. The proof of Lemma 3.2 also shows that if $u^{(5)}(x) = 0$, $x \in [a,b]$, then the term $O(h^4)$ in (45) and the terms $O(h^5)$ in (44), (46) drop out.

In order to show the stability of the system (30)–(32), we introduce the following sets of points in R,

$$\omega_1 = \{x_{i_*+1}, \ldots, x_{i^*-1}\}, \quad \omega_2 = \{x_{i_*}, x_{i^*}\}, \quad \omega = \omega_1 \cup \omega_2.$$

For any real-valued function v on ω, let the real-valued function $\widetilde{L}v$ on ω be defined by (cf. the left-hand sides of (30)–(32))

$$\widetilde{L}v(x_i) = \begin{cases} 2\dfrac{h+h_l}{h}v(x_i) - 2\dfrac{h_l}{h}v(x_{i+1}), & i = i_*, \\[2ex] \dfrac{-v(x_{i-1}) + 2v(x_i) - v(x_{i+1})}{h^2}, & x_i \in \omega_1, \\[2ex] -2\dfrac{h_r}{h}v(x_{i-1}) + 2\dfrac{h+h_r}{h}v(x_i), & i = i^*. \end{cases} \tag{62}$$

Lemma 3.3. *If real-valued functions v and w on ω are such that*

$$\widetilde{L}v(x_i) = w(x_i), \quad x_i \in \omega, \tag{63}$$

where $\widetilde{L}v$ on ω is defined by (62), then

$$\max_{x_i \in \omega}|v(x_i)| \leq \left[\frac{1}{2} + \frac{(b-a)^2}{8}\right]\max_{x_i \in \omega}|w(x_i)|.$$

Proof. Consider the case when $h_l \neq 0$ and $h_r \neq 0$. Then dividing (63) by $h_l(h+h_l)$ and $h_r(h+h_r)$ for $i = i_*$ and $i = i^*$, respectively, and using (62), we see that (63) is equivalent to

$$Lv(x_i) = z(x_i), \quad x_i \in \omega, \tag{64}$$

where

$$Lv(x_i) = \begin{cases} \dfrac{-v(x_{i-1}) + 2v(x_i) - v(x_{i+1})}{h^2}, & x_i \in \omega_1, \\[2ex] \dfrac{2}{hh_l}v(x_i) - \dfrac{2}{h(h+h_l)}v(x_{i+1}), & i = i_*, \\[2ex] -\dfrac{2}{h(h+h_r)}v(x_{i-1}) + \dfrac{2}{hh_r}v(x_i), & i = i^*, \end{cases} \tag{65}$$

$$z(x_i) = w(x_i), \quad x_i \in \omega_1, \quad z(x_{i_*}) = \frac{w(x_{i_*})}{h_l(h+h_l)}, \quad z(x_{i^*}) = \frac{w(x_{i^*})}{h_r(h+h_r)}. \tag{66}$$

Note that $Lv(x_i)$ is of the form (A.1) and that the assumptions (A.2) are satisfied. Since

$$\frac{-1+2-1}{h^2} = 0,$$

$$(67)\qquad \frac{2}{hh_l} - \frac{2}{h(h+h_l)} = \frac{2}{h_l(h+h_l)} > 0, \quad \frac{2}{hh_r} - \frac{2}{h(h+h_r)} = \frac{2}{h_r(h+h_r)} > 0,$$

the assumptions (A.3) hold. For $P = x_i \in \omega_1$, the assumption (A.4) is satisfied with $S = x_{i_*} \in \omega_2$ and $P_1 = x_{i-1}$, $P_2 = x_{i-2}$, $\ldots$, $P_m = x_{i_*+1}$. It follows from Corollary A.1 that there exist real-valued functions $v^{(1)}$ and $v^{(2)}$ on ω satisfying

$$(68)\qquad Lv^{(1)}(x_i) = \begin{cases} 0, & x_i \in \omega_1, \\ z(x_i), & x_i \in \omega_2, \end{cases} \qquad Lv^{(2)}(x_i) = \begin{cases} z(x_i), & x_i \in \omega_1, \\ 0, & x_i \in \omega_2. \end{cases}$$

Using (68) we have

$$L\left[v^{(1)} + v^{(2)}\right](x_i) = z(x_i), \quad x_i \in \omega_1.$$

Hence (64) and Corollary A.1 imply that $v = v^{(1)} + v^{(2)}$, which gives

$$(69)\qquad |v(x_i)| \le |v^{(1)}(x_i)| + |v^{(2)}(x_i)|, \quad x_i \in \omega.$$

It follows from the first equation in (68), Theorem A.3, (65) with $i = i_*, i^*$, (67), and the last two equations of (66) that

$$(70)\qquad \max_{x_i\in\omega} |v^{(1)}(x_i)| \le \frac{1}{2}\max\{h_l(h+h_l)|z(x_{i_*})|, h_r(h+h_r)|z(x_{i^*})|\} = \frac{1}{2}\max\{|w(x_{i_*})|, |w(x_{i^*})|\}.$$

To bound $\max_{x_i\in\omega}|v^{(2)}(x_i)|$, we introduce

$$(71)\qquad \psi(x) = (x-a)(b-x) = -x^2 + (a+b)x - ab, \quad x \in [a,b],$$

for which we have

$$(72)\qquad \max_{x\in[a,b]} \psi(x) \le \psi\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{4}.$$

Using (65) and (71) we verify that

$$(73)\qquad L\psi(x_i) = \frac{x_{i-1}^2 - 2x_i^2 + x_{i+1}^2}{h^2} = \frac{2h^2}{h^2} = 2, \quad x_i \in \omega_1,$$

$$L\psi(x_{i_*}) = \frac{2}{hh_l}h_l(b - x_{i_*}) - \frac{2}{h(h_l+h)}(h+h_l)(b - x_{i_*+1}) = \frac{2}{h}(x_{i_*+1} - x_{i_*}) = 2,$$

$$L\psi(x_{i^*}) = -\frac{2}{h(h+h_r)}(x_{i^*-1} - a)(h+h_r) + \frac{2}{hh_r}(x_{i^*} - a)h_r$$

$$(74)\qquad = \frac{2}{h}(x_{i^*} - x_{i^*-1}) = 2,$$

so that

$$(75)\qquad L\psi(x_i) = 2, \quad x_i \in \omega.$$

With

$$(76)\qquad K = \frac{1}{2}\max_{x_i\in\omega_1} |z(x_i)|,$$

the second equation in (68) and (75) yield

$$(77)\qquad |Lv^{(2)}(x_i)| \le L(K\psi)(x_i), \quad x_i \in \omega.$$

It follows from (77), Theorem A.2, (76), (72), and the first equation in (66) that

$$(78)\qquad \max_{x_i\in\omega}|v^{(2)}(x_i)| \le K\max_{x_i\in\omega}\psi(x_i) \le \frac{(b-a)^2}{8}\max_{x_i\in\omega_1}|z(x_i)| = \frac{(b-a)^2}{8}\max_{x_i\in\omega_1}|w(x_i)|.$$

Hence the desired bound follows from (69), (70), and (78).

Next, consider the case $h_l = 0$, $h_r \neq 0$. (Other cases such as $h_l \neq 0$, $h_r = 0$ and $h_l = 0$, $h_r = 0$ can be treated in a similar way.) Then dividing (63) by $h_r(h + h_r)$ for $i = i^*$ and using (62) with $h_l = 0$, we see that (63) is equivalent to (64), where

$$(79) \qquad Lv(x_i) = \begin{cases} \dfrac{-v(x_{i-1}) + 2v(x_i) - v(x_{i+1})}{h^2}, & x_i \in \omega_1, \\ 2v(x_{i_*}), & i = i_*, \\ -\dfrac{2}{h(h+h_r)}v(x_{i-1}) + \dfrac{2}{hh_r}v(x_i), & i = i^*, \end{cases}$$

$$(80) \qquad z(x_i) = w(x_i), \quad x_i \in \omega_1, \quad z(x_{i_*}) = w(x_{i_*}), \quad z(x_{i^*}) = \frac{w(x_{i^*})}{h_r(h+h_r)}.$$

Once again $Lv(x_i)$ is of the form (A.1) and the assumptions (A.2)–(A.4) are satisfied. Therefore, we again have (68) and (69). It follows from the first equation in (68), Theorem A.3, (79) with $i = i_*, i^*$, the second equation in (67), and the last two equations in (80) that (cf. (70))

$$(81) \quad \max_{x_i \in \omega} |v^{(1)}(x_i)| \le \frac{1}{2}\max\{|z(x_{i_*})|, h_r(h+h_r)|z(x_{i^*})|\} = \frac{1}{2}\max\{|w(x_{i_*})|, |w(x_{i^*})|\}.$$

To bound $\max_{x_i \in \omega} |v^{(2)}(x_i)|$, we now use (cf. (71))

$$(82) \qquad \psi(x) = (x - x_{i_*})(b - x) = -x^2 + (x_{i_*} + b)x - x_{i_*}b,$$

for which we have (cf. (72))

$$(83) \qquad \max_{x \in [x_{i_*}, b]} \psi(x) \le \psi\left(\frac{x_{i_*} + b}{2}\right) = \frac{(b - x_{i_*})^2}{4} \le \frac{(b-a)^2}{4}.$$

Using (79) with $x_i \in \omega_1$ and (82), we have (73). Using (79) with $i = i^*$ and (82), we have (cf. (74))

$$L\psi(x_{i^*}) = -\frac{2}{h(h+h_r)}(x_{i^*-1} - x_{i_*})(h + h_r) + \frac{2}{hh_r}(x_{i^*} - x_{i_*})h_r = \frac{2}{h}(x_{i^*} - x_{i^*-1}) = 2.$$

Hence

$$(84) \qquad L\psi(x_i) = 2, \quad x_i \in \omega_1, \quad i = i^*.$$

Using (79) with $i = i_*$ and (82), we have

$$(85) \qquad L\psi(x_{i_*}) = 2\psi(x_{i_*}) = 0.$$

With K of (76), the second equation in (68), (84), and (85) yield (77). Hence, as before, using (77), Theorem A.2, (76), (83), and the first equation in (80), we obtain (78). The desired result follows again from (69), (81), and (78). □

Remark 3.3. *Lemma 3.3 implies that (63) has a unique solution and that if T is the matrix corresponding to the tridiagonal system (30)–(32), then* $\|T^{-1}\|_\infty \le \frac{1}{2} + \frac{(b-a)^2}{8}$.

Theorem 3.1. *If the exact solution u of (1)–(2) with $r(x) = 0$ is sufficiently smooth and the OSC approximate solution $U \in V$ is defined by (6) and (8), then*

$$(86) \qquad \max_{i_* \le i \le i^*} |u(x_i) - U(x_i)| \le Ch^4.$$

Proof. We set

$$v(x_i) = u(x_i) - U(x_i), \quad i = i_*, \dots, i^*.$$

Then it follows from (62), the first equation in (18), (30)–(32), and (41)–(43) that

$$\widetilde{L}v(x_i) = \widetilde{L}u(x_i) - \widetilde{L}U(x_i) = \epsilon_i, \quad x_i \in \omega,$$

and hence (86) follows from Lemmas 3.3 and 3.2, and (4). □

In the following theorem we show that the error in approximating the exact solution $u(x)$ by the OSC approximation $U(x)$ on $[a, b]$ is of optimal fourth order accuracy.

Theorem 3.2. *If the exact solution u of (1)–(2) with $r(x) = 0$ is sufficiently smooth and the OSC approximate solution $U \in V$, defined by (6) and (8), is extended onto $[a, b]$ using (26)–(29), then*

$$\max_{x\in[a,b]} |u(x) - U(x)| \leq Ch^4.$$

Proof. It follows from (34), the first equation in (18), and (86) that

$$\beta_i = u(x_{i+1}) - u(x_i) + \frac{h^2}{4\sqrt{3}} \left[(\sqrt{3}+1)f(\xi_{i,1}) + (\sqrt{3}-1)f(\xi_{i,2})\right] + O(h^4), \quad i = i_*, \ldots, i^*-1. \tag{87}$$

Taylor's theorem gives

$$u(x_{i+1}) - u(x_i) = hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{6}u^{(3)}(x_i) + O(h^4), \quad i = i_*, \ldots, i^* - 1. \tag{88}$$

It follows from (57) and (58) that

$$\begin{aligned}
&(\sqrt{3}+1)f(\xi_{i,1}) + (\sqrt{3}-1)f(\xi_{i,2}) \\
&= -(\sqrt{3}+1)\left[u''(x_i) + \frac{3-\sqrt{3}}{6}hu^{(3)}(x_i)\right] - (\sqrt{3}-1)\left[u''(x_i) + \frac{3+\sqrt{3}}{6}hu^{(3)}(x_i)\right] \\
&= -2\sqrt{3}u''(x_i) - \frac{4\sqrt{3}}{6}hu^{(3)}(x_i) + O(h^2), \quad i = i_*, \ldots, i^* - 1.
\end{aligned} \tag{89}$$

Using (87)–(89), we obtain

$$\beta_i - hu'(x_i) = O(h^4), \quad i = i_*, \ldots, i^* - 1. \tag{90}$$

It follows from (33), the first equation in (18), and (86) that

$$\beta_i = u(x_i) - u(x_{i-1}) - \frac{h^2}{4\sqrt{3}} \left[(\sqrt{3}-1)f(\xi_{i-1,1}) + (\sqrt{3}+1)f(\xi_{i-1,2})\right] + O(h^4),\ i = i_*+1, \ldots, i^*. \tag{91}$$

Taylor's theorem gives

$$u(x_i) - u(x_{i-1}) = hu'(x_i) - \frac{h^2}{2}u''(x_i) + \frac{h^3}{6}u^{(3)}(x_i) + O(h^4), \quad i = i_* + 1, \ldots, i^*. \tag{92}$$

It follows from (59) and (60) that

$$\begin{aligned}
&(\sqrt{3}-1)f(\xi_{i-1,1}) + (\sqrt{3}+1)f(\xi_{i-1,2}) \\
&=(1-\sqrt{3})\left[u''(x_i) - \frac{3+\sqrt{3}}{6}hu^{(3)}(x_i)\right] - (\sqrt{3}+1)\left[u''(x_i) - \frac{3-\sqrt{3}}{6}hu^{(3)}(x_i)\right] \\
&= -2\sqrt{3}u''(x_i) + \frac{4\sqrt{3}}{6}hu^{(3)}(x_i) + O(h^2), \quad i = i_* + 1, \ldots, i^*.
\end{aligned} \tag{93}$$

Using (91)–(93), we obtain

$$\beta_i - hu'(x_i) = O(h^4), \quad i = i_* + 1, \ldots, i^*. \tag{94}$$

Let $\widetilde{U}$ be the Hermite cubic interpolant of u on $[x_{i_*}, x_{i^*}]$, that is,

$$\widetilde{U}(x) = \sum_{i=i_*}^{i^*} \left[u(x_i)\phi_i(x) + hu'(x_i)\psi_i(x)\right], \quad x \in [x_{i_*}, x_{i^*}].$$

Then (3.6.15) in [1] and $[x_{i_*}, x_{i^*}] \subset [a, b]$ yield

$$|u(x) - \widetilde{U}(x)| \leq \frac{1}{384} \max_{t\in[x_{i_*},x_{i^*}]} |u^{(4)}(t)|h^4 \leq \frac{1}{384} \max_{t\in[a,b]} |u^{(4)}(t)|h^4, \quad x \in [x_{i_*}, x_{i^*}]. \tag{95}$$

Using (15), the first equation in (18), the triangle inequality, (86), (90), and (94), we have

$$|U(x) - \widetilde{U}(x)| = \left| \sum_{i=i_*}^{i^*} \left\{ [U(x_i) - u(x_i)]\, \phi_i(x) + \left[\beta_i - hu'(x_i)\right] \psi_i(x) \right\} \right|$$
$$\leq \sum_{i=i_*}^{i^*} \left[\, |U(x_i) - u(x_i)||\phi_i(x)| + |\beta_i - hu'(x_i)||\psi_i(x)|\, \right]$$
$$\leq Ch^4 \sum_{i=i_*}^{i^*} \left[|\phi_i(x)| + |\psi_i(x)|\right], \quad x \in [x_{i_*}, x_{i*}].$$

It follows from (16) and (17) that

$$|\phi_i(x)| \leq 3, \quad |\psi_i(x)| \leq 1, \qquad x \in [x_{i_*}, x_{i*}].$$

Since for given $x \in [x_k, x_{k+1}]$, the sum $\sum_{i=i_*}^{i^*}$ reduces to $\sum_{i=k}^{k+1}$, we arrive at

$$|U(x) - \widetilde{U}(x)| \leq Ch^4, \quad x \in [x_{i_*}, x_{i*}]. \tag{96}$$

The triangle inequality, (95), and (96) yield

$$|u(x) - U(x)| \leq Ch^4, \quad x \in [x_{i_*}, x_{i*}]. \tag{97}$$

Next, we bound the error in the interval $[x_{i^*}, b]$. It follows from (28), (29), and (18) that

$$U(x) = \alpha_{i^*-1} q_1(x) + \beta_{i^*-1} q_2(x) + \alpha_{i^*} q_3(x) + \beta_{i^*} q_4(x), \quad x \in [x_{i^*}, b],$$

where the q_j in P_3 are such that

$$q_1(x_{i^*-1}) = 1, \quad q_1'(x_{i^*-1}) = 0, \quad q_1(x_{i^*}) = 0, \quad q_1'(x_{i^*}) = 0,$$
$$q_2(x_{i^*-1}) = 0, \quad q_2'(x_{i^*-1}) = h^{-1}, \quad q_2(x_{i^*}) = 0, \quad q_2'(x_{i^*}) = 0,$$
$$q_3(x_{i^*-1}) = 0, \quad q_3'(x_{i^*-1}) = 0, \quad q_3(x_{i^*}) = 1, \quad q_3'(x_{i^*}) = 0,$$
$$q_4(x_{i^*-1}) = 0, \quad q_4'(x_{i^*-1}) = 0, \quad q_4(x_{i^*}) = 0, \quad q_4'(x_{i^*}) = h^{-1}.$$

Equations (16) and (17) imply that

$$q_1(x) = h^{-3}[h + 2(x - x_{i^*-1})](x - x_{i^*})^2, \quad q_2(x) = h^{-3}(x - x_{i^*-1})(x - x_{i^*})^2,$$
$$q_3(x) = h^{-3}[h - 2(x - x_{i^*})](x - x_{i^*-1})^2, \quad q_4(x) = h^{-3}(x - x_{i^*-1})^2(x - x_{i^*}).$$

Using the second equation in (4), we have

$$|q_1(x)| \leq 5, \quad |q_2(x)| \leq 2, \quad |q_3(x)| \leq 4, \quad |q_4(x)| \leq 4, \quad x \in [x_{i^*}, b]. \tag{98}$$

Let $\widetilde{U}$ be the Hermite cubic interpolant of u with the nodes x_{i^*-1} and x_{i^*}, that is, $\widetilde{U} \in P_3$

$$\widetilde{U}(x_{i^*-1}) = u(x_{i^*-1}), \quad \widetilde{U}'(x_{i^*-1}) = u'(x_{i^*-1}), \quad \widetilde{U}(x_{i^*}) = u(x_{i^*}), \quad \widetilde{U}'(x_{i^*}) = u'(x_{i^*}).$$

Then (3.6.14) in [1] yields

$$\begin{aligned} |u(x) - \widetilde{U}(x)| &\leq \frac{1}{24}(x - x_{i^*-1})^2 (x - x_{i^*})^2 \max_{t \in [x_{i^*-1}, b]} |u^{(4)}(t)| \\ &\leq \frac{1}{6} \max_{t \in [a,b]} |u^{(4)}(t)| h^4, \quad x \in [x_{i*}, b]. \end{aligned} \tag{99}$$

Using (15), the first equation in (18), the triangle inequality, (86), (90), (94), and (98), we also have

$$\begin{aligned} \left|U(x) - \widetilde{U}(x)\right| &= \big|\, [U(x_{i^*-1}) - u(x_{i^*-1})]\, q_1(x) + \left[\beta_{i^*-1} - hu'(x_{i^*-1})\right] q_2(x) \\ &\qquad + [U(x_{i^*}) - u(x_{i^*})]\, q_3(x) + \left[\beta_{i^*} - hu'(x_{i^*})\right] q_4(x) \,\big| \\ &\leq Ch^4, \quad x \in [x_{i^*}, b]. \end{aligned} \tag{100}$$

It follows from the triangle inequality, (99) and (100) that

$$|u(x) - U(x)| \leq Ch^4, \quad x \in [x_{i^*}, b].$$

TABLE 1

N	10	20	30	40	50	60	70	80	90	100
$\|\epsilon_i\|_N$	5.3–06	1.7–06	9.1–07	6.2–07	4.7–07	3.9–07	3.3–07	1.2–09	3.9–09	7.2–09
γ_N	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0001	1.0000	1.0000
δ_N		1.6616	1.4822	1.3396	1.2225	1.1245	1.0410	42.107	−10.080	−5.8106
$\|T_N^{-1}\|_\infty$	0.7496	0.7498	0.7498	0.7498	0.7499	0.7499	0.7499	0.7500	0.7500	0.7500

Similarly, it can be shown that

$$|u(x) - U(x)| \le Ch^4, \quad x \in [a, x_{i_*}].$$

The last two inequalities and (97) yield the desired result. □

4. Numerical Results

Our theoretical results given in Theorems 3.1 and 3.2 indicate that the OSC scheme, comprising (6) and (8), for approximating sufficiently smooth u of (1)–(2) with $r(x) = 0$ is fourth order accurate in the discrete and global maximum norms over $[a, b]$. However, we do not observe the convergence rate four when using the standard formula for computing convergence rate on a sequence of uniform partitions. In Example 1 we explain why this is the case. In all of our examples we use $N = 10, 20, \dots, 90, 100$ and $h = h_N = 1/N$.

Example 1. With $[a, b] = [0, \sqrt{2}]$ and $r(x) = 0$, we take the exact solution of (1)–(2) to be $u(x) = x^4$. For each N, we have $h_l = h_{N,l} = 0$ and $h_r = h_{N,r} \ne 0$. Since $u^{(5)}(x) = 0$, $u^{(4)}(x) = 24$, $x \in [a, b]$, and since $h_{N,l} = 0$, Lemma 3.2 and Remark 3.2 imply that

$$[\epsilon_{i_*}, \epsilon_{i_*+1} \dots, \epsilon_{i^*-1}, \epsilon_{i^*}]^T = [0, 0, \dots, 0, -2h_{N,r}^2 (h_N + h_{N,r})^2]^T. \tag{101}$$

Hence the discrete norm of the truncation errors (41)–(43), defined by

$$\|\epsilon_i\|_N = \max_{i=i_*,\dots,i^*} |\epsilon_i|, \tag{102}$$

behaves according to the formula

$$\|\epsilon_i\|_N = 2h_{N,r}^2 (h_N + h_{N,r})^2. \tag{103}$$

We compute $\|\epsilon_i\|_N$ using (102) and (41)–(43), the ratios

$$\gamma_N = \|\epsilon_i\|_N / [2h_{N,r}^2 (h_N + h_{N,r})^2], \tag{104}$$

and the convergence rates for $\|\epsilon_i\|_N$ using the formula (commonly used for uniform partitions)

$$\delta_N = \frac{\log(\|\epsilon_i\|_N / \|\epsilon_i\|_{N-1})}{\log(h_N / h_{N-1})}. \tag{105}$$

The computed values of γ_N presented in Table 1 confirm (103). In Table 1, we observe erratic behavior of δ_N since $\|\epsilon_i\|_N$ behaves according to the formula in (103) rather than $\|\epsilon_i\|_N = Ch_N^4$. In the last row of Table 1 we compute $\|T_N^{-1}\|_\infty$, where T_N is the matrix T in the tridiagonal system (30)–(32) for $h = h_N$. It follows from Remark 3.3 that for our interval $[a, b]$, $\|T_N^{-1}\|_\infty \le 3/4$. Hence the corresponding results in Table 1, indicate that the upper bound on $\|T^{-1}\|_\infty$ in Remark 3.3 is sharp. In Table 2, we present the norm of the error at the nodes given by

$$\|u(x_i) - U(x_i)\|_N = \max_{i=i_*,\dots,i^*} |u(x_i) - U(x_i)| \tag{106}$$

for which the convergence rate is computed using

$$\delta_N = \frac{\log(\|u(x_i) - U(x_i)\|_N / \|u(x_i) - U(x_i)\|_{N-1})}{\log(h_N / h_{N-1})}.$$

It follows from the proof of Theorem 3.1 and (101) that

$$\begin{aligned} \|[u_{i_*} - U(x_{i_*}), \dots, u_{i^*} - U(x_{i^*})]^T\|_\infty &= \|T_N^{-1} [\epsilon_{i_*}, \epsilon_{i_*+1} \dots, \epsilon_{i^*-1}, \epsilon_{i^*}]^T\|_\infty \\ &= 2h_{N,r}^2 (h_N + h_{N,r})^2 \|T_N^{-1}(:, l)\|_\infty. \end{aligned} \tag{107}$$

TABLE 2

N	10	20	30	40	50	60	70	80	90	100
$\|u-U\|_N$	2.6–06	8.2–07	4.5–07	3.1–07	2.3–07	1.9–07	1.6–07	5.9–10	1.9–09	3.6–09
$\|T_N^{-1}(:,l)\|_\infty$	0.4950	0.4950	0.4950	0.4950	0.4950	0.4950	0.4950	0.4994	0.4989	0.4985
δ_N		1.6616	1.4822	1.3396	1.2225	1.1245	1.0410	42.040	−10.071	−5.8031

TABLE 3

N	10	20	30	40	50	60	70	80	90	100
$\|u-U\|_\infty$	2.6–06	8.2–07	4.5–07	3.1–07	2.3–07	1.9–07	1.6–07	5.9–10	1.9–09	3.6–09
δ_N		1.6616	1.4822	1.3396	1.2225	1.1245	1.0410	42.041	−10.072	−5.8035

where $l = i^* - i_* + 1$. Numerical results in Table 2 suggest that $\|T_N^{-1}(:,l)\|_\infty \approx 0.5$. Hence it follows from (106), (107), and (103) that

$$\|u(x_i) - U(x_i)\|_N \approx 0.5\|\epsilon_i\|_N,$$

which shows that, except for the constant multiple 0.5, the norm of the error at the nodes and the discrete norm of the truncation error behave alike. This explains the same erratic behavior of the δ_N in Tables 1 and 2. In Table 3, we present the L^∞ norm of the error on $[a,b]$ given by

$$\|u-U\|_{L^\infty[a,b]} = \max_{a\le x\le b} |u(x) - U(x)|,$$

which is approximated using 10 equally spaced points in each subinterval $[x_i, x_{i+1}]$, $i = i_*, \dots, i^*-1$, and $[x_{i^*}, b]$ and for which the convergence rate is computed using

$$\delta_N = \frac{\log(\|u-U_N\|_{L^\infty[a,b]}/\|u-U_{N-1}\|_{L^\infty[a,b]})}{\log(h_N/h_{N-1})},$$

where U_N is U for $h = h_N$.

The results in Table 3 are similar to those in Table 2 even though our theoretical analysis does not imply directly that the behavior of the L^∞ norm of the error on $[a,b]$ should be the same as that of the norm of the error at the nodes.

Example 2. In this example, we show that similar numerical results are obtained for a finite difference scheme corresponding to our OSC scheme. The finite difference scheme [13, pg. 243] is given by

$$\begin{aligned}
\frac{h+h_l}{h}u_{i_*} - \frac{h_l}{h}u_{i_*+1} &= h_l h f(x_{i_*}) + u_a,\\
\frac{-u_{i-1}+2u_i-u_{i+1}}{h^2} &= f(x_i), \quad i = i_*+1,\dots,i^*-1,\\
-\frac{h_r}{h}u_{i^*-1} + \frac{h+h_r}{h}u_{i^*} &= h_r h f(x_{i^*}) + u_b,
\end{aligned}$$

for which, assuming that u is sufficiently smooth, we have [13, pg. 244]

$$\max_{i_*\le i\le i^*} |u(x_i) - u_i| \le Ch^2.$$

We take (1)–(2) with $[a,b] = [0,\sqrt{2}]$, $r(x) = 0$, $u(x) = x^2$, and compute

$$\|u(x_i) - u_i\|_N = \max_{i_*\le i\le i^*} |u(x_i) - u_i|,$$

$$\delta_N = \frac{\log(\|u(x_i)-u_i\|_N/\|u(x_i)-u_i\|_{N-1})}{\log(h_N/h_{N-1})}.$$

The numerical results presented in Table 4 are similar, if not worse, than those in Table 2. The errors in Table 2 are approximately the squares of the corresponding errors in Table 4.

Example 3. In this example, we use the OSC scheme to solve (1)–(2) with $[a,b] = [-\sqrt{2}/2, \sqrt{2}-1]$, $r(x) = \sin(x) + 1$, and $u(x) = e^x$. For each N, we have $h_l = h_{N,l} \ne 0$

TABLE 4

N	10	20	30	40	50	60	70	80	90	100
$\|u(x_i) - u_i\|_N$	1.2–03	5.0–04	2.7–04	1.5–04	8.1–05	3.5–05	1.0–06	1.8–05	2.5–05	2.4–05
δ_N		1.2613	1.5460	1.9898	2.7909	4.7069	22.876	-21.722	−2.5035	0.18677

TABLE 5

N	10	20	30	40	50	60	70	80	90	100
$\|u - U_N\|_{L^\infty[a,b]}$	1.6–07	5.0–08	2.8–08	1.9–08	1.4–08	1.2–08	1.0–08	4.0–10	3.5–10	3.1–10
δ_N		1.6552	1.4732	1.3319	1.2160	1.1188	1.0361	24.103	1.2395	1.1784

and $h_r = h_{N,r} \neq 0$. Not surprisingly, the numerical results presented in Table 5 show a behavior similar to that in Table 3.

5. Conclusion

We have formulated a new OSC scheme for a TPBVP without the boundary subintervals. The motivation for such a problem and the OSC scheme, which uses transfer of the Dirichlet boundary values, is the ADI OSC solution of parabolic problems on arbitrary domains. We have proved theoretically that our OSC scheme has optimal fourth order accuracy in the maximum norm. By numerical examples we have confirmed the theoretical results. The numerical results also indicate that one should not expect to observe the convergence rate four when computing the convergence rate of our scheme using the standard convergence rate formula (cf. (105)) which assumes that the error is proportional to h^4.

Appendix

Following [13, 14, 15], we present a general approach, based on the discrete maximum principle, for showing stability of finite difference schemes. Our presentation follows closely that in [15].

In what follows we assume that ω is a finite set of points in R^n $(n = 1, 2, 3)$ and $C(\omega)$ is the set of real valued function on ω, that is, $v \in C(\omega)$ if and only if $v : \omega \to R$. For $v \in C(\omega)$, let $Lv \in C(\omega)$ be defined by

$$Lv(P) = \sum_{Q\in N(P)} a(P,Q)v(Q), \quad P \in \omega, \tag{A.1}$$

where $P \in N(P) \subset \omega$,

$$a(P,P) > 0, \quad a(P,Q) < 0, \quad P \in \omega, \quad Q \in N'(P) \equiv N(P) - \{P\}. \tag{A.2}$$

We set

$$b(P) = \sum_{Q\in N(P)} a(P,Q), \quad P \in \omega.$$

We assume that $\omega = \omega_1 \cup \omega_2$, $\omega_1 \cap \omega_2 = \emptyset$, $\omega_2 \neq \emptyset$,

$$b(P) \geq 0, \quad P \in \omega_1, \qquad b(P) > 0, \quad P \in \omega_2, \tag{A.3}$$

and that for every $P \in \omega_1$ there exist $S \in \omega_2$ and a sequence $\{P_k\}_{k=1}^m \subset \omega_1$ such that

$$P_1 \in N'(P), \quad P_2 \in N'(P_1), \quad \cdots \quad , \quad P_m \in N'(P_{m-1}), \quad S \in N'(P_m). \tag{A.4}$$

Theorem A.1. *If $v \in C(\omega)$ and $Lv(P) \geq 0$, $P \in \omega$, then $v(P) \geq 0$, $P \in \omega$.*

Proof. Assume the claim is not true. Then there is $P \in \omega$ such that

$$v(P) = \min_{Q\in\omega} v(Q) < 0.$$

We consider two cases: $P \in \omega_2$ or $P \in \omega_1$. In both cases

$$Lv(P) = \sum_{Q \in N(P)} a(P,Q)\,[v(Q) - v(P) + v(P)] = \sum_{Q \in N'(P)} a(P,Q)\,[v(Q) - v(P)] + b(P)v(P). \tag{A.5}$$

In the first case, the first term on the left-hand side of (A.5) is ≤ 0 and the second term is < 0. Therefore $Lv(P) < 0$ which gives a contradiction.

In the second case there exist $S \in \omega_2$ and a sequence $\{P_k\}_{k=1}^m \subset \omega_1$ such that (A.4) holds. The second term on the right-hand side of (A.5) is ≤ 0. Since $P_1 \in N'(P)$, we must have $v(P_1) = v(P)$ since otherwise we would have $v(P_1) > v(P)$ which would lead to the contradiction $Lv(P) < 0$. Repeating this argument we have

$$v(P) = v(P_1) = v(P_2) = \ldots = v(S).$$

Since $S \in \omega_2$ and $v(S) = \min_{Q \in \omega} v(Q)$, the second case reduces to the first case. □

Corollary A.1. *For any $g \in C(\omega)$ there is a unique $v \in C(\omega)$ such that*

$$Lv(P) = g(P), \quad P \in \omega.$$

Proof. It is sufficient to show that if $v \in C(\omega)$ is such that $Lv(P) = 0$, $P \in \omega$, then $v(P) = 0$, $P \in \omega$. Since $Lv(P) = 0$, $P \in \omega$, implies $L(-v)(P) = 0$, $P \in \omega$, it follows from Theorem A.1 that $v(P) \geq 0$ and $v(P) \leq 0$, $P \in \omega$. Hence $v(P) = 0$, $P \in \omega$. □

Theorem A.2. *Assume v and z in $C(\omega)$ are such that*

$$|Lv(P)| \leq Lz(P), \quad P \in \omega. \tag{A.6}$$

Then

$$|v(P)| \leq z(P), \quad P \in \omega. \tag{A.7}$$

Proof. It follows from (A.6) that

$$-Lz(P) \leq Lv(P) \leq Lz(P), \quad P \in \omega,$$

and hence

$$L(v+z)(P) \geq 0, \quad L(z-v)(P) \geq 0, \quad P \in \omega.$$

Applying Theorem A.1 we have

$$-z(P) \leq v(P) \leq z(P), \quad P \in \omega,$$

which gives (A.7). □

Theorem A.3. *If $v \in C(\omega)$ is such that $Lv(P) = 0$, $P \in \omega_1$, then*

$$\max_{Q \in \omega} |v(Q)| \leq \max_{Q \in \omega_2} \frac{|Lv(Q)|}{b(Q)}.$$

Proof. Let $z \in C(\omega)$ be such that

$$Lz(P) = |Lv(P)|, \quad P \in \omega.$$

It follows from Corollary A.1 and Theorem A.1 that z exists and that $z(P) \geq 0$, $P \in \omega$. Let $P \in \omega$ be such that

$$z(P) = \max_{Q \in \omega} z(Q).$$

Consider two cases: $P \in \omega_2$ or $P \in \omega_1$. In the first case, using (A.5) with z replacing v, we have

$$\sum_{Q \in N'(P)} a(P,Q)\,[z(Q) - z(P)] + b(P)z(P) = Lz(P) = |Lv(P)|.$$

Since the first term on the left-hand side is ≥ 0, we get

$$z(P) \leq \max_{Q \in \omega_2} \frac{|Lv(Q)|}{b(Q)},$$

and hence the required inequality follows from Theorem A.2.

In the second case there exist $S \in \omega_2$ and a sequence $\{P_k\}_{k=1}^m \subset \omega_1$ such that (A.4) holds. Using (A.5) with z replacing v, we also have

$$\sum_{Q \in N'(P)} a(P,Q)\,[z(Q) - z(P)] + b(P)z(P) = 0.$$

The second term on the left-hand side is ≥ 0. Since $P_1 \in N'(P)$, we must have $z(P_1) = z(P)$ since otherwise we would have $z(P_1) < z(P)$ which would make the left-hand side > 0. Repeating this argument we have

$$z(P) = z(P_1) = z(P_2) = \ldots = z(S).$$

Since $S \in \omega_2$ and $z(S) = \max_{Q \in \omega} z(Q)$, the second case reduces to the first case. $\square$

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