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FINITE DIFFERENCE SCHEMES FOR THE KORTEWEG-DE VRIES-KAWAHARA EQUATION

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Abstract. We are concerned with the convergence of fully discrete finite difference schemes for the Korteweg-de Vries-Kawahara equation, which is a transport equation perturbed by dispersive terms of third and fifth order. It describes the evolution of small but finite amplitude long waves in various problems in fluid dynamics. Both the decaying case on the full line and the periodic case are considered. If the initial data $u|_{t=0} = u_0$ are of high regularity, $u_0 \in H^5(\mathbb{R})$, the schemes are shown to converge to a classical solution. Finally, the convergence is illustrated by an example.

Key words. Kawahara Equation, finite difference scheme, implicit schemes, convergence, existence.

1. Introduction

1.1. The Equation. This paper is concerned with the Korteweg-de Vries-Kawahara (Kawahara) equation, which reads

$$(1)\qquad \begin{cases} u_t + uu_x + \partial_x^3 u = \partial_x^5 u, & (x,t) \in \Pi_T, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $\Pi_T = \mathbb{R} \times (0,T]$ with fixed $T > 0$, u_0 the given initial data, and $u : \Pi_T \mapsto \mathbb{R}$ is the unknown scalar map. It is well known that the one-dimensional waves of small but finite amplitude in dispersive systems (e.g., the magneto-acoustic waves in plasmas, the shallow water waves, the lattice waves, etc.) can be described by the Korteweg-de Vries (KdV) equation, given by

$$(2)\qquad u_t + uu_x + \partial_x^3 u = 0,$$

which admits either compressive or rarefactive steady solitary wave solution (by a solitary water wave, we mean a travelling wave solution of the water wave equations for which the free surface approaches a constant height as $|x| \to \infty$) according to the sign of the dispersion term (the third order derivative term). In fact, in the galaxy of dispersive equations used to model waves phenomena, KdV equation is undoubtedly the brightest star.

However, under certain circumstances, it might happen that the coefficient of the third order derivative in the KdV equation becomes significantly small or even zero. In such a scenario, it is customary to take account of the higher order effect of dispersion in order to balance the nonlinear effect. As a result one may obtain a generalized nonlinear dispersive equation, known as Kawahara equation, which has a form of the KdV equation with an additional fifth order derivative term, given by (1). The Kawahara equation, an important nonlinear dispersive wave equation, describes solitary wave propagation in media in which the first order dispersion is anomalously small. A more specific physical background of this equation was introduced by Hunter and Scheurle [11], where they used it to describe the evolution of solitary waves in fluids in which the Bond number is less than but close to $\frac{1}{3}$ and

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the Froude number is close to 1. In the literature, this equation is also referred to as the fifth order KdV equation or singularly perturbed KdV equation. The fifth order term $\partial_x^5 u$ is called the Kawahara term.

1.2. Mathematical Background. There exists a fairly satisfactory well posedness theory for both KdV and Kawahara equations. The literature herein is substantial, and we will here only give a non-exhaustive overview. Within the existing framework, we mention the remarkable paper by Kenig et al., where the authors provide the local existence theory for the KdV equation in the Sobolev Space H^s, for $s > -3/4$. For a completely satisfactory well posedness theory for KdV equation, we refer to the monograph of Tao [23], and references therein.

Over the past four decades, there has been an increased interest to understand the solitary wave solutions of the Kawahara equation [6, 14, 16, 17]. It is found that, similar to the KdV equation, the Kawahara equation also has solitary wave solutions which decay rapidly to zero as $t \to \infty$, but unlike the KdV equation whose solitary wave solutions are non-oscillating, the solitary wave solutions of the Kawahara equation have oscillatory trails. This shows that the Kawahara equation is not only similar but also different from the KdV equation in the properties of solutions. The strong physical background of the Kawahara equation and such similarities and differences between it and the KdV equation in both the form and the behavior of the solution render the mathematical treatment of this equation particularly interesting. The Cauchy problem given by (1) has been studied by a few authors [3,7,15,24,25]. In that context, we mention the paper [3], where authors have shown that the problem (1) has a local solution $u \in C([-T,T]; H^r(\mathbb{R}))$ if $u_0 \in H^r(\mathbb{R})$ and $r > -1$. This local result combined with the energy conservation law yields that (1) has a global solution $u \in C([-\infty,\infty]; L^2(\mathbb{R}))$ if $u_0 \in L^2(\mathbb{R})$. Furthermore, the above mentioned results for (1) has been improved can be found in [25]. They even managed to prove local existence of solutions for $u_0 \in H^r(\mathbb{R})$, for $r \geq -7/5$ and global existence for $u_0 \in H^r(\mathbb{R})$, for $r > -1/2$. For the well posedness theory of (1), we refer to [7] and for the regularity results of such solutions, we refer to [20].

1.3. Numerical Approaches. There has been a number of papers involving the numerical computation of solutions of the Cauchy problem (1). For the KdV equation, a galore of numerical schemes available in literature. We just mention an interesting fact, and rarely referred to in the current literature, is that the first mathematical proof of existence and uniqueness of solutions of the KdV equation, was accomplished by Sjöberg [22] in 1970, using a finite difference approximation. His approach is based on a semi-discrete approximation where one discretizes the spatial variable, thereby reducing the equation to a system of ordinary differential equations. However, we stress that for numerical computations also this set of ordinary differential equations will have to be discretized in order to be solved. Therefore, to have a completely satisfactory numerical method, one seeks a fully discrete scheme that reduces the actual computation to a solution of a finite set of algebraic equations. In fact, this is accomplished in a recent paper by Holden et al. [8], both in the periodic case and on the full line.

A popular numerical approach has been the application of various spectral methods. Fourier-Galerkin spectral method for the KdV and Kawahara equations has been studied in [1, 18, 19]. Pseudospectral method or spectral collocation method have been used to solve PDEs like KdV, Kawahara equations in [4, 5]. On the other hand, in [18], an error estimate for a simple spectral fully discrete scheme for Kawahara equation has been proved. The equation was discretized in space by the standard Fourier-Galerkin spectral method and in time by the explicit leap-frog scheme. For the resulting fully discrete, conditionally stable scheme they prove an L^2-error bound of spectral accuracy in space and of second-order accuracy in time. Furthermore, finite difference schemes for Kawahara equation are also available in literature [2, 21]. But, as far as we are concerned, there is no rigorous proof of convergence for such schemes.

Finally, we mention that the numerical computation of solutions of the Kawahara equation is rather capricious. Two competing equations are involved, namely nonlinear convective term uu_x, which in the context of the equation $u_t = uu_x$ yields an infinite gradient in finite time even for smooth data, and the linear dispersive terms u_{xxx}, u_{xxxxx}, which in the context of the equation $u_t = u_{xxx} + u_{xxxxx}$ produces hard to compute dispersive waves, and these two effects combined makes it difficult to obtain accurate and fast numerical methods. Most of the finite difference schemes will consist of a sum of two terms, one discretizing the convective term and one discretizing the dispersive terms. These two effects will have to balance each other, as it is known that the Kawahara equation itself keeps the Sobolev norm $H^s(s > -1)$ bounded.

1.4. Scope of this Paper. In this paper, we focus on the derivation of convergent finite difference numerical methods for (1). The problem of analyzing convergent numerical schemes of course intimately is connected with the mathematical properties of the Cauchy problem for the Kawahara equation, which is well developed in literature.

First part of the paper deals with the convergence analysis of the following semi implicit (explicit discretization for the "nonlinear" term and implicit discretization for the "dispersive" terms) finite difference scheme

$$u_j^{n+1} = \overline{u}_j^n - \Delta t\, \overline{u}_j^n D u_j^n - \Delta t\, D_+^2 D_- u_j^{n+1} + \Delta t\, D_+^3 D_-^2 u_j^{n+1}, \quad n \in \mathbb{N}_0,\ j \in \mathbb{Z}, \tag{3}$$

where $u_j^n \approx u(j\Delta x, n\Delta t)$, and $\Delta x, \Delta t$ are small discretization parameters. Furthermore, D and $D_\pm$ denote symmetric and forward/backward (spatial) finite differences, respectively, and $\overline{u}$ denotes a spatial average. We remark that, this scheme is an extension of the scheme for KdV equation, analyzed in [8], to the Kawahara equation. We prove the following result, both for the full line and the periodic case: If the initial data $u_0 \in H^5(\mathbb{R})$, we show (see Theorem 3.2 and Remark 3.1) that the approximation (3) converges uniformly as $\Delta x \to 0$ with $\Delta t = \mathcal{O}\left(\Delta x^{\frac{3}{2}}\right)$ in $C(\mathbb{R} \times [0, \bar{T}])$, for any positive $\bar{T}$ to the unique solution of the Kawahara equation. Moreover, global existence of solutions has been proved. The above result gives a positive answer to the quest for a numerical scheme for Kawahara equation, for which we can prove convergence rigorously.

Having said this, however, the above mention scheme is clearly dissipative (cf. Figure 1), and as a result, produces below par numerical results. In fact, both phase and amplitude errors are evident from Figure 1. To overcome such drawbacks, we propose the following Crank-Nicolson fully implicit difference scheme

$$u_j^{n+1} = u_j^n - \Delta t\, \widetilde{u}_j^{n+\frac{1}{2}} D u_j^{n+\frac{1}{2}} - \Delta t\, D_+ D D_- u_j^{n+\frac{1}{2}} + \Delta t\, D_+^2 D D_-^2 u_j^{n+\frac{1}{2}}, \quad n \in \mathbb{N}_0,\ j \in \mathbb{Z}, \tag{4}$$

where we have used the notation $u^{n+\frac{1}{2}} = \frac{1}{2}(u^n + u^{n+1})$, and $\widetilde{u}$ denotes a spatial average. Moreover, in the second part of the paper, we prove the following convergence result: If the initial data $u_0 \in H^5(\mathbb{R})$, we show (see Theorem 4.2) that the approximation (3) converges uniformly as $\Delta x \to 0$ with $\Delta t = \mathcal{O}(\Delta x)$ in $C(\mathbb{R} \times [0, \bar{T}])$ for any positive $\bar{T}$ to the unique solution of the Kawahara equation.

To sum up, the proposed Crank-Nicolson scheme (4) has the following advantages:

- The scheme is *conservative*, i.e., it preserves the L^2-norm of the solution.
- The Figure 1 shows that the Crank-Nicolson scheme performs better than the dissipative scheme (3). In fact, this scheme doesn't entertain phase or dissipation errors, while phase error and substantial dissipation error are evident for the other scheme.
- An improvement of CFL condition helps to run the scheme for large time efficiently.

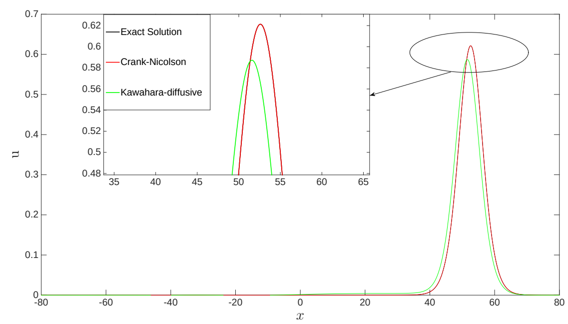

Figure 1. Comparison of exact and numerical solutions for different schemes at $T = 200$, with initial data (60) and $c = 10$.

The rest of the paper is organized as follows: In Section 2, we present the necessary notation and a semi implicit difference scheme for the Kawahara equation (1). Next, in Section 3, we show the convergence of the scheme for an initial data u_0 in $H^5(\mathbb{R})$ both for the full line and the periodic case. In Section 4, we propose a Crank-Nicolson type difference scheme and give details about the convergence proof of such schemes. In Section 5, we have shown the uniqueness of the solution, while in Section 6 we exhibit a numerical experiment showing the convergence.

2. Semi Implicit Finite Difference Scheme

As we mentioned earlier, following Holden et al. [8], we first analyze a semi implicit finite difference scheme for Kawahara equation. Here and in the sequel, we use the letters C, K to denote various generic constants. There are situations where constants may change from line to line, but the notation is kept unchanged so long as it does not impact the central idea. We start by introducing the necessary notation needed to define the scheme. Thought this paper, we reserve Δx, and Δt to denote small positive numbers that represent spatial and temporal discretization parameter of the numerical scheme respectively. Derivatives will be approximated by finite differences, and the basic quantities are as follows. For any function $p\colon \mathbb{R} \to \mathbb{R}$ we set

$$D_{\pm} p(x) = \pm \frac{1}{\Delta x}\big(p(x \pm \Delta x) - p(x)\big), \text{ and } D = \frac{1}{2}\left(D_+ + D_-\right).$$

Next, we introduce the average operators

$$\overline{p}(x) = \frac{1}{2}\left(p(x+\Delta x) + p(x-\Delta x)\right), \quad \widetilde{p}(x) = \frac{1}{3}\left(p(x+\Delta x) + p(x) + p(x-\Delta x)\right).$$

A simple use of Leibnitz rule allows us to verify the following identities readily:

(5a) $$D(pq) = \overline{p}Dq + \overline{q}Dp,$$

(5b) $$D_{\pm}(pq) = S^{\pm}pD_{\pm}q + qD_{\pm}p = S^{\pm}qD_{\pm}p + pD_{\pm}q,$$

(5c) $$D_+D_-(pq) = D_-p\,D_+q + S^-q\,D_-D_-p + D_+p\,D_+q + q\,D_+D_-p.$$

Here we have defined the shift operator

$$S^{\pm}p(x) = p(x \pm \Delta x).$$

We discretize the real axis using Δx and set $x_j = j\Delta x$ for $j \in \mathbb{Z}$. For a given function p we define $p_j = p(x_j)$. We will consider functions in ℓ^2 with the usual inner product and norm

$$(p, q) = \Delta x \sum_{j\in\mathbb{Z}} p_j q_j, \quad \|p\| = \|p\|_2 = (p, p)^{1/2}, \qquad p, q \in \ell^2.$$

In the periodic case with period J, the sum over $\mathbb{Z}$ is replaced by a finite sum $j = 0, \dots, J-1$. Furthermore, we define h^5-norm of a lattice function p as

$$\|p\|_{h^5} := \|p\| + \|D_+p\| + \|D_+D_-p\| + \|D_+DD_-p\| + \left\|D_+^2D_-^2p\right\| + \left\|D_+^2DD_-^2p\right\|.$$

The various difference operators enjoy the following properties:

$$(p, D_{\pm}q) = -(D_{\mp}p, q), \quad (p, Dq) = -(Dp, q), \qquad p, q \in \ell^2.$$

Further useful properties include

(6) $$\begin{aligned}(u, D_+^2D_-u) &= \frac{1}{2}(u, D_-D_+^2u) - \frac{1}{2}(u, D_+^2DD_-^2u) = \frac{1}{2}\big(u, (D_-D_+^2 - D_+D_-^2)u\big)\\ &= \frac{1}{2}\big(u, D_-D_+(D_- - D_+)u\big) = \frac{\Delta x}{2}\|D_+D_-u\|^2\end{aligned}$$

In a similar fashion, we find that

(7) $$\begin{aligned}(u, D_+^3D_-^2u) &= \frac{1}{2}(u, D_+^3D_-^2u) - \frac{1}{2}(u, D_-^3D_+^2u)\\ &= \frac{1}{2}\left(u, D_-^2D_+^2(D_+ - D_-)u\right) = -\frac{\Delta x}{2}\left\|D_-D_+^2u\right\|^2,\end{aligned}$$

and

(8) $$(D_+^2D_-u, D_+^3D_-^2u) = -\left\|D_-^2D_+^2u\right\|^2.$$

We also need to discretize in the time direction. Introduce (a small) time step $\Delta t > 0$, and use the notation

$$D_+^t p(t) = \frac{1}{\Delta t}\big(p(t + \Delta t) - p(t)\big),$$

for any function $p\colon [0, T] \to \mathbb{R}$. Write $t_n = n\Delta t$ for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A fully discrete grid function is a function $u_{\Delta x}\colon \Delta t\,\mathbb{N}_0 \to \mathbb{R}^{\mathbb{Z}}$, and we write $u_{\Delta x}(x_j, t_n) = u_j^n$. (A CFL condition will enforce a relationship between Δx and Δt, and hence we only use Δx in the notation.)

We are now ready to propose the following semi implicit scheme (based upon a six-point stencil) to generate approximate solutions to the Kawahara equation (1)

(9) $$u_j^{n+1} = \overline{u}_j^n - \frac{\Delta t}{3}[\overline{u}_j^n Du_j^n + D(u_j^n)^2] - \Delta t D_-D_+^2u_j^{n+1} + \Delta t D_+^3D_-^2u_j^{n+1}, \quad n \in \mathbb{N}_0,\ j \in \mathbb{Z}.$$

Keeping in mind that $Du_j^2 = 2\overline{u}_jDu_j$, we can rewrite the above scheme as

(10) $$u_j^{n+1} = \overline{u}_j^n - \Delta t\overline{u}_j^n Du_j^n - \Delta t D_-D_+^2u_j^{n+1} + \Delta t D_+^3D_-^2u_j^{n+1}.$$

For the initial data we have

$$u_j^0 = u_0(x_j), \quad j \in \mathbb{Z}.$$

Remark 2.1. *This scheme can be reformulated as an operator splitting scheme as follows. Set*

$$u_j^{n+1/2} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(\frac{1}{2}\left(u_{j+1}^n\right)^2 - \frac{1}{2}\left(u_{j-1}^n\right)^2\right),$$

i.e., $u^{n+1/2}$ is solution operator of the Lax–Friedrichs scheme for Burgers' equation, applied to u^n. Then

$$\frac{u^{n+1} - u^{n+1/2}}{\Delta t} = -D_+^2 D_- u^{n+1} + D_+^3 D_-^2 u^{n+1},$$

i.e., u^{n+1} is the approximate solution operator of a first-order implicit scheme for the equation $u_t + u_{xxx} = u_{xxxxx}$. If we write these two approximate solution operators as $S_{\Delta t}^B$, and $S_{\Delta t}^A$, respectively, the updated formula (10) reads

$$u^{n+1} = \left(S_{\Delta t}^A \circ S_{\Delta t}^B\right) u^n.$$

The convergence of this type of operator splitting using exact solution operators have been shown in [9], with severe restrictions on the initial data. The results in this paper can be viewed as a convergence result for operator splitting using approximate operators with less restrictions on the initial data, but with specified ratios between the temporal and spatial discretizations (CFL-like conditions).

Finally, we conclude this section by stating the following lemma, which essentially gives a relation between discrete and continuous Sobolev norms. Since we shall use this lemma frequently, for the sake of completeness, we present a proof of this lemma in the full line case.

Lemma 2.1. *There exists a generic constant C such that for all $u \in H^5(\mathbb{R})$*

$$\|u\|_{h^5} \leq C \|u\|_{H^5},$$

where we identify u with its discrete evaluation $\{u(x_j)\}_j$.

Proof. To begin with, observe that the discrete operator $D_+^2 D D_-^2$ commutes with the continuous operator ∂_x^5. A simple use of the Hölder estimate reveals that

$$\begin{aligned}
\left\|D_+^2 D D_-^2 u\right\|^2 &= \Delta x \sum_j \left(\frac{1}{\Delta x}\left(D_+ D D_-^2 u(x_{j+1}) - D_+ D D_-^2 u(x_j)\right)\right)^2 \\
&= \Delta x \sum_j \left(\int_{x_j}^{x_{j+1}} \frac{1}{\Delta x} \partial_x D_+ D D_-^2 u(x)\, dx\right)^2 \\
&\leq \Delta x \sum_j \left(\|1/\Delta x\|_{L^2([x_j, x_{j+1}])} \left\|\partial_x D_+ D D_-^2 u(x)\right\|_{L^2([x_j, x_{j+1}])}\right)^2 \\
&= \left\|D_+ D D_-^2 \partial_x u\right\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
\left\|D_+ D D_-^2 \partial_x u\right\|_{L^2(\mathbb{R})}^2 &\leq \left\|D D_-^2 \partial_x^2 u\right\|_{L^2(\mathbb{R})}^2, \cdots\cdots\cdots\cdots, \\
\left\|D\, \partial_x^4 u\right\|_{L^2(\mathbb{R})}^2 &\leq \left\|\partial_x^5 u\right\|_{L^2(\mathbb{R})}^2 \leq \|u\|_{H^5(\mathbb{R})}^2.
\end{aligned}$$

Furthermore, similar arguments can be used to show

$$\|u\|^2, \|D_+u\|^2, \|D_+D_-u\|^2, \|D_+DD_-u\|^2, \left\|D_+^2D_-^2u\right\|^2 \le \|u\|^2_{H^5(\mathbb{R})}.$$

Combining above results, we conclude the proof. □

3. Convergence Analysis

In this section, we present convergence analysis of the scheme given by (10). In what follows, we take preliminary motivation from [8] to carry out the convergence analysis. As we will see, in the sequel, extracting a strongly convergent subsequence from the approximating sequence $\{u_{\Delta x}\}_{\Delta x>0}$ mainly relies on the classical Arzelà–Ascoli type compactness. In what follows, we first show that the implicit scheme can be solved with respect to u_j^{n+1}. In fact, we can rewrite the scheme (10) as

$$(1 + \Delta t D_+^2D_- - \Delta t D_+^3D_-^2)u_j^{n+1} = \overline{u}_j^n - \Delta t\, \overline{u}_j^n Du_j^n,$$

and by taking inner product of the scheme with u^{n+1}, we conclude

$$\begin{aligned}
&\left((1 + \Delta t D_+^2D_- - \Delta t D_+^3D_-^2)u^{n+1}, u^{n+1}\right) \\
&= \left\|u^{n+1}\right\|^2 + \Delta t(D_+^2D_-u^{n+1}, u^{n+1}) - \Delta t(D_+^3D_-^2u^{n+1}, u^{n+1}) \\
&= \left\|u^{n+1}\right\|^2 + \frac{1}{2}\Delta t\Delta x\left\|D_+D_-u^{n+1}\right\|^2 + \frac{1}{2}\Delta t\Delta x\left\|D_+^2D_-u^{n+1}\right\|^2 \ge \left\|u^{n+1}\right\|^2,
\end{aligned}$$

thus

$$\left\|u^{n+1}\right\| \le \left\|(1 + \Delta t D_+^2D_- - \Delta t D_+^3D_-^2)u^{n+1}\right\| = \|\overline{u}^n - \Delta t\, \overline{u}^n Du^n\|.$$

Next, we state the following fundamental stability lemma, which is the first step towards exhibiting the convergence proof of the scheme. For the sake of completeness, we also furnish a sketch of the proof below.

Lemma 3.1. *Let u_j^n be the approximate solution generated by the difference scheme (10). Then the following estimate holds*

$$\begin{aligned}
&\left\|u^{n+1}\right\|^2 + \Delta t\Delta x^{1/2}\big(\Delta x\lambda\left\|D_+^2D_-u^{n+1}\right\|^2 \\
&\quad + \Delta x^{1/2}\left\|D_+^2D_-u^{n+1}\right\|^2 + \frac{\delta}{\lambda}\|Du^n\|^2 + \Delta x\lambda\left\|D_+^3D_-^2u^{n+1}\right\|^2 \\
&\quad + 2\Delta x\lambda\left\|D_+^2D_-^2u^{n+1}\right\|^2 + \Delta x^{1/2}\left\|D_+D_-u^{n+1}\right\|^2\big) \le \|u^n\|^2.
\end{aligned} \tag{11}$$

provided the CFL condition

$$\lambda\left\|u^0\right\|\left(\frac{1}{3} + \frac{1}{2}\lambda\left\|u^0\right\|\right) < \frac{1-\delta}{2}, \quad \delta \in (0,1), \tag{12}$$

holds where $\lambda = \Delta t/\Delta x^{3/2}$.

Proof. Following Holden et al. [8] and setting $w := \overline{u} - \Delta t\, \overline{u}Du$, we conclude that

$$\|w\|^2 + \delta\Delta x^2\|Du\|^2 \le \|u\|^2, \tag{13}$$

provided the CFL condition (12) is satisfied.

Next we study the full difference scheme by adding the "Airy term" $\Delta t\, D_+^2D_-u_j^{n+1}$ and the "Kawahara term" $\Delta t\, D_+^3D_-^2u_j^{n+1}$. Thus the full difference scheme (10) can be written as

$$v = w - \Delta t\, D_+^2D_-v + \Delta t\, D_+^3D_-^2v.$$

Furthermore, a simple application of the identities (6)–(8) implies, for the function u^n

$$\begin{aligned}\|w\|^2 &= \left\|u^{n+1}\right\|^2 + \Delta t \Delta x^{1/2}\big(\Delta x\lambda \left\|D_+^2 D_- u^{n+1}\right\|^2 + \Delta x^{1/2}\left\|D_+^2 D_- u^{n+1}\right\|^2 \\ &\quad + \Delta x\lambda \left\|D_+^3 D_-^2 u^{n+1}\right\|^2 + 2\Delta x\lambda \left\|D_+^2 D_-^2 u^{n+1}\right\|^2 + \Delta x^{1/2}\left\|D_+ D_- u^{n+1}\right\|^2\big) \\ &\le \|u^n\|^2 - \delta \Delta x^2 \|Du\|^2, \end{aligned} \tag{14}$$

and finally addition of (13) and (14) completes the proof. □

To proceed further, we note that the Arzelà–Ascoli compactness framework demands an estimate for the temporal derivative of the solutions. To achieve that, we consider the equation satisfied by the discrete time derivative. In what follows, we have the following lemma:

Lemma 3.2. *Let u_j^n be an approximate solution generated by the the difference scheme (10). Then the following estimate holds*

$$\begin{aligned} &\left\|D_+^t u^n\right\|^2 + \Delta t^2 \left\|D_+^2 D_- D_+^t u^n\right\|^2 + \Delta t^2 \left\|D_+^2 D_- D_+^t u^n\right\|^2 \\ &+ \Delta t^2 \left\|D_+^3 D_-^2 D_+^t u^n\right\|^2 + 2\Delta t^2 \left\|D_+^2 D_-^2 D_+^t u^n\right\|^2 \\ &+ \Delta t \Delta x \left\|D_+ D_- D_+^t u^n\right\|^2 + \tilde{\delta}\Delta x^2 \left\|D D_+^t u^{n-1}\right\|^2 \\ &\le \left\|D_+^t u^{n-1}\right\|^2 \left(1 + 3\Delta t \|Du^n\|_\infty\right), \end{aligned} \tag{15}$$

provided Δt is chosen such that

$$6\|u_0\|^2 \lambda^2 + \|u_0\|\lambda < \frac{1-\tilde{\delta}}{2}, \qquad \tilde{\delta} \in (0,1). \tag{16}$$

Proof. The proof uses same type of arguments as in Lemma 3.1. For a complete argument, consult Holden et al. [8]. □

Next, we rewrite the difference scheme (10) in the form

$$\alpha^{n+1} = D_+^t u^n = \frac{1}{2\mu} D_+ D_- u^n - \overline{u}^n D u^n - D_+^2 D_- u^{n+1} + D_+^3 D_-^2 u^{n+1}, \tag{17}$$

where $\mu = \Delta t/\Delta x^2 = \lambda/\Delta x^{1/2}$. Therefore, following Sjoberg [22], we proceed as below:

$$\begin{aligned} \left\|D_+^3 D_-^2 u^{n+1}\right\| &\le \left\|\alpha^{n+1}\right\| + \|\overline{u}^n D u^n\| + \frac{1}{2\mu}\|D_+ D_- u^n\| + \left\|D_+^2 D_- u^n\right\| \\ &\le \left\|\alpha^{n+1}\right\| + \|Du^n\|_\infty \|u_0\| + \frac{1}{2\mu}\left(\varepsilon \left\|D_+^3 D_-^2 u^n\right\| + C(\varepsilon)\|u_0\|\right) \\ &\qquad + \left(\varepsilon_2 \left\|D_+^3 D_-^2 u^n\right\| + C(\varepsilon_2)\|u_0\|\right) \\ &\le \left\|\alpha^{n+1}\right\| + \|u_0\|\left(\varepsilon_1 \left\|D_+^3 D_-^2 u^n\right\| + C(\varepsilon_1)\|u_0\|\right) \\ &\quad + \frac{1}{2\mu}\varepsilon \left\|D_+^3 D_-^2 u^n\right\| + \frac{1}{2\mu} C(\varepsilon)\|u_0\| + \varepsilon_2 \left\|D_+^3 D_-^2 u^n\right\| + C(\varepsilon_2)\|u_0\| \\ &\le \left\|\alpha^{n+1}\right\| + \left(\varepsilon_2 + \varepsilon_1\|u_0\| + \frac{\Delta x^{1/2}}{2\lambda}\varepsilon\right)\left\|D_+^3 D_-^2 u^n\right\| \\ &\qquad \underbrace{+ C(\varepsilon_1)\|u_0\|^2 + \left(\frac{\Delta x^{1/2}}{2\lambda} C(\varepsilon) + C(\varepsilon_2)\right)\|u_0\|}_{\mathcal{A}(\varepsilon_1,\varepsilon_2,\varepsilon)} \end{aligned}$$

$$
\begin{aligned}
&\le \left\|\alpha^{n+1}\right\| + \frac{1}{2}\left\|D_+^3 D_-^2 u^n\right\| + \mathcal{A} \ \ \text{(choosing } \varepsilon_1,\, \varepsilon_2 \text{ and } \varepsilon \text{ appropriately)}\\
&= \left\|\alpha^{n+1}\right\| + \frac{1}{2}\left\|D_+^3 D_-^2\left(u^{n+1} - \Delta t \alpha^{n+1}\right)\right\| + \mathcal{A}\\
&\le \left\|\alpha^{n+1}\right\| + \frac{1}{2}\left\|D_+^3 D_-^2 u^{n+1}\right\| + \frac{1}{2}\Delta t\left\|D_+^3 D_-^2 \alpha^{n+1}\right\| + \mathcal{A}\\
&\le \left\|\alpha^{n+1}\right\| + \frac{1}{2}\left\|D_+^3 D_-^2 u^{n+1}\right\| + \frac{1}{2}\left\|\alpha^n\right\|\left(1 + 3\Delta t\left\|Du^n\right\|_\infty\right)^{1/2} + \mathcal{A}\\
&\le \left\|\alpha^{n+1}\right\| + \frac{1}{2}\left\|D_+^3 D_-^2 u^{n+1}\right\| + \frac{1}{2}\left\|\alpha^n\right\|\left(1 + 3\lambda\left\|u_0\right\|\right)^{1/2} + \mathcal{A}.
\end{aligned}
$$

Hence, we conclude

$$\left\|D_+^3 D_-^2 u^{n+1}\right\| \le c_0 + c_1\left\|\alpha^{n+1}\right\| + c_2\left\|\alpha^n\right\|, \tag{18}$$

for some constants c_0, c_1 and c_2 that are independent of Δx. Exploiting this and the interpolation inequalties, we get

$$
\begin{aligned}
\left\|\alpha^{n+1}\right\|^2 &\le \left\|\alpha^n\right\|^2 + \Delta t\left(\varepsilon\left\|D_+^2 D_- u^n\right\| + C(\varepsilon)\left\|u^n\right\|\right)\left\|\alpha^n\right\|^2\\
&\le \left\|\alpha^n\right\|^2 + C\Delta t\left(\varepsilon\left(c_0 + c_1\left\|\alpha^n\right\| + c_2\left\|\alpha^{n-1}\right\|\right) + C(\varepsilon)\left\|u_0\right\|\right)\left\|\alpha^n\right\|^2.
\end{aligned}
$$

Since $\|u^n\|$ is bounded by $\|u_0\|$,

$$\left\|\alpha^{n+1}\right\|^2 \le \left\|\alpha^n\right\|^2 + \Delta t\left(d_1\left\|\alpha^n\right\|^2 + d_2\left(\left\|\alpha^n\right\|^3 + \left\|\alpha^n\right\|^2\left\|\alpha^{n-1}\right\|\right)\right), \tag{19}$$

for constants d_1 and d_2 which only depend on $\|u_0\|$ and λ. Set $a_n = \|\alpha^n\|^2$, so that

$$a_{n+1} \le a_n + \Delta t\left(d_1 a_n + d_2\left(a_n^{3/2} + a_n a_{n-1}^{1/2}\right)\right).$$

Now let $A = A(t)$ be the solution of the differential equation

$$\frac{dA}{dt} = d_1 A + 2d_2 A^{3/2}, \qquad A\left(t_1\right) = a_1 > 0.$$

This solution has a blow-up time

$$T^\infty = t_1 + \frac{2}{d_1}\ln\left(1 + \frac{d_1}{d_2\sqrt{a_1}}\right).$$

Furthermore, following [8], we conclude that for $t_n < T^\infty$, we have

$$a_n \le A(t_n), \quad n \in \mathbb{N}.$$

Therefore, we can follow Sjöberg [22] to prove convergence of the scheme for $t < \bar{T}$. We proceed as follows: We construct the piecewise quintic continuous interpolation $u_{\Delta x}(x,t)$ in two steps. First we make a spatial interpolation for each t_n:

$$
\begin{aligned}
u^n(x) &= \frac{1}{6}(u_{j-1}^n + 4u_j^n + u_{j+1}^n) + (x - x_j)Du_j^n + \frac{1}{2}(x - x_j)^2 D_+ D_- u_j^n\\
&\quad + \frac{1}{6}(x - x_j)^3 D_+^2 D_- u_j^n + \frac{1}{24}(x - x_j)^4 D_+^2 D_-^2 u_j^n\\
&\quad + \frac{1}{120}(x - x_j)^5 D_+^3 D_-^2 u_j^n, \ x \in [x_j, x_{j+1}).
\end{aligned} \tag{20}
$$

Next we interpolate in time:

$$u_{\Delta x}(x,t) = u^n(x) + (t - t^n)D_+^t u^n(x), \quad x \in \mathbb{R},\, t \in [t_n, t_{n+1}],\, (n+1)t_{n+1} \le \bar{T}. \tag{21}$$

Notice that

$$u_{\Delta x}(x_j, t_n) = \frac{1}{6}(u_{j-1}^n + 4u_j^n + u_{j+1}^n), \qquad j \in \mathbb{Z}, \quad n \in \mathbb{N}_0.$$

Moreover, observe that $u_{\Delta x}$ is continuous everywhere and four times continuously differentiable in space, and the function $u_{\Delta x}$ satisfies for $x \in [x_j, x_{j+1})$ and $t \in [t_n, t_{n+1}]$

$$\begin{aligned}\partial_x u_{\Delta x}(x,t) &= Du_j^n + (x-x_j)D_+D_-u_j^n + \frac{1}{2}(x-x_j)^2 D_+^2 D_- u_j^n \\ &\quad + \frac{1}{6}(x-x_j)^3 D_+^2 D_-^2 u_j^n + \frac{1}{24}(x-x_j)^4 D_+^3 D_-^2 u_j^n \\ &\quad + (t-t^n) D_+^t \Big(Du_j^n + (x-x_j)D_+D_-u_j^n \\ &\quad + \frac{1}{2}(x-x_j)^2 D_+^2 D_- u_j^n \Big) \\ &\quad + (t-t^n) D_+^t \Big(\frac{1}{6}(x-x_j)^3 D_+^2 D_-^2 u_j^n \\ &\quad + \frac{1}{24}(x-x_j)^4 D_+^3 D_-^2 u_j^n \Big),\end{aligned} \tag{22}$$

$$\begin{aligned}\partial_x^2 u_{\Delta x}(x,t) &= D_+D_-u_j^n + (x-x_j)D_+^2 D_- u_j^n \\ &\quad + \frac{1}{2}(x-x_j)^2 D_+^2 D_-^2 u_j^n + \frac{1}{6}(x-x_j)^3 D_+^3 D_-^2 u_j^n \\ &\quad + (t-t^n) D_+^t \Big(D_+D_-u_j^n + (x-x_j) D_+^2 D_- u_j^n \Big) \\ &\quad + (t-t^n) D_+^t \Big(\frac{1}{2}(x-x_j)^2 D_+^2 D_-^2 u_j^n + \frac{1}{6}(x-x_j)^3 D_+^3 D_-^2 u_j^n \Big),\end{aligned} \tag{23}$$

$$\begin{aligned}\partial_x^3 u_{\Delta x}(x,t) &= D_+^2 D_- u_j^n + (x-x_j) D_+^2 D_-^2 u_j^n + \frac{1}{2}(x-x_j)^2 D_+^3 D_-^2 u_j^n \\ &\quad + (t-t^n) D_+^t \Big(D_+^2 D_- u_j^n + (x-x_j) D_+^2 D_-^2 u_j^n \\ &\quad + \frac{1}{2}(x-x_j)^2 D_+^3 D_-^2 u_j^n \Big),\end{aligned} \tag{24}$$

$$\begin{aligned}\partial_x^4 u_{\Delta x}(x,t) &= D_+^2 D_-^2 u_j^n + (x-x_j) D_+^3 D_-^2 u_j^n \\ &\quad + (t-t^n) D_+^t \Big(D_+^2 D_-^2 u_j^n + (x-x_j) D_+^3 D_-^2 u_j^n \Big),\end{aligned} \tag{25}$$

$$\partial_x^5 u_{\Delta x}(x,t) = D_+^3 D_-^2 u_j^n + (t-t^n) D_+^t D_+^3 D_-^2 u_j^n, \tag{26}$$

$$\partial_t u_{\Delta x}(x,t) = D_+^t u^n(x), \tag{27}$$

which implies

$$\|u_{\Delta x}(\cdot,t)\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})}, \tag{28}$$

$$\|\partial_x u_{\Delta x}(\cdot,t)\|_{L^2(\mathbb{R})} \le C, \tag{29}$$

$$\|\partial_t u_{\Delta x}(\cdot,t)\|_{L^2(\mathbb{R})} \le C, \tag{30}$$

$$\left\|\partial_x^3 u_{\Delta x}(\cdot,t)\right\|_{L^2(\mathbb{R})} \le C, \tag{31}$$

$$\left\|\partial_x^5 u_{\Delta x}(\cdot,t)\right\|_{L^2(\mathbb{R})} \le C, \tag{32}$$

for $t \le \bar{T}$ and for a constant C which is independent of Δx. The first bound (28), follows by exact integration of the square of (21) over each interval $[x_j, x_{j+1})$ and summation over j. The last bound (32) follows from

$$\left\|D_+^3 D_-^2 u^n\right\| \le \left\|D_+^t u^n\right\| + \|\bar{u}^n\|_\infty \|Du^n\| + \left\|D_+^2 D_- u^n\right\| \le C. \tag{33}$$

The bound on $\partial_t u_{\Delta x}$ also implies that $u_{\Delta x} \in \mathrm{Lip}([0,\bar{T}]; L^2(\mathbb{R}))$. Then an application of the Arzelà–Ascoli theorem using (28) shows that the set $\{u_{\Delta x}\}_{\Delta x>0}$ is sequentially compact in $C([0,\bar{T}]; L^2(\mathbb{R}))$, such that there exists a sequence $\left\{u_{\Delta x_j}\right\}_{j\in\mathbb{N}}$ which converges uniformly in $C([0,\bar{T}]; L^2(\mathbb{R}))$ to some function u. Then we can apply a straightforward modification of the proof of the Lax–Wendroff like result in [10] to conclude that u is a weak solution. For the benefit of the reader we formulate the appropriate theorem here.

Theorem 3.1. *Suppose that $u_0 \in L^2(\mathbb{R})$. Consider the approximations $u_{\Delta x}$ given by (20) and (21). Suppose that $u_{\Delta x}$ converges strongly in $L^2_{\mathrm{loc}}(\mathbb{R}\times[0,T])$ to u as $\Delta x \to 0$. Then $u \in L^\infty([0,T]; L^2(\mathbb{R}))$ is a weak solution of the Cauchy problem (1), that is, it satisfies*

$$\int_0^T\int_{-\infty}^{\infty}\Big(\phi_t u + \phi_x u^2 + \phi_{xxx}u - \phi_{xxxxx}u\Big)\,dxdt + \int_{-\infty}^{\infty}\phi(x,0)u_0(x)\,dx = 0, \tag{34}$$

for all $\phi \in C_0^\infty(\mathbb{R}\times[0,T])$.

Proof. The proof follows the standard approach as in [10, Thm. 2.1]. □

The bounds (29), (30), (31) and (32) mean that u is actually a strong solution such that (1) holds as an L^2-identity. Thus the limit u is the unique solution to the Kawahara equation taking the initial data u_0.

Summing up, we have proved the following theorem:

Theorem 3.2. *Assume that $u_0 \in H^5(\mathbb{R})$. Then there exists a finite time $\bar{T}$, depending only on $\|u_0\|_{H^5(\mathbb{R})}$, such that for $t \le \bar{T}$, the difference approximations defined by (10) converge uniformly in $C(\mathbb{R}\times[0,\bar{T}])$ to the unique solution of the Kawahara equation (1) as $\Delta x \to 0$ with $\Delta t = \mathcal{O}\left(\Delta x^{\frac{3}{2}}\right)$.*

3.1. Global Existence. Inspired by [22], we now proceed to conclude the existence of a solution of the equation (1) for all time. We begin with following Lemma.

Lemma 3.3. *Let $u(x,t)$ be a solution of the problem (1). Then there exist a constants α_1, α_2 such that*

$$\int_{\mathbb{R}} u^2(x,t)\,dx = \int_{\mathbb{R}} u^2(x,0)\,dx = \int_{\mathbb{R}} u_0^2\,dx = \alpha_1 \tag{35}$$

$$\int_{\mathbb{R}}\left(\frac{1}{3}u^3 - u_x^2 - u_{xx}^2\right)dx = \int_{\mathbb{R}}\left(\frac{1}{3}u_0^3 - (u_0')^2 - (u_0'')^2\right)dx = \alpha_2 \tag{36}$$

Proof. To prove (35), we start by multyplying the equation (1) by u and integrate by parts in space

$$\begin{aligned}\int_{\mathbb{R}} uu_t\,dx &= \int_{\mathbb{R}} -u^2u_x - uu_{xxx} + uu_{xxxxx}\,dx\\ &= -\int_{\mathbb{R}}(\frac{1}{3}u^3)_x\,dx - \int_{\mathbb{R}}(uu_{xx} - \frac{1}{2}u_x^2)_x\,dx - \int_{\mathbb{R}} u_x u_{xxxx}\,dx\\ &= -\int_{\mathbb{R}}(\frac{1}{3}u^3)_x\,dx - \int_{\mathbb{R}}(uu_{xx} - \frac{1}{2}u_x^2)_x\,dx - \int_{\mathbb{R}}(u_xu_{xxx} - \frac{1}{2}u_{xx}^2)_x\,dx = 0.\end{aligned}$$

To prove (36), we start by multyplying (1) by u^2 and integrate by parts in space

$$\int_{\mathbb{R}} u^2u_t\,dx = \int_{\mathbb{R}} -u^3u_x - u^2u_{xxx} + u^2u_{xxxxx}\,dx$$

$$= -\int_{\mathbb{R}} (\frac{1}{4}u^4)_x\, dx + 2\int_{\mathbb{R}} (uu_x)u_{xx}\, dx - 2\int_{\mathbb{R}} (uu_x)u_{xxxx}\, dx$$
$$= 2\int_{\mathbb{R}} \left(-u_t - u_{xxx} + u_{xxxxx}\right) u_{xx} dx - 2\int_{\mathbb{R}} \left(-u_t - u_{xxx} + u_{xxxxx}\right) u_{xxxx} dx$$
$$= 2\int_{\mathbb{R}} u_{tx}u_x\, dx - 2\int_{\mathbb{R}} u_{xx}u_{xxx}\, dx + 2\int_{\mathbb{R}} u_{xx}u_{xxxxx}\, dx$$
$$- 2\int_{\mathbb{R}} u_{tx}u_{xxx}\, dx + 2\int_{\mathbb{R}} u_{xxx}u_{xxxx}\, dx - 2\int_{\mathbb{R}} u_{xxxx}u_{xxxxx}\, dx$$
$$= 2\int_{\mathbb{R}} u_{tx}u_x\, dx + 2\int_{\mathbb{R}} u_{txx}u_{xx}\, dx.$$

From this we can conclude that

$$\frac{d}{dt}\int_{\mathbb{R}} \left(\frac{1}{3}u^3 - u_x^2 - u_{xx}^2\right)\, dx = 0.$$

Hence (36) follows. □

Lemma 3.4. *Let $u(x,t)$ be a solution of the problem (1). Then there exists a constant α such that*

$$\max |u_x(x,t)| \le \alpha \tag{37}$$

$$\|v\|^2 \le e^{\gamma t} \left\| -u_0u_0^{'} - u_0^{'''} + u_0^{'''''} \right\|^2, \qquad v = \frac{\partial u}{\partial t}. \tag{38}$$

Proof. From (36), it follows that

$$\|u_{xx}\|^2 \le \frac{1}{3}\int_{\mathbb{R}} |u^3|\, dx + \|u_x\|^2 + |\alpha_2|$$
$$\le \frac{1}{3}\left(c(\epsilon)\|u\| + \epsilon\|u_{xx}\|\right)\|u\|^2 + \left(c(\epsilon)\|u\|^2 + \epsilon\|u_{xx}\|^2\right) + |\alpha_2|$$
$$= \frac{1}{3}\left(c(\epsilon)\sqrt{\alpha_1} + \epsilon\|u_{xx}\|\right)\alpha_1 + \left(c(\epsilon)\alpha_1 + \epsilon\|u_{xx}\|^2\right) + |\alpha_2|.$$

Now we can rewrite the above inequality in the following form

$$a\|u_{xx}\|^2 - b\|u_{xx}\| - c \le 0, \tag{39}$$

for some constants a, b, c, where $a = 1-\epsilon$, $b = \frac{1}{3}\alpha_1\epsilon$ and $c = \frac{1}{3}c(\epsilon)\sqrt{\alpha_1}+c(\epsilon)\alpha_1+|\alpha_2|$. Now it is easy to see that (39) gives,

$$\left(\sqrt{a}\|u_{xx}\| - \frac{b}{2\sqrt{a}}\right)^2 \le c + \frac{b^2}{4a}.$$

From the above relation, it is clear that $\|u_{xx}\| \le \alpha_3$, for some constant α_3. Again using the interpolation inequality, $\|u_x\| \le (c(\epsilon)\|u\| + \epsilon\|u_{xx}\|)$, we can conclude that $\|u_x\| \le \alpha_4$, for some constant α_4. Also we can use a similar type interpolation inequality to conclude that (37) holds.

Now the function $v(x,t) = \frac{\partial u}{\partial t}$ satisfies,

$$\frac{dv}{dt} = -vu_x - uv_x - v_{xxx} + v_{xxxxx}.$$

Multiplying the above equation by v and integrating in space yields,

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 = -(v, vu_x) - (v, uv_x) - (v, v_{xxx}) + (v, v_{xxxxx}) \le (v^2, u_x).$$

Now (37) gives,

$$\frac{d}{dt}\|v\|^2 \le \max|u_x|\,\|v\|^2 \le C\,\|v\|^2\,,$$

which implies

$$\|v\|^2 = \left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2 \le e^{\gamma t}\left\|\frac{\partial u(\cdot,0)}{\partial t}\right\|^2 = e^{\gamma t}\left\|-u_0u_0' - u_0^{'''} + u_0^{'''''}\right\|^2$$

Hence, we have established (38). □

Now to get a bound on $\|u_{xxx}\|$, we proceed as follows:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u_{xxx}^2\,dx &= \int_{\mathbb{R}} u_{xxx}u_{xxxt}\,dx = -\int_{\mathbb{R}} u_{xxxx}u_{xxt}\,dx = \int_{\mathbb{R}} u_{xxxxx}u_{xt}\,dx\\
&= \int_{\mathbb{R}} u_{xt}(u_t + uu_x + u_{xxx})\,dx\\
&= \int_{\mathbb{R}} vv_x\,dx + \int_{\mathbb{R}} uu_xu_{xt}\,dx - \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u_{xx}^2\,dx\\
&= \int_{\mathbb{R}} (\frac{1}{2}v^2)_x\,dx + \int_{\mathbb{R}} uu_xu_{xt}\,dx - \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u_{xx}^2\,dx\\
&= \int_{\mathbb{R}} vu_x^2\,dx + \int_{\mathbb{R}} vuu_{xx}\,dx - \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u_{xx}^2\,dx\\
&\le (\max|u_x|)(\|u_x\|\,\|v\|) + (\max|u|)(\|u_{xx}\|\,\|v\|) - \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u_{xx}^2\,dx
\end{aligned}$$

A simple use of interpolation inequality implies all the terms $\max|u|, \max|u_x|$, $\|u_x\|\,, \|u_{xx}\|$ are bounded by some constant. Hence, the above expression implies

$$\frac{d}{dt}\int_{\mathbb{R}}\left(\frac{1}{2}u_{xxx}^2 + \frac{1}{2}u_{xx}^2\right)\,dx \le \alpha\,\|v\|\,.$$

Therefore we conclude that,

$$\|u_{xxx}\|^2 \le C_1e^{\gamma t}\left(\left\|-u_0u_0' - u_0^{'''} + u_0^{'''''}\right\|\right). \tag{40}$$

Furthermore, we use the Kawahara equation (1) and triangular inequality to conclude that

$$\|u_{xxxxx}\| \le C_1e^{\gamma t}\left(\left\|-u_0u_0' - u_0^{'''} + u_0^{'''''}\right\|\right) + C_2, \tag{41}$$

where C_1 and C_2 are constants. One can see that the bound (38) guarantees that $\frac{\partial u}{\partial t}$ is square integrable for every t and (40), (41) that the problem (1) with the initial function $u(x,T_1)$ instead of $u_0(x)$ has a solution for $T_1 \le t \le T_2 = 2T_1$. Consequently, we get a solution of the Kawahara equation for $0 \le t \le T$. Now to obtain the existence of sloutions for all $t > 0$, we will repeat the extension procedure. For this purpose suppose that existence can be proven only in $0 < T < \infty$. Now if we look at the expression for T^{∞}, we find that only $A(t_1)$ depends on t. But $A(t_1)$ can, because of (38), be chosen to hold in the whole interval $0 \le t \le T$. Consequently, if we consider problem (1) with $u_0(x) = u(x,\tau)$ for some τ sufficiently close to T, we can, by using the local procedure, get existence for values of t lying outside $0 \le t \le T$.

Remark 3.1. *To keep the presentation fairly short we have only provided details in the full line case. However, we note that the same proofs apply, mutatis mutandis, also in the periodic case.*

4. Crank-Nicolson Finite Difference Scheme

We recall that, being dissipative, the fully discrete scheme (10) for the Kawahara equation (1) has several disadvantages. For example, as the Figure 1 indicates, semi implicit scheme given by (10) is clearly inferior to the other schemes, since both the phase error and substantial dissipation error is evident for the scheme given by (10). This observation, in fact, encourage us to quest for a conservative, provably convergent, scheme for the kawahara equation. To sum up, as we will see in the sequel, the scheme (42) ensures such properties.

To this end, we propose the following fully implicit Crank-Nicolson scheme (based upon a six-point stencil) to generate approximate solutions of the Kawahara equation (1)

(42)
$$u_j^{n+1} = u_j^n - \Delta t\, \mathbb{G}(u_j^{n+\frac{1}{2}}) - \Delta t\, D_+DD_- u_j^{n+\frac{1}{2}} + \Delta t\, D_+^2DD_-^2 u_j^{n+\frac{1}{2}}, \quad n \in \mathbb{N}_0,\ j \in \mathbb{Z},$$

where we have used the following notations:

$$u^{n+\frac{1}{2}} := \frac{1}{2}(u^n + u^{n+1}), \text{ and } \mathbb{G}(u) := \widetilde{u}\, Du.$$

For the initial data we have

$$u_j^0 = u_0(x_j), \quad j \in \mathbb{Z}.$$

Note that since the scheme (42) is implicit in nature, we must guarantee that the scheme is well-defined, i.e., it admits a solution. This has been addressed in next subsection (cf. Remark 4.1)

4.1. Convergence analysis. To begin with, we show that the Crank-Nicolson scheme is L^2-conservative. For that, we simply take inner product of the scheme (42) and $u_j^{n+\frac{1}{2}}$, to obtain

$$\begin{aligned}&\frac{1}{2\Delta t}(u^{n+1} - u^n, u^{n+1} + u^n)\\ &= -\,(u^{n+\frac{1}{2}}, \mathbb{G}u^{n+\frac{1}{2}}) - (u^{n+\frac{1}{2}}, D_+DD_-u^{n+\frac{1}{2}}) + (u^{n+\frac{1}{2}}, D_+^2DD_-^2u^{n+\frac{1}{2}})\end{aligned}$$

A simple straightforward calculation, making use of summation-by-parts formula, reveals that

$$(D_+DD_-u, u) = 0, \quad (D_+^2DD_-^2u, u) = 0, \text{ and } (\mathbb{G}(u), u) = 0.$$

Thus, we conclude that

$$\left\|u^{n+1}\right\| = \left\|u^n\right\|.$$

To solve (42), we use a simple fixpoint iteration, and define the sequence $\{w_\ell\}_{\ell\geq 0}$ by letting $w_{\ell+1}$ be the solution of the linear equation

(43)
$$\begin{cases} w_{l+1} = v - \Delta t\, \mathbb{G}\left(\frac{v+w_l}{2}\right) - \frac{1}{2}\Delta t\, D_+DD_-\left(v + w_{l+1}\right) + \frac{1}{2}\Delta t\, D_+^2DD_-^2\left(v + w_{l+1}\right), \\ w^0 = v := u^n. \end{cases}$$

To this end, we first prove the following stability lemma which essentially serves as a building block for the subsequent convergence analysis.

Lemma 4.1. *Choose a constant L such that $0 < L < 1$ and set*

$$K = \frac{6-L}{1-L} > 6.$$

We consider the iteration (43) *with* $w^0 = u^n$, *and assume that the following CFL condition holds*

$$\lambda \le L/\left(K\left\|u^n\right\|_{h^5}\right), \text{ with } \lambda = \Delta t/\Delta x. \tag{44}$$

Then there exists a function u^{n+1} *which solves* (42), *and* $\lim_{\ell\to\infty} w_\ell = u^{n+1}$. *Furthermore the following estimate holds:*

$$\left\|u^{n+1}\right\|_{h^5} \le K\left\|u^n\right\|_{h^5}. \tag{45}$$

Proof. To begin with, note that by setting $\Delta w_l := w_{l+1} - w_l$, a straightforward calculation using the iterative scheme (43) returns

$$\begin{aligned}&\left(1 + \frac{1}{2}\Delta t\, D_+DD_- - \frac{1}{2}\Delta t\, D_+^2DD_-^2\right)\Delta w_l\\ =&\Delta t\left[\mathbb{G}\left(\frac{v+w_l}{2}\right) - \mathbb{G}\left(\frac{v+w_{l-1}}{2}\right)\right] := \Delta t\Delta\mathbb{G}.\end{aligned} \tag{46}$$

Next, applying the discrete operator $D_+^2DD_-^2$ to (46), then multiplying the resulting equation by $\Delta x D_+^2DD_-^2\Delta w_l$, and subsequently summing over $j \in \mathbb{Z}$, we conclude

$$\begin{aligned}\left\|D_+^2DD_-^2\Delta w_l\right\|^2 &= \Delta t\left(D_+^2DD_-^2\Delta\mathbb{G}, D_+^2DD_-^2\Delta w_l\right)\\ &\le \Delta t\left\|D_+^2DD_-^2\Delta\mathbb{G}\right\|\left\|D_+^2DD_-^2\Delta w_l\right\|.\end{aligned}$$

After some manipulations, we find that

$$\Delta\mathbb{G} = \frac{1}{4}\left[\widetilde{\Delta w_{l-1}}D\left(v + w_{l-1}\right) + \widetilde{(v+w_l)}D\left(\Delta w_{l-1}\right)\right].$$

Next, in order to calculate $D_+^2DD_-^2\Delta\mathbb{G}$, we use the identities (5a), (5b), and (5c) repeatedly, and the discrete Sobolev inequalities (cf. [8, Lemma A.1]). This results

$$\left\|D_+^2DD_-^2\left(\widetilde{\Delta w_{l-1}}D\left(v+w_{l-1}\right)\right)\right\| \le \frac{C}{\Delta x}\max\left\{\left\|v\right\|_{h^5}, \left\|w_{l-1}\right\|_{h^5}\right\}\left\|\Delta w_{l-1}\right\|_{h^5}.$$

A similar argument shows that

$$\left\|D_+^2DD_-^2\left(\widetilde{(v+w_l)}D\left(\Delta w_{l-1}\right)\right)\right\| \le \frac{C}{\Delta x}\max\left\{\left\|v\right\|_{h^5}, \left\|w_l\right\|_{h^5}\right\}\left\|\Delta w_{l-1}\right\|_{h^5}.$$

Combining the above two results, we obtain

$$\left\|D_+^2DD_-^2\Delta w_l\right\| \le C\,\lambda\max\left\{\left\|v\right\|_{h^5}, \left\|w_l\right\|_{h^5}, \left\|w_{l-1}\right\|_{h^5}\right\}\left\|\Delta w_{l-1}\right\|_{h^5}. \tag{47}$$

Observe that, appropriate inequality like (47) can be obtained for $\left\|\Delta w_l\right\|$, $\left\|D_+\Delta w_l\right\|$, $\left\|D_+D_-\Delta w_l\right\|$, $\left\|D_+DD_-\Delta w_l\right\|$, and $\left\|D_+^2D_-^2\Delta w_l\right\|$, which in turn can be used, along with (47), to conclude

$$\left\|\Delta w_l\right\|_{h^2} \le \lambda\max\left\{\left\|v\right\|_{h^2}, \left\|w_l\right\|_{h^2}, \left\|w_{l-1}\right\|_{h^2}\right\}\left\|\Delta w_{l-1}\right\|_{h^2}.$$

To proceed further, we need to estimate $\left\|D_+^2DD_-^2w_l\right\|$. To that context, we first observe that w_1 satisfies the following equation

$$w_1 = v + \Delta t\,\mathbb{G}(v) - \frac{1}{2}\Delta t\, D_+DD_-\left(v+w_1\right) + \frac{1}{2}\Delta t\, D_+^2DD_-^2\left(v+w_1\right). \tag{48}$$

Again, we apply the discrete operator $D_+^2DD_-^2$ to the above equation (48) satisfied by w_1, and subsequently taking the inner product with $D_+^2DD_-^2(v+w_1)$, we get

$$\begin{aligned}\left\|D_+^2DD_-^2w_1\right\|^2 &= \left\|D_+^2DD_-^2v\right\|^2 + \Delta t\left(D_+^2DD_-^2\mathbb{G}(v), D_+^2DD_-^2\left(v+w_1\right)\right)\\ &= \left\|D_+^2DD_-^2v\right\|^2 + \Delta t\left(D_+^2DD_-^2\mathbb{G}(v), D_+^2DD_-^2w_1\right)\\ &\le \left\|D_+^2DD_-^2v\right\|^2 + \Delta t^2\left\|D_+^2DD_-^2\mathbb{G}(v)\right\|^2 + \frac{1}{4}\left\|D_+^2DD_-^2w_1\right\|^2.\end{aligned}$$

Next, a simple calculation using identities (5a)–(5c) along with the discrete Sobolev inequalities (cf. [8, Lemma A.1]) confirm that

$$\left\|D_+^2 D D_-^2\, \mathbb{G}(v)\right\| = \left\|D_+^2 D D_-^2\,(\tilde{v}\, Dv)\right\| \le \frac{2}{\Delta x}\, \|v\|_{h^5}^2 .$$

Hence

$$\left\|D_+^2 D D_-^2\, w_1\right\| \le \frac{4}{3}\left(1 + 4\lambda^2 \|v\|_{h^5}^2\right)^{1/2} \|v\|_{h^5} . \tag{49}$$

Now choose a constant $L \in (0,1)$, and define K by

$$K = \frac{6-L}{1-L} > 6.$$

Moreover, the CFL condition (44) confirms that

$$\frac{4}{3}\sqrt{1 + 4\lambda^2 \|v\|_{h^5}^2} \le 4.$$

In view of the inequality (49), the above estimate implies that

$$\left\|D_+^2 D D_-^2\, w_1\right\| \le K\, \|v\|_{h^5} .$$

Observe that similar estimates can be obtained for the lower order discrete derivative of w_l. Hence, we conclude

$$\|w_1\|_{h^5} \le K\, \|v\|_{h^5} .$$

Furthermore, assume inductively that

$$\|w_l\|_{h^5} \le K\, \|v\|_{h^5} , \quad \text{for } l = 1, \dots, m, \tag{50a}$$

$$\|\Delta w_l\|_{h^5} \le L\, \|\Delta w_{l-1}\|_{h^5} , \quad \text{for } l = 2, \dots, m. \tag{50b}$$

We have already shown (50a) for $m = 1$. To show (50b) for $m = 2$, note that

$$\|\Delta w_2\|_{h^5} \le \lambda \max\left\{ \|v\|_{h^5} , \|w_1\|_{h^5} \right\} \|\Delta w_1\|_{h^5} \le 4\lambda\, \|v\|_{h^5}\, \|\Delta w_1\|_{h^5} \le L\, \|\Delta w_1\|_{h^5} ,$$

by CFL condition (44). To show (50a) for $m > 1$,

$$\begin{aligned}\|w_{m+1}\|_{h^5} &\le \sum_{l=0}^{m} \|\Delta w_l\|_{h^5} + \|v\|_{h^5} \le \|(w_1 - v)\|_{h^5} \sum_{l=0}^{m} L^l + \|v\|_{h^5} \\ &\le \left(\|w_1\|_{h^5} + \|v\|_{h^5}\right) \frac{1}{1-L} + \|v\|_{h^5} \le \frac{4+2-L}{1-L}\, \|v\|_{h^5} = K\, \|v\|_{h^5} .\end{aligned}$$

Then

$$\|\Delta w_{m+1}\|_{h^5} \le \lambda K\, \|v\|_{h^5}\, \|\Delta w_m\|_{h^5} \le L\, \|\Delta w_m\|_{h^5} ,$$

if the CFL condition (44) holds.

To sum up, if $L \in (0,1)$, and K is defined by $K = (6-L)/(1-L)$, and λ satisfies the CFL-condition

$$\lambda \le \frac{L}{K\, \|v\|_{h^5}},$$

then we have the desired estimate (45). Finally, using (50b), one can show that $\{w_\ell\}$ is Cauchy, hence $\{w_\ell\}$ converges. This completes the proof. □

Remark 4.1. *We remark that, the existence of solution for the implicit scheme* (42) *is guaranteed, by virtue of the above Lemma 4.1. Having said this, however, the above result only guarantees that the iteration scheme converges for one time step under CFL condition* (44)*, where the ratio between temporal and spatial mesh sizes must be smaller than an upper bound that depends on the computed solution at that time, i.e.,* u^n*. Since we want the CFL-condition to only depend on the initial data*

u_0, we have to derive local a priori bounds for the computed solution u^n. This has been achieved in the sequel (cf. Theorem 4.1) to conclude that the iteration scheme (43) converges for sufficiently small Δt.

The following lemma plays an important role in the convergence analysis:

Lemma 4.2. *Let the approximate solution u^n be generated by the Crank-Nicholson scheme (42). Then, there exists a polynomial $\mathcal{Q}(X)$, with positive coefficients, such that*

$$D_+^t\left(\|u^n\|_{h^5}\right) \le \mathcal{Q}\left(\left\|u^{n+\frac{1}{2}}\right\|_{h^5}\right) \le \mathcal{Q}\left(K\|u^n\|_{h^5}\right).$$

Proof. Applying the discrete operator $D_+^2 D D_-^2$ to (42), and subsequently taking inner product with $D_+^2 D D_-^2 u^{n+\frac{1}{2}}$ yields

$$\frac{1}{2}\left\|D_+^2 D D_-^2 u^{n+1}\right\|^2 = \frac{1}{2}\left\|D_+^2 D D_-^2 u^n\right\|^2 + \Delta t(D_+^2 D D_-^2 \mathbb{G}(u^{n+\frac{1}{2}}), D_+^2 D D_-^2 u^{n+\frac{1}{2}}),$$

which implies

$$D_+^t\left(\left\|D_+^2 D D_-^2 u^n\right\|\right) \le 2\frac{(D_+^2 D D_-^2 \mathbb{G}(u^{n+\frac{1}{2}}), D_+^2 D D_-^2 u^{n+\frac{1}{2}})}{\left\|D_+^2 D D_-^2 u^{n+1}\right\| + \left\|D_+^2 D D_-^2 u^n\right\|}.$$

For the moment we drop the index $n+\frac{1}{2}$ from our notation, and use the notation u for $u^{n+\frac{1}{2}}$, where n is fixed. We proceed to calculate

$$\begin{aligned}
(D_+^2 D D_-^2 \mathbb{G}(u), D_+^2 D D_-^2 u) &:= (D_+^2 D D_-^2\left(\overline{u}\, Du\right), D_+^2 D D_-^2 u) \\
&= (D_-\overline{u}\, D_+(Du), D_+^2 D D_-^2 u) + (S^-\overline{u}\, D_+^2 D D_-^2(Du), D_+^2 D D_-^2 u) \\
&\quad + (D_+\overline{u}\, D_+(Du), D_+^2 D D_-^2 u) + (D_+^2 D D_-^2 \overline{u}\, Du, D_+^2 D D_-^2 u) \\
&:= \mathcal{E}^1(u) + \mathcal{E}^2(u) + \mathcal{E}^3(u) + \mathcal{E}^4(u).
\end{aligned}$$

To begin with, we see that

$$\begin{aligned}
\left|\mathcal{E}^1(u)\right| \le C\left\|D_-\overline{u}\right\|_\infty \left\|D_+^2 D D_-^2 u\right\|^2 &\le C(\left\|D_+^2 D D_-^2 u\right\|^2 + \|u\|^2)\left\|D_+^2 D D_-^2 u\right\|^2 \\
&\le C\left\|D_+^2 D D_-^2 u\right\| \mathcal{Q}\left(\|u\|_{h^5}\right).
\end{aligned}$$

Similar arguments shows that

$$\left|\mathcal{E}^3(u)\right| \le C\left\|D_+^2 D D_-^2 u\right\| \mathcal{Q}\left(\|u\|_{h^5}\right), \text{ and } \left|\mathcal{E}^4(u)\right| \le C\left\|D_+^2 D D_-^2 u\right\| \mathcal{Q}\left(\|u\|_{h^5}\right).$$

To estimate the last term, we proceed as follows:

$$\begin{aligned}
\mathcal{E}^2(u) &:= (S^-\overline{u}\, D_+^2 D D_-^2(Du), D_+^2 D D_-^2 u) \\
&\overset{v:=D_+^2 D D_-^2 u}{=\!=\!=} (S^-\overline{u}\, D\, v, v) = -(D\left(v\, S^-\overline{u}\right), v) \\
&= \frac{\Delta x}{2}(D_+(S^-\overline{u})\, D\, v, v) + \frac{1}{2}(S^- v\, D(S^-\overline{u}), v).
\end{aligned}$$

Therefore, we conclude

$$\left|\mathcal{E}^2(u)\right| \le C\left\|D_+^2 D D_-^2 u\right\| \mathcal{Q}\left(\|u\|_{h^5}\right).$$

Using all the above estimates, we have

$$\begin{aligned}
2\frac{(D_+^2 D D_-^2 \mathbb{G}(u^{n+\frac{1}{2}}), D_+^2 D D_-^2 u^{n+\frac{1}{2}})}{\left\|D_+^2 D D_-^2 u^{n+1}\right\| + \left\|D_+^2 D D_-^2 u^n\right\|} &\le 2\frac{C\left\|D_+^2 D D_-^2 u^{n+\frac{1}{2}}\right\| \mathcal{Q}\left(\left\|u^{n+\frac{1}{2}}\right\|_{h^5}\right)}{\left\|D_+^2 D D_-^2 u^{n+1}\right\| + \left\|D_+^2 D D_-^2 u^n\right\|} \\
&\le \mathcal{Q}\left(\left\|u^{n+\frac{1}{2}}\right\|_{h^5}\right) \le \mathcal{Q}\left(K\|u^n\|_{h^5}\right).
\end{aligned}$$

Hence, we conclude

$$D_+^t\left(\left\|D_+^2 D D_-^2 u^n\right\|\right) \leq \mathcal{Q}\left(K\left\|u^n\right\|_{h^5}\right).$$

Moreover, even easier calculations help us to conclude

$$D_+^t\left(\left\|u^n\right\|\right)=0, D_+^t\left(\left\|D_+ u^n\right\|\right), D_+^t\left(\left\|D_+ D_- u^n\right\|\right) \leq \mathcal{Q}\left(K\left\|u^n\right\|_{h^5}\right)$$
$$D_+^t\left(\left\|D_+^2 D_- u^n\right\|\right), D_+^t\left(\left\|D_+^2 D_-^2 u^n\right\|\right) \leq \mathcal{Q}\left(K\left\|u^n\right\|_{h^5}\right).$$

This essentially finishes the proof. □

We can now state the following stability-result:

Theorem 4.1. *If the initial function u_0 is H^5-regular, then there exist constants $C, T>0$ only depending on $\|u_0\|$ such that for small enough λ:*

$$\left\|u^n\right\|_{h^5} \leq C, \quad \forall t_n \leq T.$$

Proof. Set $y_n=\left\|u^n\right\|_{h^5}$. By Lemma 4.2, there is a polynomial $\mathcal{Q}(X)$ with positive coefficients such that $D_+^t y_n \leq \mathcal{Q}(K\, y_n)$.

Next, we consider the following ordinary differential equation

$$\begin{cases} y'(t)=\mathcal{Q}(K\, y(t)), \\ y(0)=C\left\|u_0\right\|_{H^5}. \end{cases}$$

Since the function $\mathcal{Q}$ is locally Lipschitz continuous for positive arguments, this ODE has a unique solution which blows up at some finite time, say at $t=T^{\infty}$. We choose $T=T^{\infty}/2$. Also, note that the solution $y(t)$ of the above differential equation is *strictly-increasing* and *convex*.

We now prove by induction that if

(51) $$\lambda \leq \frac{L}{K\, y(T)},$$

then

$$\forall t_n \leq T: \quad y_n \leq y(t_n)$$

Since $y(0)=C\left\|u_0\right\|_{H^5}$, in view of the Lemma 4.3, the claim follows for $n=0$. We assume that the claim holds for $n=0,1,2,\ldots,m$. As $0<y_m \leq y(T)$, (51) implies that λ satisfies the CFL condition (44). So, from Lemma 4.1, we have $y_{m+1} \leq K y_m$.

Therefore, using the induction hypothesis, we conclude

$$\begin{aligned} y_n & \leq y_{n-1}+\Delta t \mathcal{Q}(K y_{n-1}) \leq y(t_{n-1})+\Delta t\, \underbrace{\mathcal{Q}(K y(t_{n-1}))}_{=y'(t_{n-1})} \\ & \leq y(t_{n-1})+\int_{t_{n-1}}^{t_n} y'(t) t=y(t_n). \end{aligned}$$

This proves the claim. □

Finally, a simple use of discrete Sobolev inequalities (cf. [8, Lemma A.1]) along with Theorem 4.1, and the scheme (42) implies that $\left\|D_+^t u^n\right\| \leq C$.

Therefore, we can follow Sjöberg [22] to prove convergence of the scheme (42) for $t<T$.

Summing up, we have proved the following theorem:

Theorem 4.2. *Assume that $u_0 \in H^5(\mathbb{R})$. Then there exists a finite time T, depending only on $\|u_0\|_{H^5(\mathbb{R})}$, such that for $t \le T$, the difference approximations defined by (42) converge uniformly in $C(\mathbb{R}\times[0,T])$ to the unique solution of the Kawahara equation (1) as $\Delta x \to 0$ with $\Delta t = \mathcal{O}(\Delta x)$.*

4.2. Error Estimate. To this end, we derive an error estimate in both space and time for the smooth solutions of the Kawahara equation (1).

First, recall that for our approximation of the nonlinear term uu_x, we choose

$$\mathbb{G}(u) = \widetilde{u}\, Du = \frac{1}{3}D(u^2) + \frac{1}{3}u\, Du,$$

so that, for u smooth, we have

$$\mathbb{G}(u) - uu_x = \mathcal{O}(\Delta x^2), \text{ as } \Delta x \mapsto 0.$$

Moreover, a straightforward truncation error analysis shows that both our approximations for $\partial_x^3 u$, and $\partial_x^5 u$ are second order accurate.

Now we are ready to present the error estimate analysis for smooth solutions of (1). In what follows, we begin with the following useful lemma.

Lemma 4.3. *If u_x is uniformly bounded, then we have*

$$|(\mathbb{G}(U) - \mathbb{G}(u), U-u)| \le C\left\|U-u\right\|^2, \text{ where } C := \frac{1}{2}\left\|u_x\right\|_\infty. \tag{52}$$

Proof. Let us set $e := U - u$. Then, observe that $U^2 - u^2 = (2u+e)e = 2ue + e^2$, and therefore

$$(D(U^2-u^2), e) = 2(D(ue), e) + (D(e^2), e).$$

A similar argument shows that

$$(UDU - uDu, e) = (eDu, e) + (uDe, e) + (eDe, e).$$

Hence

$$(\mathbb{G}(U) - \mathbb{G}(u), e) = -\frac{1}{3}(uDe, e) + \frac{1}{3}(eDu, e) + (\mathbb{G}(e), e).$$

Furthermore, since $eDe = \frac{1}{2}D_-(eS^+e)$, we conclude

$$-(uDe, e) = -\frac{1}{2}(u, D_-(eS^+e)) = \frac{1}{2}(D_-u, eS^+e) \le C\left\|e\right\|^2.$$

Finally, since $(eDu, e) \le 2C\left\|e\right\|^2$, and $(\mathbb{G}(e), e) = 0$, we conclude the lemma. □

We now state the following error estimate result.

Theorem 4.3. *Let u be a smooth solution of the Kawahara equation (1), and U^n be the approximate solution generated by the difference scheme (10). Then*

$$\|U^n - u(t_n)\| \le C_T(u)(\Delta x^2 + \Delta t^2), \text{ for } t_n \le T. \tag{53}$$

Proof. To begin with, we first introduce $e^n := U^n - u^n$, with $u^n := u(t_n)$. Then, since u satisfies the Kawahara equation (1), we have

(54)

$$\frac{e^{n+1}-e^n}{\Delta t} + D_+DD_-e^{n+\frac{1}{2}} - D_+^2DD_-^2e^{n+\frac{1}{2}} := -\mathbb{G}(U^{n+\frac{1}{2}}) + \mathbb{G}(u^{n+\frac{1}{2}}) - K^n,$$

where

$$\begin{aligned} K^n :=& \frac{u^{n+1}-u^n}{\Delta t} + \mathbb{G}(u^{n+\frac{1}{2}}) + D_+DD_-u^{n+\frac{1}{2}} - D_+^2DD_-^2u^{n+\frac{1}{2}} \\ &- \left(u_t + uu_x + \partial_x^3 u - \partial_x^5 u\right)^{n+\frac{1}{2}}, \end{aligned}$$

where we have used the notation $u^{n+\frac{1}{2}} := u(t_n + \frac{1}{2}\Delta t)$. Next, a simple truncation error analysis shows that

$$\left\| \frac{u^{n+1} - u^n}{\Delta t} - u_t^{n+\frac{1}{2}} \right\| + \left\| \mathbb{G}(u^{n+\frac{1}{2}}) - (uu_x)^{n+\frac{1}{2}} \right\| \leq C(u)(\Delta x^2 + \Delta t^2),$$
$$\left\| D_+DD_-u^{n+\frac{1}{2}} - (\partial_x^3 u)^{n+\frac{1}{2}} \right\| + \left\| D_+^2DD_-^2u^{n+\frac{1}{2}} - (\partial_x^5 u)^{n+\frac{1}{2}} \right\| \leq C(u)\Delta x^2.$$

Therefore, multiplying the equation (54) by $e^{n+\frac{1}{2}}$ returns

$$\begin{aligned}\Big(\frac{e^{n+1} - e^n}{\Delta t}, e^{n+\frac{1}{2}}\Big) + \big(D_+DD_-e^{n+\frac{1}{2}}, e^{n+\frac{1}{2}}\big) - \big(D_+^2DD_-^2e^{n+\frac{1}{2}}, e^{n+\frac{1}{2}}\big)\\ = \big(\mathbb{G}(u^{n+\frac{1}{2}}) + \mathbb{G}(U^{n+\frac{1}{2}}), e^{n+\frac{1}{2}}\big) - \big(K^n, e^{n+\frac{1}{2}}\big)\end{aligned}$$

Hence, by virtue of Lemma 4.3, we conclude

$$\begin{aligned}\left\|e^{n+1}\right\|^2 - \left\|e^n\right\|^2 &\leq C\Delta t \left\|e^{n+\frac{1}{2}}\right\|^2 + C\Delta t \left\|K^n\right\| \left\|e^{n+\frac{1}{2}}\right\| \\ &\leq C\Delta t \left(\left\|e^n\right\|^2 + \left\|e^{n+1}\right\|^2\right) + C\Delta t \left\|K^n\right\|^2.\end{aligned}$$

This gives us, for sufficiently small Δt

$$\left\|e^{n+1}\right\|^2 \leq (1 + C\Delta t) \left\|e^n\right\|^2 + C\Delta t(\Delta x^2 + \Delta t^2)^2.$$

Finally, a repeated use of the above inequality along with the observation $e^0 = 0$ helps us to conclude

$$\left\|e^{n+1}\right\|^2 \leq e^{CT} \left\|e^0\right\|^2 + Cn\Delta t(\Delta x^2 + \Delta t^2)^2 \leq CT(\Delta x^2 + \Delta t^2)^2, \text{ for } t_n \leq T.$$

This finishes the proof of the theorem. □

5. Uniqueness

To prove the uniqueness, let us assume that there exists two solutions $u(x,t)$ and $v(x,t)$ of the problem (1). Then the function $w = u - v$ satisfies the following equation

$$\begin{aligned} w_t &= -(uu_x - vv_x) - w_{xxx} + w_{xxxxx} \\ &= -wu_x - vw_x - w_{xxx} + w_{xxxxx} \\ w(x,0) &= 0. \end{aligned} \tag{55}$$

Hence, by taking inner product of the above equation with w, we have

$$\begin{aligned}(w, w_t) = \frac{1}{2}\frac{\partial \left\|w\right\|^2}{\partial t} &= -(w, wu_x) - (w, vw_x) - (w, w_{xxx}) + (w, w_{xxxxx}) \\ &= -(w^2, u_x) + (w^2, v_x)/2.\end{aligned}$$

Now use of estimate (37) implies,

$$\frac{\partial \left\|w\right\|^2}{\partial t} \leq C \left\|w\right\|^2,$$

for some constant C. Now as $\left\|w(\cdot,0)\right\|^2 = 0$, it is clear that $\left\|w(\cdot,t)\right\|^2 = 0$ for all t which consequently implies uniqueness.

6. Numerical Experiment

Fully discrete schemes given by (10) and (42) have been tested on suitable test case, one-soliton interaction, in order to demonstrate its effectiveness. It is well known that a soliton is a self-reinforcing solitary wave that maintains its shape while it travels at a constant speed. Solitons are caused by a cancellation of nonlinear and dispersive effects in the medium. In recent years the soliton solutions of the fifth order KdV-type highly nonlinear equations

$$u_t + u^p u_x + u_{xxx} + \gamma u_{xxxxx} = 0, \tag{56}$$

(with $p > 0$) have received considerable attention in the literature [12, 13]. In particular, attention has been focused on the role of the last term in this equation, which describes higher order dispersive effects and may have an important influence on the properties of the solitons. It has been shown (cf. [13]) that the stationary solitary waves can exist only at $\gamma < 0$. The solitons of equation (56) with $\gamma > 0$ are radiating and they are unstable. It has also been numerically demonstrated that the solitary waves at $\gamma < 0$ exist and the fifth order derivative term with $\gamma < 0$ plays a stabilizing role. Several authors [2, 21] have studied the soliton experiments in the context of both KdV and Kawahara equation. We shall compare our schemes with the schemes given in [2] and [18].

6.1. The Method of Ceballos, Sepulveda and Villagran. We mention that all the numerical experiments performed in [2] are based on the equation given by

$$u_t + u_x + uu_x + u_{xxx} = u_{xxxxx}. \tag{57}$$

However, it is very easy to see that (1) and (57) are completely equivalent by way of simple change of variables. In [2], authors have considered the following scheme for the Kawahara equation (1)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{1}{2} D_-[u_i^n]^2 + A u_i^{n+1} = 0,$$

where $A = D_+D_+D_- - D_+D_+D_+D_-D_-$. We shall refer to this scheme as JMO scheme.

6.2. Spectral Method. We also compare our result with the highly accurate spectral scheme given in [18]. This is a Fourier–Galerkin scheme, integrated in time using an explicit leap-frog scheme. For a positive integer N, consider the space S_N defined by

$$S_N = \operatorname{span}\Big\{ \exp(ikx) : k \in \mathbb{Z} \cap [-N, N] \Big\}. \tag{58}$$

The fully discrete Fourier–Galerkin (spectral) approximation to (1) is a map $\mathbf{U}$ from $[0, \infty)$ to the real-valued elements of S_N such that, for all $\varphi \in S_N$:

$$\Big(\mathbf{U}^{n+1} - \mathbf{U}^{n-1}, \varphi\Big) + 2\Delta t \left(\mathbf{U}^n \mathbf{U}^n_x + \mathbf{U}^n_{xxx} - \mathbf{U}^n_{xxxxx}, \varphi\right) = 0. \tag{59}$$

We shall refer to this scheme as the spectral scheme.

6.3. One Soliton solution. In the case of Kawahara equation given by (1), if we consider the initial profile of the form

$$u(x,0) = \frac{105}{169} \operatorname{sech}^4 \left(\frac{1}{2\sqrt{13}}(x - c) \right), \tag{60}$$

then it is known (cf. [4]) that the explicit solution is given by the following travelling wave

$$u(x,t) = \frac{105}{169}\text{sech}^4\left(\frac{1}{2\sqrt{13}}(x - \frac{36t}{169} - c)\right).$$

This result can be verified through substitution. We have applied all numerical methods to simulate the periodic single soliton solution with initial data $u_0(x) = u(x,0)$. In Figure 2, we show the exact solutions at $T = 100$ and $T = 250$ as well as the numerical solutions computed using 4096 grid points in the interval $[-80, 80]$. We have also computed numerically the error for a range of Δx, where the L^2-error

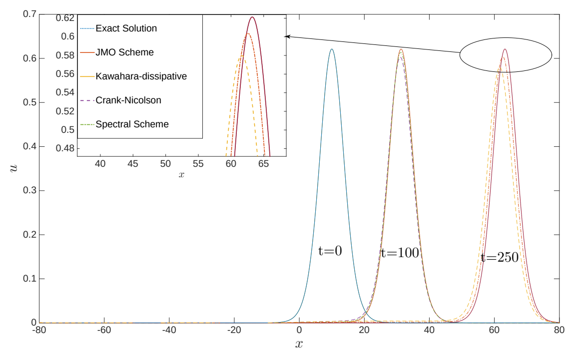

FIGURE 2. Comparison of exact and numerical solutions with initial data (60) and $c = 10$.

at time T is defined by

$$\mathcal{E}^1_{\Delta x}(T) = \sqrt{\frac{1}{N}\sum_{j=1}^{N}|u(x_j,T) - u_{\Delta x}(x_j,T)|^2},$$

and the relative error is defined by

$$\mathcal{E}^2_{\Delta x}(T) = 100 \times \frac{\sum_{j=1}^{N}|u(x_j,T) - u_{\Delta x}(x_j,T)|}{\sum_{j=1}^{N}u(x_j,T)}.$$

In Table 6.3, and Table 6.3, we show L^2-errors and relative errors respectively for this example at time $T = 10$. An expected first and second order convergence has been observed for the schemes (10) and (42) respectively.

TABLE 1. L^2 errors for the one-soliton solution at time $T = 10$.

N	$\mathcal{E}^1_{\Delta x}(10)/JMO$	$\mathcal{E}^1_{\Delta x}(10)/(10)$	Rate	$\mathcal{E}^1_{\Delta x}(10)/(42)$	Rate	$\mathcal{E}^1_{\Delta x}(10)/Spectral$
512	1.50e-3	3.40e-3		2.63e-4		3.47e-08
1024	7.56e-4	1.70e-3	1.00	2.40e-4	0.13	3.51e-10
2048	3.81e-4	8.32e-4	1.03	6.11e-5	1.97	1.66e-11
4096	1.19e-4	4.17e-4	0.99	1.97e-5	1.63	8.47e-13
8192	9.62e-5	2.09e-4	0.97	5.49e-6	1.84	5.89e-14

Figure 2 confirms that all the difference methods have resolved the main features of the solution. In particular, we see that both spectral scheme and Crank-Nicolson scheme have maintained the shape (e.g., speed, height) of the soliton. However, the phase error as well as significant amplitude error are evident for both Kawahara-dissipative (10) and JMO schemes after long time. Nevertheless, we observe that the qualitative features of both schemes are "right", but neither their heights nor their positions are correct.

TABLE 2. Relative errors for the one-soliton solution at time $T = 10$.

N	$\mathcal{E}^2_{\Delta x}(10)/JMO$	$\mathcal{E}^2_{\Delta x}(10)/(10)$	$\mathcal{E}^2_{\Delta x}(10)/(42)$	$\mathcal{E}^2_{\Delta x}(10)/Spectral$
512	1.42e0	3.33e0	2.99e-1	8.32e-05
1024	7.25e-1	1.65e0	2.52e-1	6.64e-07
2048	3.70e-1	8.37e-1	6.46e-2	3.24e-08
4096	1.88e-1	4.22e-1	6.14e-2	1.59e-09
8192	9.54e-2	2.12e-1	1.54e-2	1.14e-10

7. Conclusion

We have considered the Kawahara equation that has applications in fluid mechanics. This is a generalized nonlinear dispersive equation which has a form of the KdV equation with an additional fifth order derivative term.

A popular numerical approach has been the application of various spectral methods. There has been a great deal of work on the Fourier-Galerkin spectral method for the nonlinear dispersive equations. In particular, convergence of spectral method for the Kawahara equation has been shown.

We proved here convergence of two fully discrete implicit finite difference schemes for the Kawahara equation. We have observed that the fully discrete conservative scheme (42) works far better than the dissipative scheme (10) in practice. In fact as Figure 2 depicts, the phase error and the dissipation error is evident for the dissipative scheme given by (10). On the other hand, the conservative scheme (42) resolves the solution very well, even after long time.

However, one of the crucial assumption in our convergence analysis was the regularity of the initial profile. In our case, we have assumed that the initial datum is in $H^5(\mathbb{R})$. In future, we plan to prove the convergence of a finite difference scheme for the Kawahara equation with reduced regularity assumptions on initial data.

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