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A LITERATURE SURVEY OF MATHEMATICAL STUDY OF METAMATERIALS

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Abstract. Since the successful construction of the so-called double negative metamaterials in 2000, there has been a growing interest in studying metamaterials across many disciplinaries. In this paper, we present a survey of recent progress in metamaterials and its applications from the mathematical point of view. Due to the great amount of papers published in this area, here we mainly discuss those issues interested to us. Our main goal is to attract more mathematicians to study this fascinating subject.

Key words. Metamaterials, Maxwell's equations, homogenization, edge elements, cloaking.

1. Introduction

In terms of metamaterials, we are specifically interested in those artificially structured composite materials with simultaneously negative electric permittivity ϵ and magnetic permeability μ.

According to Solymar and Shamonina [1, p.317], four seminar papers made the birth of the subject of metamaterials. The first one is by Russian physicist Victor Veselago [2], who wrote the fundamental paper on metamaterial (he called the left-hand material). In this paper, he investigated many properties unique to substances with both negative permittivity ϵ and negative permeability μ, even though nobody knew how to construct such a material at that time. The second important paper is due to the paper published in 2000 by David Smith *et al.* [3]. In this paper, through a physical experiment they demonstrated a composite medium (formed by a periodic array of interspaced conducting nonmagnetic split ring resonators and continuous wires) exhibits a frequency region in the microwave regime with simultaneously negative values of effective permeability $\mu_{eff}(\omega)$ and permittivity $\epsilon_{eff}(\omega)$, where ω is the frequency of incident radiation. This split ring structure forms the first successfully constructed left-handed medium, or double negative metamaterial. The third seminar paper is due to Shelby, Smith and Schultz [4], who in early 2001 presented experimental data at microwave frequencies on a structured metamaterial that there exists a frequency band where the effective index of refraction (normally defined as $n = \sqrt{\epsilon\mu}$) is negative when both ϵ and μ are negative. The other landmark work is due to John Pendry's perfect lens paper published in 2000 [5], in which he proposed the idea to use a slab of negative refractive index material to bring light to a perfect focus without the usual constraints imposed by wavelength. The principle behind this is that the negative refractive index material can restore not only the phase of propagating waves but also the amplitude of evanescent waves.

Since 2000, there has been a tremendous growing interest in studying metamaterial across many disciplines due to its potential revolutions in areas such as

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communications, sensing, radar technology, sub-wavelength imaging, data storage, and invisible cloak device.

2. Metamaterial Invisibility Cloaks

In June 23, 2006's issue of Science magazine, Leonhardt [6] and Pendry *et al.* [7] independently published their works on electromagnetic cloaking. Leonhardt used a conformal mapping to describe how an inhomogeneous indices of refraction $n(\boldsymbol{x})$ in two dimensions can cause light rays to go around a region and emerge on the other side as if they had passed through a free space where $n = 1$. While Pendry's idea is to enclose a finite size object to be cloaked by a specially designed metamaterial coating, which can control the electromagnetic field by mimicking the heterogeneous anisotropic nature of the matrices of permittivity and permeability obtained through changes of coordinates. In Nov.10, 2006's Science, Schurig *et al.* [8] demonstrated the first practical realization of such a cloak with the use of a metamaterial consisting of concentric layers of split ring resonators (SRRs) and made a copper cylinder invisible to an incident plane wave at a specific microwave frequency (8.5 GHz).

Actually, there are many other cloaking ideas with metamaterials. For example, Alu and Engheta [9] proposed an idea of employing a plasmonic or metamaterial cover to drastically reduce the overall scattering from moderately sized objects by means of a scattering cancellation effect. Later they [10] extended the idea to using a plasmonic coating to render an electromagnetic sensor almost invisible to detection by incident waves, while the sensor can remain effective as a device to receive, measure, and observe incident waves.

The idea of "cloaking by anomalous localized resonance" was proposed in 2006 by Milton and Nicorovici [11] and has been further developed by many researchers such as Bruno and Lintner[12], Bouchitte and Schweizer [13], Ammari *et al.* [14], Kohn *et al.* [15], and Nguyen [16]. As Kohn *et al.* [15] mentioned that the mathematical problem of this type cloaking boils down to the investigation of the behavior of the elliptic problem $\nabla \cdot (a(\boldsymbol{x})\nabla u(\boldsymbol{x})) = f(\boldsymbol{x})$, where $a(\boldsymbol{x})$ is a complex coefficient with a matrix-shell-core structure, with real part equal to 1 in the matrix and the core, and -1 in the shell. The interesting problem is to understand the resonant behavior of the solution when the imaginary part of $a(\boldsymbol{x})$ goes to zero, and how the location of the source f plays in the resonance. Many papers are restricted to radial geometries except [14, 15].

Among many proposed cloaking techniques with metamaterials, Pendry *et al.*'s cloaking technique [7] seems to be the most popular one, which is nicknamed as *transformation optics/electromagnetics* (e.g., [17, 18]). It is agreed now that the transformation based cloaking was first discovered back in 2003 by mathematicians Greenleaf *et al.* [19, 20, 21] for nondetectability examples in the context of the Calderón problem.

The principle behind transformation optics is to use a coordinate transformation to derive the spatial dependent permittivity and permeability to guide the wave. For electromagnetic wave, the derivation boils down to the important property that Maxwell's equations are form invariant under coordinate transformations.

Theorem 2.1. [22, Appendix A] *Consider the time-harmonic Maxwell's equations (assuming time harmonic variation of* $\exp(j\omega t)$*):*

$$\nabla \times \boldsymbol{E} + j\omega\mu\boldsymbol{H} = 0, \quad \nabla \times \boldsymbol{H} - j\omega\epsilon\boldsymbol{E} = 0, \tag{1}$$

where $\boldsymbol{E}(\boldsymbol{x})$ *and* $\boldsymbol{H}(\boldsymbol{x})$ *are the electric and magnetic fields in the frequency domain, and* ϵ *and* μ *are the permittivity and permeability of the material.*

Under a coordinate transformation $\boldsymbol{x}' = \boldsymbol{x}'(\boldsymbol{x})$*, the equations (1) keep the same form in the transformed coordinate system:*

$$\nabla' \times \boldsymbol{E}' + j\omega\mu'\boldsymbol{H}' = 0, \quad \nabla' \times \boldsymbol{H}' - j\omega\epsilon'\boldsymbol{E}' = 0, \tag{2}$$

where all new variables are given by

$$\boldsymbol{E}'(\boldsymbol{x}') = A^{-T}\boldsymbol{E}(\boldsymbol{x}),\ \boldsymbol{H}'(\boldsymbol{x}') = A^{-T}\boldsymbol{H}(\boldsymbol{x}),\ A = (a_{ij}),\ a_{ij} = \frac{\partial x_i'}{\partial x_j}, \tag{3}$$

and

$$\mu'(\boldsymbol{x}') = A\mu(\boldsymbol{x})A^T/det(A), \quad \epsilon'(\boldsymbol{x}') = A\epsilon(\boldsymbol{x})A^T/det(A). \tag{4}$$

A simple proof of Theorem 2.1 can be found in Appendix A of [22]. Due to the singularity of the parameters of the perfect cloaking, mathematicians proposed the approximate cloaking concept by incorporating regularization into the cloaking construction. In mathematical community, the approximate/near cloaking has recently been extensively studied. For example, near cloaking schemes were developed for electric impedance tomography [23, 24], for scalar waves governed by the Helmholtz equation (e.g., [48, 26, 27]), for full Maxwell equations [28, 29, 30], and for the second-order wave equations [31].

3. Well-posedness analysis

The recent development of metamaterials raised some issues in the theoretical and numerical study of time-harmonic Maxwells equations. There are some concerns about the effects on the well-posedness of the problem and the regularity of the solution caused by the possible sign-change of the dielectric permittivity and/or of the magnetic permeability. For example, at the optical frequencies sign-change of the permittivity does happen at the interface between a metal and a classical medium. Another case is that sign-change happens for both permittivity and permeability at the interface between a dielectric and double negative metamaterials.

In a series of papers [32, 33, 34, 35, 36, 37], Dhia *et al.* studied the well-posedness of time harmonic Maxwells equations with sign-changing coefficients. For example, in [35], they proved that in a bounded domain Ω the equation

$$-\nabla \cdot (\mu^{-1}\nabla\phi) - \omega^2\epsilon\phi = f,$$

with $f \in L^2(\Omega)$ and Dirichlet boundary conditions, may be strongly ill-posed in the usual H^1 framework for a sign-changing function μ.

Well-posedness results have been obtained by Fernandes & Raffetto for anisotropic metamaterials [38], and for bianisotropic media [39]. In 2014, Fernandes and Raffetto [40] deduced many regularity results for the time-harmonic Maxwell's equations

$$\nabla \times \boldsymbol{H} = j\omega\boldsymbol{D} + \boldsymbol{J}_e, \quad \nabla \times \boldsymbol{E} = -j\omega\boldsymbol{B} - \boldsymbol{J}_m, \quad \text{in } \Omega$$

satisfying the constitutive relations of a bianisotropic material

$$\boldsymbol{E} = \epsilon\boldsymbol{D} + \chi\boldsymbol{B}, \quad \boldsymbol{H} = \xi\boldsymbol{D} + \mu\boldsymbol{B}, \quad \text{in } \Omega$$

where ϵ, χ, ξ, μ are 3×3 matrix-valued functions of the space, $j = \sqrt{-1}$, and $\omega > 0$ is the angular frequency. Here $\boldsymbol{D}$ and $\boldsymbol{B}$ are the electric and magnetic flux densities, respectively. They obtained the local regularity of the electromagnetic fields by considering that the physical domain Ω is filled by just a general bianisotropic material.

In [41], Cocquet *et al.* proved some well-posedness results for first order symmetric systems describing wave propagation in electromagnetic, acoustic, and linear elastic metamaterials, where they assume that the coefficients of the governing equations are frequency dependent. As for electromagnetics, they considered two models. The first model describes wave propagation in bi-anisotropic metamaterials: Find $(\boldsymbol{E},\boldsymbol{H}) \in H(curl;\Omega)^2$ such that

$$\begin{aligned}
&\omega\epsilon(\omega,\boldsymbol{x})\boldsymbol{E} + \omega\xi(\omega,\boldsymbol{x})\boldsymbol{H} - \nabla\times\boldsymbol{H} = -\boldsymbol{J}(\boldsymbol{x}),\ \textit{in}\ \Omega,\\
&\omega\eta(\omega,\boldsymbol{x})\boldsymbol{E} + \omega\mu(\omega,\boldsymbol{x})\boldsymbol{H} + \nabla\times\boldsymbol{E} = -\boldsymbol{M}(\boldsymbol{x}),\quad \textit{in}\ \Omega,\\
&\boldsymbol{n}\times(\boldsymbol{E} + \Lambda(\boldsymbol{n}\times\boldsymbol{H})) = 0,\ \boldsymbol{x}\in\partial\Omega,\quad \text{impedance boundary condition}
\end{aligned}$$

where ξ and η are coupling coefficients. The second model is for wave propagation in chiral media described by the Drude-Born-Fedorov system: Find $(\boldsymbol{E},\boldsymbol{H}) \in H(curl;\Omega)^2$ such that

$$\begin{aligned}
&\omega\epsilon(\omega,\boldsymbol{x})\boldsymbol{E} + \omega\beta(\omega,\boldsymbol{x})\epsilon(\omega,\boldsymbol{x})\nabla\times\boldsymbol{E} - \nabla\times\boldsymbol{H} = -\boldsymbol{J}(\boldsymbol{x}),\ \textit{in}\ \Omega,\\
&\omega\mu(\omega,\boldsymbol{x})\boldsymbol{H} + \omega\beta(\omega,\boldsymbol{x})\mu(\omega,\boldsymbol{x})\nabla\times\boldsymbol{H} + \nabla\times\boldsymbol{E} = -\boldsymbol{M}(\boldsymbol{x}),\ \textit{in}\ \Omega,\\
&\boldsymbol{n}\times(\boldsymbol{E} + \Lambda(\boldsymbol{n}\times\boldsymbol{H})) = 0,\ \boldsymbol{x}\in\partial\Omega,\quad \text{impedance boundary condition}
\end{aligned}$$

where β is the chirality of the material. Generic well-posedness results are proved for both models with either scalar physical parameters or 3×3 tensor coefficients.

As for time-dependent problems, Nicaise [42] obtained the existence result for the following model:

$$\begin{aligned}
&\frac{\partial\boldsymbol{D}}{\partial t} - \text{curl}\boldsymbol{H} = 0,\ \textit{in}\ \Omega\times(0,\infty),\\
&\frac{\partial\boldsymbol{B}}{\partial t} + \text{curl}\boldsymbol{E} = 0,\ \textit{in}\ \Omega\times(0,\infty),\\
&\boldsymbol{H}\times\boldsymbol{n} = \boldsymbol{J},\ \boldsymbol{E}\times\boldsymbol{n} = 0,\ \textit{on}\ \partial\Omega\times(0,\infty),\\
&\boldsymbol{E}(0) = \boldsymbol{E}_0,\ \boldsymbol{H}(0) = \boldsymbol{H}_0,\ \textit{in}\ \Omega,
\end{aligned}$$

with the Drude-Born-Fedorov constitutive relations

$$\boldsymbol{D} = \epsilon(\boldsymbol{E} + \beta\text{curl}\boldsymbol{E}),\ \boldsymbol{B} = \mu(\boldsymbol{H} + \beta\text{curl}\boldsymbol{H}),$$

and another model:

$$\begin{aligned}
&\epsilon\frac{\partial}{\partial t}(\boldsymbol{E} + \beta\text{curl}\boldsymbol{E}) - \text{curl}\boldsymbol{H} = 0,\ \textit{in}\ \Omega\times(0,\infty),\\
&\mu\frac{\partial}{\partial t}(\boldsymbol{H} + \beta\text{curl}\boldsymbol{H}) + \text{curl}\boldsymbol{E} = 0,\ \textit{in}\ \Omega\times(0,\infty),\\
&\boldsymbol{H}\cdot\boldsymbol{n} = \boldsymbol{E}\cdot\boldsymbol{n} = 0,\ \textit{on}\ \partial\Omega\times(0,\infty),\\
&\boldsymbol{E}(0) = \boldsymbol{E}_0,\ \boldsymbol{H}(0) = \boldsymbol{H}_0,\ \textit{in}\ \Omega,
\end{aligned}$$

In [43], Liaskos obtained some well-posedness results for the following initial boundary value problem:

$$\begin{aligned}
&\epsilon\frac{\partial}{\partial t}(\boldsymbol{E} + \beta(t)\text{curl}\boldsymbol{E}) - \text{curl}\boldsymbol{H} = -J_e,\ \textit{in}\ \Omega\times(0,\infty),\\
&\mu\frac{\partial}{\partial t}(\boldsymbol{H} + \beta(t)\text{curl}\boldsymbol{H}) + \text{curl}\boldsymbol{E} = -J_m,\ \textit{in}\ \Omega\times(0,\infty),\\
&div\boldsymbol{E} = div\boldsymbol{H} = 0,\ \textit{in}\ \Omega\times(0,\infty),\\
&\boldsymbol{H}\cdot\boldsymbol{n} = \boldsymbol{E}\cdot\boldsymbol{n} = curl\boldsymbol{E}\cdot\boldsymbol{n} = curl\boldsymbol{H}\cdot\boldsymbol{n} = 0,\ \textit{on}\ \partial\Omega\times(0,\infty),\\
&\boldsymbol{E}(0) = \boldsymbol{E}_0,\ \boldsymbol{H}(0) = \boldsymbol{H}_0,\ \textit{in}\ \Omega.
\end{aligned}$$

Many other related results can be found in papers (cf. [44, 45, 46]) and a recent book by Roach *et al.* [47].

4. Homogenization of metamaterials

Rigorous derivation of the effective properties of metamaterials is quite challenging and is still in its early stage. In [48], Kohn and Shipman made a great effort in clarifying the meaning of the effective dielectric permittivity and magnetic permeability in the quasi static limit, which assumes that the scale of the microstructure is sufficiently small compared to the free space wavelength of the fields. They consider the 2D ring-type resonators, and the rings must have high conductivity as the inverse of the characteristic length of the microstructure. For a microstructured composite material satisfying the relations

$$D = \epsilon E, \quad B = \mu H,$$

they obtain the homogenized coefficients ϵ^* and μ^* on the macroscopic level:

$$D_{av} = \epsilon^* E_{av}, \quad B_{av} = \mu^* H_{av}.$$

The homogenized Maxwell system is

$$\text{curl } E_{av} - j\omega\mu^* h_e^0 = 0, \ \text{curl } h_e^0 - j\omega\epsilon^* E_{av} = 0,$$

where E_{av} is a 2D vector, scalar h_e^0 represents the exterior value of the magnetic field, and the two curl operators are in the 2D meaning. We like to remark that the homogenization of microresonators is different from the standard homogenization of composites, since here H and B fields (also E and D fields) are averaged over different parts of a unit cell.

In [49], Bouchitté and Schweizer presented a rigorous derivation of the effective properties of a metamaterial containing split rings. More specifically, they consider the Maxwell system

$$\text{curl } E_\eta = j\omega\mu_0 H_\eta, \ \text{curl } H_\eta = -j\omega\epsilon_\eta\epsilon_0 E_\eta,$$

with relative permittivity $\epsilon_\eta = 1 + j\frac{k}{\eta^2}$ in $\sum_\eta$ and $\epsilon_\eta = 1$ in $R^3 \setminus \sum_\eta$. The domain $\sum_\eta$ is occupied by the split rings with diameters of order η, the distances between ring centers are η, the circular cross section of each ring has radius in the order of η, and the upper part of each ring has a slit of size $O(\eta^2)$. They proved that the homogenized system is

$$\text{curl } E_{av} = j\omega\mu_0\mu^* H_{av}, \ \text{curl } H_{av} = -j\omega\epsilon_0\epsilon^* E_{av},$$

where $\mu^* = \epsilon^* = 1$ in $R^3 \setminus \sum_\eta$, while $\mu^* = \mu_{eff}(\omega)$ and $\epsilon^* = \epsilon_{eff}$, i.e., the effective permittivity ϵ_{eff} is real, positive and frequency independent, while the effective permeability $\mu_{eff}(\omega)$ is frequency dependent and its eigenvalues can have negative or positive real parts.

In [50], Lamacz and Schweizer generalized the results of [13] to a more complicated ring geometry, which satisfies the conditions: many (order η^3), small (order η), thin (order η^2), and highly conductive (order η^3) metallic objects.

In [51], Chen and Lipton constructed metamaterials made from subwavelength periodic nonmagnetic coated cylinders immersed in a nonmagnetic host. The coated cylinders are parallel to the z axis and made from a frequency independent high dielectric core and a frequency dependent dielectric plasmonic coating. The effective medium parameters are derived as leading order terms of an explicit multiscale expansion for the solution of Maxwells equations.

We like to remark that although many homogenization techniques have been proposed for metamaterials over the last decade (e.g., [52, 53, 54, 55, 56, 57]), unfortunately, rigorous mathematical analysis are still restrict to geometries that are much simpler [50, p.1462] than nested split rings or fish-net designs.

5. Metamaterial applications

Here we mention some interesting potential applications of metamaterials to inspire mathematicians to pursue some mathematical studies in these subjects.

The refocusing property of metamaterial slabs can generate focusing spots in biological tissue as required in microwave hyperthermia treatment. Moreover, the heating spot in tissue can be adjusted easily by moving the heating source around, which overcomes the complex deployment and control system as required in the conventional array applicator. Study [58] claims that use of metamaterial lenses can result in higher power deposition and thus can achieve more effective microwave hyperthermia in cancerous tissue.

Biosensors play a very important role in many areas such as food safety, disease diagnostics, environmental monitoring, and investigation of biological phenomena. In recent years, many researchers proposed various metamaterial-based sensors, the main principle behind this is that the refractive index is rapidly changed by adsorption and desorption of analyte particles. Alu *et al.* [59] proposed a dielectric sensing method by using ϵ near zero narrow waveguide channels. By using metamaterial lens, Shreiber *et al.* [60] developed a novel microwave nondestructive evaluation sensor to detect material defects at the wavelength level. Thin-film sensor with metamaterials was proposed by Labidi *et al.* [61]. Xu *et al.* [62] reported a flexible metamaterial based photonic device operating in the visible-IR regime, and they claimed that this device has potential applications in high sensitivity strain, biological and chemical sensing.

Metamaterials provide a new path forward for the construction of absorbers. The first experimental demonstration of a metamaterial perfect absorbers (MPA) was in 2008 [63], and the structure consists of two metamaterial resonators which can absorb all incident radiation within a single unit cell layer. The experiments demonstrated a peak absorbance greater than 88% at 11.5 GHz. This seminal work in the microwave regime inspired many continuous works in other frequency regimes with high absorptivity. In 2010, Liu *et al.* [64] demonstrated a spatially dependent metamaterial perfect absorber operating in the THz regime. The experimental absorption of 97% was achieved at 6.0μm wavelength, and matched well with numerical full-wave simulations carried out by using the commercial program CST Microwave Studio 2009. In 2011, Aydin *et al.* [65] demonstrated an ultrathin (260 nm) plasmonic super absorber consisting of a metal-insulator-metal stack with a nanostructured top silver film composed of crossed trapezoidal arrays. Their super absorber yields broadband and polarization-independent resonant light absorption over the entire visible spectrum (400–700 nm) with an average measured absorption of 0.71 and simulated absorption of 0.85.

In 2000, Pendry [5] made a theoretical proposal of using metamaterials to build the superlens which overcome the so-called "diffraction limit" in optics: whenever an object is imaged by an optical system, fine features (i.e., those smaller than half the wavelength of the light) are permanently lost in the image. The key feature needed in a superlens is its ability to enhance the evanescent waves, resulting in a sharper image. Since 2000, superlenses have been realized in both microwave and optical frequencies with different designs. In 2005, the optical superlensing effect was observed using a thin slab of silver (a ϵ-negative material), that could effectively image 60-nm features ($\lambda/6$), well below the diffraction limit [66, 67].

The sub-diffraction-limited image was recorded by optical lithography at 365 nm wavelength. Using optical phonon resonance enhancement, a SiC superlens at mid-infrared frequency demonstrated a $\lambda/20$ resolution in terms of wavelength [68].

Cherenkov radiation (CR) is emitted when the charged particles move faster than the speed of light in a material. The CR was experimentally observed in the normal media by Cherenkov back in 1934. In his 1968 seminar paper, Veselago [2] made the statement that a double negative metamaterial could induce reversal of CR. Until 2003, Lu *et al.* [69] firstly verified Veselago's view in both dispersive and dissipative metamaterials. The reversed CR provides a potential application for Cherenkov detectors, since the detection sensitivity can be greatly improved due to the natural separation between the detectors for the particle and the detectors for the reversed CR in metamaterials. Many continuous works on using metamaterials to improve the Cherenkov detectors have appeared later. For example, Averkov *et al.* [70] investigated the case that an electron bunch moving in a vacuum above an isotropic metamaterial produced the reversed CR. Duan *et al.* studied the CR in a waveguide filled with anisotropic metamaterials [71], and in a waveguide filled with anisotropic metamaterials [72].

6. Numerical modelling of metamaterials

Due to many interesting potential applications of metamaterials, numerical modelling plays a very important role in the study of metamaterials and applications by providing a cheap justification of the expensive physical experiments or as a replacement for some experiments that cannot be even carried out under current situation.

Since the proposal of Yee scheme back in 1966 [73], the so-called finite-difference time-domain (FDTD) method has been one of the most useful electromagnetic modeling tools because of its simplicity and versatility. Since the discovery of metamaterials in 2000, the FDTD method has been one of the dominate simulation tools in modelling of metamaterials (see [74] and references therein). Due to our own research interest and the shortcoming of FDTD method for solving problems on complex geometric domains, here we mainly focus on the finite element methods [75].

To model metamaterials, two popular dispersive media models are often used. One is the lossy Drude model, whose permittivity and permeability in frequency domain are described by

$$\epsilon(\omega) = \epsilon_0\left(1 - \frac{\omega_{pe}^2}{\omega(\omega - j\Gamma_e)}\right), \ \mu(\omega) = \mu_0\left(1 - \frac{\omega_{pm}^2}{\omega(\omega - j\Gamma_m)}\right), \tag{5}$$

where ω_{pe} and ω_{pm} are the electric and magnetic plasma frequencies, Γ_e and Γ_m are the electric and magnetic damping frequencies, and ω is the incident wave frequency.

Using a time-harmonic variation of $exp(j\omega t)$, we can obtain the governing equations for modeling the wave propagation in metamaterials described by the Drude model (5):

$$\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{H} - \boldsymbol{J}, \tag{6}$$

$$\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} = -\nabla \times \boldsymbol{E} - \boldsymbol{K}, \tag{7}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}\frac{\partial \boldsymbol{J}}{\partial t}+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\boldsymbol{J}=\boldsymbol{E}, \tag{8}$$

$$\frac{1}{\mu_0\omega_{pm}^2}\frac{\partial \boldsymbol{K}}{\partial t}+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}\boldsymbol{K}=\boldsymbol{H}, \tag{9}$$

where $\boldsymbol{J}$ and $\boldsymbol{K}$ are the induced electric and magnetic currents, respectively.

Another popular model used for modeling wave propagation in metamaterials is described by the so-called Lorentz model, which in frequency domain is given by

$$\epsilon(\omega)=\epsilon_0(1-\frac{\omega_{pe}^2}{\omega^2-\omega_{e0}^2-j\Gamma_e\omega}),\quad \mu(\omega)=\mu_0(1-\frac{\omega_{pm}^2}{\omega^2-\omega_{m0}^2-j\Gamma_m\omega}), \tag{10}$$

where $\omega_{pe},\omega_{pm},\Gamma_e$ and Γ_m have the same meaning as the Drude model. Furthermore, ω_{e0} and ω_{m0} are the electric and magnetic resonance frequencies, respectively.

Transforming (10) into time domain, we obtain the governing equations for modeling the wave propagation in metamaterials described by the Lorentz model (10):

$$\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t}+\frac{\partial \boldsymbol{P}}{\partial t}-\nabla\times\boldsymbol{H}=0, \tag{11}$$

$$\mu_0\frac{\partial \boldsymbol{H}}{\partial t}+\frac{\partial \boldsymbol{M}}{\partial t}+\nabla\times\boldsymbol{E}=0, \tag{12}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}\frac{\partial^2 \boldsymbol{P}}{\partial t^2}+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\frac{\partial \boldsymbol{P}}{\partial t}+\frac{\omega_{e0}^2}{\epsilon_0\omega_{pe}^2}\boldsymbol{P}-\boldsymbol{E}=0, \tag{13}$$

$$\frac{1}{\mu_0\omega_{pm}^2}\frac{\partial^2 \boldsymbol{M}}{\partial t^2}+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}\frac{\partial \boldsymbol{M}}{\partial t}+\frac{\omega_{m0}^2}{\mu_0\omega_{pm}^2}\boldsymbol{M}-\boldsymbol{H}=0, \tag{14}$$

where $\boldsymbol{P}$ and $\boldsymbol{M}$ are the induced electric and magnetic polarizations, respectively.

In the last ten years, the author and his collaborators have developed various finite element methods (FEMs) for solving the above models, For example, [76, 77] developed a fully-discrete FEM solving (6)-(9) by edge elements directly. We like to point out that we can solve for $\boldsymbol{J}$ and $\boldsymbol{K}$ from (8) and (9) in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$ respectively, then substitute into (6)-(7), which leads to integral-differential type Maxwell's equations. Some FEMs have been developed for solving these integral-differential equations [78, 79, 80]. Considering some advantages of the discontinuous Galerkin (DG) methods (e.g., much simpler in programming nodal basis functions than edge elements), we also explored some DG methods for solving the Drude model [81, 82, 83]. Compared to the Drude model, there are much fewer FEMs developed for solving the Lorentz model [84, 85]. Recently, we also carried out the superconvergence investigation of edge elements on rectangular elements [86], cubic elements [87], triangular elements [88], and tetrahedral elements [89]. We like to mention that the superconvergence analysis of higher order edge elements on triangular and tetrahedral elements are still open.

In the last few years, we also started working on simulation of metamaterial cloaks by FEMs. We like to mention that the cloaking models are much more complicated than the Maxwell's equations in metamaterials described by the Drude model or the Lorentz model above, since the permittivity and permeability in cloaking are not only dispersive but also highly anisotropic. Initially we used the multiphysics commercial finite element package COMSOL to solve some famous cloak models [90]. After we feel comfortable about the physics of metamaterial cloaking, we developed our own codes to solve those cloak models using adaptive FEMs for steady-state problems [91, 92], and edge elements for time-domain cloaking models [93, 94]. Very recently, we also used COMSOL to simulate the total transmission (equivalent to cloaking) and total reflection phenomena happened in the zero-index metamaterials [95, 96].

In addition to our own works on numerical modeling of metamaterials mentioned above and numerous FDTD papers published on metamaterials (too many to be cited here), Li, Liang and Lin [97] proposed an interesting energy conserved splitting FDTD scheme for solving the Drude model (6)-(9). The scheme is unconditionally stable, and converges first order in time and second order in space.

Chung and Ciarlet [98] proposed a staggered discontinuous Galerkin method for wave propagation in media with dielectrics and metamaterials by solving the problem:

$$\begin{aligned}&\mathrm{div}(\mu^{-1}\nabla u) + \omega^2 \epsilon u = f \text{ in } \Omega \subset R^d,\ d = 1, 2, 3,\\&u = 0 \quad \text{on } \partial\Omega.\end{aligned}$$

A posteriori error estimator is derived by Nicaise and Venel [99] for transmission problems with sign changing coefficients:

$$\begin{aligned}&-\mathrm{div}(a\nabla u) = f \text{ in } \Omega \subset R^2,\\&u = 0 \quad \text{on } \partial\Omega,\end{aligned}$$

where $a \geq \epsilon_0$ in Ω_+ and $a \leq -\epsilon_0$ in Ω_-. Here Ω_+ and Ω_- are the two subdomains of Ω such that $\overline{\Omega} = \overline{\Omega}_+ \cup \overline{\Omega}_-$ and $\Omega_+ \cap \Omega_- = \emptyset$. Both lower and upper bounds are obtained.

Very recently, Yang and Wang [100] proposed a spectral element method for solving circular and elliptic cylindrical cloaks in the frequency domain. Brenner, Gedicke and Sung [101] used the Hodge decomposition approach to solve 2D time-harmonic Maxwell's equations with anisotropic electric permittivity and sign-changing magnetic permeability. Numerical results are presented for experiments that involve metamaterials and electromagnetic cloaking.

Generally speaking, there are not many papers published in regards of developing and analyzing numerical methods for solving Maxwell's equations in metamaterials. Hopefully in the near future, many excellent numerical methods proposed for solving Maxwell's equations in simple media (e.g., papers [102, 103, 104, 105, 106, 107], and books [108, 109, 110]) can be extended to metamaterials.

7. Summary

The literature on metamaterials and its potential applications in various areas such as cloaking, subwavelength imaging, and transformation optics has grown enormously in the last decade. Due to our limited knowledge, here we only touched on a very limited number of papers and subjects. Many interesting subjects such as acoustic metamaterials (e.g.,[111, 112, 113]) and hyperbolic metamaterials [114] are untouched. Interested readers can consult numerous books published in this area (e.g., [115, 116, 117, 118, 119]).

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