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NUMERICAL SOLUTION OF NONSTATIONARY PROBLEMS FOR A CONVECTION AND A SPACE-FRACTIONAL DIFFUSION EQUATION

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Abstract. Convection-diffusion equations provide the basis for describing heat and mass transfer phenomena as well as processes of continuum mechanics. An unsteady problem is considered for a convection and a space-fractional diffusion equation in a bounded domain. A first-order evolutionary equation containing a fractional power of an elliptic operator of second order is studied for general boundary conditions of Robin type. Finite element approximation in space is employed. To construct approximation in time, regularized two-level schemes are used. The numerical implementation is based on solving the equation with the fractional power of the elliptic operator using an auxiliary Cauchy problem for a pseudo-parabolic equation. The results of numerical experiments are presented for a model two-dimensional problem.

Key words. Convection-diffusion problem, fractional partial differential equations, elliptic operator, fractional power of an operator, two-level difference scheme.

1. Introduction

Convection-diffusion problems are typical for mathematical models of fluid mechanics. Heat transfer as well as impurities spreading are occurred not only due to diffusion, but result also from medium motion. Principal features of physical and chemical phenomena observing in fluids and gases [2, 20] are generated by media motion resulting from various forces. Computational algorithms for the numerical solution of such problems are of great importance; they are discussed in many publications (see, e.g., [13, 25]).

In considering the second-order parabolic equations, the convective terms may be written in divergent, nondivergent, and skew-symmetric forms [29]. Transient problems of convection-diffusion are governed by evolutionary operator equations in the appropriate spaces. Their study is based on examining the corresponding properties of the differential operators of convective and diffusive transport. In constructing their discrete analogs, we focus on the approximations that inherit the basic properties of these operators.

Nowadays, non-local applied mathematical models based on the use of fractional derivatives in time and space are actively discussed [1, 8, 18]. Many models, which are used in applied physics, biology, hydrology and finance, involve both subdiffusion (fractional in time) and supper-diffusion (fractional in space) operators. Supper-diffusion problems are treated as evolutionary problems with a fractional power of an elliptic operator. The mathematical models of convection with fractional diffusion are considered with many works (see., e.g., [3, 11, 21–23, 36]).

To solve problems with fractional powers of elliptic operators, we can apply finite volume and finite element methods oriented to using arbitrary domains and irregular computational grids [19, 26]. The computational realization is associated with

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the implementation of the matrix function-vector multiplication. For such problems different approaches [10] are available. Problems of using Krylov subspace methods with the Lanczos approximation when solving systems of linear equations associated with the fractional elliptic equations are discussed in [16]. A comparative analysis of the contour integral method, the extended Krylov subspace method, and the preassigned poles and interpolation nodes method for solving space-fractional reaction-diffusion equations is presented in [6]. The simplest variant is associated with the explicit construction of the solution using the known eigenvalues and eigenfunctions of the elliptic operator with diagonalization of the corresponding matrix [5, 14, 15]. Unfortunately, all these approaches demonstrates too high computational complexity for multidimensional problems.

We have proposed [34] a computational algorithm for solving an equation for fractional powers of elliptic operators on the basis of a transition to a pseudo-parabolic equation. For the auxiliary Cauchy problem, the standard two-level schemes are applied. The computational algorithm is simple for practical use, robust, and applicable to solving a wide class of problems. A small number of time steps is required to find a solution. This computational algorithm for solving equations with fractional powers of operators is promising when considering transient problems.

To solve numerically evolutionary equations of first order, as a rule, two-level difference schemes are used for approximation in time. Investigation of stability for such schemes in the corresponding finite-dimensional (after discretization in space) spaces is based on the general theory of operator-difference schemes [27, 28]. In particular, the backward Euler scheme and Crank-Nicolson scheme are unconditionally stable for a non-negative operator. As for one-dimensional problems for the space-fractional diffusion equation, an analysis of stability and convergence for this equation was conducted in [17] using finite element approximation in space. A similar study for the Crank-Nicolson scheme was considered earlier in [31] using finite difference approximations in space. We separately note the works [12, 24, 30], where the numerical methods for solving one-dimensional in spaces nonstationary problems for a convection and a space-fractional diffusion equation are considered.

In this paper, we propose two-level difference schemes for numerical solution of multidimensional nonstationary problems for a convection and a space-fractional diffusion equation. The main computational complexity of such problems is associated with solving a fractional diffusion problem on the new time level. An algorithm that uses an auxiliary problem for a pseudo-parabolic equation is applied [34]. The stability of the proposed two-level difference schemes is investigated. Results of calculations for a model two-dimensional problem based on finite-element approximations in space demonstrate efficiency of the proposed approach.

The paper is organized as follows. The formulation of an unsteady problem for a convection and a space-fractional diffusion equation is given in Section 2. Finite element approximations in space is discussed in Section 3. In Section 4, we construct a special additive difference scheme for time and investigate its stability. The results of numerical experiments are described in Section 5.

2. Problem formulation

In a bounded polygonal domain $\Omega \subset R^m$, $m = 1, 2, 3$ with the Lipschitz continuous boundary $\partial\Omega$, we search the solution for a comvection-diffusion problem. The diffusion process is described using a fractional power of an elliptic operator.

Define the elliptic operator as

$$\mathcal{D}u = -\mathrm{div} k(\boldsymbol{x}) \mathrm{grad}\, u + c(\boldsymbol{x}) u \tag{1}$$

with coefficients $0 < k_1 \leq k(\boldsymbol{x}) \leq k_2$, $c(\boldsymbol{x}) \geq 0$. The operator $\mathcal{D}$ is defined on the set of functions $u(\boldsymbol{x})$ that satisfy on the boundary $\partial\Omega$ the following conditions:

$$k(\boldsymbol{x})\frac{\partial u}{\partial n} + \mu(\boldsymbol{x})u = 0, \quad \boldsymbol{x} \in \partial\Omega, \tag{2}$$

where $\mu(\boldsymbol{x}) \geq \mu_1 > 0$, $\boldsymbol{x} \in \partial\Omega$.

In the Hilbert space $H = L_2(\Omega)$, we define the scalar product and norm in the standard way:

$$< u, w >= \int_\Omega u(\boldsymbol{x})w(\boldsymbol{x})d\boldsymbol{x}, \quad \|u\| =< u, u >^{1/2} .$$

In the spectral problem

$$\mathcal{D}\varphi_k = \lambda_k\varphi_k, \quad \boldsymbol{x} \in \Omega,$$
$$k(\boldsymbol{x})\frac{\partial \varphi_k}{\partial n} + \mu(\boldsymbol{x})\varphi_k = 0, \quad \boldsymbol{x} \in \partial\Omega,$$

we have

$$\lambda_1 \leq \lambda_2 \leq ...,$$

and the eigenfunctions φ_k, $\|\varphi_k\| = 1$, $k = 1, 2, ...$ form a basis in $L_2(\Omega)$. Therefore,

$$u = \sum_{k=1}^{\infty}(u, \varphi_k)\varphi_k.$$

Let the operator $\mathcal{D}$ be defined in the following domain:

$$D(\mathcal{D}) = \{u \mid u(x) \in L_2(\Omega), \ \sum_{k=0}^{\infty}|(u, \varphi_k)|^2\lambda_k < \infty\}.$$

Under these conditions $\mathcal{D} : L_2(\Omega) \to L_2(\Omega)$ and the operator $\mathcal{A}$ is self-adjoint and positive defined:

$$\mathcal{D} = \mathcal{D}^* \geq \delta I, \quad \delta > 0, \tag{3}$$

where I is the identity operator in H. For δ, we have $\delta = \lambda_1$. In applications, the value of λ_1 is unknown (the spectral problem must be solved). Therefore, we assume that $\delta \leq \lambda_1$ in (3). Let us assume for the fractional power of the operator $\mathcal{D}$:

$$\mathcal{D}^\alpha u = \sum_{k=0}^{\infty}(u, \varphi_k)\lambda_k^\alpha\varphi_k.$$

More general and mathematically complete definition of fractional powers of elliptic operators is given in [35].

The convection is provied by nonstationary field of medium velocity $\boldsymbol{v}(\boldsymbol{x}, t)$. On the boundary the following condition is satisfied

$$\boldsymbol{v}(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \partial\Omega. \tag{4}$$

We take the convection operator in the simmetric form [7, 33]:

$$\mathcal{C}u = \operatorname{div}(\boldsymbol{v}\, u) - \frac{1}{2}\operatorname{div}\boldsymbol{v}\, u. \tag{5}$$

Using the restrictions (4) the convection operator is skew-symmetric:

$$\mathcal{C} = -\mathcal{C}^*. \tag{6}$$

We seek the solution of the Cauchy problem for the convection-diffusion equation. The solution $u(\boldsymbol{x}, t)$ satisfies the equation

$$\frac{du}{dt} + \mathcal{C}u + \mathcal{D}^\alpha u = f(t), \quad 0 < t \leq T, \tag{7}$$

and the initial condition

$$u(0) = u_0, \tag{8}$$

under the restriction $0 < \alpha < 1$. The main feature of the problem is that the evolutionary first-order equation has the fractional power of the operator $\mathcal{D}$.

3. Discretization in space

To solve numerically the problem (4), (5), we employ finite element approximations in space [4, 32]. For (1) and (2), we define the bilinear form

$$a(u, w) = \int_\Omega (k \operatorname{grad} u \operatorname{grad} w + c\, uw)\, d\boldsymbol{x} + \int_{\partial\Omega} \mu\, uw\, d\boldsymbol{x}.$$

By (3), we have

$$a(u, u) \ge \delta \|u\|^2.$$

Define a subspace of finite elements $V^h \subset H^1(\Omega)$. Let $\boldsymbol{x}_i,\ i = 1, 2, ..., M_h$ be triangulation points for the domain Ω. Define pyramid function $\chi_i(\boldsymbol{x}) \subset V^h,\ i = 1, 2, ..., M_h$, where

$$\chi_i(\boldsymbol{x}_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \ne j. \end{cases}$$

For $w \in V_h$, we have

$$w(\boldsymbol{x}) = \sum_{i=i}^{M_h} w_i \chi_i(\boldsymbol{x}),$$

where $w_i = w(\boldsymbol{x}_i),\ i = 1, 2, ..., M_h$.

We define the discrete elliptic operator A as

$$a(u, w) = < Au, w >, \quad \forall\ u, w \subset V^h,$$

where, similarly to (3),

$$A = A^* \ge \delta I, \quad \delta > 0. \tag{9}$$

The discrete convection operator is introduced similarly. Taking into account (4), (5) we set

$$c(u, w) = \int_\Omega \operatorname{div}(\boldsymbol{v}\, u)\, w\, d\boldsymbol{x} - \frac{1}{2} \int_\Omega \operatorname{div}\boldsymbol{v}\, uw\, d\boldsymbol{x}.$$

We define the operator C as

$$c(u, w) = < Cu, w >, \quad \forall\ u, w \subset V^h.$$

Similarly to (6) we have

$$C = -C^*. \tag{10}$$

For the problem (7), (8), we put into the correspondence the operator equation for $w(t) \subset V^h$:

$$\frac{dw}{dt} + Cw + D^\alpha w = \psi(t), \quad 0 < t \le T, \tag{11}$$

$$w(0) = w_0, \tag{12}$$

where $\psi(t) = Pf(t)$, $w_0 = Pu_0$ with P denoting L_2-projection onto V^h.

Now we will obtain an elementary a priori estimate for the solution of (11), (12) assuming that the solution of the problem, coefficients of the elliptic operator, the right-hand side and initial conditions are sufficiently smooth.

Let us multiply equation (11) by w and integrate it over the domain Ω. Taking into account the skew-symmetry (see (10)) of the operator C we have

$$\left\langle \frac{dw}{dt}, w \right\rangle + < D^\alpha w, w > = < \psi, w > .$$

In view of the self-adjointness and positive definiteness of the operator D^α, the right-hand side can be evaluated by the inequality

$$< \psi, w > \leq < D^\alpha w, w > + \frac{1}{4} < D^{-\alpha}\psi, \psi > .$$

By virtue of this, we have

$$\frac{d}{dt}\|w\|^2 \leq \frac{1}{2}\|\psi\|^2_{D^{-\alpha}},$$

where $\|\psi\|_{D^{-\alpha}} = < D^{-\alpha}\psi, \psi >^{1/2}$. The latter inequality leads us to the desired a priori estimate:

$$\|w(t)\|^2 \leq \|w_0\|^2 + \frac{1}{2}\int_0^t \|\psi(\theta)\|^2_{D-\alpha} d\theta. \quad (13)$$

Taking into account (9), the estimate (13) can be simplified:

$$\|w(t)\|^2 \leq \|w_0\|^2 + \frac{1}{2}\delta^{-\alpha}\int_0^t \|\psi(\theta)\|^2 d\theta. \quad (14)$$

We will focus on the estimates (13), (14) for the stability of the solution with respect to the initial data and the right-hand side in constructing discrete analogs of the problem (11), (12).

4. Approximation in time

To solve numerically the problem (11), (12), we use the simplest implicit two-level scheme. Let τ be a step of a uniform grid in time such that $w^n = w(t^n)$, $t^n = n\tau$, $n = 0, 1, ..., N$, $N\tau = T$. It seems reasonable to begin with the simplest explicit scheme

$$\frac{w^{n+1} - w^n}{\tau} + (C + D^\alpha)w^n = \psi^n, \quad n = 0, 1, ..., N-1, \quad (15)$$

$$w^0 = w_0. \quad (16)$$

Advantages and disadvantages of explicit schemes for the standard parabolic problem ($\alpha = 1$) are well-known, i.e., these are a simple computational implementation and a time step restriction (see, e.g., [27, 28]). In our case ($\alpha \neq 1$), the main drawback (conditional stability) remains, whereas the advantage in terms of implementation simplicity does not exist. The approximate solution at a new time level is determined via (15) as

$$w^{n+1} = w^n - \tau(C + D^\alpha)w^n + \tau\psi^n. \quad (17)$$

Thus, we must calculate $D^\alpha w^n$. In view of these problems, considering the scheme (15), it is more correct to speak of the scheme with the explicit approximations in time in contrast to the standard fully explicit scheme.

Let us approximate equation (10) by the backward Euler scheme:

$$\frac{w^{n+1} - w^n}{\tau} + (C + D^\alpha)w^{n+1} = \psi^{n+1}, \quad n = 0, 1, ..., N-1. \quad (18)$$

The main advantage of the implicit scheme (18) in comparison with (15) is its absolute stability. Let us derive for this scheme the corresponding estimate for stability.

Multiplying equation (18) scalarly by τw^{n+1}, we obtain

$$\begin{aligned} < w^{n+1}, w^{n+1} > \; + \; \tau < D^{\alpha} w^{n+1}, w^{n+1} > \\ = < w^{n}, w^{n+1} > + \tau < \psi^{n+1}, w^{n+1} > . \end{aligned} \tag{19}$$

The terms on the right side of (19) are estimated using the inequalities:

$$< w^{n}, w^{n+1} > \leq \frac{1}{2} < w^{n+1}, w^{n+1} > + \frac{1}{2} < w^{n}, w^{n} >,$$

$$< \psi^{n+1}, w^{n+1} > \leq < D^{\alpha} w^{n+1}, w^{n+1} > + \frac{1}{4} < D^{-\alpha} \psi^{n+1}, \psi^{n+1} > .$$

The substitution into (19) leads to the following level-wise estimate:

$$\|w^{n+1}\|^2 \leq \|w^{n}\|^2 + \frac{1}{2} \tau \|\psi^{n+1}\|^2_{D^{-\alpha}}.$$

This implies the desired estimate for stability:

$$\|w^{n+1}\|^2 \leq \|w_0\|^2 + \frac{1}{2} \sum_{k=0}^{n} \tau \|\psi^{k+1}\|^2_{D^{-\alpha}}, \tag{20}$$

which is a discrete analog of the estimate (12). Similarly to (13), in view of (9), from (20), we get

$$\|w^{n+1}\|^2 \leq \|w_0\|^2 + \frac{1}{2} \delta^{-\alpha} \sum_{k=0}^{n} \tau \|\psi^{k+1}\|^2. \tag{21}$$

To obtain the solution at the new time level, it is necessary to solve the problem

$$(I + \tau (C + D^{\alpha})) w^{n+1} = w^{n} + \tau \psi^{n}.$$

A more complicated situation arises in the implementation of the Crank-Nicolson scheme:

$$\frac{w^{n+1} - w^{n}}{\tau} + (C + D^{\alpha}) \frac{w^{n+1} + w^{n}}{2} = \psi^{n+1/2}, \quad n = 0, 1, ..., N-1.$$

In this case, we have

$$\left(I + \frac{\tau}{2} (C + D^{\alpha}) \right) w^{n+1} = w^{n} - \frac{\tau}{2} (C + D^{\alpha}) w^{n} + \tau \psi^{n+1/2}.$$

The numerical implementation of the above-mentioned approximations in time for the standard parabolic problems ($\alpha = 1$ in (11)) is based on calculating the values of $\Phi(A) b$, $A = C + D$ for $\Phi(z) = (1 + \sigma \tau z)^{-1}$, $\sigma = 0.5, 1$ and $\Phi(z) = z$. For problems with fractional powers of elliptic operators, we apply the approach proposed early in our paper [34]. It is based on the computation of $\Phi(D) b$ for $\Phi(z) = z^{-\beta}$, $0 < \beta < 1$.

For the explicit approximation in time, we rewrite (17) in the form

$$w^{n+1} = w^{n} - \tau C w^{n} - \tau D D^{-\beta} w^{n} + \tau \psi^{n}, \quad \beta = 1 - \alpha.$$

Therefore, the computational implementation is based on the evaluation of $\Phi(D) b$ for $\Phi(z) = z^{-\beta}$ and $\Phi(z) = z$. A similar approach is not valid for the backward Euler scheme (16), (18) and moreover for the Crank-Nicolson scheme. To construct a more appropriate from a computational point of view approximations in time for the Cauchy problem (11), (12), we apply the principle of regularization for operator-difference schemes proposed by A.A. Samarskii [27].

For a regularizing operator $R > 0$, the simplest regularized scheme for solving (10), (11) has the form (see, e.g., [33]):

$$(I + \tau R)\frac{w^{n+1} - w^n}{\tau} + (C + D^\alpha)w^n = \psi^{n+1}, \quad n = 0, 1, ..., N - 1. \tag{22}$$

Now we will derive the stability conditions for the regularized scheme (16), (22) and after that we will select the appropriate regularizing operator R itself.

Rewrite equation (22) in the form

$$\left(I + \tau\left(R - \frac{1}{2}(C + D^\alpha)\right)\right)\frac{w^{n+1} - w^n}{\tau} + (C + D^\alpha)\frac{w^{n+1} + w^n}{2} = \psi^{n+1}. \tag{23}$$

Let us denote

$$G = I + \tau\left(R - \frac{1}{2}(C + D^\alpha)\right). \tag{24}$$

We choose the regularizing operator R such that the operator G is self-adjoint. Taking into account (24) we set

$$R = \frac{1}{2}C + Q, \quad Q = Q^*,$$

so

$$G = I + \tau\left(Q - \frac{1}{2}D^\alpha\right). \tag{25}$$

The regularized scheme (22) takes the following form

$$(I + \tau Q)\frac{w^{n+1} - w^n}{\tau} + C\frac{w^{n+1} + w^n}{2} + D^\alpha w^n = \psi^{n+1}, \quad n = 0, 1, ..., N - 1. \tag{26}$$

Thus, for the convection term the Crank-Nicolson approximation is taken, and the regularization (the operator Q) is only associated with the fractional diffusion (the operator D^α).

Taking inot account (25) and $G = G^*$ from (23) we have

$$G\frac{w^{n+1} - w^n}{\tau} + (C + D^\alpha)\frac{w^{n+1} + w^n}{2} = \psi^{n+1}.$$

Multiplying it scalarly by $\tau(w^{n+1} + w^n)$, we get

$$< Gw^{n+1}, w^{n+1} > - < Gw^n, w^n > + \frac{\tau}{2} < D^\alpha(w^{n+1} + w^n), w^{n+1} + w^n >$$
$$= \tau < \psi^{n+1}, w^{n+1} + w^n >,$$

For

$$Q \geq \frac{1}{2}D^\alpha \tag{27}$$

we have $G = G^* \geq I$, and $G = I + O(\tau)$. Under these conditions, we obtain the inequality

$$\|w^{n+1}\|_G^2 \leq \|w^n\|_G^2 + \frac{1}{2}\tau\|\psi^{n+1}\|_{D^{-\alpha}}^2.$$

Thus, for the regularized difference scheme (16), (26), under the condition (27), the following estimate for stability with respect to the initial data and the right-hand side holds:

$$\|w^{n+1}\|_G^2 \leq \|w_0\|_G^2 + \frac{1}{2}\sum_{k=0}^{n}\tau\|\psi^{k+1}\|_{D^{-\alpha}}^2. \tag{28}$$

To select an appropriate regularizing operator Q, we should take into account two conditions, i.e., first, to satisfy the inequality (27), and secondly, to simplify calculations. Our choice is based on the inequality

$$D^\alpha \leq \alpha D + (1-\alpha)I, \tag{29}$$

which is the simplest version of Young's inequality for positive operators (see, e.g., [9]). In the scheme (26), we put

$$Q = \sigma(\alpha D + (1-\alpha)I). \tag{30}$$

For $\sigma \geq 0.5$, in view of (29), the inequality (28) holds.

The result of our analysis is the following statement.

Theorem 1. *The regularized scheme (16), (26) with the regularizer Q selected according to (30) is unconditionally stable for $\sigma \geq 0.5$. The approximate solution satisfies the a priori estimate (27), (28).*

The transition to a new time level is performed via the formula

$$\begin{aligned} \Big(I + \frac{\tau}{2}C + \sigma\tau(\alpha D + (1-\alpha)I\Big) w^{n+1} \\ = \Big(I - \frac{\tau}{2}C + \sigma\tau(\alpha D + (1-\alpha)I\Big) w^{n} \\ - \tau D D^{-\beta} w^n + \tau \psi^{n+1}, \quad \beta = 1-\alpha. \end{aligned}$$

Therefore, it is necessary to solve the discrete elliptic problem with the operator

$$B = I + \frac{\tau}{2}C + \sigma\tau(\alpha D + (1-\alpha)I$$

and compute $D^{-\beta}w^n$.

The main peculiarity of solving the Cauchy problem (11), (12) is the necessity to evaluate values

$$g^n = D^{-\beta}w^n, \quad n = 0, 1, ..., N-1, \quad 0 < \beta = 1-\alpha < 1.$$

The computational algorithm is based on the consideration of the auxiliary Cauchy problem [34].

Assume that

$$y(s) = \delta^\beta (s(D-\delta I) + \delta I)^{-\beta} y(0),$$

then for the determination of g^n, we can put

$$g^n = y(1), \quad y(0) = \delta^{-\beta} w^n. \tag{31}$$

The function $y(s)$ satisfies the evolutionary equation

$$(sS + \delta I)\frac{dy}{ds} + \beta S y = 0, \tag{32}$$

where $S = D - \delta I \geq 0$. Thus, the calculation of $D^{-\beta}w^n$ is based on the solution of the Cauchy problem (31), (32) within the unit interval for the pseudo-parabolic equation.

To solve numerically the problem (31), (32), we use the simplest implicit two-level scheme. Let η be a step of a uniform grid in time such that $y_k = y(s_k)$, $s_k = k\eta$, $k = 0, 1, ..., K$, $K\eta = 1$. Let us approximate equation (32) by the backward Euler scheme

$$(s_{k+1}S + \delta I)\frac{y_{k+1} - y_k}{\eta} + \beta S y_{k+1} = 0, \quad k = 0, 1, ..., K-1, \tag{33}$$

$$y_0 = \delta^{-\beta} w^n. \tag{34}$$

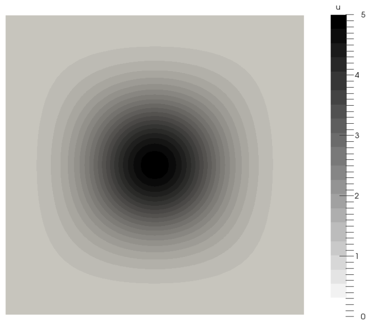

FIGURE 1. The stationary solution with $\gamma = 75$, $\alpha = 1$.

For the Crank-Nicolson scheme, we have

$$(s_{k+1/2}G + \delta I)\frac{y_{k+1} - y_k}{\eta} + \beta G \frac{y_{k+1} + y_k}{2} = 0, \quad k = 0, 1, ..., K-1. \tag{35}$$

The difference scheme (34), (35) approximates the problem (31), (32) with the second order by η, whereas for scheme (33), (34) we have only the first order. The above two-level schemes are unconditionally stable.

5. Numerical experiments

We illustrate the possibilities of the proposed computational algorithm by results of numerical solution of a model two-dimentional problem ($\Omega \subset R^2$). The computational domain is the unit square:

$$\Omega = \{\boldsymbol{x} = (x_1, x_2) \mid 0 < x_i < 1, \ i = 1, 2\}.$$

We seek the solution of the equation

$$\frac{\partial u}{\partial t} + \operatorname{div}(\boldsymbol{v} u) + (-\Delta)^\alpha u = 1,$$

where $\mu(\boldsymbol{x}) = 10$ in (2). We assume that the medium is incompressible ($\operatorname{div} \boldsymbol{v} = 0$) and

$$\boldsymbol{v} = (v_1, v_2), \quad v_1 = \frac{\partial \psi}{\partial x_2}, \quad v_2 = -\frac{\partial \psi}{\partial x_1}.$$

Stream function $\psi(\boldsymbol{x})$ is specified by the expression

$$\psi(\boldsymbol{x}) = \gamma x_1(1 - x_1) x_2 (1 - x_2),$$

where the parameter γ define the intensity of convection. We use the simplest computational mesh with uniform partition of the square Ω. The computations are carried out on the mesh (51×51) with mesh step $h = 0.02$.

Firstly, note that peculiarities of solving stationary the convection-diffusion problem without taking into account the fractional effect when $\alpha = 1$. In our model problem the solution in the center of computational domain grows when the velocity of the medium is increased. Figure 1 shows the solution with $\gamma = 75$. For comparison, in Figure 2 we provide the solution of the diffusion problem with $\gamma = 0$. The dependence of the solution on the intensity of circulation is illustrated in Figure 3, where we present the solution in the mid-section with various γ.

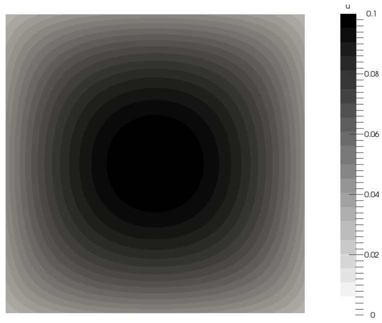

FIGURE 2. The stationary solution of the diffusion problem ($\gamma = 0$, $\alpha = 1$).

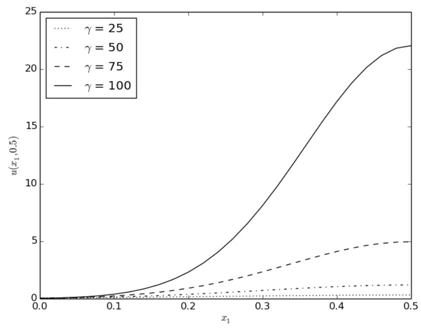

FIGURE 3. The solution in the mid-section $x_2 = 0.5$ with various γ.

The nonstationary convection-diffusion problem ($\alpha = 1$) is considered under the following initial condition

$$u(\boldsymbol{x}, 0) = 0.$$

Figure 4 shows the dynamics of numerical solution with $\gamma = 75$ and using the scheme (26), (30) with $\sigma = 1$ on different time grids. We observe the first order accuracy in time under th chosen impicit approximations for the diffusion. When using the Crank-Nicolson scheme ($\sigma = 0.5$ in (26), (30)) the accuracy of numerical solution significantly increases. The solutions of nonstationary convection-diffusion problem in the mid-section at different time levels are shown in Figure 5.

Let us provide some results of numerical solution of the convection problem with the fractional diffusion. Figure 6 shows the dependence of the maximum solution on time with $\alpha = 0.75$ when using different time grids. Here, we use the regularized scheme (26), (30)) with $\sigma = 0.5$.

The comparison with the standard diffusion (see Figure 4) shows that the stabilization pf solution is faster, the effects of diffusion smoothing are weaken. The solution even more strongly assembles in the center of the domain with significant increase in the amplitude (cf. Figure 1 and Figure 7). This effect can be traced to the purely diffusion problem (Figure 2 and Figure 8).

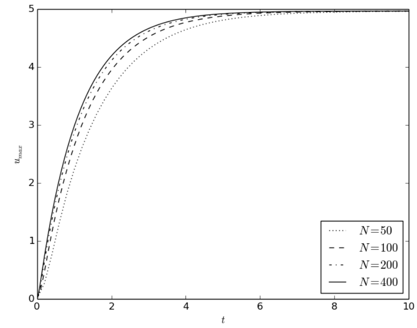

FIGURE 4. The maximum value of solution on different time grids ($\alpha = 1$, $\sigma = 1$).

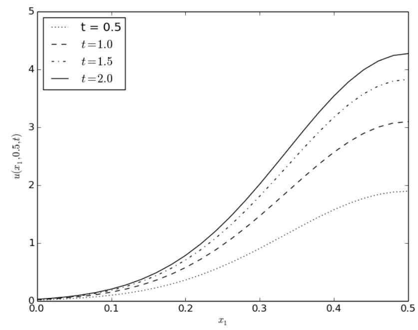

FIGURE 5. The solution in the mid-section $x_2 = 0.5$ at the different time levels.

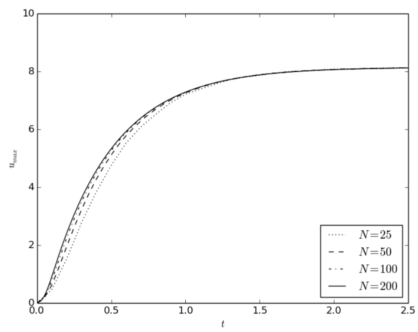

FIGURE 6. The maximum value of solution with different time grids ($\alpha = 0.75$, $\sigma = 0.5$).

The principal effect of the parameter α is shown in Figure 9, where the profile of the stationary solution with the intense convection ($\gamma = 75$) and different values of α are given. The similar solutions for the problem without convection are shown in Figure 10.

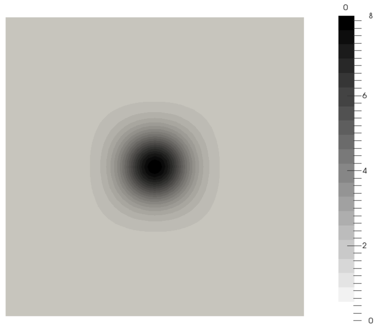

FIGURE 7. The stationary solution with $\gamma = 75$, $\alpha = 0.75$.

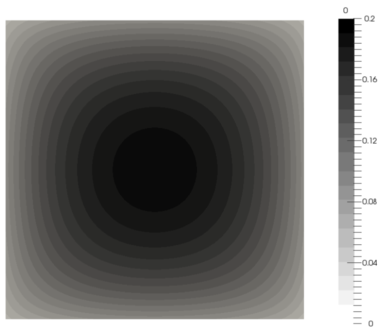

FIGURE 8. The stationary solution of the diffusion problem ($\gamma = 0$, $\alpha = 0.75$).

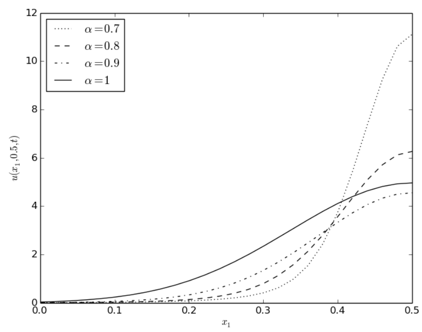

FIGURE 9. The stationary solution in the section $x_2 = 0.5$ with different α ($\gamma = 75$).

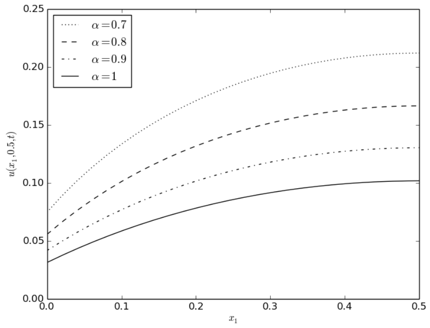

FIGURE 10. The stationary solution in the section $x_2 = 0.5$ with different α ($\gamma = 0$).

Acknowledgments

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