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PERTURBATION ANALYSIS OF INPUT-OUTPUT COEFFICIENTS ON ECONOMIC MODULE IN THE MRICE-E MODEL

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Abstract. MRICE-E model is a new integrated assessment model (IAM) applied on evaluating climate change and the loss of economic welfare. Its economic module for China adopts a dynamic, nonlinear and multi-sectional CGE model. In this paper, we are concerned with the effects of perturbations in input-output coefficients in the CGE model. In the analytical framework, some concepts such as the Lyapunov exponent and the condition number from dynamic system and numerical linear algebra are employed to measure the errors brought by perturbations of the I-O coefficients. We finally derive the upper bound estimation of errors growth through time. To reduce the effects of the possible perturbations, some suggestions about categorization of the industrial sectors are given in the end.

Key words. climate change, IAM, CGE model, input-output analysis, dynamic system, Lyapuonv exponent.

1. Introduction

Multi-factor Regional Integrated Model of Climate and Economy System Equilibrium (MRICE-E) model established by [18] is one of IAMs to evaluate interaction between the climate change and the economic system. There are two mainlines of IAM researches, one kind of which is the programming models represented by DICE and RICE (see [22] and [23]). The other kind is the computable general equilibrium (CGE) models. MRICE-E model integrates these two kinds of models, absorbing their respective advantages. It can reveal the impact of climate change on economy as DICE and RICE models. In the meantime, it can allow great details of sectoral disaggregation as CGE models can. Also, it is an extension of MRICES system by [21], reflecting the idea of equilibrium.

MRICE-E model has four interactive systems: the climate system, the economic system, GDP spillover mechanism system and the policy adjustment system as showed in Figure 1. The economic system is the center part of the model and interacts with the other three systems.

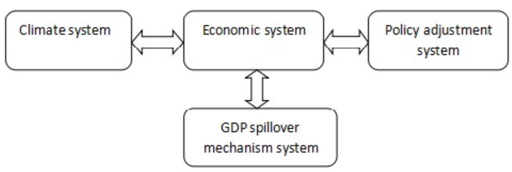

Figure 1. The framework of MRICE-E model.

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The model divides the world into eight regions: China, the US, Japan, the European Union (EU), the high-developing countries, the medium-developing countries, the low-developing countries and the developed countries. Each region has its own economic system and shares the global climate system. The climate change is added into the economic system as a factor in production function. The emission mitigation strategies can change the climate in return as investment activities in economy. Consequently, the economic system and climate system are re-integrated in the MRICE-E model as showed in Figure 2.

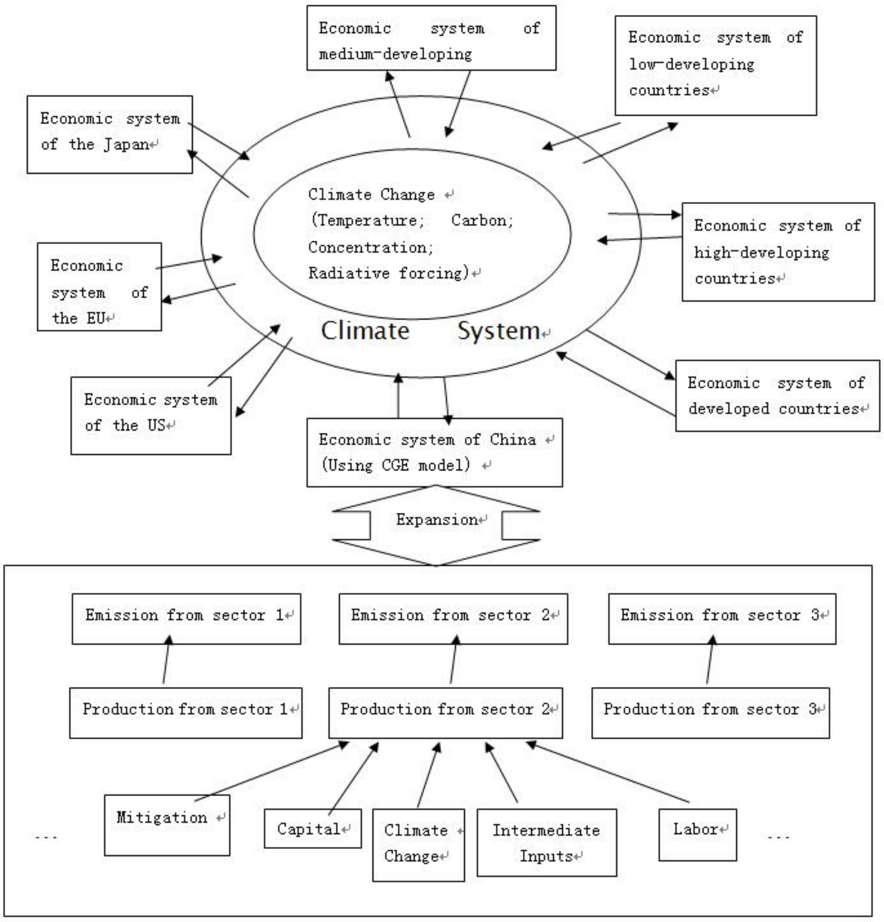

Figure 2. The static structure of economic and climate systems in MRICE-E model.

While macroeconomic model is used in other regions in the world, the dynamic CGE model is used to describe the economic activities in China. Usually, a CGE model consists dozens of equations in order to describe the details of the sectoral variables. Accordingly, there are hundreds of parameters consistent with the equations. It is a well-recognized issue that these parameters may diverge from the real values. In this paper, we focus on the perturbations in one kind of very important parameters: input-output coefficients. They represent the transactions among the different industrial sectors. Their perturbations may stem from technological change, structural change and the general effects of economic growth and development. For example, the input-output coefficients based upon data in year 2000 can not reflect the situations after the year 2005. In order to estimate the coefficients in these years, nonsurvey and partial-survey methods [20, Chapter 7] are needed. In the process of constructing the coefficients, the errors are made inevitably.

The effects of perturbations in parameters in the input-output models or CGE models have been widely discussed for a long time, both empirically and analytically. In the empirical framework, the sensitivity analysis is a common way to find how the results respond to the perturbations. It treats the perturbations as uncertainty and the Monte Carlo experiments are adopted especially involving a large number of parameters. For instance, we refer to [2], [17] for its application in input-output models and [1], [9] in CGE models. Also, it can deal with many complicated models. However, similar examples can lead to very different conclusions under the empirical analysis and hence the convincing theoretical explanations can not been made. In addition, various aspects of the problem have been studied in the analytical framework. Back to the early 1950s, the effects of individual and multiple coefficients changes in static input-output model have begun to be considered, such as in [25] and [10]. [16] has provided a brief review of most of the literature about this problem up to recent years. For dynamic economic system, there are fewer researches about this problem due to its complicated mathematical structure. We refer to [5] for example, which has used the dominant eigenvalue of the matrix to measure the effects of perturbation on balanced growth paths in the I-O system.

We consider the perturbation problem of MRICE-E model in the analytical view. Although we are only concerned with the economic system, the structure is still very complicated. To sum up, it is a dynamic, multidimensional and nonlinear system, containing dozens even hundreds of equations. To obtain effective theoretical results, we simplify the original model by setting some parameters exogenous. By this way, the main structure of the models has been kept and the number of equations has been reduced. Instead of only considering one or few elements as the former literatures, we treat the perturbations as an ensemble, measured by norm. Some other concepts are also adopted from numerical analysis and dynamic system such as condition number, spectrum radius, Lyapunov exponent, and so on. The aim of this paper under mathematical view is to provide the effective upper bound estimation of error growth brought by the perturbations in input-output coefficients. Under the economical view, we finally provide some suggestions about industrial sector categorization according to our numerical results.

The remainder of the paper proceeds as follows. In section 2, we describe the simplified economic system in mathematical form. The effects of perturbations of input-output coefficients will be discussed in sections 3, 4 and 5. In section 3, the input-output linear system of equations is analyzed. Its upper bound of condition number has been derived. In section 4, the stability of dynamic economic system has been considered and Lyapunov exponent is introduced to measure the growth of the initial error. Using the conclusions in section 3 and 4, we derive the final result about upper bound estimation of error growth in section 5.

2. The economic system: the dynamic CGE model

2.1. The simplified dynamic CGE model. The computable general equilibrium (CGE) models are a tool of empirical economic analysis established by [13]. It uses the actual economic data to estimate how production behavior, commodity markets, households and social welfare react to changes in policies, technology or environment. The models contain dozens of equations, many of which are nonlinear. The dynamic CGE model is the extension of the static model by using period-to-period system. It can generate a time path for model simulations. Since the 1990s, CGE models have been extensively used in IAM of environmental economics, such as FUND model established by [28], WIAGEM model established by [14] and etc. In recent years, the application of the dynamic model has attracted some attentions, such as [8], [26] and [27].

In MRICE-E model, the economic system in China is dynamic CGE model with multiple industrial sectors. The main structure of system contains three parts: the part of production, the part of commodity demand and the part of capital updating. In the rest of the paper, the industry is categorized into n sectors and the time period is set as T years. The subscript i, t means that the variable is in the i's sector and at the t year.

The structure of production part has two layers, see [18] and [19]. At the top level, the quantities of value-added are defined as a function over factors of productivity, labor and capital. At the bottom level, each industrial sector uses commodities of other sectors as intermediate inputs. We use Cobb-Douglas production function to describe the value-added $Y_{i,t}$ as follows:

$$Y_{i,t} = b_{i,t} K_{i,t}^{\alpha_{i,t}} L_{i,t}^{1-\alpha_{i,t}}, \tag{1}$$

where $K_{i,t}$ is capital and $L_{i,t}$ is labor; $\alpha_{i,t}$ and $1 - \alpha_{i,t}$ represent the elasticity of capital and labor; $b_{i,t}$ is a measure of productivity which is affected by temperature and emissions mitigation rate. For the concrete definition of $b_{i,t}$, we refer to [21]. The capital variable $K_{i,t}$ and the value-added $Y_{i,t}$ are endogenous and the others are exogenous.

The input-output coefficients appear at the bottom level and represent the intermediate inputs. Let $X_{j,t}$ be the sector j's total commodity output, which satisfies that

$$X_{j,t} = \sum_{i=1}^{n} a_{i,j,t} X_{i,t} + Y_{j,t}. \tag{2}$$

At the t year, the sector j purchases inputs from the sector i and $a_{i,j,t}$ represents the distribution proportion of sector i's outputs across sector j. For instance, the i sector is defined as agriculture, the j sector as manufacturing. When 1 unit of goods of agriculture is produced, $a_{i,j,t}$ unit of the goods transfers to manufacturing sector as intermediate inputs. Different from traditional input-output coefficients which are called as technical coefficients or direct input coefficients used in the demand-driven model, $a_{i,j,t}$ is called direct output coefficients or allocation coefficients used in the supply-side model which is also called Ghosh model ([11]). [6] has proved that Ghosh model is equal to demand-driven model when it is interpreted as a price model. See Chapter 12 of [20] for details. According to the real economic significance, we have that

$$0 \leqslant a_{i,j,t} < 1, \quad \sum_{j=1}^{n} a_{i,j,t} < 1. \tag{3}$$

The structure of commodity demand can be written as

$$X_{i,t} = \rho_{i,t} X_{i,t} + C_{i,t} + G_{i,t} + S_{i,t} + I_{i,t}. \tag{4}$$

By the market clearing, $X_{i,t}$ can also be viewed as the total commodity demand. On the right hand side of equation (4), $I_{i,t}$ denotes the demand for investment, $C_{i,t}$ the demand of household, $G_{i,t}$ the demand of the government, and $S_{i,t}$ the demand for stock. $\rho_{i,t}X_{i,t}$ represents intermediate consumptions of sector i and $\rho_{i,t} = \sum_{j=1}^{n} a_{i,j,t}$. We assume that

$$I_{i,t} = \eta_{i,t}X_{i,t}, \tag{5}$$

where $0 \leqslant \eta_{i,t} < 1-\rho_{i,t}$ is an exogenous parameter which represents the proportion of investment demand in the total demand.

In order to explain the production and the demand parts, let us see the input-output table showed in Figure 3. In input-output analysis, the equations (2) represent the columns of the table which describe the composition of inputs required by a particular industrial sector. The equations (4) represent the rows of table which describe distribution of output throughout the economy. The sums of the i-th row and i-th column are both equal to $X_{i,t}$, i.e. the total output or input of sector i. See [20, Chapter 1] for details.

	Input of section 1	Input of section 2	……	Input of section n	Con.	Gov.	Sto.	Inv.	
Ouput of sector 1	$a_{1,1,t}X_{1,t}$	$a_{1,2,t}X_{1,t}$	……	$a_{1,n,t}X_{1,t}$	$C_{1,t}$	$G_{1,t}$	$S_{1,t}$	$I_{1,t}$	$X_{1,t}$
Ouput of sector 2	$a_{2,1,t}X_{2,t}$	$a_{2,2,t}X_{2,t}$	……	$a_{2,n,t}X_{2,t}$	$C_{2,t}$	$G_{2,t}$	$S_{1,t}$	$I_{1,t}$	$X_{2,t}$
……	……	……	……	……	……	……	……	……	……
Ouput of sector n	$a_{n,1,t}X_{n,t}$	$a_{n,2,t}X_{n,t}$	……	$a_{n,n,t}X_{n,t}$	$C_{n,t}$	$G_{n,t}$	$S_{n,t}$	$I_{n,t}$	$X_{n,t}$
Value added	$Y_{1,t}$	$Y_{2,t}$	……	$Y_{n,t}$					
	$X_{1,t}$	$X_{2,t}$	……	$X_{n,t}$					

FIGURE 3. The expanded input-output table.

The capital stock is updated between $t-1$ and t years. At the first place, the investment will be re-allocated among all the sectors. $Is_{i,t}$ is the investment derived by the i's sector, satisfying that

$$Is_{i,t} = \gamma_{i,t}\sum_{i=1}^{n} I_{i,t}, \tag{6}$$

where $\gamma_{i,t} \geq 0$ is the investment allocation proportion. Since the total derived investment should be equal to the total demand of investment, we have $\sum\limits_{i=1}^{n} \gamma_{i,t} = 1$. Finally, the growth of the capital stock is expressed as

$$K_{i,t+1} = Is_{i,t} + (1-\delta_{i,t})K_{i,t}, \tag{7}$$

where $0 < \delta_{i,t} \leqslant 1$ is the capital depreciation rate.

2.2. The mathematical description using vectors and matrices. In this subsection, we adopt vectors and matrices to describe the dynamic system. Firstly, we define the following n-dim vectors:

$$Y(t) = (Y_{1,t}, Y_{2,t}, \cdots, Y_{n,t})^T,\ X(t) = (X_{1,t}, X_{2,t}, \cdots, X_{n,t})^T,$$
$$I(t) = (I_{1,t}, I_{2,t}, \cdots, I_{n,t})^T,\ \text{and } K(t) = (K_{1,t}, K_{2,t}, \cdots, K_{n,t})^T.$$

Particularly, set

$$K_\alpha(t) = (K_{1,t}^{\alpha_{1,t}}, K_{2,t}^{\alpha_{2,t}}, \cdots, K_{n,t}^{\alpha_{n,t}})^T.$$

Then, we define the $n \times n$-dim matrices as follows:

$$B(t) = \text{diag}\ (b_{1,t}, b_{2,t}, \cdots, b_{n,t}),\ \eta(t) = \text{diag}\ (\eta_{1,t}, \eta_{2,t}, \cdots, \eta_{n,t}),$$

$$\delta(t) = \text{diag}\ (\delta_{1,t}, \delta_{2,t}, \cdots, \delta_{n,t}),\ \text{and}\ L_{1-\alpha}(t) = \text{diag}\ (L_{1,t}^{1-\alpha_{1,t}}, L_{2,t}^{1-\alpha_{2,t}}, \cdots, L_{n,t}^{1-\alpha_{n,t}}).$$

The matrices $A(t)$ and $\Gamma(t)$ are set as

$$A(t) = \begin{pmatrix} a_{1,1,t} & a_{1,2,t} & \cdots & a_{1,n,t} \\ a_{2,1,t} & a_{2,2,t} & \cdots & a_{2,n,t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1,t} & a_{n,2,t} & \cdots & a_{n,n,t} \end{pmatrix} \ \text{and} \ \Gamma(t) = \begin{pmatrix} \gamma_{1,t} & \gamma_{1,t} & \cdots & \gamma_{1,t} \\ \gamma_{2,t} & \gamma_{2,t} & \cdots & \gamma_{2,t} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n,t} & \gamma_{n,t} & \cdots & \gamma_{n,t} \end{pmatrix}.$$

After some simple computations, (1), (2), (5), (6, and (7) can be rewritten as:

$$Y(t) = B(t)L_{1-\alpha}(t)K_\alpha(t), \tag{8}$$

$$X(t) = A^T(t)X(t) + Y(t), \tag{9}$$

$$K(t+1) = \Gamma(t)\eta(t)X(t) + (E - \delta(t))K(t), \tag{10}$$

where E is the $n \times n$-dim identity matrix and $t = 0, 1, 2, \cdots, T-1$.

In the dynamic system (8)-(10), the productivity $B(t)$, the labor $L_{1-\alpha}(t)$, the proportion of investment demand $\eta(t)$, the investment allocation rate $\Gamma(t)$, the capital depreciation rate $\delta(t)$ and the direct output coefficients matrix $A(t)$ are exogenous parameters. The value-added $Y(t)$, the total commodity output $X(t)$ and the capital $K(t)$ are endogenous variables, updating year by year.

Set the perturbation of direct output coefficients matrix $A(t)$ as $\epsilon A(t)$. The disturbed matrix of $A(t)$ is $\tilde{A}(t)$, satisfying

$$\tilde{A}(t) = A(t) + \epsilon A(t).$$

The rest of the paper is around one topic: How does the perturbation matrix $\epsilon A(t)$ affect the solutions of the dynamic system (8)-(10).

3. Perturbation analysis of the input-output equations

In this section, we focus on the input-output equation (9). Using the same value-add $Y(t)$, the total commodity output $X(t)$ and its disturbed value $\tilde{X}(t)$ can be solved by the original equation

$$(E - A^T(t))X(t) = Y(t), \tag{11}$$

and its disturbed form

$$(E - \tilde{A}^T(t))\tilde{X}(t) = Y(t). \tag{12}$$

Set $\epsilon X(t) = \tilde{X}(t) - X(t)$. Now we try to find the relation between $\epsilon A(t)$ and $\epsilon X(t)$.

We adopt the norm denoted by $\|\cdot\|$ to measure the vectors and matrices. The common norms contain 1-norm denoted by $\|\cdot\|_1$ and ∞-norm denoted by $\|\cdot\|_\infty$. Let $x = (x_1, x_2, \cdots, x_n)^T$, $M = (m_{i,j})_{n\times n}$. The 1-norm and ∞-norm are defined as (see. e.g., [12, Chapter 5]):

$$\|x\|_1 = \sum_i |x_i|, \qquad \|x\|_\infty = \max_i |x_i|, \tag{13}$$

$$\|M\|_1 = \max_j \sum_i |m_{ij}| \text{ and } \|M\|_\infty = \max_i \sum_j |m_{ij}|. \tag{14}$$

The perturbation analysis of linear equations is one of the basic concerns in numerical linear algebra. In numerical linear algebra, the condition number usually characterizes the impact of the small perturbations of coefficients. The condition number of matrix M is defined by

$$\kappa(M) = \|M\|\|M^{-1}\|.$$

Obviously, the condition number is related to the form of the norm. The following lemma, which is common in textbooks such as [4], shows that the condition number offers an effective upper bound of the impact of perturbations on the solution.

Lemma 3.1. *Let $X(t)$ and $\tilde{X}(t)$ be solutions of equations (11) and (12). Then we have*

$$\frac{\|\epsilon X(t)\|}{\|\tilde{X}(t)\|} \leqslant \kappa(E - A^T(t)) \frac{\|\epsilon A^T(t)\|}{\|E - A^T(t)\|}. \tag{15}$$

It can see from (15) that the condition number $\kappa(E-A^T(t))$ measures the relative change $\frac{\|\epsilon X(t)\|}{\|\tilde{X}(t)\|}$ in the answer as multiple of the relative change $\frac{\|\epsilon A^T(t)\|}{\|E-A^T(t)\|}$ in the data.

The following result is specific to our problem. Based on the properties of $A(t)$, the upper bound of the condition number $\kappa(E-A^T(t))$ is estimated in the following theorem.

Theorem 3.2. *For any $\epsilon > 0$, there exists a norm $\|\cdot\|_\epsilon$ related to ϵ so that the corresponding condition number $\kappa_\epsilon(\cdot)$ satisfies*

$$\kappa_\epsilon(E - A^T(t)) \leqslant \frac{\max\limits_i |1 - a_{i,i,t} + \sum\limits_{j\neq i} a_{i,j,t}|}{\min\limits_i |1 - \sum\limits_{j=1}^{n} a_{i,j,t}|} + O(\epsilon). \tag{16}$$

Proof. See Appendix. □

In fact, we can also derive the following corollary according to proof of the above theorem.

Corollary 3.3. *For any $\epsilon > 0$, there exists a norm $\|\cdot\|_\epsilon$ related to ϵ satisfying*

$$\|(E - A^T(t))^{-1}\|_\epsilon \leqslant \frac{1}{\min\limits_i |1 - \sum\limits_{j=1}^{n} a_{i,j,t}|} + \epsilon. \tag{17}$$

Remark 3.4. *Now we try to interpret the upper bound estimation (16). When the condition number is too large, the problem is called ill-conditioned. If a problem is ill-conditioned, a little disturbance of the coefficients will cause the totally wrong results. Observe the estimation at the right hand side of (16). Since (3), we have*

$$\max_i |1 - a_{i,i,t} + \sum_{j\neq i} a_{i,j,t}| \leqslant 2.$$

The ill-conditioned problem only happens when

$$\min_i |1 - \sum_{j=1}^{n} a_{i,j,t}| \to 0 \quad \text{or} \quad \max_i \sum_{j=1}^{n} a_{i,j,t} \to 1. \tag{18}$$

This means that if the output of one industrial sector is almost used as intermediate inputs, the problem could be ill-conditions. This is the situation that we should avoid.

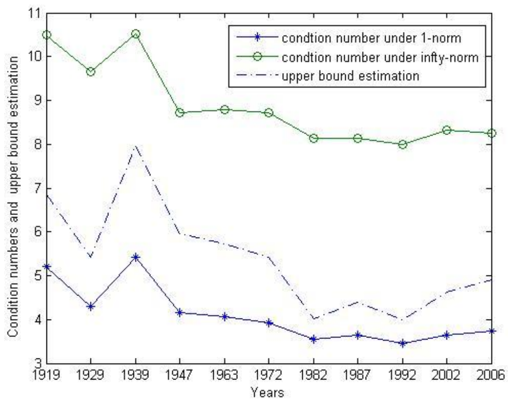

FIGURE 4. Comparison between condition numbers and their estimation (US Technical coefficients).

The effectiveness of the estimation will be verified in a simple example. The technical coefficients matrix denoted by $TC(t) = (tc_{i,j})_{n\times n}$ is similar to $A^T(t)$. Similar to Theorem 3.2, we have

$$\kappa_\epsilon(E - TC(t)) \leqslant \frac{\max\limits_j |1 - tc_{j,j,t} + \sum\limits_{i\neq j} tc_{i,j,t}|}{\min\limits_j |1 - \sum\limits_{i=1}^{n} tc_{i,j,t}|} + O(\epsilon). \tag{19}$$

The definition of κ_ϵ is involved in the norm $\|\cdot\|_\epsilon$ and has no concrete form. In linear algebra, it can be proved that all the norms on finite dimensions are equivalent, which means that for any two norms $\|\cdot\|_A$ and $\|\cdot\|_B$ defined in $R^{n\times n}$, there exists constants c and C such that

$$c\|x\|_A \leqslant \|x\|_B \leqslant C\|x\|_A, \quad \forall x \in R^{n\times n}. \tag{20}$$

We use the condition number κ_1 and κ_∞ corresponding to 1-norm and ∞-norm, and consider the technical coefficients in US through 1919 to 2006 (see Appendix B in [20]). From Figure 4, we can see that the tendency of our estimation is consistent with the tendency of two kinds of condition numbers under 1-norm and ∞-norm, which implies that at least our estimation can been used to judge whether the problem is ill-conditioned or not.

4. The Lyapunov stability and the Lyapunov exponent of the dynamic system

In the rest of the paper, we consider the whole dynamic system. The system (8)-(10) can be written as

$$K(t+1) = f(t, K(t)), \quad t = 0, 1, \cdots, T-1, \tag{21}$$

where

$$f(t, K(t)) = \Gamma(t)\eta(t)(E - A^T(t))^{-1}B(t)L_{1-\alpha}(t)K_\alpha(t) + (E - \delta(t))K(t).$$

A dynamic system (see e.g., [15]) is an evolution rule that describes what future state follows from the current state. The states for all future are called a trajectory or orbit. The initial state of the system will determine all the future states by solving the system, iterating the relation step by step.

The property of stability or chaos is of fundamental importance for a dynamic system, since the initial state of a system may not be known precisely. The system is called Lyapunov stable if the small enough initial separation in states implies that future states will be close enough forever. On the other hand, chaos means the small separation in initial states will yield diverging values in the future states and makes the long-term prediction difficult. As a useful study of the system and important step to derive the final results, we measure how the initial errors grow over time, using the concept of the Lyapunov exponent.

To be specific, we define the general discrete dynamic system as

$$x(t+1) = f(t, x(t)), \quad t = 0, 1, \cdots, T-1. \tag{22}$$

The disturbed state is $\tilde{x}(t)$ and the disturbed system is

$$\tilde{x}(t+1) = f(t, \tilde{x}(t)), \quad t = 0, 1, \cdots, T-1. \tag{23}$$

The initial separation between the two systems $\tilde{x}(0) - x(0)$ is denoted by $\epsilon x(0)$. The evolutionary separations over time $\tilde{x}(t) - x(t)$ are denoted by $\epsilon x(t)$. They are determined by the initial separation $\epsilon x(0)$ and the form of dynamics. We cite the definition of Lyapunov exponent from [7], which is defined in finite time system.

Definition 4.1. *Finite time Lyapunov exponent*
Let $x(t)$ and $\tilde{x}(t)$ defined in (22) and (23). The separations between the two systems are denoted by

$$\epsilon x(t) = \tilde{x}(t) - x(t), \quad t = 0, 1, 2, \cdots, T-1.$$

Then, the finite time Lyapunov exponent at time T is

$$\lambda_T(x(0), \epsilon x(0)) = \frac{1}{T} \log \frac{\|\epsilon x(T)\|}{\|\epsilon x(0)\|}, \tag{24}$$

which depends on the initial state $x(0)$, initial error $\epsilon x(0)$, and time interval T.

The Lyapunov exponents can be positive or negative. Negative Lyapunov exponents indicate Lyapunov stability, while positive Lyapunov exponents demonstrate chaos. Their values measure the convergence or divergence rates of evolution in separation.

For a specific dynamic system with fixed initial state, different initial errors causes different Lyapunov exponents. However, the value and the structure of possible initial errors can not be determined in general. Therefore, it is reasonable to define an upper bound of Lyapunov exponents under all possible initial errors.

Definition 4.2. *For the dynamic system $x(t+1) = f(t, x(t))$ with the initial state $x(0)$, the upper bound of finite time Lyapunov exponent at time T is defined as*

$$\lambda_{\max} = \sup_{\epsilon x(0)} \lambda_T(x(0), \epsilon x(0)).$$

Next we turn to the dynamic CGE system (21). The disturbed system of (21) is set as

$$\tilde{K}(t+1) = f(t, \tilde{K}(t)), \quad t = 0, 1, 2, \cdots, T-1, \tag{25}$$

where $\tilde{K}(t) = (\tilde{K}_{1,t}, \tilde{K}_{2,t}, \cdots, \tilde{K}_{n,t})^T$.

Consequently, the separations of states between (21) and (25) are

$$\epsilon K(t) = \tilde{K}(t) - K(t), \quad t = 0, 1, 2, \cdots, T-1,$$

where $\epsilon K(t) = (\epsilon K_{1,t}, \epsilon K_{2,t}, \cdots, \epsilon K_{n,t})^T$.

The next lemma gives an upper bound estimation of $\lambda_{\max}$ of the dynamic system (21).

Lemma 4.3. *Assume that* $\alpha_{i,t}K_{i,t}^{\alpha_{i,t}-1} < \sigma$ *and* $\alpha_{i,t}\tilde{K}_{i,t}^{\alpha_{i,t}-1} < \sigma$ *for all* i, t*, where* σ *is a positive constant. For* $t = 0, 1, 2, \cdots, T-1$*, set*

$$C_t = \sigma n \max_i \gamma_{i,t} \max_i \eta_{i,t} \max_i b_{i,t} l_{i,t}^{1-\alpha_{i,t}} \|(E - A^T(t))^{-1}\|_\infty + \max_i (1 - \delta_{i,t}). \tag{26}$$

It can be verified that

$$\frac{\|\epsilon K(t+1)\|_\infty}{\|\epsilon K(t)\|_\infty} \leqslant C_t. \tag{27}$$

Consequently, the upper bound of finite time Lyapunov exponent corresponding to ∞*-norm satisfies:*

$$\lambda_{\max} \leqslant \frac{1}{T}\sum_{t=0}^{T-1} \log C_t. \tag{28}$$

Proof. Using the properties of ∞-norm, we have from (21) and (25) that

$$\begin{aligned}
& \|\tilde{K}(t+1) - K(t+1)\|_\infty = \|f(\tilde{K}(t), t) - f(K(t), t)\|_\infty \\
\leqslant \;& \|\Gamma(t)\eta(t)(E - A^T(t))^{-1}B(t)L_{1-\alpha}(t)\|_\infty \|\tilde{K}_\alpha(t) - K_\alpha(t)\|_\infty \\
+ \;& \|E - \delta(t)\|_\infty \|\tilde{K}(t) - K(t)\|_\infty \\
\leqslant \;& \|\Gamma(t)\|_\infty \|\eta(t)\|_\infty \|(E - A^T(t))^{-1}\|_\infty \|B(t)L_{1-\alpha}(t)\|_\infty \|\tilde{K}_\alpha(t) - K_\alpha(t)\|_\infty \\
+ \;& \|E - \delta(t)\|_\infty \|\tilde{K}(t) - K(t)\|_\infty. \qquad (29)
\end{aligned}$$

It is from the definition (13)-(14) that

$$\begin{aligned}
& \|E - \delta(t)\|_\infty = \max_i (1 - \delta_i), \;\; \|B(t)L_{1-\alpha}(t)\|_\infty = \max_i b_{i,t} l_{i,t}^{1-\alpha_{i,t}}, \\
& \|\Gamma(t)\|_\infty = n \max_i \gamma_{i,t}, \quad \|\eta(t)\|_\infty = \max_i \eta_{i,t}. \qquad (30)
\end{aligned}$$

On the other hand,

$$\|\tilde{K}_\alpha(t) - K_\alpha(t)\|_\infty = \max_i |(K_{i,t} + \epsilon K_{i,t})^\alpha - K_{i,t}^\alpha|.$$

Let i_0 satisfy

$$|(K_{i_0,t} + \epsilon K_{i_0,t})^\alpha - K_{i_0,t}^\alpha| = \max_i |(K_{i,t} + \epsilon K_{i,t})^\alpha - K_{i,t}^\alpha|.$$

According to the Lagrange mean value theorem,

$$\begin{aligned}
\|\tilde{K}_\alpha(t) - K_\alpha(t)\|_\infty &= |(K_{i_0,t} + \epsilon K_{i_0,t})^\alpha - K_{i_0,t}^\alpha| \\
&\leqslant |\alpha_{i_0} \xi^{\alpha_{i_0}-1}||\epsilon K_{i_0,t}|,
\end{aligned}$$

where $K_{i_0,t} \leqslant \xi \leqslant K_{i_0,t} + \epsilon K_{i_0,t}$ if $\epsilon K_{i_0,t} \geq 0$ or $K_{i_0,t} + \epsilon K_{i_0,t} \leqslant \xi \leqslant K_{i_0,t}$ if $\epsilon K_{i_0,t} \leqslant 0$. From the assumption of this theorem, we have

$$\|\tilde{K}_\alpha(t) - K_\alpha(t)\|_\infty \leqslant \sigma \|\epsilon K(t)\|_\infty. \tag{31}$$

From (29), (30) and (31), (27) can be derived. It implies that

$$\lambda_{\max} = \frac{1}{T}\log\frac{\|\epsilon K(T)\|}{\|\epsilon K(0)\|} = \frac{1}{T}\sum_{t=0}^{T-1}\log\frac{\|\epsilon K(t+1)\|}{\|\epsilon K(t)\|} \leqslant \frac{1}{T}\sum_{t=0}^{T-1} C_t.$$

We complete the proof. □

From this theorem, C_t offers upper bound of the error growth rate at the t year. Correspondingly, $\lambda_{\max}$ represents the upper bound of the average error growth rate through T years. Now we decompose C_t and explain it from the economic view.

There are two terms in C_t: $\sigma n \max_i \gamma_{i,t} \max_i \eta_{i,t} \max_i b_{i,t} l_{i,t}^{1-\alpha_{i,t}} \|(E - A^T(t))^{-1}\|_\infty$ and $\max_i(1-\delta_{i,t})$. The value of $\max_i(1-\delta_{i,t})$ belongs to $[0,1]$, depending on the capital depreciation rate. Now we focus on the first term and divide it into three multipliers: $\sigma \max_i b_{i,t} l_{i,t}^{1-\alpha_{i,t}}$, $n \max_i \gamma_{i,t} \max_i \eta_{i,t}$ and $\|(E - A^T(t))^{-1}\|_\infty$.

The multiplier $\sigma \max_i b_{i,t} l_{i,t}^{1-\alpha_{i,t}}$ depends on the choice of Cobb-Douglas (C-D) functions. In fact, it is the upper bound of derivatives of the C-D functions. Since $\sigma \leqslant \alpha_{i,t} K_{i,t}^{\alpha_{i,t}-1}$ and $0 < \alpha_{i,t} < 1$, $K_{i,t}$ should be away from 0 to make sure that σ is not too large. This implies that there should not exist a sector whose capital is too small compared to others.

The value of multiplier $n \max_i \gamma_{i,t} \max_i \eta_{i,t}$ belongs to $[0, n]$. $n \max_i \gamma_{i,t}$ represents how the investments are reallocated. If they are reallocated equally, the value can achieve to its lower bound 1. If the situation is far from allocation equal, it can be large up to n. $\max_i \eta_{i,t}$ represents the maximal proportion of investment needed in the total need among all the sections. Its value belongs to $[0,1]$.

Let us see the third multiplier $\|(E - A^T(t))^{-1}\|_\infty$, which reflects the stability of input-output equations (9). From (17) and (20), we have the following result.

Remark 4.4. *For any $\epsilon > 0$, there exist the norm $\|\cdot\|_\epsilon$ and the constant C_ϵ related to ϵ, satisfying*

$$\|(E - A^T(t))^{-1}\|_\infty \leqslant \frac{C_\epsilon}{\min_i |1 - \sum_{j=1}^{n} a_{i,j,t}|} + O(\epsilon). \tag{32}$$

This means that we should avoid the situation that $\max \sum_{j=1}^{n} a_{i,j,t} \to 1$. Otherwise, the value of upper bound of $\|(E - A^T(t))^{-1}\|_\infty$ can not be controlled.

5. Perturbation analysis of the I-O coefficients in the dynamic system

In this section, we derive the final conclusion about how the errors grow due to the perturbation of $A(t)$ through time. To underline the coefficients matrix $A(t)$, we rewrite (21) as follows:

$$K(t+1) = g(t, A(t), K(t)), \quad t = 0, 1, 2, \cdots, T-1, \tag{33}$$

where

$$g(t, A(t), K(t)) = \Gamma(t)\eta(t)(E - A^T(t))^{-1} B(t) L_{1-\alpha}(t) K_\alpha(t) + (E - \delta(t))K(t).$$

Correspondingly, the dynamic system with the disturbed direct output coefficients $\tilde{A}(t)$ is

$$\tilde{K}(t+1) = g(t, \tilde{A}(t), \tilde{K}(t)), \quad t = 0, 1, 2, \cdots, T-1, \tag{34}$$

where $\tilde{K}(0) = K(0)$, $\tilde{A}(t) = A(t) + \epsilon A(t)$ and $\tilde{K}(t) = (\tilde{K}_{1,t}, \tilde{K}_{2,t}, \cdots, \tilde{K}_{n,t})^T$.

Consequently, the separations of states between (33) and (34) are

$$\epsilon K(t) = \tilde{K}(t) - K(t), \quad t = 0, 1, 2, \cdots, T-1,$$

where $\epsilon K(t) = (\epsilon K_{1,t}, \epsilon K_{2,t}, \cdots, \epsilon K_{n,t})^T$.

Now we are in the position to derive the upper bound estimation of $\epsilon K(T)$ in the following theorem.

Theorem 5.1. *Assume that* $a_{i,t}K_{i,t}^{a_{i,t}-1} < \sigma$, $a_{i,t}\tilde{K}_{i,t}^{a_{i,t}-1} < \sigma$, $K_{i,t}^{a_{i,t}} < M$, *and* $\tilde{K}_{i,t}^{\alpha_{i,t}} < M$, *where* σ *and* M *are positive constants for all* i, t. *For* $t = 0, 1, \cdots, T-1$, *set*

$$C_t = \alpha\sigma n \max_i \gamma_{i,t} \max_i \eta_{i,t} \max_i b_{i,t} l_{i,t}^{1-\alpha_{i,t}} \|(E - A^T(t))^{-1}\|_\infty + \max_i (1 - \delta_{i,t}),$$

$$D_t = nM \max_i \gamma_{i,t} \max_i \eta_{i,t} \max_i b_{i,t} l_{i,t}^{1-\alpha_{i,t}} \frac{\|(E - A^T(t))^{-1}\|_\infty^2}{1 - \|\epsilon A^T(t)\|_\infty \|(E - A^T(t))^{-1}\|_\infty}.$$

If $\|\epsilon A^T(t)\|_\infty$ *is small enough so that*

$$\|(E - A^T(t))^{-1})\|_\infty \|\epsilon A^T(t)\|_\infty < 1, \tag{35}$$

then it can be verified that

$$\|\epsilon K(t+1)\|_\infty \leqslant C_t \|\epsilon K(t)\|_\infty + D_t \|\epsilon A^T(t)\|_\infty. \tag{36}$$

Consequently, we have

$$\|\epsilon K(T)\|_\infty \leqslant \sum_{t=0}^{T-1} \prod_{\tau=0}^{t-1} C_\tau D_t \|\epsilon A^T(t)\|_\infty. \tag{37}$$

Proof. From (33), (34) and the properties of norm, we have

$$\begin{aligned} &\|\tilde{K}(t+1) - K(t+1)\|_\infty = \|g(t, \tilde{A}(t), \tilde{K}(t)) - g(t, A(t), K(t))\|_\infty \\ \leqslant\ & \|\Gamma(t)\eta(t)(E - A^T(t))^{-1} B(t) L_{1-\alpha}(t)(\tilde{K}_\alpha(t) - K_\alpha(t))\|_\infty \\ +\ & \|\Gamma(t)\eta(t)((E - \tilde{A}^T(t))^{-1} - (E - A^T(t))^{-1}) B(t) L_{1-\alpha}(t) \tilde{K}_\alpha(t)\|_\infty \\ +\ & \|(E - \delta(t))(\tilde{K}(t) - K(t))\|_\infty. \end{aligned} \tag{38}$$

It implies from the result of Theorem 4.3 that

$$\begin{aligned} & \|\Gamma(t)\eta(t)(E - A^T(t))^{-1} B(t) L_{1-\alpha}(t)(\tilde{K}_\alpha(t) - K_\alpha(t))\|_\infty \\ +\ & \|(E - \delta(t))(\tilde{K}(t) - K(t))\|_\infty \leqslant C_t \|\epsilon K(t)\|_\infty. \end{aligned} \tag{39}$$

From (30) and the assumption of theorem, the second term of (38) satisfies that

$$\begin{aligned} & \|\Gamma(t)\eta(t)((E - \tilde{A}^T(t))^{-1} - (E - A^T(t))^{-1}) B(t) L^\alpha(t) \tilde{K}^\alpha(t)\|_\infty \\ \leqslant & \|\Gamma(t)\|_\infty \|\eta(t)\|_\infty \|B(t) L^\alpha(t)\|_\infty \\ & \times \|\tilde{K}^\alpha(t)\|_\infty \|(E - \tilde{A}^T(t))^{-1} - (E - A^T(t))^{-1}\|_\infty \\ \leqslant & nM \max_i \gamma_{i,t} \max_i \eta_{i,t} \max_i b_{i,t} l_{i,t}^{1-\alpha_{i,t}} \\ & \times \|(E - \tilde{A}^T(t))^{-1} - (E - A^T(t))^{-1}\|_\infty. \end{aligned} \tag{40}$$

Here, it is easy to see that

$$\begin{aligned} & \|(E - \tilde{A}^T(t))^{-1} - (E - A^T(t))^{-1}\|_\infty \\ =\ & \|(E - \tilde{A}^T(t))^{-1} \epsilon A^T(t)(E - A^T(t))^{-1}\|_\infty \\ \leqslant\ & \|(E - \tilde{A}^T(t))^{-1}\|_\infty \|(E - A^T(t))^{-1}\|_\infty \|\epsilon A^T(t)\|_\infty. \end{aligned}$$

According to [3] and the assumption (35), we have

$$\|(E - \tilde{A}^T(t))^{-1}\|_\infty \leqslant \frac{\|(E - A^T(t))^{-1}\|_\infty}{1 - \|\epsilon A^T(t)\|_\infty \|(E - A^T(t))^{-1}\|_\infty},$$

which results in

$$\begin{aligned} & \|(E - \tilde{A}^T(t))^{-1} - (E - A^T(t))^{-1}\|_\infty \\ \leqslant & \frac{\|(E - A^T(t))^{-1}\|_\infty^2}{1 - \|\epsilon A^T(t)\|_\infty \|(E - A^T(t))^{-1}\|_\infty} \|\epsilon A^T(t)\|_\infty. \end{aligned} \tag{41}$$

Thus, it follows from combining the above inequalities (40) and (41) that

$$\begin{aligned} &\|\Gamma(t)\eta(t)((E-\tilde{A}^T(t))^{-1}-(E-A^T(t))^{-1})B(t)L(t)^{\alpha}\tilde{K}^{\alpha}(t)\|_{\infty} \\ \leqslant & nM\max_i \gamma_{i,t}\max_i \eta_{i,t}\max_i b_{i,t}l_{i,t}^{1-\alpha_{i,t}}\frac{\|(E-A^T(t))^{-1}\|_{\infty}^2}{1-\|\epsilon A^T(t)\|_{\infty}\|(E-A^T(t))^{-1}\|_{\infty}}. \end{aligned} \tag{42}$$

To sum up (38) and (42), we have (36), from which (37) follows. □

The inequality (36) indicates the error of capital at the $t+1$ year has two sources: the error at the former year $\epsilon K(t)$ and the perturbation matrix $\epsilon A(t)$. C_t represents the upper bound of the initial error growth rate and D_t represents the upper bound of the effects of perturbation matrix $\epsilon A(t)$. We has talked about the structure of C_t in section 4.

Now we discuss D_t and divide it into three multipliers:

$$M\max_i b_{i,t}l_{i,t}^{1-\alpha_{i,t}},\ n\max_i \gamma_{i,t}\max_i \eta_{i,t},\ \text{and}\ \frac{\|(E-A^T(t))^{-1}\|_{\infty}^2}{1-\|\epsilon A^T(t)\|_{\infty}\|(E-A^T(t))^{-1}\|_{\infty}}.$$

The multiplier $M\max_i b_{i,t}l_{i,t}^{1-\alpha_{i,t}}$ is the upper bound of Cobb-Douglas functions. Since $K_{i,t}^{a_{i,t}} < M$, $K_{i,t}$ should not be too large. There should not exist a sector whose the capital value is too large compared to others'. The second multiplier $n\max_i \gamma_{i,t}\max_i \eta_{i,t}$ has been discussed in section 4.

Before we talk about the third multiplier, we discuss the condition (35):

$$\|(E-A^T(t))^{-1})\|_{\infty}\|\epsilon A^T(t)\|_{\infty} < 1.$$

This implies that the result is effective only under the condition that the perturbation $\|\epsilon A^T(t)\|_{\infty}$ is small enough. In fact, the large $\|\epsilon A^T(t)\|_{\infty}$ means that the mathematical model has changed a lot, and hence it is difficult to evaluate the situations. Furthermore, when $1-\|\epsilon A^T(t)\|_{\infty}\|(E-A^T(t))^{-1}\|_{\infty}$ is not close to 0, the important part of third multiplier is $\|(E-A^T(t))^{-1}\|_{\infty}^2$. As showed in the last section, we should avoid the situation that $\max_i \sum_{j=1}^{n} a_{i,j,t} \to 1$ to keep its upper bound small.

6. The conclusion

In this paper, we discuss the effects of perturbations in input-output coefficients of dynamic CGE module in MRICE-E model. The concepts of norm, condition number, and Lyapunov exponent are introduced to measure the rate of error growth. We obtain the final result in Theorem 5.1 about the error propagation deduced by the perturbations in input-output coefficients. Some suggestions about industrial sector categorization have been given in the context. We sum them up as follows:

- We should avoid the situation that there exists a sector i such that

$$\sum_{j=1}^{n} a_{i,j,t} \to 1.$$

  This situation means that the output of the sector i's production is almost used as intermediate inputs.
- Make sure that every sector's capital value $K_{i,t}$ should not been too small or too big compared with others' capitals.
- The investments should be reallocated as equally as possible.

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Appendix A

To prove Theorem 3.2, Gershgorin Circle Theorem should be introduced. The following definition A.1, lemma A.2, and lemma A.3 are common in numerical linear algebra, and we refer them to [12] for example.

Definition A.1. *Let $M \in R^{n\times n}$. If a scalar $\lambda \in C$ and a nonzero vector $x \in C^n$ satisfy the equation*

$$Mx = \lambda x, \tag{A.1}$$

then λ is called an eigenvalue of matrix M. The spectrum of M is the set of all eigenvalues of M, denoted by $\lambda(M)$. The spectral radius of M is defined as $\rho(M)$, satisfying

$$\rho(M) = \max\{|\lambda| : \lambda \in \lambda(M)\}. \tag{A.2}$$

Lemma A.2. *Let $M \in R^{n\times n}$. We have two inequalities between spectral radius and norms of M*

(1) *For any norm $\|\cdot\|$, we have*

$$\rho(M) \leqslant \|M\|. \tag{A.3}$$

(2) *For any $\epsilon > 0$, there exists a norm $\|\cdot\|_\epsilon$ related to ϵ such that*

$$\|M\|_\epsilon \leqslant \rho(M) + \epsilon. \tag{A.4}$$

Lemma A.3. *Gershgorin Circle Theorem*
The eigenvalues of $M = (m_{ij})_{n\times n}$ are in the union of n discs

$$\bigcup_{j=1}^{n} \left\{ z \in C : |z - m_{jj}| \leqslant \sum_{i\neq j} |m_{ij}| \right\}. \tag{A.5}$$

Now, we are in the position to prove our Theorem 3.2.

Proof. According to Lemma A.3, we have for any $\lambda \in \lambda(E - A^T(t))$ that

$$\begin{aligned} \lambda &\in \bigcup_{i=1}^{n} \left\{ z \in C : |z - 1 + a_{i,i,t}| \leqslant \sum_{j\neq i} |a_{i,j,t}| \right\} \\ &\subseteq \bigcup_{i=1}^{n} \left\{ z \in C : |1 - \sum_{j=1}^{n} a_{i,j,t}| \leqslant |z| \leqslant |1 - a_{i,i,t} + \sum_{j\neq i} a_{i,j,t}| \right\}, \end{aligned} \tag{A.6}$$

which implies that

$$\begin{aligned} \rho(E - A^T(t)) &= \max\{|\lambda| : \lambda \in \lambda(E - A(t))\} \\ &\leqslant \max_i |1 - a_{i,i,t} + \sum_{j\neq i} a_{i,j,t}|. \end{aligned} \tag{A.7}$$

Using (A.6) and the property (3) of $A(t)$, we have

$$\min\{|\lambda| : \lambda \in \lambda(E - A^T(t))\} \geq \min_i |1 - \sum_{j=1}^{n} a_{i,j,t}| > 0. \tag{A.8}$$

Therefore, $|\lambda| > 0$. This means that $E - A^T(t)$ is invertible and

$$(E - A^T(t))x = \lambda x \Leftrightarrow (E - A^T(t))^{-1} y = \frac{1}{\lambda} y, \tag{A.9}$$

where $y = (E - A^T(t))x$. Accordingly,

$$\lambda((E - A^T(t))^{-1}) = \left\{ \frac{1}{\lambda} \mid \lambda \in \lambda(E - A^T(t)) \right\}. \tag{A.10}$$

From (A.8) and (A.10), we have

$$\begin{aligned} \rho((E - A^T(t))^{-1}) &= \max\left\{|\lambda| : \lambda \in \lambda((E - A^T(t))^{-1})\right\} \\ &= \frac{1}{\min\{|\lambda| : \lambda \in \lambda(E - A^T(t))\}} \\ &\leqslant \frac{1}{\min\limits_i |1 - \sum\limits_{j=1}^{n} a_{i,j,t}|}. \end{aligned} \tag{A.11}$$

Finally, it is from (A.4), (A.7) and (A.11) that

$$\begin{aligned} \kappa_\epsilon(E - A^T(t)) &= \|E - A^T(t)\|_\epsilon \|(E - A^T(t))^{-1}\|_\epsilon \\ &\leqslant \rho((E - A^T(t)))\rho((E - A^T(t))^{-1}) + O(\epsilon) \\ &\leqslant \frac{\max\limits_i |1 - a_{i,i,t} + \sum\limits_{j \neq i} a_{i,j,t}|}{\min\limits_i |1 - \sum\limits_{j=1}^{n} a_{i,j,t}|} + O(\epsilon). \end{aligned}$$

The proof is completed. □

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