INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 13, Number 2, Pages 310–317

FINITE VOLUME METHOD ON HYBRID MESHES FOR COASTAL OCEAN MODEL

ZHIGANG LAI, BIN WU, AND QINGSONG ZOU

Abstract. In this paper we design a modified version of FVCOM by adopting hybrid meshes and shifting the placement of velocity variables from the centroids of elements to the middle points of edges. A simplified version of geostrophic equation is solved to test the new scheme, and illustrates a nearly uniform error distribution.

Key words. FVCOM, finite volume methods, hybrid meshes.

1. Introduction

The present study is a step toward the formulation of an unstructured-grid, primitive equation, 3D ocean model which features horizontal mixed triangular-quadrilateral meshes and vertically hybrid coordinate that is flexible in ocean applications with scales crossing river-estuary-shelf-basin-global. The model framework is partially adopted from the Finite Volume Coastal Ocean Model (FVCOM) which is an unstructured-triangular-grid open-source community ocean model and has been successfully applied to many scientific and engineering problems [2, 3, 4, 5].

The model development starts with the aim of modifying FVCOM to support a mixture of triangular and quadrilateral elements. Since at least two triangles are required to match the same area of a quadrilateral element, this modification allows the model domain be partitioned with an optimal number of non-overlapping elements for given grid resolution and could significantly relieve the computational cost of FVCOM, especially in high-resolution numerical modelling. Several potential benefits of using hybrid meshes in ocean modelling were also considered which includes but not limited to: 1) flexibility in either h (mesh size) or p (polynomial order) type of model refinement; 2) easily implementing velocity radiation boundary condition which may be a difficult task in an unstructured-triangular-grid model; 3) nesting an unstructured- and structured-grid model with the exactly matched boundary cells to reduce numerical instabilities due to interpolation between two meshes; 4) better numerical performance by reducing certain mesh shape related instability issues. These motivate us to examine the possibility of migrating the discretization method of FVCOM to the new hybrid mesh and try to understand its numerical properties and identify the potential aspects for future improvement.

In addition, the numerical design of FVCOM is based on finite volume methods in which the velocity variables (u, v) are placed at the centroid of each triangle

Received by the editors March 15, 2015.

2000 *Mathematics Subject Classification.* 65M08,65D05,65D30.

The first author was partially supported by the Natural Science Foundation of China under grant 41206005. The second author was partially supported by the National Natural Science Foundation of China through grants 11401601 and 11501106. The third author was partially supported by the National Natural Science Foundation of China through grants 11571384 and 11428103, and by Guangdong Provincial Natural Science Foundation of China under grant through grant 2014A030313179.

while the pressure variables are at three vertices. This placement of variables simplifies the model by avoiding the specification of boundary conditions for velocities. However, it also brings some numerical difficulties which have to be treated with special attentions. First, without the velocity information along the boundary curves, the velocity gradient normal to the solid wall cannot be expressed explicitly. Therefore, in FVCOM, ghost cells are used to mimic the viscous boundary layer properly. Second, the velocities close to the open boundary can only be determined by the specified pressure conditions, which in most cases is the surface elevations caused by tidal fluctuations. To consider non-tidal components of velocities such as wind-driven flows or western boundary currents at open boundaries, one has to use mean-flow option in FVCOM, which actually is not very useful for real applications.

To avoid the above mentioned issues, the velocity variables in the new model are to be placed at the middle points of edges of the elements. It should be noted that this placement of variables is different from the lowest-order Raviart-Thomas element [9] as no constraint on the velocity is needed, while the latter requires the normal velocity be specified at edges and this is essentially the triangular C-grids that are widely used in unstructured-grid ocean modelling [1, 7]. Actually, our variable placement is very similar to the non-conforming linear elements of velocities that were used in a finite element scheme for two-layer shallow water equations [8]. Combining with linear conforming elements of surface elevations, the resulting discrete scheme is free of two-grid oscillations and appears suitable for coastal semi-enclosed basins circulation problems [8].

While the mesh is constructed in different way, and the placement of velocity and pressure variables changes, the numerical scheme of finite volume discretization changes accordingly. In this paper a FVCOM-like low order finite volume approach is described to resolve the velocities based on a mixed triangular-quadrilateral mesh. A simple geostrophic problem that determines the large-scale, steady-state ocean circulation is used to validate the model, discuss its numerical properties and identify the potential aspects for future improvement.

The remainder of this paper is organized as follows. In section 2, the numerical discretization of the geostrophic problem is introduced. In section 3, a numerical experiment is presented, followed by a detailed description of the results. The major finding and conclusions are summarized in section 4.

2. The new scheme with numerical treatments

Under the Boussinesq approximation, the incompressible, hydrostatic Navier-Stokes equation for the momentum in the model is

$$\frac{dU}{dt} + fk \times U = -g\nabla\eta - \frac{1}{\rho_0}\nabla B + \nabla \cdot A_h \nabla U + \partial_z A_v \partial_z U, \tag{1}$$

where $U = (u, v)$ represents the horizontal velocity components, ρ_0 is the reference density, B is the baroclinic pressure obtained through integrating the hydrostatic relation $\partial_z p = -g\rho$ from $z = 0$ with in-situ density ρ, g is the gravitational acceleration, η is the sea surface elevation, f is the Coriolis parameter, k is the vertical unit vector, A_h and A_v are lateral and vertical turbulent eddy viscosity respectively, and ∇ stands for 2D gradient or divergence operators.

The geostrophic equation is the lowest order of equation (1) for rapidly rotating fluids in the ocean interior with ignorable frictional effects and over large (> 100 km) spatial and long (> 2 days) temporal scales [6]. Recently, an unstructured grid, finite volume, 3D primitive equation, turbulent closure coastal ocean model is developed by Chen et al. [2, 3, 4, 5], which is called Finite Volume Coastal Ocean

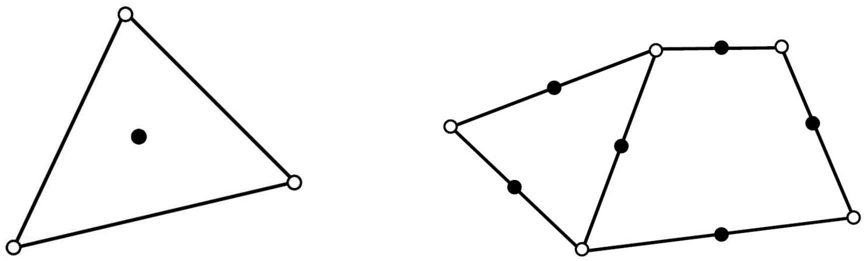

FIGURE 1. The placement of velocity and pressure variables in FVCOM (left) and our scheme (right).

Model (FVCOM). It is a prognostic model, which uses a transformation in the vertical to convert irregular bottom topography into a rectangular computational domain for a simple numerical approach. FVCOM is composed of external and internal modes that are computed separately using two split steps, and is solved numerically by the flux calculation in the integral form of primitive equations over non-overlapping, unstructured triangular grids. This numerical approach combines the best of the finite element method for accurate coastal geometric fit and the finite difference method for simple structure of the model code and computational efficiency. In addition, the flux calculation method with an integral form of equations provides a better representation of momentum, mass, salt, and heat conservation.

In FVCOM, the velocity variables (u, v) are placed at the centroid of each triangle element, while pressure variables are at the vertices. This setting can avoid the specification of boundary conditions for velocities, but brings some numerical difficulties at the same time, as we have described in the previous section. Different from FVCOM, we construct the mesh by using both triangular and quadrilateral elements. In the region away from the boundary, the use of quadrilateral elements provides us more flexibility in numerical treatments. Moreover, we place the velocity variables at the middle points of the edges. Fig. 1 makes comparison for the placement of velocity and pressure variables in FVCOM and our scheme, in which the circles indicate the positions of pressure variables and the points indicate the velocity variables. A typical local snapshot of our mesh is shown in Fig. 2.

In order to illustrate our numerical treatments, a simplified version of the geostrophic equation for homogeneous (without density variation) fluids is considered, which is written as

$$-fv = -g\frac{\partial \eta}{\partial x}, \quad fu = -g\frac{\partial \eta}{\partial y}, \tag{2}$$

where x, y are Cartesian coordinates in eastward and northward directions, u and v are velocity components in x and y direction respectively, g is the gravitational acceleration and η is the surface elevation (or pressure) with respect to the reference level $z = 0$. The Coriolis parameter, f, is the component of the earth's rotation perpendicular to the ocean surface and set to be constant, the value of which is specified according to the center latitude of a local region.

Integrating both sides of (2) on a typical cell Ω, we obtain

$$\begin{cases} \displaystyle\iint_\Omega fv\,dxdy = \iint_\Omega g\frac{\partial \eta}{\partial x}dxdy, \\ \displaystyle\iint_\Omega fu\,dxdy = -\iint_\Omega g\frac{\partial \eta}{\partial y}dxdy. \end{cases}$$

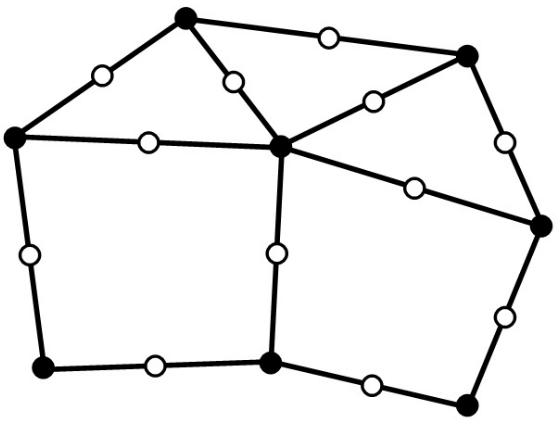

FIGURE 2. The triangular and quadrilateral elements together with the placement of velocity and pressure variables.

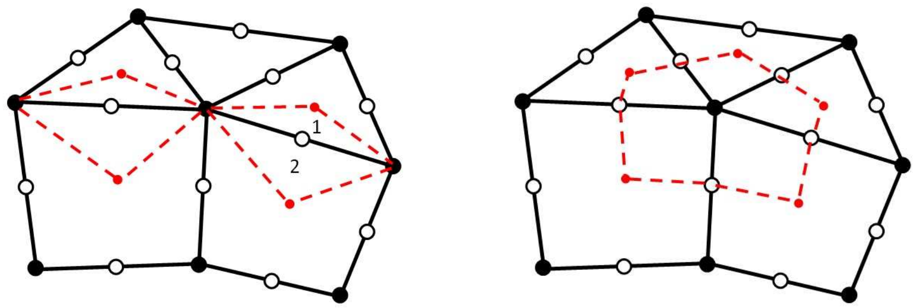

FIGURE 3. The control cells associated with velocity (left) and pressure (right).

The region of Ω is quadrilateral, which is created by locating the centroids of triangular and quadrilateral elements, and connecting the centroids with vertices, as shown in Fig. 3. The right hand side of the equations can be transformed into boundary integral. For example,

$$\iint_{\Omega} g\frac{\partial \eta}{\partial x}dxdy = \oint_{\partial\Omega} g\eta dy.$$

Note that the boundary of Ω is composed of four line segments. On each linear segment, the trapezoid rule can be applied for the calculation of integral, as long as the values of η on both end points are known. Assume that the values of η at all vertices are given. It is remained to specify the values at the centroids. In each triangle element, the value of η at the centroid can be approximately found by constructing linear interpolation with respect to the values at the three vertices and then evaluating it at the centroid. In this way, the numerical error for computing $\oint_{\partial\Omega} g\eta dy$ is $O(h^3)$, where h is the maximum diameter of the elements. For quadrilateral elements, the value of η at the centroid can be approximated by bilinear interpolation.

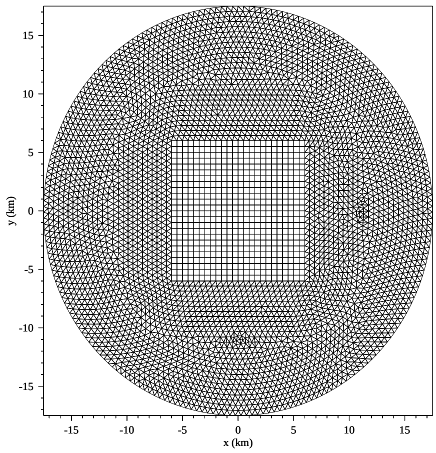

FIGURE 4. The hybrid mesh for circular ocean basin.

There are two ways to compute $\iint_\Omega fvdxdy$. The first way is approximating the integrand by constant function and then integrating, i.e.,

$$\iint_\Omega fvdxdy \approx \mathbf{fv}|\Omega|,$$

in which $|\Omega|$ stands for the area of Ω. Notice that on the right hand side the symbols $\mathbf{f}$ and $\mathbf{v}$ both represent the function values of f and v at some specified point. It is easily seen that the error of this rule is $O(h^3)$. The second way is finding an appropriate linear polynomial to approximate the function v in the triangles "1" and "2", as shown in Fig. 3. Notice that the triangle "1" is located in a triangle element and there are three control points of velocity variables which are located at the middle points of the three edges of the element. It is natural to construct a linear interpolating polynomial and then integrate it in triangle "1". The integral on triangle "2" is calculated similarly. The difference is that a bilinear polynomial is constructed since it is in a quadrilateral element. The second way leads to better numerical accuracy than the first, but the cost is that more than one velocity variables are used to approximate the integral, which forces us to solve a linear system.

As shown in Fig. 3, the control cell corresponding to a pressure variable can be formed by successively connecting the centroids of elements and middle points of edges. The idea for calculating the integrals on the domain or along the edge is similar to that used in the numerical treatments associated with velocity variables.

Although equation (2) is simple, it dictates the fundamentally dynamical feature in geophysical fluids. Therefore, it forms a basic test case to examine the performance of a numerical method when applying it to ocean modelling. In the next section the effectiveness of our new scheme will be illustrated through the equation.

3. Numerical experiments

The geostrophic flows in a circular ocean basin with a radius of 17.5 km are evaluated on hybrid triangular-quadrilateral meshes, as shown in Fig. 4. In this experiment, the precise value of the Coriolis parameter has little impact on the results, thus a constant value of 1.0×10^{-4} (evaluated at 45oN) is given. The

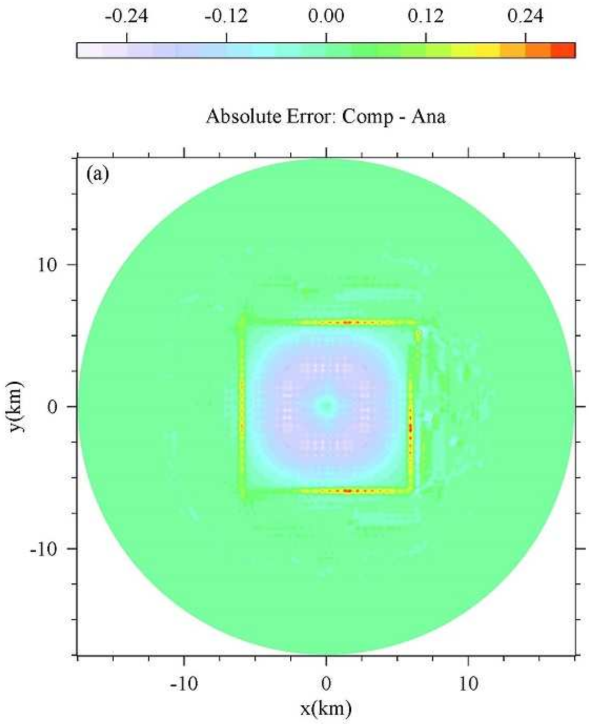

FIGURE 5. The absolute error of numerical solution.

surface elevation assumes a specified Gaussian distribution which is defined by

$$\eta = A \exp\left(-\frac{x^2+y^2}{\sigma^2}\right), \tag{3}$$

where $A = 2.0\text{m}$ and $\sigma = 2.5 \times 10^7\text{m}$. Taking derivative in equation (3) with respect to x and y respectively and inserting them into equation (2), one can find the analytical solution for the geostrophic flows, which is

$$\begin{cases} v = -gA\exp\left(-\dfrac{x^2+y^2}{\sigma^2}\right)\left(\dfrac{2x}{\sigma^2}\right)/f, \\ u = gA\exp\left(-\dfrac{x^2+y^2}{\sigma^2}\right)\left(\dfrac{2y}{\sigma^2}\right)/f. \end{cases} \tag{4}$$

It should be noted that in the above solution, no boundary conditions could be considered at the coastline of the closed basin. But in the numerical discretization, a slippery condition with zero velocity crossing the boundary is specified along the coastline. This different treatment of boundary conditions will cause a discrepancy between the analytical and numerical solutions as shown later. But it has no relevance to the accuracy of the numerical discretization.

The numerical experiment suggests that the finite volume approach in FVCOM can be used in the new discretization of velocities with a consistent numerical error over triangular and quadrilateral elements. By subtracting the analytical from the numerical solutions, the absolute errors are obtained and then illustrated in Fig. 5. The figure shows a tendency of the absolute error to increase toward the center

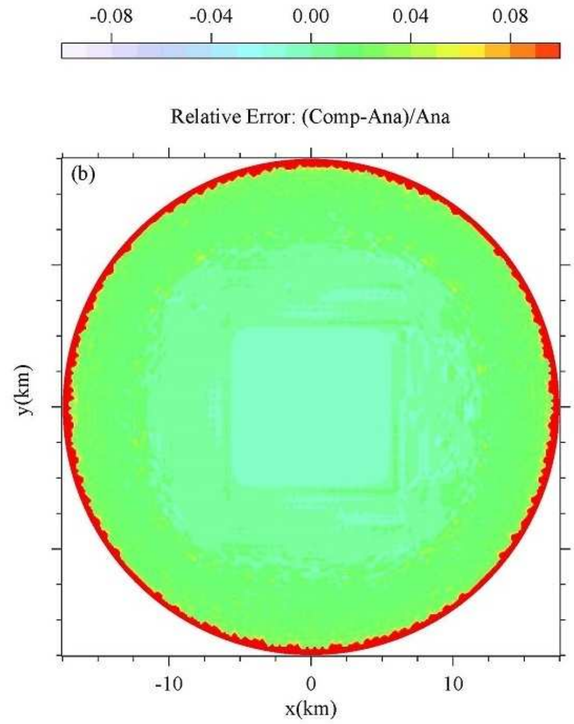

FIGURE 6. The relative error of numerical solution.

of the basin. However, it is worth noting that the magnitude of the velocities (4) turns to be stronger toward the center of the basin. As a result, it is not surprising that the error grows toward the center. To reveal the relation between magnitudes of numerical error and analytic solution, we explore the ratio of them and find in Fig. 6 a more or less uniform error pattern except close to the coastline where the discrepancy should be related to the inability to consider the boundary condition in the analytical solution, as pointed out in the previous paragraph. Since homogeneous boundary condition is applied, it is easily found that the relative error at the boundary is 1. Meanwhile, it is observed from Fig. 6 that the error decays rapidly to less than 0.04.

4. Concluding remarks

The FVCOM is a popular ocean model, but still contains some numerical difficulties. We modify FVCOM by adopting hybrid meshes and changing the placement of velocity and pressure variables, and explore the possibility of overcoming the difficulties. The new scheme illustrates satisfactory numerical properties through experiments on a simplified version of geostrophic equation.

The new scheme still needs rigorous theoretical analysis, which is our future project of this topic. For general equation, more numerical treatments are required during the discretization process.

References

[1] Casulli, V. and Walters, R.A., An unstructured grid, three-dimensional modelbased on the shallow water equations, Int. J. Numer. Meth. Fluids, 32 (2000), 331C348.

[2] Chen, C. and Beardsley, R.C., Cross-frontal water exchange on Georges Bank: modeling exploration of the US GLOBEC/Georges Bank phase III study, Journal of Oceanography, 58 (2002), 403-420.

[3] Chen, C., Beardsley, R.C. and Cowles, G., An unstructured grid, finite-volume coastal ocean model (FVCOM) system, Special Issue entitled Advances in Computational Oceanography, Oceanography, 19 (2006), 78-89.

[4] Chen, C., Beardsley, R.C. and Cowles, G., An unstructured grid, finite-volume coastal ocean model FVCOM user manual. SMAST/UMASSD, 2006.

[5] Chen, C., Liu, H. and Beardsley, R.C., An unstructured, finite-volume, three-dimensional, primitive equation ocean model: application to coastal ocean and estuaries, J. of Atmo. And Ocean. Tech., 20 (2003), 159–186.

[6] Cushman-Roisin, B. and Beckers, J.M., Introduction to geophysical fluid dynamics: physical and numerical aspects, Academic Press, (2011).

[7] Fringer, O.B., Gerritsen, M. and Street, R.L., An unstructured-grid, finite-volume, nonhydrostatic, parallel coastal ocean simulator, Ocean Model, 14 (2006), 139-173.

[8] Hua, B.L. and Thomasset, F., A noise-free finite element scheme for the two-layer shallow water equations, Tellus A, 36 (1984), 157-165.

[9] Raviart, P.A. and Thomas, J.M., A mixed finite element method for 2nd orderelliptic problems, Magenes, E. (Ed.), Mathematical aspects of the finite element methods, Lecture Notes in Mathematics, Springer, Berlin, (1977).

School of Marine Sciences, Sun Yat-sen University, Guangzhou, 510275, P. R. China
E-mail: laizhig@mail.sysu.edu.cn

School of Mathematics and Computational Science and Guangdong Province Key Laboratory of Computational Science, Sun Yat-sen University, Guangzhou 510275, P. R. China
E-mail: wubin@mail.sysu.edu.cn

School of Data and Computers and Guangdong Province Key Laboratory of Computational Science, Sun Yat-sen University, Guangzhou 510275, P. R. China
E-mail: mcszqs@mail.sysu.edu.cn