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HIGH ORDER GALERKIN METHODS WITH GRADED MESHES FOR TWO-DIMENSIONAL REACTION-DIFFUSION PROBLEMS

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Abstract. We develop high-order Galerkin methods with graded meshes for solving the two-dimensional reaction-diffusion problem on a rectangle. With the help of the comparison principle, we establish upper bounds for high order partial derivatives of an arbitrary order of its exact solution. According to prior information of the high order partial derivatives of the solution, we design both implicit and explicit graded meshes which lead to numerical solutions of the problem having an *optimal* convergence order. Numerical experiments are presented to confirm the theoretical estimate and to demonstrate the outperformance of the proposed meshes over the Shishkin mesh.

Key words. Singularly perturbation, reaction-diffusion problem, priori estimates, graded meshes, Galerkin method.

1. Introduction

The singular perturbation problem is an important class of boundary value problems which have broad applications. When the perturbation parameter is sufficiently small, the solution of the problem will have significantly large (partial) derivatives near the boundary, which illustrates *boundary layers*. The existence of boundary layers brings difficulty to numerical solutions of the problem, making the standard numerical methods unstable and fail to yield accurate results (cf. [26, 32]). In order to overcome the difficulty, various special meshes were constructed for singularity perturbation problems, among which the Bakhvalov type mesh and Shishkin type mesh are frequently used. Layer-adapted meshes (cf. [18]) are growing popular. Graded meshes were investigated in [7, 22, 23]. In particular, meshes proposed in [22, 23] for the one dimensional problem based on prior estimates of high order derivatives of the exact solution lead to uniform convergent solutions with optimal convergence rates for high order singular perturbation problems. A number of numerical schemes [19, 34, 37, 39, 41, 42] were established based on special meshes, providing accurate numerical solutions to one-dimensional singular perturbation problems.

Useful meshes may be constructed according to the behavior of the true solution. As a result, it is important to estimate the derivatives of the exact solution of the problem. For one-dimensional singular perturbation problems, the solutions were characterized clearly (cf. [28]), which provides valuable information to the analysis of numerical methods. For the two-dimensional problems, estimating the high order (mixed) derivatives of the solutions is much more involved. The authors of [15] utilized the maximum principle to establish the upper bounds of partial derivatives

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(with respect to a single variable x or y) of the solution of two-dimensional reaction-diffusion problems. With the aid of the estimates, the convergence properties of the numerical methods based on the Shishkin meshes were analyzed in [13, 14]. Asymptotic expansion is another tool to investigate the behavior of the solution. In [3], an asymptotic expansion of the solution was constructed, which contains boundary layer terms for edges and corner layer terms for vertices of the domain, and a uniform bound for the remainder was established to validate the uniform convergence of the expansion. The authors of [8] used the Btuzov expansion to expand the partial derivatives of the solution in a form that shows explicitly both the traditional corner singularities and boundary layers of the solution. In [21] the solution was decomposed into four terms, two of which were estimated pointwise and the other two were estimated with respect to a function norm. Based on knowledge of the solution of the singular perturbation problem on a rectangle, several popular numerical methods including the finite difference methods [16, 25, 35], the standard finite element methods [33, 38, 40] and the streamline diffusion finite element methods [20, 27, 36] were discussed. The finite volume methods [2, 10] and some adaptive schemes [4, 29] were also considered. More results can be found in [24] and the surveys [11, 12].

The goal of this paper is to construct numerical schemes of high accuracy for solving the reaction-diffusion problem on a rectangle based on prior information of high order derivatives of its exact solution. For this purpose, we present pointwise estimates of the high order (mixed) partial derivatives of the solution. Most existing results in the literature estimate the derivatives with respect to special function norms, and pointwise estimates are presented only for derivatives of order lower than two. Although the knowledge of mixed partial derivatives is not necessary in all circumstances, it is definitely crucial for some numerical methods such as sparse grid schemes. For both theoretical interest and computational purpose, we establish pointwise estimates for all types of derivatives of arbitrary orders. The upper bounds we give in this paper have a unified form, which illustrates a comprehensive view of the solution. The tool we use to establish the upper bounds is the maximum principle with a help of the solutions of related auxiliary problems.

The upper bounds of the exact solution of the problem suggest using the tensor product form of the one dimensional meshes to construct our approximate solutions. There are meshes leading to numerical solutions with optimal convergence rate for one-dimensional problems [7, 22, 23, 31]. Nevertheless, the meshes were constructed *implicitly*, which may bring inconvenience to the implementation of numerical schemes. We propose an *explicit realization* of the mesh from [22]. The solutions of the numerical methods based on the explicit realization are proved to converge at optimal order. Numerical results are presented to demonstrate the theoretical estimates and compare the proposed graded mesh with the well-known Shishkin mesh. The numerical results show that the proposed graded mesh outperforms the Shishkin mesh especially for high-order elements.

The paper is organized in four sections plus an appendix. In section 2 we present the upper bounds of the (mixed) partial derivatives of the solution. We describe in section 3 the Galerkin methods associated with graded rectangular meshes, the solutions of which converge uniformly at optimal rates. The positions of the knots are identified explicitly. In section 4 we present numerical examples to demonstrate the theoretical estimates on convergence of the numerical solutions. In the appendix, we provide details of proofs of two technical results necessary for establishing the upper bounds of the mixed partial derivatives of the exact solution.

2. Upper Bounds of Derivatives

The goal of this paper is to develop Galerkin methods having the optimal convergence order for solving the two-dimensional singularly perturbed boundary value problems on a rectangular domain. The boundary value problem to be considered has the form

$$(1)\qquad \begin{cases} Lu(x,y) := -\epsilon^2 \triangle u(x,y) + a(x,y)u(x,y) = f(x,y), & (x,y) \in \Omega, \\ u(x,y) = 0, & (x,y) \in \partial\Omega, \end{cases}$$

where $\Omega := (0,1)^2$ is the unit square, $0 < \epsilon \ll 1$, and a and f are both sufficiently smooth functions on $\overline{\Omega}$, the closure of Ω. Moreover, we assume that there exists a positive constant α such that

$$a(x,y) \geq \alpha^2, \qquad (x,y) \in \overline{\Omega}.$$

The solution of boundary value problem (1), in general, exhibits boundary layers due to the singular perturbation. As a result, standard computational methods fail to generate satisfactory numerical results. Aiming at constructing accurate numerical methods for solving the problem, we shall propose meshes based on prior information of the high order mixed partial derivatives of its exact solution, based on which piecewise polynomial spaces are constructed to approximate the solution. To this end, we devote this section to establishing upper bounds of the derivatives of the solution. Specifically, we will give upper bounds of partial derivatives (of an arbitrary order) of the solution. The bounds on the high order derivatives will guide the design of the meshes for the piecewise polynomial space that will be used to approximate the solution and serve for the convergence analysis of the proposed numerical methods.

For the sake of brevity, for a smooth function v, we use the notation

$$v_{x^i y^j} := \frac{\partial^{i+j} v}{\partial x^i \partial y^j}$$

to denote the mixed partial derivatives of v, and the cases when $i = 0$ or $j = 0$ is denoted by simple notation v_{y^j} or v_{x^i}, respectively. We shall use the function

$$\Upsilon(t) := e^{\frac{-\alpha t}{(1+\sigma)\epsilon}}, \quad t \in [0,1]$$

to describe the boundary layers of the solution of (1), where the parameter $\sigma > \sqrt{2} - 1$. We observe that $\Upsilon(t) \in (0,1]$ for all $t \in [0,1]$ and $\Upsilon(0) = 1$. Moreover, since $\epsilon \ll 1$, $\Upsilon(t)$ decays rapidly to 0 as t grows. By differentiating Υ, we find that for any $j > 0$, the derivative $\Upsilon^{(j)}$ behaves like Υ and

$$\Upsilon^{(j)}(0) = \mathcal{O}(\epsilon^{-j}).$$

Therefore, the function Υ characterizes a special type of singularity at 0.

The goal of this section is to establish the result that there exist positive constants $c_{i,j}$ such that for all $0 < \epsilon \ll 1$ and for all $(x,y) \in \overline{\Omega}$,

$$(2)\qquad |u_{x^i y^j}(x,y)| \leq c_{i,j} \mathcal{E}_i(x) \mathcal{E}_j(y),$$

where

$$\mathcal{E}_i(x) := 1 + \epsilon^{-i}[\Upsilon(x) + \Upsilon(1-x)].$$

These upper bounds characterize the singularity of the derivatives of u near the corners and edges of the domain. Precisely, they show that $u_{x^i y^j} = \mathcal{O}(\epsilon^{-(i+j)})$ near the four corners of Ω, $u_{x^i y^j} = \mathcal{O}(\epsilon^{-i})$ along the left and right edges of Ω, and $u_{x^i y^j} = \mathcal{O}(\epsilon^{-j})$ along the upper and lower edges.

The tool we use to establish the above inequality is the so-called *comparison principle*, a direct consequence of the maximum principle (cf. Theorem 3.1 of [6]) of the differential operator L. We state below the comparison principle for convenient reference.

Lemma 2.1. *If $w, \phi \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $|w(x,y)| \leq \phi(x,y)$ for all $(x,y) \in \partial\Omega$ and $|Lw(x,y)| \leq L\phi(x,y)$ for all $(x,y) \in \Omega$, then $|w(x,y)| \leq \phi(x,y)$ for all $(x,y) \in \overline{\Omega}$.*

The function ϕ appearing in the above lemma is called a *barrier function*. When we use the comparison principle to prove (2), the function on the right hand side of the inequality is treated as a barrier function. For later reference, we define

$$\phi_{i,j}(x,y) := \mathcal{E}_i(x)\mathcal{E}_j(y) \tag{3}$$

for $(x,y) \in \overline{\Omega}$ and observe by direct calculation that

$$\begin{aligned} L\phi_{i,j}(x,y) = & \left[a(x,y) - \frac{\alpha^2}{(1+\sigma)^2}\right](\epsilon^{-i}\kappa(x) + \epsilon^{-j}\kappa(y)) \\ & + \left[a(x,y) - \frac{2\alpha^2}{(1+\sigma)^2}\right]\epsilon^{-(i+j)}\kappa(x)\kappa(y) + a(x,y), \end{aligned} \tag{4}$$

where $\kappa(t) := \Upsilon(t) + \Upsilon(1-t)$. Since $a(x,y) \geq \alpha^2$, we have that

$$\begin{aligned} L\phi_{i,j}(x,y) \geq \alpha^2\big[\ & 1 + \left(1 - (1+\sigma)^{-2}\right)(\epsilon^{-i}\kappa(x) + \epsilon^{-j}\kappa(y)) \\ & + \left(1 - 2(1+\sigma)^{-2}\right)\epsilon^{-(i+j)}\kappa(x)\kappa(y)\big]. \end{aligned}$$

Let

$$\rho := 1 - \frac{2}{(1+\sigma)^2}.$$

When $\sigma > \sqrt{2} - 1$, there hold $1 - (1+\sigma)^{-2} > 1/2$ and $\rho > 0$. Thus, in this case, for all $(x,y) \in \Omega$,

$$L\phi_{i,j}(x,y) \geq \alpha^2\left[1 + \frac{1}{2}(\epsilon^{-i}\kappa(x) + \epsilon^{-j}\kappa(y)) + \rho\epsilon^{-(i+j)}\kappa(x)\kappa(y)\right]. \tag{5}$$

This inequality will be used later.

Before estimating the derivatives of u, we comment on its regularity by citing a theorem from [8].

Theorem 2.2. *Let $K \geq 1$ be a positive integer and u be the solution of (1). Let $P_l, 1 \leq l \leq 4$ denote the four vertices of $\overline{\Omega}$. For $\mu = 1, 2$, $u \in C^{2K-\mu+2}(\overline{\Omega})$ if and only if $f \in C^{2K-\mu}(\overline{\Omega})$, and*

$$\sum_{i=1}^{K}(-1)^{i-1}f_{x^{2(K-i)}y^{2(i-1)}}(P_l) = 0, \quad \text{for} \ \ 1 \leq i \leq K, \ 1 \leq l \leq 4.$$

Even though we present our results with a general K for theoretical interest, in practice, a relatively small value of K is acceptable. For estimating the derivatives of the solution u, a somewhat stronger assumption was imposed in the literature [5, 13, 30, 40] as a sufficient condition for the regularity of u. We cite the result below.

Theorem 2.3. *Let $K \geq 1$ be a positive integer and u be the solution of (1). Let $P_l, 1 \leq l \leq 4$ denote the four vertices of $\overline{\Omega}$. For $\mu = 1, 2$, if $f \in C^{2K-\mu}(\overline{\Omega})$, and*

$$f_{x^{2(K-i)}y^{2(i-1)}}(P_l) = 0, \quad \text{for} \ \ 1 \leq i \leq K, \quad 1 \leq l \leq 4, \tag{6}$$

then $u \in C^{2K-\mu+2}(\overline{\Omega})$.

Equation (6) is called the *compatibility condition*. This compatibility condition is rather restrictive. It might not be reasonable to expect that the solution u is smooth at a corner of the domain. One may need to study the singularity of the solution at the corners of the domain and develop numerical algorithms accordingly. In this paper, we shall stick to the assumption as the literature [5, 13, 30, 40] did.

In the remaining part of this section, we prove (2) holds for all $i, j > 0$ by induction on $i + j$. As the initial step of the induction, we prove the following result.

Proposition 2.4. *If*

$$c_{0,0} \geq \alpha^{-2} \|f\|_\infty, \tag{7}$$

then inequality (2) holds for $(i, j) = (0, 0)$.

Proof. It follows from (7) that $c_{0,0}\alpha^2 \geq \|f\|_\infty \geq f(x, y)$, for $(x, y) \in \Omega$. On the other hand, we obtain from (5) that $L\phi_{0,0}(x, y) \geq \alpha^2$, for $(x, y) \in \Omega$. Hence

$$|Lu(x, y)| = |f(x, y)| \leq c_{0,0}(L\phi_{0,0})(x, y), \qquad (x, y) \in \Omega.$$

For each $(x, y) \in \partial\Omega$, we have that $u(x, y) = 0$ and $\phi_{0,0}(x, y)$ is nonnegative. Thus

$$|u(x, y)| \leq \phi_{0,0}(x, y), \qquad (x, y) \in \partial\Omega.$$

The comparison principle proves the desired result. □

Next, we establish two theorems to complete the induction procedure. To this end, for $i, j > 0$, we define two index sets

$$\mathcal{N}_{(i,j)} := \{(m, n) : 0 \leq m \leq i, 0 \leq n \leq j\} \quad \text{and} \quad \tilde{\mathcal{N}}_{(i,j)} := \mathcal{N}_{(i,j)} \setminus \{(i, j)\}.$$

For $i + j = k$, we introduce

$$\lambda_1(i, j) := 1 + \sum_{(m,n)\in\tilde{\mathcal{N}}_{(i,j)}} \binom{i}{m}\binom{j}{n} c_{m,n},$$

$$\lambda_2(i, j) := \sum_{(m,n)\in\tilde{\mathcal{N}}_{(i,j)}} \binom{i}{m}\binom{j}{n} c_{m,n}\epsilon^{i-m},$$

$$\lambda_3(i, j) := \sum_{(m,n)\in\tilde{\mathcal{N}}_{(i,j)}} \binom{i}{m}\binom{j}{n} c_{m,n}\epsilon^{j-n},$$

$$\lambda_4(i, j) := \sum_{(m,n)\in\tilde{\mathcal{N}}_{(i,j)}} \binom{i}{m}\binom{j}{n} c_{m,n}\epsilon^{(i+j)-(m+n)},$$

and

$$\Theta := \max_k \{\|f\|_{k,\infty}, \|a\|_{k,\infty}\},$$

where $\|\cdot||_{k,\infty}$ denotes the maximum norm for the derivative of order not greater than k. We have the following upper bounds for the differential operator L on Ω.

Theorem 2.5. *Suppose that (2) holds for all* i, j *satisfying* $i+j < k$. *If for* $i, j \geq 0$ *with* $i + j = k$, *a positive constant* $c_{i,j}$ *is chosen to satisfy the condition*

$$c_{i,j}\alpha^2 \geq \Theta \cdot \max\{\lambda_1(i, j), 2\lambda_2(i, j), 2\lambda_3(i, j), \rho^{-1}\lambda_4(i, j)\}, \tag{8}$$

then for all $(x, y) \in \Omega$,

$$|Lu_{x^iy^j}(x, y)| \leq c_{i,j}L\phi_{i,j}(x, y). \tag{9}$$

Proof. We obtain from (1) that

$$Lu_{x^iy^j} = f_{x^iy^j} - \sum_{(m,n)\in\tilde{\mathcal{N}}_{(i,j)}} \binom{i}{m}\binom{j}{n} a_{x^{i-m}y^{j-n}} u_{x^my^n}.$$

If (2) holds for $i+j<k$, then for $i+j=k$ and for $(x,y)\in\Omega$, we have that

$$\begin{aligned}|Lu_{x^iy^j}(x,y)| \le \Theta\big[\quad & \lambda_1(i,j)+\lambda_2(i,j)\epsilon^{-i}\kappa(x) \\ & +\lambda_3(i,j)\epsilon^{-j}\kappa(y)+\lambda_4(i,j)\epsilon^{-(i+j)}\kappa(x)\kappa(y)\big].\end{aligned} \tag{10}$$

Comparing (10) with (5), we observe that (9) holds for $i+j=k$ when $c_{i,j}$ satisfies condition (8). Thus, we obtain the desired estimate. □

Note that in (8), the expression of lower bound of $c_{i,j}$ involves ϵ. However, since the exponent of ϵ is nonnegative, we have even better lower bound of $c_{i,j}$ when ϵ is small. Actually, we can let $\epsilon = 1$ to obtain a uniform but not-so-accurate lower bound.

In order to use the comparison principle to obtain upper bounds of $u_{x^iy^j}$, we shall need their upper bounds on $\partial\Omega$. Expressions of high order derivatives of u are not available directly from equation (1). We will establish them with the help of *auxiliary problems*.

We need the notion of the binary string. We call $(\alpha_1\alpha_2\ldots\alpha_m)$ with $\alpha_i = 0$ or 1 for $1\le i\le m$ a *binary string*. Here m is the length of the string. The empty string (), which contains no digit, is denoted by θ. Let Ξ be the set of all binary strings. For $\omega = 0,1$ and $\alpha\in\Xi$, we define $\phi_\omega(\alpha)$ as the string concatenating ω and α. For example, if $\alpha = (01)$, then $\phi_0(\alpha) = (001)$ and $\phi_1(\alpha) = (101)$. With $\alpha\in\Xi$, we associate a function $F^{[\alpha]}$, which is defined recursively as follows. Let $F^{[\theta]} := f$. When $F^{[\alpha]}$ is defined, we let $v^{[\alpha]}$ be the unique solution of boundary value problem

$$\begin{cases} Lv^{[\alpha]}(x,y) = F^{[\alpha]}(x,y), & (x,y)\in\Omega, \\ v^{[\alpha]}(x,y) = 0, & (x,y)\in\partial\Omega. \end{cases} \tag{11}$$

Then $F^{[\phi_0(\alpha)]}$ is defined by

$$F^{[\phi_0(\alpha)]} := F^{[\alpha]}_{x^2} - ag^{[\phi_0(\alpha)]} + \epsilon^2\big(g^{[\phi_0(\alpha)]}_{x^2} + g^{[\phi_0(\alpha)]}_{y^2}\big) - a_{x^2}v^{[\alpha]} - 2a_xv^{[\alpha]}_x \tag{12}$$

with

$$\begin{aligned} g^{[\phi_0(\alpha)]}(x,y) := \quad & -\epsilon^{-2}[\Upsilon(x) - x\Upsilon(1)]F^{[\alpha]}(0,y) \\ & -\epsilon^{-2}[\Upsilon(1-x)-(1-x)\Upsilon(1)]F^{[\alpha]}(1,y).\end{aligned} \tag{13}$$

Likewise, the function $F^{[\phi_1(\alpha)]}$ is defined by

$$F^{[\phi_1(\alpha)]} := F^{[\alpha]}_{y^2} - ag^{[\phi_1(\alpha)]} + \epsilon^2\big(g^{[\phi_1(\alpha)]}_{x^2} + g^{[\phi_1(\alpha)]}_{y^2}\big) - a_{y^2}v^{[\alpha]} - 2a_yv^{[\alpha]}_y \tag{14}$$

with

$$\begin{aligned} g^{[\phi_1(\alpha)]}(x,y) := \quad & -\epsilon^{-2}[\Upsilon(y) - y\Upsilon(1)]F^{[\alpha]}(x,0) \\ & -\epsilon^{-2}[\Upsilon(1-y)-(1-y)\Upsilon(1)]F^{[\alpha]}(x,1).\end{aligned} \tag{15}$$

We have the following property for $v^{[\phi_w(\alpha)]}$ and $v^{[\alpha]}$.

Lemma 2.6. *If $F^{[\alpha]}$ satisfies the compatibility condition $F^{[\alpha]}(P_l) = 0$, $1\le l\le 4$, where P_l, $1\le l\le 4$, denote the four vertices of $\overline{\Omega}$, then*

$$v^{[\phi_0(\alpha)]} + g^{[\phi_0(\alpha)]} = v^{[\alpha]}_{x^2}, \qquad v^{[\phi_1(\alpha)]} + g^{[\phi_1(\alpha)]} = v^{[\alpha]}_{y^2}.$$

Proof. We provide a proof for the first equality only since the second one can be similarly handled. Let $\omega := v^{[\phi_0(\alpha)]} + g^{[\phi_0(\alpha)]} - v^{[\alpha]}_{x^2}$. We will prove that ω is the solution of the corresponding homogeneous problem of (11) and thus, $\omega = 0$.

It follows from the boundary condition of (11) that $v^{[\alpha]}_{x^2}(x,0) = v^{[\alpha]}_{x^2}(x,1) = 0$, and $v^{[\alpha]}_{y^2}(0,y) = v^{[\alpha]}_{y^2}(1,y) = 0$. Therefore,

$$v^{[\alpha]}_{x^2}(0,y) = -v^{[\alpha]}_{y^2}(0,y) + \epsilon^{-2}a(0,y)v^{[\alpha]}(0,y) - \epsilon^{-2}F^{[\alpha]}(0,y) = -\epsilon^{-2}F^{[\alpha]}(0,y).$$

Similarly, we may show that

$$v^{[\alpha]}_{x^2}(1,y) = -\epsilon^{-2}F^{[\alpha]}(1,y).$$

Making use of (13) and the compatibility condition, we obtain

$$g^{[\phi_0(\alpha)]}(x,0) = g^{[\phi_0(\alpha)]}(x,1) = 0,$$

$$g^{[\phi_0(\alpha)]}(0,y) = -\epsilon^{-2}F^{[\alpha]}(0,y), \quad g^{[\phi_0(\alpha)]}(1,y) = -\epsilon^{-2}F^{[\alpha]}(1,y).$$

Moreover, $v^{[\phi_0(\alpha)]}$ vanishes on $\partial\Omega$. Thus ω vanishes on $\partial\Omega$.

Differentiating (11) twice with respect to x leads to

$$Lv^{[\alpha]}_{x^2} = F^{[\alpha]}_{x^2} - a_{x^2}v^{[\alpha]} - 2a_x v^{[\alpha]}_x.$$

On the other hand,

$$Lg^{[\phi_0(\alpha)]} = -\epsilon^2(g^{[\phi_0(\alpha)]}_{x^2} + g^{[\phi_0(\alpha)]}_{y^2}) + ag^{[\phi_0(\alpha)]}.$$

Thus, we obtain that

$$L(v^{[\alpha]}_{x^2} - g^{[\phi_0(\alpha)]})(x,y) = F^{[\phi_0(\alpha)]}(x,y), \qquad (x,y) \in \Omega.$$

Noting that $Lv^{[\phi_0(\alpha)]} = F^{[\phi_0(\alpha)]}$, we conclude that $L\omega = 0$ in Ω.

The above analysis shows that ω is the solution of the corresponding homogeneous problem of (11). This leads to $\omega = 0$ on $\overline{\Omega}$. □

For an $\alpha \in \Xi$, we denote by $|\alpha|$ the length of α. Let $\varrho_0(\alpha)$ and $\varrho_1(\alpha)$ represent, respectively, the number of 0 and 1 emerging in α. Moreover, let $\varpi(\alpha)$ denote the first character of α. Then we have the following result.

Proposition 2.7. *Suppose that $f \in C^{2K-2}(\overline{\Omega})$ for $K \geq 1$ and satisfies the compatibility condition (6). For $\alpha_1, \alpha_2 \in \Xi_K := \{\alpha \in \Xi : |\alpha| < K\}$, if $\varrho_0(\alpha_1) = \varrho_0(\alpha_2)$ and $\varrho_1(\alpha_1) = \varrho_1(\alpha_2)$, then*

$$F^{[\alpha_1]} = F^{[\alpha_2]}. \tag{16}$$

Moreover, for any $\alpha \in \Xi_K$, $F^{[\alpha]} \in C^{2(K-1-|\alpha|)}(\overline{\Omega})$ and satisfies the compatibility condition

$$F^{[\alpha]}_{x^{2(K-|\alpha|-i)}y^{2(i-1)}}(P_l) = 0, \qquad 1 \leq i \leq K - |\alpha|, \tag{17}$$

where P_l, $1 \leq l \leq 4$, denote the four vertices of $\overline{\Omega}$.

Proof. We prove equation (16) by induction on the length of α. The cases with $|\alpha| \leq 1$ are trivial. Assume that equation (16) holds for $|\alpha| \leq k$. Let $\alpha_1, \alpha_2 \in \Xi$ such that $|\alpha_1| = |\alpha_2| = k+1$ and $\varrho_\omega(\alpha_1) = \varrho_\omega(\alpha_2)$, $\omega = 0, 1$. We only need to consider the cases with $\varrho_\omega(\alpha_j) > 0$ for $\omega = 0, 1$ and $j = 1, 2$. If $\varpi(\alpha_1) = \varpi(\alpha_2)$, we then conclude (16) by the induction hypothesis. Otherwise, we assume without loss of generality that $\varpi(\alpha_1) = 0$ and $\varpi(\alpha_2) = 1$. In this case we assume that $\alpha_1 = \phi_0(\alpha_1')$ and $\alpha_2 = \phi_1(\alpha_2')$ for some α_1' and α_2'.

We first consider the case that $\alpha_1' = \phi_1(\overline{\alpha})$ and $\alpha_2' = \phi_0(\overline{\alpha})$ for some string $\overline{\alpha}$, that is, α_1 is the concatenation of the string (01) and $\overline{\alpha}$, and α_2 is the concatenation of (10) and $\overline{\alpha}$. Now we verify that $g^{[\alpha_1]} = g^{[\alpha_2']}_{y^2}$. Note that $g^{[\alpha_1]} = g^{[\phi_0(\alpha_1')]}$ and $g^{[\alpha_2']} = g^{[\phi_0(\overline{\alpha})]}$, according to (13), it suffices to prove for all $y \in [0,1]$ that

$$F^{[\alpha_1']}(t,y) = F^{[\overline{\alpha}]}_{y^2}(t,y), \qquad t = 0,1.$$

Note that $\alpha_1' = \phi_1(\overline{\alpha})$, and it follows from the boundary conditions of (11) and compatibility condition of $F^{[\overline{\alpha}]}$ that all but the first term of right hand side of (14) vanish. Thus the above equalities are verified. Similarly, we have $g^{[\alpha_2]} = g^{[\alpha_1']}_{x^2}$.

By applying (12) to $F^{[\alpha_1]}$ and then applying (14) to $F^{[\alpha_1']}$, we obtain that

$$\begin{aligned} F^{[\alpha_1]} = \; & F^{[\overline{\alpha}]}_{x^2y^2} - ag^{[\alpha_1]} - (ag^{[\alpha_1']})_{x^2} + \epsilon^2(g^{[\alpha_1]}_{x^2} + g^{[\alpha_1]}_{y^2} + g^{[\alpha_1']}_{x^4} + g^{[\alpha_1']}_{x^2y^2}) \\ & -(a_{x^2}v^{[\alpha_1']} + 2a_x v^{[\alpha_1']}_x) - (a_{y^2}v^{[\overline{\alpha}]} + 2a_y v^{[\overline{\alpha}]}_y)_{x^2}. \end{aligned}$$

Making use of the equalities

$$g^{[\alpha_1]} = g^{[\alpha_2']}_{y^2}, \quad v^{[\alpha_1']} + g^{[\alpha_1']} = v^{[\overline{\alpha}]}_{y^2},$$

where the second equation is obtained directly from Lemma 2.6, we conclude that

$$\begin{aligned} F^{[\alpha_1]} = \; & F^{[\overline{\alpha}]}_{x^2y^2} - a(g^{[\alpha_1']}_{x^2} + g^{[\alpha_2']}_{y^2}) + \epsilon^2(g^{[\alpha_1']}_{x^4} + g^{[\alpha_1']}_{x^2y^2} + g^{[\alpha_2']}_{x^2y^2} + g^{[\alpha_2']}_{y^4}) \\ & -(a_{x^2y^2}v^{[\overline{\alpha}]} + a_{x^2}v^{[\overline{\alpha}]}_{y^2} + a_{y^2}v^{[\overline{\alpha}]}_{x^2} + 2a_{x^2y}v^{[\overline{\alpha}]}_y + 2a_{xy^2}v^{[\overline{\alpha}]}_x \\ & +2a_x v^{[\overline{\alpha}]}_{xy^2} + 2a_y v^{[\overline{\alpha}]}_{x^2y} + 4a_{xy}v^{[\overline{\alpha}]}_{xy}). \end{aligned} \tag{18}$$

Similarly, if we apply (14) to $F^{[\alpha_2]}$ and then apply (12) to $F^{[\alpha_2']}$, with the help of

$$g^{[\alpha_2]} = g^{[\alpha_1']}_{x^2}, \quad v^{[\alpha_2']} + g^{[\alpha_2']} = v^{[\overline{\alpha}]}_{x^2},$$

we observe that $F^{[\alpha_2]}$ has the same expansion as $F^{[\alpha_1]}$ as described above. Thus, we have proved equation (16) in this case.

Now we consider the general case. Since $\varrho_1(\alpha_1') > 0$, there exists a string α_1'' such that $\varpi(\alpha_1'') = 1$ and $\varrho_\omega(\alpha_1'') = \varrho_\omega(\alpha_1')$ for $\omega = 0,1$. Since $\varpi(\alpha_1'') = 1$, there exists a string $\hat{\alpha}$ such that $\alpha_1'' = \phi_1(\hat{\alpha})$. We let $\alpha_2'' := \phi_0(\hat{\alpha})$. It follows from (12) that

$$F^{[\alpha_1]} = F^{[\alpha_1']}_{x^2} - ag^{[\alpha_1]} + \epsilon^2(g^{[\alpha_1]}_{x^2} + g^{[\alpha_1]}_{y^2}) - a_{x^2}v^{[\alpha_1']} - 2a_x v^{[\alpha_1']}_x.$$

We obtain from the induction hypothesis that $F^{[\alpha_1'']} = F^{[\alpha_1']}$ and $v^{[\alpha_1'']} = v^{[\alpha_1']}$. Expanding $F^{[\alpha_1'']}$ with (14) leads to

$$\begin{aligned} F^{[\alpha_1]} = \; & F^{[\hat{\alpha}]}_{x^2y^2} - ag^{[\alpha_1]} - (ag^{[\alpha_1'']})_{x^2} + \epsilon^2(g^{[\alpha_1]}_{x^2} + g^{[\alpha_1]}_{y^2} + g^{[\alpha_1'']}_{x^4} + g^{[\alpha_1'']}_{x^2y^2}) \\ & -(a_{x^2}v^{[\alpha_1'']} + 2a_x v^{[\alpha_1'']}_x) - (a_{y^2}v^{[\hat{\alpha}]} + 2a_y v^{[\hat{\alpha}]}_y)_{x^2}. \end{aligned}$$

Note that $\alpha_1 = \phi_0(\alpha_1'')$ and $\alpha_2'' = \phi_0(\hat{\alpha})$. Moreover,

$$F^{[\alpha_1'']}(t,y) = F^{[\hat{\alpha}]}_{y^2}(t,y), \qquad t = 0,1.$$

Thus $g^{[\alpha_1]} = g^{[\alpha_2'']}_{y^2}$. Finally, we lead to the expansion of $F^{[\alpha_1]}$, which is the right hand side of (18) with $\overline{\alpha}$, α_1', α_2' replaced by $\hat{\alpha}$, α_1'', α_2'', respectively. When we use (14) to expand $F^{[\alpha_2]}$, and note that $\varrho_\omega(\alpha_2'') = \varrho_\omega(\alpha_2')$, $\omega = 0,1$, we follow the same way to yield the expansion of $F^{[\alpha_2]}$ which is the same as $F^{[\alpha_1]}$. We prove equation (16) in this case. The induction principle ensures that equation (16) holds in general.

The compatibility condition (17) for $F^{[\alpha]}$, $\alpha \in \Xi_K$ can be obtained directly from the compatibility condition (6) for f. $\square$

As shown in the above proposition, the binary strings with the same number of 0's and 1's correspond to the same function. This allows us to breviate the notation. We use the notation $F^{[2m,2n]} := F^{[\alpha]}$ with $m = \varrho_0(\alpha)$ and $n = \varrho_1(\alpha)$. Accordingly, we denote $v^{[2m,2n]} := v^{[\alpha]}$, and rewrite the auxiliary problem as

$$\begin{cases} Lv^{[2m,2n]}(x,y) = F^{[2m,2n]}(x,y), & (x,y) \in \Omega, \\ v^{[2m,2n]}(x,y) = 0, & (x,y) \in \partial\Omega. \end{cases} \tag{19}$$

When $\varpi(\alpha) = 0$, we let $g^{[2m,2n]} := g^{[\alpha]}$, and when $\varpi(\alpha) = 1$ we let $\tilde{g}^{[2m,2n]} := g^{[\alpha]}$. With the above notation, we have the identities

$$v^{[2m,2n]} = v^{[2m-2,2n]}_{x^2} - g^{[2m,2n]} \quad \text{and} \quad v^{[2m,2n]} = v^{[2m,2n-2]}_{y^2} - \tilde{g}^{[2m,2n]}.$$

Noting $F^{[0,0]} = f$, we have $v^{[0,0]} = u$. Repeatedly applying the identities yields

$$v^{[2m,2n]} = u_{x^{2m}y^{2n}} - G^{[2m,2n]} \tag{20}$$

for any $m, n > 0$ with

$$G^{[2m,2n]} := \sum_{m'=1}^{m} g^{[2m',2n]}_{x^{2m-2m'}} + \sum_{n'=1}^{n} \tilde{g}^{[0,2n']}_{x^{2m}y^{2n-2n'}}.$$

With (20), estimates for high order derivatives of u may be obtained from those for lower order derivatives of $v^{[2m,2n]}$ and $G^{[2m,2n]}$. Actually, it suffices to establish upper bounds for $v^{[2m,2n]}_{x^p y^q}$ with $p + q \le 2$.

Now we assume (2) holds for $i + j \le 2M$ with $M \ge 0$ and establish the following two propositions.

Proposition 2.8. *If $f \in C^{2K-2}(\overline{\Omega})$, $K \ge 1$ and satisfies the compatibility condition (6), then for any $m, n \ge 0$ with $m + n \le M - 1$ and $p, q \ge 0$ with $p + q \le 2(M - m - n) - 1$, there exist positive constants $\vartheta^{[2m,2n]}_{p,q}$, $\tilde{\vartheta}^{[2m,2n]}_{p,q}$, $\ell^{[2m,2n]}_{p,q}$ and $\theta^{[2m,2n]}_{p,q}$, such that*

$$\left| g^{[2m,2n]}_{x^p y^q}(x,y) \right| \le \vartheta^{[2m,2n]}_{p,q} \epsilon^{-(2m+p)} \kappa(x)(1 + \epsilon^{-(2n+q)} \kappa(y)), \tag{21}$$

$$\left| \tilde{g}^{[2m,2n]}_{x^p y^q}(x,y) \right| \le \tilde{\vartheta}^{[2m,2n]}_{p,q} \epsilon^{-(2n+q)} \kappa(y)(1 + \epsilon^{-(2m+p)} \kappa(x)), \tag{22}$$

$$\left| F^{[2m,2n]}_{x^p y^q}(x,y) \right| \le \theta^{[2m,2n]}_{p,q} (1 + \epsilon^{-(2m+p)} \kappa(x))(1 + \epsilon^{-(2n+q)} \kappa(y)) \tag{23}$$

and

$$\left| G^{[2m,2n]}_{x^p y^q}(x,y) \right| \le \ell^{[2m,2n]}_{p,q} (1 + \epsilon^{-(2m+p)} \kappa(x))(1 + \epsilon^{-(2n+q)} \kappa(y)) \tag{24}$$

hold for all $(x,y) \in \overline{\Omega}$.

Proposition 2.9. *For any nonnegative integers m, n, p, q satisfying $m + n = M$ and $0 < p + q \le 2$, there exists a positive constant $c^{[2m,2n]}_{p,q}$ such that for all $\epsilon > 0$*

$$|v^{[2m,2n]}_{x^p y^q}(x,y)| \le c^{[2m,2n]}_{p,q} \phi_{2m+p,2n+q}(x,y), \quad (x,y) \in \partial\Omega.$$

Proofs of the above two propositions will be given in the appendix. Now we are ready to establish the upper bounds of high order derivatives of u on the boundary.

Theorem 2.10. *If (2) holds for all i, j satisfying $i + j \leq 2M$ for a nonnegative integer M, then for nonnegative integers m, n, p, q satisfying $m + n = M$ and $0 < p + q \leq 2$, there holds*

$$|u_{x^{2m+p}y^{2n+q}}(x, y)| \leq c_{2m+p,2n+q}\phi_{2m+p,2n+q}(x, y) \tag{25}$$

for any $(x, y) \in \partial\Omega$ with positive constants $c_{2m+p,2n+q} \geq c_{p,q}^{[2m,2n]} + \ell_{p,q}^{[2m,2n]}$.

Proof. It follows from (20) that

$$u_{x^{2m+p}y^{2n+q}} = u_{x^p y^q}^{[2m,2n]} + G_{x^p y^q}^{[2m,2n]}.$$

We then obtain the desired result from Propositions 2.8 and 2.9. □

Making use of Theorems 2.5 and 2.10, we prove by induction the following theorem.

Theorem 2.11. *If $f \in C^{2K-2}(\overline{\Omega})$, $K \geq 1$ and satisfies the compatibility condition (6), and u is the solution of (1), then for each pair of i, j with $i + j \leq 2K$ there exists a positive constant $c_{i,j}$ such that for all $(x, y) \in \overline{\Omega}$ and for all $\epsilon > 0$*

$$|u_{x^i y^j}(x, y)| \leq c_{i,j}\mathcal{E}_i(x)\mathcal{E}_j(y). \tag{26}$$

Proof. The theorem is proved by induction on $i+j$. First of all, we use Proposition 2.4 to verify the case $i+j = 0$. Then suppose that the inequality holds for $i+j \leq 2M$ with $M \geq 0$. We conclude from Theorems 2.5 and 2.10 that for $i+j = 2M+1, 2M+2$, there exist positive constant c_{ij} such that

$$|Lu_{x^i y^j}(x, y)| \leq c_{i,j}L\phi_{ij}(x, y)$$

for $(x, y) \in \Omega$ and

$$|u_{x^i y^j}(x, y)| \leq c_{i,j}\phi_{ij}(x, y)$$

for $(x, y) \in \partial\Omega$. It follows from the comparison principle that (26) holds for $i + j = 2M + 1, 2M + 2$. Therefore, we establish (26) for all i, j with $i + j \leq 2K$. □

Theorem 2.11 provides upper bounds for the derivatives of u of all orders. Moreover, the upper bounds are in a tensor product form, which illustrates different singularity behaviors of the solution at the corners and along the edges. We observe from (26) that the estimates are *pointwise*, but not with respect to a norm of u. This is crucial for the design of our meshes. In passing, we point that in the special case when a is a constant function, [8] makes use of the Butzov expansion to obtain a pointwise estimate for derivatives of u of all order. However, the upper bounds given in that paper differ much from those in (26) and have complicated expressions.

3. Galerkin methods with graded meshes

In this section, we develop the Galerkin methods for solving (1). The approximation subspace will be chosen as a piecewise polynomial space on a specially designed mesh. In order to obtain numerical solutions with an optimal convergence rate, the mesh will be chosen as a graded mesh according to the behavior of the derivatives of the exact solution of (1). We shall present both implicit and explicit meshes for the approximation subspaces.

For the domain Ω, we denote by $L^2(\Omega)$ the space of square integrable real-valued functions on Ω with the norm $\|\cdot\|$ and the associated inner product $(\cdot, \cdot)$. Let $H^1(\Omega)$ be the Sobolev space on Ω with the norm $\|\cdot\|_1$, and let $H_0^1(\Omega)$ be the closure of the set $\{v \in C^1(\Omega) : v|_{\partial\Omega} = 0\}$ with respect to the norm $\|\cdot\|_1$. Moreover, we define the energy norm

$$\|v\|_\epsilon := [\epsilon^2(\|v_x\|^2 + \|v_y\|^2) + \|v\|^2]^{1/2}, \qquad v \in H^1(\Omega).$$

By defining the bilinear form

$$A(v,w) := \epsilon^2(v_x, w_x) + \epsilon^2(v_y, w_y) + (av, w), \qquad v, w \in H^1(\Omega),$$

we rewrite (1) as a variational form, in which we seek $u \in H_0^1(\Omega)$ such that

$$A(u,v) = (f,v), \qquad v \in H_0^1(\Omega). \tag{27}$$

The solution of (27) is called a weak solution of (1).

We describe a rectangular mesh to partition Ω. Specifically, we let N be a positive integer and

$$0 = x_0 < x_1 < \cdots < x_N = 1, \qquad 0 = y_0 < y_1 < \cdots < y_N = 1.$$

Then the corresponding rectangular mesh is formed by drawing lines parallel to the x-axes and y-axes through the points (x_i, y_j), $0 \le i, j \le N$. We use the notation π_N to refer to the rectangular mesh. For any positive integer r, we say the function $v(x,y)$ is a bi-rth order polynomial if $v(x,\cdot)$ and $v(\cdot,y)$ are both polynomials of order r. Denote

$$\tau_{i,j} := (x_i, x_{i+1}) \times (y_j, y_{j+1}), \qquad 0 \le i, j < N,$$

and for any domain τ, we define $Q_r(\tau)$ to be the set of all bi-rth order polynomials on τ. The approximation subspace is chosen as

$$V_N := \{v : v \in H_0^1(\Omega), v|_{\tau_{i,j}} \in Q_r(\tau_{i,j}), 0 \le i, j < N\}.$$

We seek $u_N \in V_N$ to satisfy

$$A(u_N, v) = (f, v), \qquad v \in V_N. \tag{28}$$

The solution u_N of equation (28) is then considered as an approximation of the solution u of equation (27).

We next consider the approximation accuracy of the approximate solution u_N. It follows from (27) and (28) that

$$A(u - u_N, v) = 0, \qquad v \in V_N.$$

Therefore, by Cea's Lemma, there exists a positive constant c such that for all $\epsilon > 0$ and for all positive integers N

$$\|u - u_N\|_\epsilon \le c \inf_{v \in V_N} \|u - v\|_\epsilon.$$

This suggests that the accuracy of u_N depends entirely on the approximation power of the subspace V_N that approximates the solution u. The approximation strength of the subspace V_N relies upon the mesh on which the subspace is based.

The rest of this section is devoted to the construction of an *explicit* mesh, which leads to piecewise polynomial approximate solutions with optimal convergence rate. The meshes associated with optimal convergence rate are introduced and studied in [7, 22, 23, 31]. The design of the graded mesh is based on the idea that the interpolation errors of the approximate solution should distribute equally in the subintervals of the mesh. We would like to review the mesh proposed in [22], in which a generating function

$$h_\lambda(x) := \frac{\lambda\epsilon}{N} e^{\frac{\alpha x}{r\epsilon}}, \quad x \in [0,1] \tag{29}$$

was introduced for the design of the mesh, where $\lambda > 0$ serves as a parameter. A mesh of $[0,1]$ with

$$0 = x_0 < x_1 < x_2 < \cdots < x_{\tilde{N}-1} < x_{\tilde{N}} = 1,$$

was then chosen to satisfy the conditions

$$x_{i+1} - x_i \le \min\{h_\lambda(x_i), h_\lambda(1-x_{i+1}), 1/N\}, \quad \text{for all } \ i \in \mathbb{Z}_{\tilde{N}} := \{0,1,\ldots,\tilde{N}-1\}, \tag{30}$$

and

$$\tilde{N} \le cN \tag{31}$$

for some positive constant c independent of N or ϵ. A mesh satisfying the above conditions associated with the generating function is called *optimal* since it generates the approximation subspace which has the optimal order of convergence. We remark that the definition of *optimal meshes* is associated with the behavior of the solution of one-dimensional reaction-diffusion problems, which has a singularity near both the end points. For example, the mesh from [23], which is related to convection-diffusion problems, is different from that proposed in [22].

For the two-dimensional problem (1), based on the estimates of derivatives of the solution presented in the previous section, we shall use the generating function

$$\tilde{h}_\lambda(x) := \frac{\lambda\epsilon}{N} e^{\frac{\alpha x}{(1+\sigma)r\epsilon}}, \quad x \in [0,1]. \tag{32}$$

We remark that the function $\tilde{h}_\lambda$ differs from h_λ in (29) because it has to inherit the constant $1+\sigma$ from the upper bounds emerging in (26). A rectangular mesh $\pi_{\tilde{N}}$ of Ω is called an *optimal rectangular mesh*, if $\{x_i : i \in \mathbb{Z}_{\tilde{N}+1}\}$ and $\{y_j : j \in \mathbb{Z}_{\tilde{N}+1}\}$ are both optimal meshes with respect to the generating function (32). An implicit mesh $\pi_{\tilde{N}}$ of Ω may be obtained if we obtain an implicit mesh of $[0,1]$ following the method described in [22]. For brevity of notations, we denote the mesh by π_N instead of $\pi_{\tilde{N}}$, but keep in mind that the subscript N is no longer the number of subintervals at each direction.

The meshes described in [7, 22, 23] were constructed *implicitly* in the sense that the definition of one knot may need information of some other knots. Implicit meshes are less convenient for programming in the following two respects. Firstly, with an implicit mesh we do not know the exact number of knots in advance, which brings us difficulty in allocating memory for the knots as well as any other variables associated with the knots. Secondly, we have to create the knots in specific order, such as to define a knot only with the information of the knots ahead of it. Therefore, we are motivated to give an *explicit construction* of optimal meshes.

Corresponding to the generating function (32), we define the *transition point*

$$\rho_\lambda := -\frac{(1+\sigma)r\epsilon}{\alpha}\log(\lambda\epsilon). \tag{33}$$

It is natural to require $\lambda < \epsilon^{-1}$ so that $\rho_\lambda > 0$. Note that there holds

$$\tilde{h}_\lambda(\rho_\lambda) = \frac{1}{N}.$$

Moreover, since $\epsilon \ll 1$, we assume without loss of generality that $\rho_\lambda < 1/2$. We next construct a mesh π_N with

$$0 = x_0 < x_1 < x_2 < \cdots < x_{4N-1} < x_{4N} = 1.$$

When $N \le i \le 3N$, we let

$$x_i := \rho_\lambda + (1-2\rho_\lambda)\frac{i-N}{2N}, \tag{34}$$

that is, these knots distribute uniformly in the subinterval $[\rho_\lambda, 1-\rho_\lambda]$. For $3N < i \le 4N$, we define $x_i = 1 - x_{4N-i}$. Hence it is left to identify the distribution of the knots $x_i,\ i \in \mathbb{Z}_N$ in the subinterval $[0, \rho_\lambda]$. According to the definition of optimal meshes, as long as the knots $x_i,\ i \in \mathbb{Z}_{N+1}$ satisfy the condition

$$x_{i+1} - x_i \le \tilde{h}_\lambda(x_i), \tag{35}$$

π_N is an optimal mesh. Our task is to allocate these knots such that they meet the requirement (35). To this end, we need the following technical lemma.

Lemma 3.1. *Suppose that the function H is defined on $[0, \rho_\lambda]$ such that H' is positive and decreasing. Let N be a fixed positive integer. For each $i \in \mathbb{Z}_{N+1}$, let x_i be the unique value that satisfies*

$$H(x_i) = H(0) + \frac{i}{N}(H(\rho_\lambda) - H(0)).$$

If there exists a generating function $\tilde{h}_{\tilde{\lambda}}$ such that

$$1/H'(x_{i+1}) \le N\tilde{h}_{\tilde{\lambda}}(x_i), \qquad i \in \mathbb{Z}_N, \tag{36}$$

then $\{x_i : i \in \mathbb{Z}_{N+1}\}$ is an optimal mesh of $[0, \rho_\lambda]$.

Proof. It follows from the mean value theorem that for each $i \in \mathbb{Z}_N$, there exists $\xi_i \in (x_i, x_{i+1})$ such that

$$H'(\xi_i) = \frac{H(x_{i+1}) - H(x_i)}{x_{i+1} - x_i} = \frac{1}{N}\frac{H(\rho_\lambda) - H(0)}{x_{i+1} - x_i}.$$

Since H' is a positive decreasing function, we have $H'(\xi_i) \ge H'(x_{i+1})$. Thus

$$x_{i+1} - x_i \le \frac{1}{N}\frac{H(\rho_\lambda) - H(0)}{H'(x_{i+1})}.$$

By hypothesis of this lemma, we observe that for any $N > 0$, $\{x_i : i \in \mathbb{Z}_{N+1}\}$ is an optimal mesh on $[0, \rho_\lambda]$ associated with the generating function $\tilde{h}_{\tilde{\lambda}[H(\rho_\lambda) - H(0)]}$. □

The inequality (36) motivates us to construct H such that $1/H'$ has the form of generating functions. Specifically, we define

$$H(x) := -\frac{(1+\sigma)r}{\alpha} e^{-\frac{\alpha x}{(1+\sigma)r\epsilon}}. \tag{37}$$

It is straightforward to get

$$H'(x) = \epsilon^{-1} e^{-\frac{\alpha x}{(1+\sigma)r\epsilon}}.$$

Thus, $\frac{1}{NH'(x)}$ is exactly the generating function $\tilde{h}_1(x)$. Moreover, it follows from the definition of x_i that $H(x_i),\ i \in \mathbb{Z}_{N+1}$ is an arithmetic sequence and all numbers in the sequence have the same sign. Therefore,

$$H(x_{i+1}) \ge \frac{1}{2}H(x_i), \qquad i \in \mathbb{Z}_N.$$

Noting that H' is a multiple of H, we conclude that

$$\frac{1}{NH'(x_{i+1})} \le \frac{2}{NH'(x_i)} = \tilde{h}_{1/2}(x_i).$$

We have verified all hypotheses of Lemma 3.1. Utilizing the function (37), we conclude from Lemma 3.1 that

$$x_i := -\frac{(1+\sigma)r\epsilon}{\alpha}\log\frac{N-i(1-\lambda\epsilon)}{N}, \qquad i\in\mathbb{Z}_{N+1}. \tag{38}$$

We construct the optimal rectangular mesh $\pi_N := \{(x_i, y_j) : i, j \in \mathbb{Z}_{4N}\}$ with

$$0 = x_0 < x_1 < \cdots < x_{4N-1} < x_{4N} = 1, \quad 0 = y_0 < y_1 < \cdots < y_{4N-1} < y_{4N} = 1$$

by choosing the knots as follows. Choose a value of $\lambda > 0$ and define the transition point ρ_λ of the generating function (32) by (33). For $i \in \mathbb{Z}_{N+1}$, we set x_i with (38). For $N < i < 3N$, we set x_i with (34), and for $i \geq 3N$, we set x_i with $x_i = 1-x_{4N-i}$. The values of y_i's are chosen in a similar way, with perhaps a different value of λ.

Note that the mesh constructed above is actually a *realization* of the optimal meshes, and it may not be the unique realization. In [16, 17], a mesh aiming at convection-diffusion problems were constructed *explicitly.* It is worth noting that it is optimal only for the piecewise linear polynomial approximation.

We next present an error estimate of numerical solutions that result from the approximation subspace associated with the optimal rectangular mesh described above.

Theorem 3.2. *Suppose that $u \in W^{r,\infty}(\Omega)$. Let N be a positive integer N. If u_N is the solution of (28) associated with the optimal rectangular mesh, then there exists a positive constant c such that for all $\epsilon > 0$ and for all N,*

$$\|u - u_N\|_\epsilon \leq cN^{-(r-1)}. \tag{39}$$

Proof. We shall prove the theorem by using Cea's Lemma with the help of a specific interpolation of u from the approximation subspace V_N. Specifically, for $i, j \in \mathbb{Z}_N$, we choose $\xi_{i,\ell}$ and $\zeta_{j,\ell}$ for $\ell \in \mathbb{Z}_r$ to satisfy

$$x_i = \xi_{i,0} < \xi_{i,1} < \cdots < \xi_{i,r-2} < \xi_{i,r-1} = x_{i+1},$$
$$y_j = \zeta_{j,0} < \zeta_{j,1} < \cdots < \zeta_{j,r-2} < \zeta_{j,r-1} = y_{j+1}.$$

Given any $v \in H_0^1(\Omega)$, we define $q_{i,j}$ to be the unique bi-rth order polynomial to interpolate u at the knots $\{(x_{i,\ell_1}, y_{j,\ell_2}) : \ell_1, \ell_2 \in \mathbb{Z}_r\}$. Let $\Pi_N u$ be the piecewise polynomial of bi-rth order such that

$$\Pi_N u|_{\tau_{i,j}} = q_{i,j},\ i, j \in \mathbb{Z}_{\tilde{N}}.$$

By the regularity property of the solution u and Theorem 3 of [1] that there exists a positive constant c such that

$$\|u - \Pi_N u\|_{\infty,\tau_{i,j}} \leq c \sum_{m+n=r} (x_{i+1} - x_i)^m (y_{j+1} - y_j)^n \|u_{x^m y^n}\|_{\infty,\tau_{i,j}},$$

$$\|(u - \Pi_N u)_x\|_{\infty,\tau_{i,j}} \leq c \sum_{m+n=r} (x_{i+1} - x_i)^{m-1} (y_{j+1} - y_j)^n \|u_{x^m y^n}\|_{\infty,\tau_{i,j}},$$

$$\|(u - \Pi_N u)_y\|_{\infty,\tau_{i,j}} \leq c \sum_{m+n=r} (x_{i+1} - x_i)^m (y_{j+1} - y_j)^{n-1} \|u_{x^m y^n}\|_{\infty,\tau_{i,j}}.$$

Analogous to the analysis in [22], we sum up the above estimates through all $\tau_{i,j}$ to conclude

$$\|u - \Pi_N u\|_\epsilon \leq cN^{-(r-1)}$$

for some positive constant c. Since $\Pi_N u \in V_N$, we have that

$$\inf_{v\in V_N} \|u - v\|_\epsilon \leq c\|u - \Pi_N u\|_\epsilon \leq cN^{-(r-1)}.$$

The desired theorem is thus proved by combining Cea's Lemma with the above estimate. □

4. Numerical Examples

In this section, we present two numerical examples to demonstrate the effectiveness of the proposed method. In these examples we use piecewise polynomial bases of three different orders and compare three meshes, namely, the Shishkin mesh, the *implicit mesh* (that is, the tensor product form of the mesh proposed in [22]) and the *explicit mesh* (that is, the explicit realization proposed in the previous section). All programs are run on a personal computer equipped with 2.5GHz Intel Core, i5 CPU and 4G memory.

Example 1: In this example we solve equation (1) with $a = 2$. For comparison purpose, we choose the right hand side f of equation (1) so that

$$u(x,y) := \left(1 - \frac{e^{-x/\epsilon} + e^{-(1-x)/\epsilon}}{1 + e^{-1/\epsilon}}\right)\left(1 - \frac{e^{-y/\epsilon} + e^{-(1-y)/\epsilon}}{1 + e^{-1/\epsilon}}\right), \quad (x,y) \in \Omega$$

is the exact solution of (1). For each mesh, we use the corresponding piecewise bilinear polynomial space to discretize the equation. The numerical results are listed in Table 1, where $e_N := u - u_N$, and "C. R." stands for the convergence rate defined by

$$\log\left(\frac{e_{N_1}}{e_{N_2}}\right) \Big/ \left(\frac{N_2}{N_1}\right),$$

where N_1 and N_2 are two successive values of N (or $\tilde{N}$).

For the bilinear polynomial approximation, we additionally compare the proposed graded mesh with the Bakhvalov-Shishkin mesh (cf. [17]), namely *B-S mesh* since in the literature this mesh was constructed for solving the linear convection-diffusion problem. Numerical results of the bilinear approximation are presented in Table 1. We also solve the same equation with the piecewise bi-quadratic polynomial approximation, and the numerical results are shown in Table 2.

TABLE 1. Numerical results for bilinear bases.

ϵ	N	Shishkin mesh		Implicit mesh		Explicit mesh		B-S mesh	
		$\|e_N\|_\epsilon$	C. R.	$\|e_{\tilde{N}}\|_\epsilon$	C. R.	$\|e_N\|_\epsilon$	C. R.	$\|e_N\|_\epsilon$	C. R.
10^{-3}	32	1.11-2		3.72-3		4.45-3		4.52-3	
	64	6.68-3	0.7311	1.83-3	1.0933	2.24-3	0.9891	2.23-3	1.0011
	128	3.91-3	0.7753	8.70-4	1.0417	1.13-3	0.9922	1.13-3	0.9958
	256	2.23-3	0.8065	5.65-4	1.0247	5.68-4	0.9954	5.68-4	0.9964
	512	1.26-3	0.8298	2.13-4	1.0119	2.83-4	0.9975	2.84-4	0.9978
10^{-4}	32	3.52-3		1.31-3		1.42-3		1.46-3	
	64	2.12-3	0.7320	1.83-3	1.1259	7.13-4	0.9950	7.20-4	1.0247
	128	1.24-3	0.7754	2.82-4	1.0541	3.58-4	0.9928	3.59-4	0.9986
	256	7.07-4	0.8065	1.38-4	1.0195	1.80-4	0.9954	1.80-4	0.9968
	512	3.98-4	0.8298	6.83-5	1.0148	9.00-5	0.9975	9.00-5	0.9978
10^{-5}	32	1.12-3		4.16-4		4.71-4		5.40-4	
	64	6.70-4	0.7424	1.84-4	1.1259	2.27-4	1.0534	2.32-4	1.2184
	128	3.91-4	0.7762	8.92-5	1.0306	1.13-4	1.0000	1.14-4	1.0267
	256	2.24-4	0.8066	4.40-5	1.0369	5.69-5	0.9963	5.69-5	1.0003
	512	1.26-4	0.8298	2.16-5	1.0178	2.85-5	0.9976	2.85-5	0.9982

TABLE 2. Numerical results for bi-quadratic bases.

ϵ	Shishkin mesh			Implicit mesh			Explicit mesh		
	N	$\|e_N\|_\epsilon$	C. R.	$\widetilde{N}$	$\|e_{\widetilde{N}}\|_\epsilon$	C. R.	N	$\|e_N\|_\epsilon$	C. R.
10^{-3}	32	2.60-3		33	1.58-4		32	3.86-4	
	64	9.83-4	1.4047	65	4.75-5	2.1310	64	9.86-5	1.9685
	128	3.41-4	1.5277	128	1.15-5	2.0877	128	2.50-5	1.9818
	256	1.12-4	1.6049	255	2.85-6	2.0300	256	6.29-6	1.9900
	512	3.55-5	1.6569	513	6.97-7	2.0141	512	1.58-6	1.9947
10^{-4}	32	8.24-4		33	6.42-4		32	1.22-4	
	64	3.11-4	1.4047	62	1.72-5	2.0911	64	3.12-5	1.9697
	128	1.08-4	1.5277	126	3.90-5	2.0919	128	7.91-6	1.9820
	256	3.55-5	1.6049	255	9.31-7	2.0310	256	1.99-6	1.9901
	512	1.12-5	1.6569	511	2.28-7	2.0259	512	4.99-7	1.9947
10^{-5}	32	2.60-4		30	2.71-5		32	3.91-5	
	64	9.84-5	1.4047	62	5.43-6	2.2127	64	9.89-6	1.9818
	128	3.41-5	1.5277	128	1.23-6	2.0465	128	2.50-6	1.9834
	256	1.12-5	1.6049	256	2.94-7	2.0657	256	6.29-7	1.9902
	512	3.56-6	1.6569	512	7.20-8	2.0316	512	1.58-7	1.9948

We observe that for both piecewise bi-linear and bi-quadratic polynomial approximations, the numerical solutions that result from the implicit mesh and explicit mesh both assume the optimal convergence rate guaranteed by Theorem 3.2.

TABLE 3. Comparison of convergence orders for bi-cubic bases.

ϵ	Shishkin mesh			Implicit mesh			Explicit mesh		
	N	$\|e_N\|_\epsilon$	C. R.	$\widetilde{N}$	$\|e_{\widetilde{N}}\|_\epsilon$	C. R.	N	$\|e_N\|_\epsilon$	C. R.
10^{-3}	32	1.56-3		32	2.23-5		32	1.12-4	
	64	4.02-4	1.9555	65	2.13-6	3.3162	64	2.14-5	2.9151
	128	8.62-5	2.2203	125	2.76-7	3.1269	128	2.75-6	2.9617
	256	1.66-5	2.3799	256	2.99-8	3.1010	256	3.48-7	2.9820
	512	2.98-6	2.4759	513	3.62-9	3.0356	512	4.38-8	2.9913
10^{-4}	32	4.93-4		32	7.07-6		32	5.12-5	
	64	1.27-4	1.9554	67	6.74-7	3.1802	64	6.79-6	2.9150
	128	2.73-5	2.2203	126	8.72-8	3.2375	128	8.71-7	2.9617
	256	5.24-6	2.3779	253	1.02-8	3.0784	256	1.10-7	2.9820
	512	9.42-7	2.4759	513	1.19-9	3.0422	512	1.39-8	2.9913
10^{-5}	32	1.56-4		32	2.24-6		32	1.62-5	
	64	4.02-5	1.9554	67	2.13-7	3.1802	64	2.15-6	2.9150
	128	8.63-6	2.2203	128	2.76-8	3.1588	128	2.76-7	2.9617
	256	1.66-6	2.3799	255	3.24-9	3.1099	256	3.49-8	2.9820
	512	2.98-7	2.4759	516	3.77-10	3.0677	512	4.38-9	2.9913

Example 2: In this example, we solve equation (1) with $a(x, y) := 1 + xy$. In this case, we choose f so that

$$u(x, y) := (1 - e^{-x/\epsilon})(1 - e^{-(1-x)/\epsilon})(1 - e^{-y/\epsilon})(1 - e^{-(1-y)/\epsilon}), \quad (x, y) \in \Omega$$

is the exact solution of the equation under consideration. In this experiment, we use the piecewise bi-cubic polynomial space as the approximate subspace. As shown in Table 3, the numerical solutions corresponding to implicit and explicit meshes both have the optimal convergence rate and outperform the solution corresponding to the Shishkin mesh.

In Tables 1, 2 and 3, we include a comparison of the proposed meshes with the well-known Shishkin mesh. The numerical results show that both implicit and explicit meshes outperform the Shishkin mesh and even more so for high-order elements. This is because the proposed graded mesh is chosen according to the principle that the errors of the approximate solution on the subintervals of the resulting mesh should be equal. As a result, the proposed graded mesh gives an optimal convergence order. While the Shishkin mesh is a piecewise uniform mesh with suited transition points and it gives a convergence order optimal up to a power of the logarithmic factor (cf. [40]). In other words, the Shishkin mesh does not give the optimal convergence order.

5. Conclusive Remarks

The high-order Galerkin methods for solving the two-dimensional reaction diffusion problem on a rectangle are proposed based on the graded meshes (implicit and explicit) which are designed according to the upper bound estimates of the high order partial derivatives of its exact solution. These methods have the optimal order of convergence and easy to implement. Numerical results confirm the theoretical estimate and show that they outperform the Shishkin method whose convergence order is known not optimal.

Appendix A. Proofs of Propositions 2.8 and 2.9

In this appendix we present proofs of Propositions 2.8 and 2.9.

Proof of Proposition 2.8:

We prove the proposition by induction on $m+n$. From (13) we obtain that

$$\left|g^{[2,0]}_{x^p y^q}(x,y)\right| \leq 2\Theta\epsilon^{-(p+2)}\alpha^p(1+\sigma)^{-p}\kappa(x), \quad (x,y)\in\overline{\Omega}.$$

Then it follows from (12) and (2) for $(x,y)\in\overline{\Omega}$ that

$$\begin{aligned}\left|F^{[2,0]}_{x^p y^q}(x,y)\right| \leq \quad & \Theta + 2\Theta\epsilon^{-(p+2)}\alpha^p(1+\sigma)^{-p}\kappa(x) \\ & +2\Theta\alpha^p(1+\sigma)^{-p}\epsilon^2(\epsilon^{-(p+2)}+\epsilon^{-p})\kappa(x) \\ & +2^{p+q+1}\Theta c(1+\epsilon^{-(p+1)}\kappa(x))(1+\epsilon^{-q}\kappa(y)).\end{aligned}$$

Thus, (23) holds with $\theta^{[2,0]}_{p,q} \geq \Theta(4\alpha^p + 2^{p+q+1}c)$ and $c := \max\{c_{m,n}, m\leq q+1, n\leq q\}$. Similarly, we could prove that

$$\left|\tilde{g}^{[0,2]}_{x^p y^q}(x,y)\right| \leq 2\Theta\epsilon^{-(q+2)}\alpha^q(1+\sigma)^{-q}\kappa(x), \quad (x,y)\in\overline{\Omega},$$

which leads to (23) for $F^{[0,2]}_{x^p y^q}$.

Now we assume the theorem holds for $m+n<k$. Given (i,j) with $m+n=k$, for all $(x,y)\in\overline{\Omega}$,

$$\begin{aligned}\left|g^{[2m,2n]}_{x^py^q}(x,y)\right| &\le 2\Theta\epsilon^{-(p+2)}\alpha^p(1+\sigma)^{-p}\kappa(x)\|F^{[2m-2,2n]}_{y^q}\|_\infty\\ &\le 2\Theta\theta^{[2m-2,2n]}_{0,q}\alpha^p(1+\sigma)^{-p}\kappa(x)\epsilon^{-(p+2)}\\ &\quad\times(1+\epsilon^{-(2m-2)}\kappa(x))(1+\epsilon^{-(2n+q)}\kappa(y)).\end{aligned}$$

Noting that

$$\kappa(x)\epsilon^{-(p+2)}(1+\epsilon^{-(2m-2)}\kappa(x))\le 3\epsilon^{-(p+2m)}\kappa(x)\quad\text{and}\quad \vartheta^{[2m,2n]}_{p,q}\ge 6\Theta\alpha^p\theta^{[2m-2,2n]}_{0,q},$$

we establish (21). It follows from the definition of $G^{[2m,2n]}$ that (24) holds with

$$\ell^{[2m,2n]}_{p,q}\ge(m+n)\cdot\max\{\vartheta^{[2m',2n]}_{2m-2m'+p,q},\tilde{\vartheta}^{[0,2n]}_{2m+p,2n-2n'+q}:1\le m'\le m,1\le n'\le n\}.$$

When $m>0$, we use (12) to conclude that $\left|F^{[2m,2n]}_{x^py^q}(x,y)\right|$ is bounded by

$$\begin{aligned}&\theta^{[2m-2,2n]}_{p+2,q}(1+\epsilon^{-(2m+p)}\kappa(x))(1+\epsilon^{-(2n+q)}\kappa(y)\\ &+2^{p+q}\Theta\vartheta^{[2m,2n]}_{p,q}\epsilon^{-(2m+p)}\kappa(x)(1+\epsilon^{-(2n+q)}\kappa(y))\\ &+\epsilon^{-(2m+p)}\kappa(x)[\vartheta^{[2m,2n]}_{p+2,q}(1+\epsilon^{-(2n+q)}\kappa(y))+\vartheta^{[2m,2n]}_{p,q+2}\epsilon^2(1+\epsilon^{-(2n+q+2)}\kappa(y))]\\ &+2^{p+q+1}\Theta(2c_{2m+p-1,2n+q}+\ell^{[2m-2,2n]}_{p,q}+\ell^{[2m-2,2n]}_{p+1,q})\\ &\times(1+\epsilon^{-(2m+p-1)}\kappa(x))(1+\epsilon^{-(2n+q)}\kappa(y))\end{aligned}$$

for $(x,y)\in\overline{\Omega}$. This implies that (23) holds for

$$\begin{aligned}\theta^{[2m,2n]}_{p,q}\ge\quad&\theta^{[2m-2,2n]}_{p+2,q}+2^{p+q}\Theta\vartheta^{[2m,2n]}_{p,q}+\vartheta^{[2m,2n]}_{p+2,q}+\vartheta^{[2m,2n]}_{p,q+2}\\ &+2^{p+q+1}\Theta(2c_{2m+p-1,2n+q}+\ell^{[2m-2,2n]}_{p,q}+\ell^{[2m-2,2n]}_{p+1,q}).\end{aligned}$$

In a similar manner, we may prove (22) with $\tilde{\vartheta}^{[2m,2n]}_{p,q}\ge 6\Theta\alpha^p\theta^{[2m,2n-2]}_{p,0}$, and make use of it to establish (23) for the case $m=0$.

The rest of the appendix will be devoted to the proof for Proposition 2.9. The proof will be done by considering three separate cases. As we will see, Proposition A.2 considers the cases $(p,q)=(0,1)$ and $(1,0)$, Proposition A.3 considers the cases $(p,q)=(0,2)$ and $(2,0)$, while Proposition A.6 is devoted to the case $(p,q)=(1,1)$.

In order to estimate $v^{[2m,2n]}_x$ and $v^{[2m,2n]}_y$, we need the following upper bounds for $v^{[2m,2n]}$. For technical consideration, we define the function $\varpi(t):=(1-\Upsilon(t))(1-\Upsilon(1-t))$, for $t\in[0,1]$.

Lemma A.1. *There exists a positive constant $\gamma\ge 2\alpha^{-2}(1+\sigma)^2\theta^{[2m,2n]}_{0,0}$, such that for all $(x,y)\in\overline{\Omega}$, there hold*

$$|v^{[2m,2n]}(x,y)|\le\gamma\epsilon^{-2m}\varpi(x)(1+\epsilon^{-2n}\kappa(y)),\tag{A.1}$$

and

$$|v^{[2m,2n]}(x,y)|\le\gamma\epsilon^{-2n}\varpi(y)(1+\epsilon^{-2m}\kappa(x)).\tag{A.2}$$

Proof. We present only the proof for (A.1), since the proof for (A.2) is similar.

Define the barrier function by

$$\varphi(x,y):=\gamma\epsilon^{-2m}\varpi(x)(1+\epsilon^{-2n}\kappa(y)),\quad (x,y)\in\overline{\Omega}.$$

Since $v^{[2m,2n]}(x,y)=0$, $(x,y)\in\partial\Omega$ and φ is a nonnegative function, we have that

$$|v^{[2m,2n]}(x,y)|\le\varphi(x,y),\quad (x,y)\in\partial\Omega.$$

On the other hand, we derive for $(x,y)\in\Omega$ that

$$\begin{aligned} L\varphi(x,y) &= \left[a(x,y)-2\frac{\alpha^2}{(1+\sigma)^2}\right]\varphi(x,y)+\frac{\gamma\alpha^2}{(1+\sigma)^2}\epsilon^{-2m}\varpi(x) \\ &\quad +\frac{\gamma\alpha^2}{(1+\sigma)^2}\epsilon^{-2m}(1+\Upsilon(1))(1+\epsilon^{-2n}\kappa(y)). \end{aligned}$$

Since the first and second terms of the right hand side are nonnegative, we have that

$$L\varphi(x,y)\ge\frac{\gamma\alpha^2}{(1+\sigma)^2}\epsilon^{-2m}(1+\Upsilon(1))(1+\epsilon^{-2n}\kappa(y)),\quad (x,y)\in\Omega.$$

Note that $1+\Upsilon(1)\ge\kappa(x)$, so that

$$\epsilon^{-2m}(1+\Upsilon(1))\ge\frac{1}{2}(\epsilon^{-2m}(1+\Upsilon(1))+\epsilon^{-2m}\kappa(x))\ge\frac{1}{2}(1+\epsilon^{-2m}\kappa(x))$$

for $x\in[0,1]$. Thus

$$L\varphi(x,y)\ge\frac{\gamma\alpha^2}{2(1+\sigma)^2}(1+\epsilon^{-2m}\kappa(x))(1+\epsilon^{-2n}\kappa(y)),\quad (x,y)\in\Omega.$$

Therefore,

$$L\varphi(x,y)\ge|F^{[2m,2n]}(x,y)|=|Lv^{[2m,2n]}(x,y)|,\quad\text{for}\quad (x,y)\in\Omega$$

when $\frac{\gamma\alpha^2}{2(1+\sigma)^2}\ge\theta_{0,0}^{[2m,2n]}$. This establishes (A.1). □

Based on the above lemma, we establish the following result.

Proposition A.2. *For any $(x,y)\in\partial\Omega$, there hold*

$$|v_x^{[2m,2n]}(x,y)|\le c_{1,0}^{[2m,2n]}\phi_{2m+1,2n}(x,y) \tag{A.3}$$

and

$$|v_y^{[2m,2n]}(x,y)|\le c_{0,1}^{[2m,2n]}\phi_{2m,2n+1}(x,y), \tag{A.4}$$

where $c_{p,q}^{[2m,2n]}\ge 2\alpha^{-1}(1+\sigma)\theta_{0,0}^{[2m,2n]}$ with $(p,q)=(0,1)$ and $(1,0)$.

Proof. Since $v^{[2m,2n]}$ vanishes on $\partial\Omega$, we have that $v_x^{[2m,2n]}(x,0)=v_x^{[2m,2n]}(x,1)=0$, $x\in[0,1]$. It follows from (A.1) that for any $y\in[0,1]$,

$$\begin{aligned} |v_x^{[2m,2n]}(0,y)| &= \lim_{x\to0^+}\tfrac{|v^{[2m,2n]}(x,y)-v^{[2m,2n]}(0,y)|}{x-0} \\ &\le \tfrac{\gamma\alpha}{1+\sigma}\epsilon^{-(2m+1)}(1+\epsilon^{-2n}\kappa(y)). \end{aligned}$$

Likewise, we have that

$$|v_x^{[2m,2n]}(1,y)|\le\frac{\gamma\alpha}{1+\sigma}\epsilon^{-(2m+1)}(1+\epsilon^{-2n}\kappa(y)),\quad y\in[0,1].$$

Thus, we conclude (A.3) from Lemma A.1. In a similar manner, we may prove (A.4) by (A.2). □

Upper bounds of $v_{x^2}^{[2m,2n]}$ and $v_{y^2}^{[2m,2n]}$ are given in the next proposition.

Proposition A.3. *For all $(x, y) \in \partial\Omega$, there hold*

$$|v_{x^2}^{[2m,2n]}(x, y)| \le c_{2,0}^{[2m,2n]} \phi_{2m+2,2n}(x, y) \tag{A.5}$$

and

$$|v_{y^2}^{[2m,2n]}(x, y)| \le c_{0,2}^{[2m,2n]} \phi_{2m,2n+2}(x, y), \tag{A.6}$$

where $c_{p,q}^{[2m,2n]} \ge 2\theta_{0,0}^{[2m,2n]}$ with $(p, q) = (2, 0)$ and $(0, 2)$.

Proof. We only prove (A.5) because the other case can be done similarly. Since $v^{[2m,2n]}$ vanishes on $\partial\Omega$, we have that $v_{x^2}^{[2m,2n]}(x, 0) = v_{x^2}^{[2m,2n]}(x, 1) = 0$, $x \in [0, 1]$. Moreover, we obtain directly from (19) that for $y \in [0, 1]$,

$$v_{x^2}^{[2m,2n]}(0, y) = -\epsilon^{-2} F^{[2m,2n]}(0, y), \quad v_{x^2}^{[2m,2n]}(1, y) = -\epsilon^{-2} F^{[2m,2n]}(1, y).$$

Hence, for $y \in [0, 1]$,

$$\begin{aligned} |v_{x^2}^{[2m,2n]}(0, y)| &\le \theta_{0,0}^{[2m,2n]} \epsilon^{-2}(1 + \epsilon^{-2m}(1 + \Upsilon(1)))(1 + \epsilon^{-2n}\kappa(y)) \\ &\le 2\theta_{0,0}^{[2m,2n]}(1 + \epsilon^{-(2m+2)}(1 + \Upsilon(1)))(1 + \epsilon^{-2n}\kappa(y)), \end{aligned}$$

since

$$\epsilon^{-2}(1 + \epsilon^{-2m}(1 + \Upsilon(1))) \le 2\epsilon^{-(2m+2)}(1 + \Upsilon(1)) < 2 + 2\epsilon^{-(2m+2)}(1 + \Upsilon(1)).$$

Therefore, for all $y \in [0, 1]$ we have that

$$|v_{x^2}^{[2m,2n]}(0, y)| \le c_{2,0}^{[2m,2n]} \phi_{2m+2,2n}(0, y)$$

when $c_{2,0}^{[2m,2n]} \ge 2\theta_{0,0}^{[2m,2n]}$. Similarly, we have that

$$|v_{x^2}^{[2m,2n]}(1, y)| \le c_{2,0}^{[2m,2n]} \phi_{2m+2,2n}(1, y), \quad y \in [0, 1].$$

Thus, (A.5) is established. □

In the last part of this appendix, we estimate $v_{xy}^{[2m,2n]}$. To this end, we need two auxiliary lemmas.

Lemma A.4. *If $\rho \ge \frac{8}{3}\Theta\alpha^{-3}(1 + \sigma)\epsilon$, then there exists a positive constant*

$$\gamma_0 \ge 8\alpha^{-2}(1 + \sigma)^2 \max\{\theta_{0,0}^{[2m,2n]}, 4\alpha^{-1}(1 + \sigma)\theta_{0,1}^{[2m,2n]}, 4\alpha^{-1}(1 + \sigma)\theta_{1,0}^{[2m,2n]}\},$$

such that for all $(x, y) \in \overline{\Omega}$,

$$|v^{[2m,2n]}(x, y)| \le \gamma_0 \epsilon^{-(2m+2n)} \varpi(x)\varpi(y).$$

Proof. Clearly, the above inequality holds on $\partial\Omega$. We only need to verify that $|F^{[2m,2n]}(x, y)| \le L\varphi_0(x, y)$ in Ω with

$$\varphi_0(x, y) := \gamma_0 \epsilon^{-(2m+2n)} \varpi(x)\varpi(y), \qquad (x, y) \in \overline{\Omega}.$$

From direct calculation we have for $(x, y) \in \Omega$ that

$$\begin{aligned} L\varphi_0(x, y) = {} & \gamma_0 \left[a(x, y) - \tfrac{2\alpha^2}{(1+\sigma)^2}\right] \epsilon^{-(2m+2n)} \varpi(x)\varpi(y) \\ & + \tfrac{\gamma_0\alpha^2}{(1+\sigma)^2} \epsilon^{-(2m+2n)} (1 + \Upsilon(1)) \left(\varpi(x) + \varpi(y)\right). \end{aligned} \tag{A.7}$$

First of all, we compare $L\varphi_0$ with $F^{[2m,2n]}$ in $\Omega_{(0,0)} := \{(x,y) : x, y \in (0,\tilde{\epsilon})\}$, the left-bottom corner of Ω, in which $\tilde{\epsilon} := (1+\sigma)\epsilon/\alpha$. To this end, we let $\psi := L\varphi_0$, and observe that for $(x,y) \in \Omega$,

(A.8)

$$\begin{aligned}\psi_x(x,y) = \quad & \gamma_0 \left[a(x,y) - \tfrac{2\alpha^2}{(1+\sigma)^2}\right] \tfrac{\alpha}{(1+\sigma)} \epsilon^{-(2m+2n+1)} (\Upsilon(x) - \Upsilon(1-x)) \varpi(y) \\ & + \gamma_0 a_x(x,y) \epsilon^{-(2m+2n)} \varpi(x) \varpi(y) \\ & + \gamma_0 (\tfrac{\alpha}{1+\sigma})^3 \epsilon^{-(2m+2n+1)} (1+\Upsilon(1))(\Upsilon(x) - \Upsilon(1-x)).\end{aligned}$$

Since ϵ is very small, without loss of generality we assume that $\tilde{\epsilon} < 1/4$. Thus,

$$\Upsilon(x) - \Upsilon(1-x) > e^{-1} - e^{-3} > \frac{1}{4}, \quad \text{for} \quad x \in (0,\tilde{\epsilon}),$$

and then

$$\left[a(x,y) - \frac{2\alpha^2}{(1+\sigma)^2}\right] \frac{\alpha}{(1+\sigma)} (\Upsilon(x) - \Upsilon(1-x)) > \frac{\rho\alpha^3}{4(1+\sigma)\epsilon}.$$

On the other hand, for all $x \in (0,\tilde{\epsilon})$ $\varpi(x) \le 1 - \Upsilon(x) \le 1 - \Upsilon(\tilde{\epsilon}) < \frac{2}{3}$. Therefore, when $\rho \ge \frac{8}{3}\frac{(1+\sigma)\epsilon}{\alpha^3}\Theta$, the sum of the first two terms of the right hand side of (A.8) is nonnegative. Consequently, we have that for $(x,y) \in \Omega_{0,0}$,

$$\begin{aligned}\psi_x(x,y) \quad & \ge \gamma_0 \left(\tfrac{\alpha}{1+\sigma}\right)^3 \epsilon^{-(2m+2n+1)} (1+\Upsilon(1))(\Upsilon(x) - \Upsilon(1-x)) \\ & \ge \tfrac{\gamma_0}{4} \left(\tfrac{\alpha}{1+\sigma}\right)^3 \epsilon^{-(2m+2n+1)} (1+\Upsilon(1)).\end{aligned}$$

Note that $\kappa(t) \le 1 + \Upsilon(1) \le 2$ and $\frac{1}{2}(1+\epsilon^{-k}\kappa(t)) \le \epsilon^{-k}(1+\Upsilon(1))$ for $t \in [0,1]$ and $k \ge 0$. Thus,

$$\psi_x(x,y) \ge \frac{\gamma_0}{32} \left(\frac{\alpha}{1+\sigma}\right)^3 (1+\epsilon^{-(2m+1)}\kappa(x))(1+\epsilon^{-2n}\kappa(y)), \quad (x,y) \in \Omega_{(0,0)}.$$

Therefore, when

$$\gamma_0 \ge 32 \left(\frac{1+\sigma}{\alpha}\right)^3 \theta_{1,0}^{[2m,2n]},$$

we have that $\psi_x(x,y) \ge \left|F_x^{[2m,2n]}(x,y)\right|$ in $\Omega_{(0,0)}$. Similarly, $\psi_y(x,y) \ge \left|F_y^{[2m,2n]}(x,y)\right|$ with

$$\gamma_0 \ge 32 \left(\frac{1+\sigma}{\alpha}\right)^3 \theta_{0,1}^{[2m,2n]}.$$

Note that $\psi(0,0) = F^{[2m,2n]}(0,0) = 0$, we conclude that $\psi(x,y) \ge |F^{[2m,2n]}(x,y)|$ in the whole $\Omega_{(0,0)}$. The above process for comparing ψ and $F^{[2m,2n]}$ can be adopted for the other three corners of Ω, namely,

$$\Omega_{(0,1)} := \{(x,y) : x \in (0,\tilde{\epsilon}), y \in (1-\tilde{\epsilon},1)\},$$

$$\Omega_{(1,0)} := \{(x,y) : x \in (1-\tilde{\epsilon},1), y \in (0,\tilde{\epsilon})\},$$

and

$$\Omega_{(1,1)} := \{(x,y) : x, y \in (1-\tilde{\epsilon},1)\}.$$

To summarize, we have proved that $L\varphi_0(x,y) = \psi(x,y) \ge \left|F^{[2m,2n]}(x,y)\right|$ at the four corners of Ω when

$$\gamma_0 \ge 32 \left(\frac{1+\sigma}{\alpha}\right)^3 \max\{\theta_{0,1}^{[2m,2n]}, \theta_{1,0}^{[2m,2n]}\}.$$

Next, we turn to considering $\Omega^c := \Omega \setminus \cup_{i,j=0}^{1} \Omega_{(i,j)}$. In Ω^c, either x or y lies in the interval $[\tilde{\epsilon}, 1-\tilde{\epsilon}]$. By symmetry, we consider only the case $x \in [\tilde{\epsilon}, 1-\tilde{\epsilon}]$. For $(x,y) \in \Omega^c$, we obtain from (A.7) that

$$L\varphi_0(x,y) \geq \frac{\gamma_0 \alpha^2}{(1+\sigma)^2} \epsilon^{-(2m+2n)} (1+\Upsilon(1)) \left[\varpi(x) + \varpi(y)\right].$$

For $t \in [\tilde{\epsilon}, 1-\tilde{\epsilon}]$, there holds $\varpi(t) \geq \varpi(\tilde{\epsilon}) = (1-e^{-1})(1-e^{1-1/\tilde{\epsilon}})$. It is not difficult to verify $1-e^{-1} \geq 0.6$ and $1-e^{1-1/\tilde{\epsilon}} \geq 1-e^{-3} \geq 5/6$. Hence, $\varpi(t) \geq 1/2$, so that

$$L\varphi_0(x,y) \geq \frac{\gamma_0 \alpha^2}{8(1+\sigma)^2} (1+\epsilon^{-2m}\kappa(x))(1+\epsilon^{-2n}\kappa(y)), \quad (x,y) \in \Omega^c.$$

Therefore, when

$$\gamma_0 \geq 8 \left(\frac{1+\sigma}{\alpha}\right)^2 \theta_{0,0}^{[2m,2n]},$$

we have that $L\varphi_0(x,y) \geq \left|F^{[2m,2n]}(x,y)\right|$ in Ω^c. The proof is completed by combining the above two parts. □

Lemma A.5. *There exists a positive constant*

$$\gamma_1 \geq \max\{\gamma_0 \alpha (1+\sigma)^{-1}, 2\alpha^{-2}(1+\sigma)^2(\theta_{0,1}^{[2m,2n]} + \theta_{0,1}^{[2m,2n]} + c_{0,0}^{[2m,2n]}\Theta)\},$$

such that for any $(x,y) \in \overline{\Omega}$,

$$|v_x^{[2m,2n]}(x,y)| \leq \gamma_1 \epsilon^{-2n} \varpi(y)(1+\epsilon^{-(2m+1)}\kappa(x)), \tag{A.9}$$

and

$$|v_y^{[2m,2n]}(x,y)| \leq \gamma_1 \epsilon^{-2m} \varpi(x)(1+\epsilon^{-(2n+1)}\kappa(y)), \tag{A.10}$$

Proof. We use the trick used in the proof of Proposition A.2 to establish the upper bound. Specifically, we conclude that $v_x^{[2m,2n]}(x,0) = v_x^{[2m,2n]}(x,1) = 0$ for $x \in [0,1]$. It follows from Lemma A.4 that

$$\left|v_x^{[2m,2n]}(0,y)\right| = \lim_{x\to 0^+} \frac{|v^{[2m,2n]}(x,y) - v^{[2m,2n]}(0,y)|}{x-0} \leq \frac{\gamma_0\alpha}{1+\sigma} \epsilon^{-(2m+2n+1)} \varpi(y),$$

and

$$|v_x^{[2m,2n]}(1,y)| \leq \frac{\gamma_0\alpha}{1+\sigma} \epsilon^{-(2m+2n+1)} \varpi(y)$$

for $y \in [0,1]$. Let

$$\varphi_1(x,y) := \gamma_1 \epsilon^{-2n} \varpi(y)(1+\epsilon^{-(2m+1)}\kappa(x)), \quad (x,y) \in \overline{\Omega}.$$

Then, φ_1 is nonnegative, and

$$\varphi_1(0,y) = \varphi_1(1,y) \geq \gamma_1 \epsilon^{-(2m+2n+1)} \varpi(y), \quad y \in [0,1].$$

Therefore, when $\gamma_1 \geq \frac{\gamma_0\alpha}{1+\sigma}$, we have that $|v_x^{[2m,2n]}(x,y)| \leq \varphi_1(x,y)$ on $\partial\Omega$.

Note that $Lv_x^{[2m,2n]} = F_x^{[2m,2n]} - a_x v^{[2m,2n]}$. Making use of the upper bound

$$|v^{[2m,2n]}(x,y)| \leq c_{0,0}^{[2m,2n]}(1+\epsilon^{-2m}\kappa(x))(1+\epsilon^{-2n}\kappa(x)), \quad (x,y) \in \overline{\Omega},$$

we have that for $(x,y) \in \Omega$

$$|Lv_x^{[2m,2n]}(x,y)| \leq (\theta_{1,0}^{[2m,2n]} + \Theta c_{0,0}^{[2m,2n]})(1+\epsilon^{-(2m+1)}\kappa(x))(1+\epsilon^{-2n}\kappa(x)).$$

By a direct computation, we observe that for $(x,y)\in\Omega$,

$$L\varphi_1(x,y) = \left[a(x,y)-2\frac{\alpha^2}{(1+\sigma)^2}\right]\varphi_1(x,y)+\frac{\gamma_1\alpha^2}{(1+\sigma)^2}\epsilon^{-2n}\varpi(y) + \frac{\gamma_1\alpha^2}{(1+\sigma)^2}\epsilon^{-2n}(1+\Upsilon(1))(1+\epsilon^{-(2m+1)}\kappa(x))$$

Using the trick used in proof of Lemma A.1, we obtain that

$$L\varphi_1(x,y)\geq \frac{\gamma_1\alpha^2}{2(1+\sigma)^2}(1+\epsilon^{-(2m+1)}\kappa(x))(1+\epsilon^{-2n}\kappa(y)),\quad (x,y)\in\Omega.$$

Therefore, there holds $|Lv_x^{[2m,2n]}(x,y)|\leq L\varphi_1(x,y)$ in Ω with

$$\gamma_1\geq 2\alpha^{-2}(1+\sigma)^2(\theta_{1,0}^{[2m,2n]}+c_{0,0}^{[2m,2n]}\Theta).$$

Finally, we establish (A.9) by the comparison principle. The inequality (A.10) can be proved in a similar way. □

With the estimates of the above two lemmas, we are now ready to establish the upper bound of $v_{xy}^{[2m,2n]}$.

Proposition A.6. *If $\rho\geq\frac{8}{3}\Theta\alpha^{-3}(1+\sigma)\epsilon$, then for all $(x,y)\in\partial\Omega$, there holds*

$$|v_{xy}^{[2m,2n]}(x,y)|\leq c_{1,1}^{[2m,2n]}\phi_{2m+1,2n+1}(x,y) \tag{A.11}$$

with

$$c_{1,1}^{[2m,2n]}\geq\max\{8\theta_{0,0}^{[2m,2n]},32\alpha^{-1}(1+\sigma)(\theta_{0,1}^{[2m,2n]}+\theta_{1,0}^{[2m,2n]}+c_{0,0}^{[2m,2n]}\Theta)\}.$$

Proof. It follows from the boundary condition of (19) that $v_x^{[2m,2n]}(x,0)=0$ for $x\in[0,1]$. Thus,

$$|v_{xy}^{[2m,2n]}(x,0)|=\lim_{y\to0^+}\frac{|v_x^{[2m,2n]}(x,y)-v_x^{[2m,2n]}(x,0)|}{|y-0|}=\lim_{y\to0^+}\frac{|v_x^{[2m,2n]}(x,y)|}{|y|}.$$

Therefore, we obtain from (A.9) that

$$|v_{xy}^{[2m,2n]}(x,0)|\leq\frac{\gamma_1\alpha}{1+\sigma}\epsilon^{-(2n+1)}(1+\epsilon^{-(2m+1)}\kappa(x)),\quad x\in[0,1].$$

The inequalities on the other three sides can be proved in a similar way. □

Since Proposition A.2, A.3 and A.6 consider all cases of nonnegative integer pairs (p,q) satisfying $0<p+q\leq 2$, we combine the three propositions to establish Proposition 2.9.

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