INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 13, Number 1, Pages 1–21

STRONG BACKWARD ERROR ANALYSIS FOR EULER-MARUYAMA METHOD

JIAN DENG

Abstract. Backward error analysis is an important tool to study long time behavior of numerical methods. The main idea of it is to use perturbed equations, namely modified equations, to represent the numerical solutions. Since stochastic backward error analysis has not been well developed so far. This paper investigates the stochastic modified equation and backward error analysis for Euler-Maruyama method with respect to strong convergence are built up. Like deterministic case, stochastic modified equations, expressed as formal series, do not converge in general. But there exists the optimal truncation of the series such that the one step error of the modified equations is sub-exponentially small with respect to time step. Moreover, the result of stochastic backward error analysis is used to study the error growth of the Euler-Maruyama method on Kubo oscillator.

Key words. backward error analysis, modified equations, strong convergence, stochastic numerical integrator.

1. Introduction

Backward error analysis is a powerful numerical analysis technique, when the qualitative behavior of numerical methods is of interest, and when statements over long time intervals are needed [4],[8],[15]. However stochastic backward error analysis is still developing, and lots of challenging problems need to be solved. One of the fundamental problems is the construction of stochastic modified equations (ME) with respect to strong convergence, which is used in backward error analysis to approximate numerical solutions. But, to the best of author's knowledge, there is no literature available for any result on strong backward error analysis. We are concerned in this paper with the construction of strong ME for the Euler-Maruyama method (EM).

For an ordinary differential equation,

$$\dot{X}_t = f(X_t), \tag{1}$$

suppose the first order numerical method $\Psi_h(X)$ with a small time step h provides an approximation to the exact solution. and it is represented in a power series of time step:

$$\Psi_h(X) = X + d_1(X)h + d_2(X)h^2 + \cdots .$$

Let ME be in form of power series of h,

$$\dot{\widetilde{X}}_t = \widetilde{f}(\widetilde{X}_t) = f(\widetilde{X}_t) + \widetilde{f}_1(\widetilde{X}_t)h + \widetilde{f}_2(\widetilde{X}_t)h^2 + \cdots ,$$

such that the the flow of ME $\widetilde{\Phi}_h$ matches with the numerical integrator Ψ_h up to arbitrary high order of h.

It is unfortunate that the power series $\widetilde{f}(\widetilde{X}_t)$ does not converge. But, under some appropriate assumption, there exists the optimal truncation such that the

Received by the editors May 20, 2015 and, in revised form, September 4, 2015.

2000 *Mathematics Subject Classification.* 65C30, 60H35.

difference of the numerical method and the flow of ME is exponentially small with respect to time step, that is

$$||\Psi_h - \widetilde{\Phi}_h|| \le Ce^{-h_0/h},$$

for some constant C and h_0. The important result allows numerical methods to be interpreted by ME. Lots of application on structure preserving method are reported, see [8][9].

To extend the idea to stochastic setting, let us consider stochastic differential equations (SDE) in the form of Stratonovich integral,

$$dX_t = f_0(X_t)dt + \sum_{r=1}^{m} f_r(X_t) \circ dW_t^r = \sum_{r=0}^{m} f_r(X_t) \circ dW_t^r, \tag{2}$$

where $X_t \in \mathbb{R}^n$, $f_r : \mathbb{R}^n \to \mathbb{R}^n$, $W_t^r, r = 1, \ldots, m$ are independent standard Wiener processes, and t is denoted as W_t^0 for notational convenience. Then we say the numerical scheme $\Psi_h(X_0)$ has strong order k if

$$\sqrt{E[||X_{nh} - \Psi_h^n(X_0)||^2]} \le Ch^k, \quad 0 < nh \le T.$$

Sometimes a weaker error is sufficient to use. If the numerical scheme $\Psi_h(X_0)$ satisfies that

$$||E[\phi(X_{nh})] - E[\phi(\Psi_h^n(X_0))]|| \le Ch^k, \quad 0 < nh \le T,$$

where $\phi(x)$ belongs to some smooth function spaces. Then we say the numerical method has a weak order k.

Shardlow [16] made an attempt by considering the perturbed functions $\widetilde{f}_r$ in stochastic ME have the form

$$\widetilde{f}_r = \sum_{i=0}^{N} \widetilde{f}_{r,i} h^i.$$

When the weak error is considered, the construction can only be performed for EM at $N = 2$ with additive noise. For multiplicative noise or higher order, there are too many conditions to determinate the coefficients of ME. In [17] [1] [2], ME with respect to a week convergence are constructed. Moreover, Debussche et al. [5] built up the weak backward error analysis via modified partial differential equations on torus for EM. Kopac [11] extended the approach to Langvin process on $\mathbb{R}^n$.

In this paper, an alternative approach to construct a perturbed function $\widetilde{f}_r$ is proposed for EM,

$$\widetilde{f}_r = \sum_{\alpha} \widetilde{f}_{r,\alpha} J_{\alpha,t},$$

where $J_{\alpha,t}$ are multiple Stratonovich integrals. Moreover, we prove that there exists the optimal truncation such that

$$\sqrt{E[||\widetilde{\Phi}_{h,N} - \Psi_h||^2]} \le Ce^{-h_0/(h^{\frac{1}{3}})}.$$

By using this result, the error growth of EM on the Kubo oscillator is investigated.

We emphasize that the proposed modified equations works for SDE with multiplicative noise. The proof given in the paper is different with those provided for ordinary differential equations [9]. We consider the implementation of EM on Kubo oscillator and discuss the error growth of it by using the stochastic backward error analysis result.

The paper is organized as follows. In the next section, we studied the product and second moments of multiple Stratonovich integrals. Then, we introduce the assumption that we need. Section 3 presents the construction of ME. Estimation of the coefficients of ME and the main theorem on optimal truncation are presented in Section 4 and 5, respectively. In the last section, the result is applied to study the long time error of EM.

2. Preliminaries

2.1. Properties of Multiple Stratonovich Integrals. Before constructing strong ME, we study the multiple Stratonovich integrals as preparation. For a multi-index $\alpha = (j_1, \ldots, j_l)$, $j_i \in \{0, 1, \ldots, m\}$ for $i = 1, \ldots, l$, the multiple Stratonovich itegral is defined as follows,

$$J_{\alpha,t} = \int_0^t \int_0^{s_l} \cdots \int_0^{s_2} \circ dw_{s_1}^{j_1} \cdots \circ dw_{s_{l-1}}^{j_{l-1}} \circ dw_{s_l}^{j_l}. \tag{3}$$

The length of the multi-index α is denoted as $l(\alpha)$. $n(\alpha)$ is the number of zeros in the multi-index. The function $p(\alpha) = l(\alpha) + n(\alpha)$ is introduced. For example, $l((2, 0, 1, 0)) = 4$, $n((2, 0, 1, 0)) = 2$ and $p((2, 0, 1, 0)) = 6$.

A multi-index of length zero v is included for completeness with $J_{v,s} := 1$ and $n(v) = 0$.

For any multi-index $\alpha = (j_1, j_2, \ldots, j_l)$ with no duplicated elements (i.e., $j_m \neq j_n$ if $m \neq n$, $m, n = 1, \ldots, l$), we define the set $R(\alpha)$ to be the empty set $R(\alpha) = \Phi$ if $l = 0, 1$ and $R(\alpha) = \{(j_m, j_n) | m < n, m, n = 1, \ldots, l\}$ if $l \geq 2$. $R(\alpha)$ defines a partial order on the set formed with the numbers included in the multi-index α, defined by $i \prec j$ if and only if $(i, j) \in R(\alpha)$. For example, $R((2, 0, 1)) = \{(2, 0), (2, 1), (0, 1)\}$. We suppose that there are no duplicated elements in or between the multi-indices $\alpha = (j_1, j_2, \ldots, j_l)$ and $\alpha' = (j'_1, j'_2, \ldots, j'_{l'})$.

Lemma 2.1. [6] [3] *If there are no duplicated elements in or between any of the multi-index* $\alpha_1 = (j_1^{(1)}, j_2^{(1)}, \ldots, j_{l_1}^{(1)})$, $\ldots$, $\alpha_n = (j_1^{(n)}, j_2^{(n)}, \ldots, j_{l_n}^{(n)})$, *then*

$$J_{\alpha_1,t} \ldots J_{\alpha_n,t} = \sum_{\beta \in \Lambda_{\alpha_1,\ldots,\alpha_n}} J_{\beta,t} \tag{4}$$

where

$$\begin{aligned} \Lambda_{\alpha_1,\ldots,\alpha_n} = &\{\beta \in \mathcal{M} | l(\beta) = \sum_{k=1}^{n} l(\alpha_k) \textit{ and } \cup_{k=1}^{n} R(\alpha_k) \subseteq R(\beta) \\ &\textit{and there are no duplicated elements in } \beta\}, \end{aligned} \tag{5}$$

and $\mathcal{M} = \{(\hat{j}_1, \hat{j}_2, \ldots, \hat{j}_{\hat{l}}) | \hat{j}_i \in \{j_1^{(1)}, j_2^{(1)}, \ldots, j_{l_1}^{(1)}, \ldots j_1^{(n)}, j_2^{(n)}, \ldots, j_{l_n}^{(n)}\}, i = 1, \ldots, \hat{l}, \hat{l} = l_1 + \cdots + l_n\}$.

To extend the lemma to multi-index with duplicated elements, we just need to assign different subscripts to each duplicated element, for example, $\Lambda_{(2,0),(0,1)} = \Lambda_{(2,0_1),(0_2,1)} = \{(2, 0_2, 1, 0_1), (0_2, 2, 1, 0_1), (0_1, 1, 2, 0_2), (0_2, 2, 0_1, 1), (2, 0_1, 0_2, 1), (2, 0_2, 0_1, 1)\}$. So we will not distinguish the cases with or without duplicated elements in the following of the paper.

For multi-indices ξ_1 and ξ_2, we define $\xi_1 \times \xi_2 := \Lambda_{\xi_1,\xi_2}$. So $\alpha \in \xi_1 \times \xi_2$ means that $J_{\alpha,t}$ can be generated by the production of $J_{\xi_1,t}$ and $J_{\xi_2,t}$. Then, we have the following properties,

Property 2.2.
- *1. If* $\xi_1 \times \xi_2 \ni \alpha$, *then* $l(\xi_1) + l(\xi_2) = l(\alpha)$ *and* $p(\xi_1) + p(\xi_2) = p(\alpha)$.
- *2.* $\#\Lambda_{\xi_1,\xi_2} = \binom{l(\xi_1)+l(\xi_2)}{l(\xi_1)} \leq \binom{p(\xi_1)+p(\xi_2)}{p(\xi_1)}$.

- *3.* $\#\{\xi_1, \xi_2 | \xi_1 \times \xi_2 \ni \alpha\} = \binom{l(\xi_1)+l(\xi_2)}{l(\xi_1)} \le \binom{p(\xi_1)+p(\xi_2)}{p(\xi_1)}$.

Proof: 1) follows from Lemma 2.1.

2) The problem is equivalent to calculate the number of multi-indices whose length is $l(\xi_1) + l(\xi_2)$, and keeping the orders of ξ_1 and ξ_2. That is equivalent to choose $l(\xi_1)$ components from the whole components of ξ_1 and ξ_2. So $\#\Lambda_{\xi_1,\xi_2} = \binom{l(\xi_1)+l(\xi_2)}{l(\xi_1)}$. The inequality follows from that $p(\xi) \ge l(\xi)$.

3) The proof is similar with that in 2). □

Lemma 2.1 induces a way to calculate the moments of multiple Stratonovich integrals. But first of all, we introduce multiple Ito integrals. For a multi-index $\alpha = (j_1, \dots, j_l)$, $j_i \in \{0, 1, \dots, m\}, i = 1, \dots, l$,

$$I_{\alpha,t}[f(\cdot,\cdot)] := \int_0^t \int_0^{s_l} \dots \int_0^{s_2} f(s_1, \cdot) dw_{s_1}^{j_1} \dots dw_{s_{l-1}}^{j_{l-1}} dw_{s_l}^{j_l}, \quad I_{\alpha,t} := I_{\alpha,t}[1],$$

where f is any appropriate process. By the martingale property of Ito integration, it is shown that

$$E[I_{\alpha,t}] = \begin{cases} 0, & \text{if } l(\alpha) \neq n(\alpha) \\ \frac{t^{l(\alpha)}}{l(\alpha)!}, & \text{if } l(\alpha) = n(\alpha). \end{cases} \tag{6}$$

The idea to calculate the moments of multiple stratonovich integral is to express the power of multiple Stratonovich integrals by a summation of other multiple Stratonovich integrals first; then transforming these Stratonovich integrals into the form of Ito by the recurrence relationship (see Chapter 5 in [10])

$$\begin{aligned} &J_{\alpha,t} = I_{\alpha,t}, \quad l(\alpha) = 1 \\ &J_{\alpha,t} = I_{(j_l),t}\left[J_{\alpha-,t}\right] + \frac{1}{2}\chi_{\{j_l=j_{l-1}\neq 0\}} I_{(0),t}\left[J_{(\alpha-)-,t}\right], \quad l(\alpha) \ge 2, \end{aligned} \tag{7}$$

where χ_A denotes the indicator function of the set A , and $\alpha-$ defines the multi-index by deleting the last component of α. In the last step, the expectations of multiple Ito integrals is obtained by (6).

We take $E[J^2_{(1,0,1),t}]$ as example to demonstrate the process. First, by Lemma 2.1, we have

$$J^2_{(1,0,1),t} = 8J_{(1,1,0,0,1,1),t} + 4J_{(1,1,0,1,0,1),t} + 4J_{(1,0,1,1,0,1),t} + 4J_{(1,0,1,0,1,1),t}.$$

Transforming the multiple Stratonovich integral to Ito integrals,

$$\begin{aligned} J_{(1,1,0,0,1,1),t} &= I_{(1,1,0,0,1,1),t} + \frac{1}{2}(I_{(1,1,0,0,0),t} + I_{(0,0,0,1,1),t}) + \frac{1}{2} I_{(0,0,0,0),t}, \\ J_{(1,1,0,1,0,1),t} &= I_{(1,1,0,1,0,1),t} + \frac{1}{2} I_{(0,0,1,0,1),t}, \\ J_{(1,0,1,1,0,1),t} &= I_{(1,0,1,1,0,1),t} + \frac{1}{2} I_{(1,0,0,0,1),t}, \\ J_{(1,0,1,0,1,1),t} &= I_{(1,0,1,0,1,1),t} + \frac{1}{2} I_{(1,0,1,0,0),t}. \end{aligned}$$

By (6),

$$E[J^2_{(1,0,1),t}] = 4E[I_{(0,0,0,0),t}] = \frac{t^4}{3!}.$$

The previous example shows that most of the expectations of the multiple Ito integrals vanishes. When calculating the moments of multiple Stratonovich integrals, we only consider the multiple Stratonovich integrals that will be transforming to the Ito integrals, $I_{(0,\dots,0)}$. Then we have the following lemma on the second moments of multiple Stratonovich integrals.

Lemma 2.3.

$$2^{n(\alpha)}\frac{h^{p(\alpha)}}{p(\alpha)!} \leq E[J_{\alpha,h}^2] \leq \binom{2l(\alpha)}{l(\alpha)} 2^{n(\alpha)-l(\alpha)}\frac{h^{p(\alpha)}}{p(\alpha)!},$$

Proof: Let

$$J_{\alpha,h}^2 = \sum_{\beta} J_{\beta,t}.$$

$J_{\beta,t}$ can be transformed to the Ito integrals $I_{(0,\ldots,0)}$, if and only if $\beta \in \mathfrak{L}$, where

$$\mathfrak{L} = \{\beta = (j_1,\ldots,j_l) \in \alpha\times\alpha | \beta^+ = v \text{ or } \beta^+ = (j_{i_1},\ldots,j_{i_{2q}}), q \in \mathbb{N},$$
$$i_{2k} = i_{2k-1}+1, j_{2k} = j_{2k-1} \text{ for } k = 1,\ldots,q\},$$

and β^+ denotes the multi-index obtained by dropping all zero components of β.

So

$$E[J_{\alpha,h}^2] = s(\alpha)2^{n(\alpha)-l(\alpha)}\frac{h^{p(\alpha)}}{p(\alpha)!}, \tag{8}$$

where $s(\alpha) = \#\mathfrak{L}$.

Then the calculation of the second moment of Stratonovich integrals is simplified to estimate $s(\alpha)$.

First, $s(\alpha)$ is not greater than $\#\Lambda_{\alpha,\alpha}$. So,

$$E[J_{\alpha,h}^2] \leq \binom{2l(\alpha)}{l(\alpha)} 2^{n(\alpha)-l(\alpha)}\frac{h^{p(\alpha)}}{p(\alpha)!}.$$

One the other hand, let $\alpha = (j_1,\ldots,j_l)$, and provide a superscript for another α, such that $\alpha' = (j_1',\ldots,j_l')$. It is noticed that there is no difference with α and α', but the superscript. So the multi-index $\beta_1 = (j_1, j_1', j_2, j_2', \ldots, j_l, j_l')$ belongs to $\mathfrak{L}$. The multi-index $\beta_2 = (j_1', j_1, j_2, j_2', \ldots, j_l, j_l')$, obtained by interchanging the first two components of β_1 , is also in $\mathfrak{L}$. Considering all the switching between j_1 and j_1', j_2 and j_2', ..., j_l and j_l', we know that $s(\alpha) \geq 2^{l(\alpha)}$, and

$$E[J_{\alpha,h}^2] \geq 2^{n(\alpha)}\frac{h^{p(\alpha)}}{p(\alpha)!} \quad \square$$

Remark 2.4. *When $l(\alpha)$ and $n(\alpha)$ are known, there exists multiple Stratonovich integrals such that the equalities in Lemma 2.3 hold. So the estimation of the second moment of multiple Stratonovich integrals is sharp.*

Corollary 2.5.

$$E[J_{\alpha,h}^2] \leq \frac{(2h)^{p(\alpha)}}{p(\alpha)!}$$

.

Proof: It follows from the fact $\binom{2l(\alpha)}{l(\alpha)} \leq 4^{l(\alpha)}$ $\square$

2.2. Assumption. Throughout the paper, we use C to denote a generic positive constant, not necessarily the same at different occurrences.

Let us assume that f_r for $r = 1,\ldots,m$ in (2) belong to $\mathcal{C}^\infty$. In addition, we suppose that

- [H1] f_r for $r = 1,\ldots,m$ satisfies one of the following conditions.

$$\sup_{\substack{|\mathbf{k}|=j\\ 0\leq r\leq m}} |\frac{\partial^{|\mathbf{k}|} f_r}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}| \leq K_j \quad \text{for } r = 0,\ldots,m, \text{ and } j = 0,1,\ldots,$$

or

$$\sup_{\substack{|\mathbf{k}|=j\\0\le r\le m}} |\frac{\partial^{|\mathbf{k}|} f_r}{\partial x_1^{k_1}\dots\partial x_n^{k_n}}| \le K_j(1+||X||)^{1-j} \quad \text{for } r=0,\dots,m, \text{ and } j=0,1,\dots.$$

- [H2] There exist a positive constant K, such that

$$K_j \le CK^j \quad \text{for } j=0,1,\dots.$$

The Hypothesis [H1] is usually needed for the numerical analysis for long time behavior [13],[5] and the convergence of high order numerical schemes [14], [6], [7]. The last hypothesis [H2] is similar with the result of Cauchy estimate, which is used in the backward error analysis for deterministic problems.

For multi-indices $\alpha=(\alpha_1,\dots,\alpha_l)$, $\beta=(\beta_1,\dots,\beta_l)$ and functions f_r in SDE (2), where $\alpha_i\in\mathbb{N}\cup\{0\}$ and $\beta_i\in\mathbb{N}$, we define the function classes

$$B_{i,j,k} := \{\sum_{\alpha,\beta} a_{\alpha,\beta}(g_1^{(\alpha_1)})^{\beta_1}\dots(g_l^{(\alpha_l)})^{\beta_l} | \quad g_n\in\{f_r\}, \quad a_{\alpha,\beta}\in\mathbb{R}, \quad l(\alpha)=l(\beta),$$
$$\sum_{n=1}^{l}\beta_n(1-\alpha_n)=k, \quad \sum_{n=1}^{l}\alpha_n\beta_n\le i, \quad \text{and} \sum_{n=1}^{l}\beta_n\le j\},$$

and

$$B_{i,j} := B_{i,j,1},$$

where $(g_1^{(\alpha_1)})^{\beta_1}$ means the β_1-th power of the α_1-th derivatives of g_1. For example,

$$f_1,\dots,f_m\in B_{0,1,1} \quad \text{and } f_0+\frac{1}{2}f_rf_r'\in B_{1,2,1}$$

If $z\in B_{i,j,k}$, the representation of z is not unique, like $z=f_0^{(1)}=2f_0^{(1)}-f_0^{(1)}$. But we only consider the representation without duplication, such that $\sum_{\alpha,\beta}|a_{\alpha,\beta}|$ achieves its minimum, and is denoted as $\mathfrak{K}(z)$. It can be shown that the functions in $B_{i,j,k}$ have the following properties:

Property 2.6.

- *If $z\in B_{i,j,k}$, then $||z||\le C\mathfrak{K}(z)K^i$, or $||z||\le C\mathfrak{K}(z)K^i(1+||X||)^k$*
- *If $z_1,z_2\in B_{i,j,k}$, then $z_1+z_2\in B_{i,j,k}$, and $\mathfrak{K}(z_1+z_2)\le\mathfrak{K}(z_1)+\mathfrak{K}(z_2)$.*
- *If $z_1\in B_{i_1,j_1,k_1}$, $z_2\in B_{i_2,j_2,k_2}$, then $z_1z_2\in B_{i_1+i_2,j_1+j_2,k_1+k_2}$ and $\mathfrak{K}(z_1z_2)\le\mathfrak{K}(z_1)\mathfrak{K}(z_2)$.*
- *If $z\in B_{i,j,k}$, then $\frac{\partial z}{\partial x_k}\in B_{i+1,j,k-1}$ and $\mathfrak{K}(\frac{\partial z}{\partial x_k})\le j\mathfrak{K}(z)$ for $k=1,\dots,n$.*

Proof: The proof of the properties is straightforward, and not shown here. □

2.3. Stratonvich Taylor expansion. The Stratonvich Taylor expansion for the SDE (2) can be expressed in form of

$$X_t = X_0+\sum_{l(\alpha)\ge0}\sum_{r=0}^{m}\mathcal{L}^\alpha f_r|_{t=0}J_{\alpha*(r),t}, \tag{9}$$

where $*$ is the concatenation operation, like $(1,0)*(2)=(1,0,2)$.

For $\alpha=(j_1,j_2,\dots,j_l)$, $\mathcal{L}^\alpha$ is defined as

$$\mathcal{L}^\alpha=\mathcal{L}^{(j_1)}\mathcal{L}^{(j_2)}\dots\mathcal{L}^{(j_l)}, \tag{10}$$

with

$$\mathcal{L}^{(j)} = f_j \cdot \bigtriangledown, \quad \text{for } j = 0, \ldots, m, \tag{11}$$

where $\cdot$ is the dot product and $\bigtriangledown$ is the gradient operator. We define $\mathcal{L}^v$ as the identity operator.

3. Construction of ME

Let the differential equation (1) or SDE (2) define the flow $\Phi_t : X_0 \to X_t$, which is approximated by numerical method Ψ_h .

Backward error analysis is to search for ME, such that the flow of the modified equation $\widetilde{\Phi}_h$ is closed to Ψ_h. Then ME, instead of Ψ_h, are used to the study the error growth and other numerical analysis topics.

3.1. Deterministic ME. Let ME for the deterministic differential equations (1) have the form of a power series

$$\dot{\widetilde{X}}_t = \widetilde{f}(\widetilde{X}_t, h) = \widetilde{f}_0(\widetilde{X}_t) + \widetilde{f}_1(\widetilde{X}_t)h + \widetilde{f}_2(\widetilde{X}_t)h^2 + \cdots .$$

Then we have,

$$\begin{aligned}
\dot{\widetilde{X}}_0 &= \widetilde{f}(\widetilde{X}_0, h) = \widetilde{f}(X_0)_0 + \widetilde{f}_1(X_0)h + \widetilde{f}_2(X_0)h^2 + \cdots , \\
\ddot{\widetilde{X}}_0 &= \frac{\partial \widetilde{f}(\widetilde{X}_t, h)}{\partial \widetilde{X}_t} \dot{\widetilde{X}}_0 = (\widetilde{f}_0'(X_0) + \widetilde{f}_1'(X_0)h + \cdots)(\widetilde{f}_0(X_0) + \widetilde{f}_1(X_0)h + \cdots).
\end{aligned}$$

So,

$$\begin{aligned}
\widetilde{X}_h &= \widetilde{X}_0 + \dot{\widetilde{X}}_h + \ddot{\widetilde{X}}_0 \frac{h^2}{2} + \cdots \\
&= x_0 + (\widetilde{f}(X_0) + \widetilde{f}_1(X_0)h + \widetilde{f}_2(X_0)h^2 + \cdots)h \\
&+ \frac{h^2}{2}(\widetilde{f}'(X_0) + \widetilde{f}_1'(X_0)h + \cdots)(\widetilde{f}(X_0) + \widetilde{f}_1(X_0)h + \cdots) + \cdots .
\end{aligned} \tag{12}$$

Assume the numerical method Ψ_h is given as a power series

$$\Psi_h(X_0) = x_0 + hf(X_0) + h^2 d_2(X_0) + \cdots . \tag{13}$$

To obtain $\widetilde{X}_{kh} = \Psi_h^k(X_0)$ for all k, we must have $\widetilde{X}_h = \Psi_h(X_0)$. Comparing the coefficients of powers of h in (12) and (13) yields the recurrence relation of $\widetilde{f}_i$

$$\begin{aligned}
&\widetilde{f}_0 = f, \\
&\widetilde{f}_1 = d_2 - \frac{1}{2!} f' f, \\
&\widetilde{f}_2 = d_3 - \frac{1}{3!}(2f'' f + f' f' f) - \frac{1}{2!}(f' \widetilde{f}_2 + \widetilde{f}_2' f) \\
&\cdots
\end{aligned}$$

Therefore, ME are constructed and uniquely determined by the numerical method Ψ_h.

3.2. Stochastic ME. In this section, the construction of stochastic ME with respect to strong convergence is studied. The recurrence relation of the coefficients in ME is provided. It shows that the existence and uniqueness of strong ME.

First, we define the stochastic processes $Y_{\alpha,t}$ for a given time step h: if $l(\alpha)>0$

$$Y_{\alpha,t} := J_{\alpha,t} - J_{\alpha,nh}, \quad \text{when } nh < t \leq (n+1)h, \quad n \in \mathbb{N}^+ \bigcup \{0\},$$

and

$$Y_{v,t} := 1.$$

It is clear that $Y_{\alpha,t} = J_{\alpha,t}$ for $0 < t \leq h$.

We suppose that $\widetilde{f}_r$ in ME has the form

$$\widetilde{f}_r = \sum_{\alpha} \widetilde{f}_{r,\alpha} Y_{\alpha,t}, \tag{14}$$

where $\widetilde{f}_r : \mathbb{R}^n \to \mathbb{R}^n$ and $\widetilde{f}_{r,\alpha} : \mathbb{R}^n \to \mathbb{R}^n$.

The whole stochastic ME are expressed as

$$\widetilde{X}_t = \widetilde{X}_0 + \sum_{r=0}^{m} \int_0^t \sum_{\alpha} \widetilde{f}_{r,\alpha}(\widetilde{X}_s) Y_{\alpha,s} \circ dW_s^r. \tag{15}$$

Remark 3.1. *It should be noticed that $Y_{\alpha,s}$, instead of $Y_{\alpha,h}$, is used to guarantee ME are non-anticipative.*

To implement the Stratonvich Taylor expansion on ME (15), we consider $Y_{\alpha,s}$ are also variable in ME. To demonstrate the idea, we consider an example with $m=1$ and $d=1$. When $0<t\leq h$, (15) is written as

$$\begin{aligned}
\widetilde{X}_t &= \widetilde{X}_0 + \int_0^t \sum_{\alpha} \widetilde{f}_{0,\alpha}(\widetilde{X}_s) Y_{\alpha,s} \circ ds + \int_0^t \sum_{\alpha} \widetilde{f}_{1,\alpha}(\widetilde{X}_s) Y_{\alpha,s} \circ dW_s, \\
Y_{(0),t} &= \int_0^t \circ ds, \\
Y_{(1),t} &= \int_0^t \circ dW_s, \\
&\vdots \\
Y_{\alpha*(0),t} &= \int_0^t Y_{\alpha,s} \circ ds, \\
Y_{\alpha*(1),t} &= \int_0^t Y_{\alpha,s} \circ dW_s, \\
&\vdots
\end{aligned} \tag{16}$$

Notice that $\widetilde{f}_r = \sum_\alpha \widetilde{f}_{r,\alpha}(\widetilde{X}_s) Y_{\alpha,s}$ are functions depend on the variables $(\widetilde{X}_t, Y_{(0),t}, Y_{(1),t}, \ldots)$. So,

$$\mathcal{L}^{(0)} \widetilde{f}_r = 1 \cdot \widetilde{f}_{r,(0)} + 0 \cdot \widetilde{f}_{r,(1)} + \cdots + Y_\alpha \cdot \widetilde{f}_{r,\alpha*(0)} + \cdots + f_0 \widetilde{f}_r',$$

$$\mathcal{L}^{(1)} \widetilde{f}_r = 0 \cdot \widetilde{f}_{r,(0)} + 1 \cdot \widetilde{f}_{r,(1)} + \cdots + Y_\alpha \cdot \widetilde{f}_{r,\alpha*(1)} + \cdots + f_1 \widetilde{f}_r'.$$

So, for ME (15) with $d \geq 1$ and $m \geq 1$,

$$\begin{aligned}
\mathcal{L}^{(j)} \widetilde{f}_r &= \sum_{\alpha} \widetilde{f}_{r,\alpha*(j)} Y_{\alpha,t} + \sum_{\alpha} f_j \cdot \triangledown \widetilde{f}_{r,\alpha} Y_{\alpha,t} \\
&= \sum_{\alpha} \widetilde{f}_{r,\alpha*(j)} J_{\alpha,t} + \sum_{\alpha} f_j \cdot \triangledown \widetilde{f}_{r,\alpha} J_{\alpha,t}.
\end{aligned} \tag{17}$$

As $\widetilde{f}_r$ is the series of multiple Stratonovich integrals, $\mathcal{L}^{(j)}\widetilde{f}_r$ is expressed as a series of multiple Stratonovich integrals. By induction, for multi-index γ, $\mathcal{L}^{\gamma}\widetilde{f}_r$ is also a series of multiple Stratonovich integrals, and $\mathcal{L}^{\gamma}\widetilde{f}_r$ can be denoted as follows,

$$\mathcal{L}^{\gamma}\widetilde{f}_r = \sum_{\alpha} \psi_{\alpha}^{\gamma*(r)} J_{\alpha,t}, \quad 0 < t \leq h, \tag{18}$$

where $\psi_{\alpha}^{\gamma*(r)}$ are functions depend on $\widetilde{X}_t$. When $\gamma = v$, $\psi_{\alpha}^{(r)} = \widetilde{f}_{r,\alpha}$ by the definition of $\widetilde{f}$.

Similar with (17), for multi-index γ which satisfies $l(\gamma) \geq 1$, and $j = 0, \ldots, m$, we obtain

$$\begin{aligned} \sum_{\alpha} \psi_{\alpha}^{(j)*\gamma} J_{\alpha,t} &= \sum_{\alpha} \mathcal{L}^{(j)} \psi_{\alpha}^{\gamma} J_{\alpha,t} = \sum_{\alpha} \psi_{\alpha*(j)}^{\gamma} J_{\alpha,t} + \sum_{\alpha} \widetilde{f}_j \cdot \nabla \psi_{\xi_1}^{\gamma} J_{\alpha,t} \\ &= \sum_{\alpha} \psi_{\alpha*(j)}^{\gamma} J_{\alpha,t} + \sum_{\xi_1,\xi_2} \psi_{\xi_1}^{(j)} \cdot \nabla \psi_{\xi_2}^{\gamma} J_{\xi_1,t} J_{\xi_2,t}. \end{aligned} \tag{19}$$

By comparing coefficients of $J_{\alpha,t}$ and Lemma 2.1, we have

$$\psi_{\alpha*(j)}^{\gamma} = \psi_{\alpha}^{(j)*\gamma} - \sum_{\xi_1 \times \xi_2 \ni \alpha} \psi_{\xi_1}^{(j)} \cdot \nabla \psi_{\xi_2}^{\gamma}, \quad \text{for } l(\gamma) \geq 1, \text{ and } l(\alpha) \geq 0. \tag{20}$$

Applying the Stratonovich Taylor expansion (9) on ME (15), we get

$$\begin{aligned} \widetilde{\Phi}_h(X_0) &= X_0 + \sum_{l(\alpha)\geq 0} \sum_{r=0}^{m} \mathcal{L}^{\alpha} \widetilde{f}_r|_{t=0} J_{\alpha*(r),t} \\ &= X_0 + \sum_{l(\alpha)\geq 0} \sum_{r=0}^{m} (\sum_{\beta} \psi_{\beta}^{\alpha*(r)}(\widetilde{X}_t) J_{\beta,t})|_{t=0} J_{\alpha*(r),h} \\ &= X_0 + \sum_{l(\alpha)\geq 0} \sum_{r=0}^{m} \psi_{v}^{\alpha*(r)}(X_0) J_{\alpha*(r),h}. \end{aligned} \tag{21}$$

Assume the numerical method Ψ_h is expressed by the summation of multiple Stratonovich integrals,

$$\Psi_h X_0 = X_0 + \sum_{l(\alpha)>0} d_{\alpha}(X_0) J_{\alpha,h}.$$

For EM, we have

$$\begin{aligned} & d_{(0)} = f_0 + \frac{1}{2} \sum_{r=1}^{m} f_r \cdot \nabla f_r, \\ & d_{(r)} = f_r \quad \text{for} \quad r = 1, \ldots, m, \\ & d_{\gamma} = 0 \quad \text{for} \quad l(\gamma) \geq 2. \end{aligned} \tag{22}$$

To obtain $\Psi_h = \widetilde{\Phi}_h$, the coefficients of $J_{\alpha,t}$ in (21) should equal to those in the numerical method (22), i.e., $d_{\gamma*(r)} = \psi_v^{\gamma*(r)}$. Therefore $\widetilde{f}_{r,\alpha} = \psi_{\alpha}^{(r)}$ and ME are uniquely defined by the recurrence relation (20).

To demonstrate the idea of construction ME, we take the linear SDE $dX_t = AX_t dt + BX_t \circ dW_t$ as example. EM is applied on the linear SDE. So it is clear that

$$d_{(0)}(X_0) = \psi_v^{(0)} = A + \frac{1}{2}B^2X_0,$$
$$d_{(r)}(X_0) = \psi_v^{(1)} = BX_0 \quad r = 1, \ldots, m,$$
$$d_\gamma(X_0) = \psi_v^\gamma = 0 \quad \text{for} \quad l(\gamma) \geq 2.$$

From the recurrence relation (20), we have

$$\psi_{(0)}^{(0)}(X_0) = -(A + \frac{1}{2}B^2)^2X_0, \quad \psi_{(1)}^{(1)}(X_0) = -B^2X_0,$$
$$\psi_{(0)}^{(1)}(X_0) = -(A + \frac{1}{2}B^2)BX_0, \quad \psi_{(1)}^{(0)}(X_0) = -B(A + \frac{1}{2}B^2)X_0.$$

Since $\psi_v^{(1,1)} = 0$ and $\psi_v^{(1,1,1)} = 0$, (20) yields that

$$\psi_{(1)}^{(1,1)}(X_0) = 0, \quad \psi_{(1,1)}^{(1)}(X_0) = 2B^3X_0.$$

Similarly,

$$\psi_{(1)}^{(1,1,1)}(X_0) = 0, \quad \psi_{(1,1)}^{(1,1)}(X_0) = 0, \quad \psi_{(1,1,1)}^{(1)}(X_0) = -6B^4X_0.$$

As a special case consider $A = -aJ$ and $B = -\sigma J$ where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, a and σ is constant. The SDE

$$dX_t = -aJX_tdt - \sigma JX_tdW_t^1 \tag{23}$$

is a linear stochastic Hamiltonian system, called Kubo oscillator.

Let $\widetilde{f}_r$ in ME (15) be truncated as

$$\widetilde{f}_{r,N} = \sum_{p(\alpha*(r))\leq N} \widetilde{f}_{r,\alpha}Y_{\alpha,s} = \sum_{p(\alpha*(r))\leq N} \psi_\alpha^{(r)}Y_{\alpha,s},$$

where N is a positive integer.

The ME with $N = 2$ is

$$\begin{aligned} \widetilde{X}_t = X_0 + \int_0^t &-(aJ + \frac{\sigma^2}{2}I)\widetilde{X}_s - [(\frac{\sigma^4}{4} - a^2)I + a\sigma^2J]\widetilde{X}_sY_{(0),s} \\ &- [\frac{\sigma^3}{2}J - a\sigma I]\widetilde{X}_sY_{(1),s}ds + \int_0^t -\sigma J\widetilde{X}_s + \sigma^2\widetilde{X}_sY_{(1),s} + 2\sigma^3J\widetilde{X}_sY_{(1,1),s} \\ &- 6\sigma^4\widetilde{X}_sY_{(1,1,1),s} - [\frac{\sigma^3}{2}J - a\sigma I]\widetilde{X}_sY_{(0),s} \circ dW_s^1. \end{aligned} \tag{24}$$

The comparison of ME (24), original equation (23) and EM is presented in Figure 1. It can been seen that an excellent agreement of the numerical solution with the exact solution of ME (24).

4. Estimation of the Coefficients of ME

Recalling that the EM is expressed as

$$\Psi_hX_0 = X_0 + \sum_{l(\alpha)>0} d_\alpha(X_0)J_{\alpha,h}, \tag{25}$$

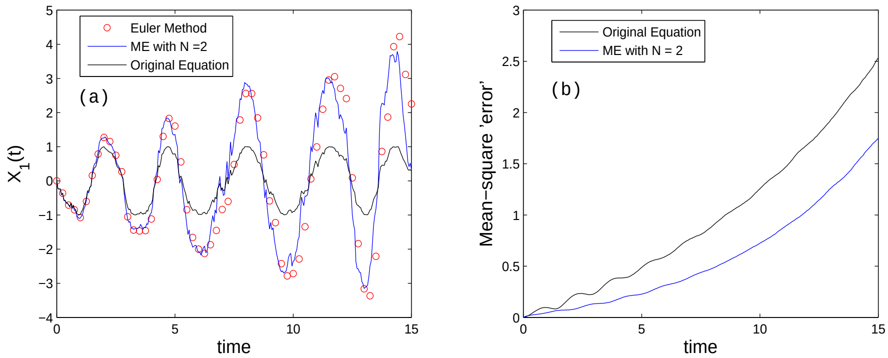

FIGURE 1. Comparison of various equations and Euler method with $a = 2$ and $\sigma = 0.5$: (a) Sample trajectories; (b) Mean-square error.

where

$$\begin{aligned} \psi_v^{(0)} &= d_{(0)} = f_0(X_k) + \frac{1}{2}\sum_{r=1}^{m} f_r \cdot \bigtriangledown f_r, \\ \psi_v^{(r)} &= d_{(r)} = f_r \quad \text{for} \quad r = 1, \ldots, m, \\ \psi_v^{\gamma} &= d_\gamma = 0 \quad \text{for} \quad l(\gamma) \geq 2. \end{aligned} \tag{26}$$

So, we have the following Lemma:

Lemma 4.1. *For EM (25), if the Hypothesis [H1] and [H2] are satisfied, then* ψ_α^γ *in (18) belongs to* $B_{p(\alpha)+p(\gamma)-1,p(\alpha)+p(\gamma)}$*, and*

$$\mathfrak{K}(\psi^\gamma_{\alpha*(j)}) \leq \mathfrak{K}(\psi_\alpha^{(j)*\gamma}) + n \sum_{\xi_1\times\xi_2\ni\alpha} [p(\gamma)+p(\xi_2)]\mathfrak{K}(\psi_{\xi_1}^{(j)})\mathfrak{K}(\psi_{\xi_2}^{\gamma}), \tag{27}$$

Proof: First $\psi_v^{(0)} = d_{(0)} \in B_{1,2}$, $\psi_v^{(r)} = d_{(r)} \in B_{0,1}$. Then the lemma follows by applying an induction on the recurrence relation (20) with Properties 2.2 and 2.6. ∎

Theorem 4.2. *For EM (25) and* $r = 0, \ldots, m$*, if the Hypothesis [H1] and [H2] are satisfied, then*

$$\mathfrak{K}(\psi_\alpha^{(r)}) \leq (\frac{m}{2}+1)l(\alpha)![n(m+2)l(\alpha)]^{l(\alpha)}.$$

Proof: Fixing an integer $N \geq 1$, and considering the ψ_α^γ satisfying $l(\gamma)+l(\alpha) \leq N$. For $j = 0, \ldots, m$, Lemma 4.1 implies

$$\mathfrak{K}(\psi^\gamma_{\alpha*(j)}) \leq \mathfrak{K}(\psi_\alpha^{(j)*\gamma}) + 2nN \sum_{\xi_1\times\xi_2=\alpha} \mathfrak{K}(\psi_{\xi_1}^{(j)})\mathfrak{K}(\psi_{\xi_2}^{\gamma}), \tag{28}$$

Applying induction on ψ_α^γ, it can be shown that

$$\mathfrak{K}(\psi^\gamma_{\alpha*(j)}) \leq (2nN)^{-1}u_{l(\gamma),l(\alpha)+1}, \quad l(\gamma)+l(\alpha) \leq N \text{ and } l(\gamma) \geq 1, \tag{29}$$

where $u_{l(\gamma),l(\alpha)}$ depends on $l(\gamma)$ and $l(\alpha)$, and is given by

$$u_{l(\gamma),l(\alpha)+1} = u_{l(\gamma)+1,l(\alpha)} + \sum_{\xi_1\times\xi_2=\alpha} u_{1,l(\xi_1)}u_{l(\gamma),l(\xi_2)}.$$

Let $l(\xi_1) = i$. Property 2.2 implies

$$u_{l(\gamma),l(\alpha)+1} = u_{l(\gamma)+1,l(\alpha)} + \sum_{i=0}^{l(\alpha)} C^i_{l(\alpha)} u_{1,i} u_{l(\gamma),l(\alpha)-i}, \tag{30}$$

Applying the exponential generating function

$$b(x,y) = \sum_{k_1,k_2 \geq 0} \frac{1}{k_1!k_2!} u_{k_1+1,k_2} x^{k_1} y^{k_2}$$

to (30), we have the first order partial differential equation:

$$\begin{aligned} &\frac{\partial b}{\partial y} = \frac{\partial b}{\partial x} + b(0,y)b(x,y), \quad \text{on} \quad [0,+\infty)\times[0,+\infty), \\ &b(x,0) = nN(2+m). \end{aligned} \tag{31}$$

The boundary condition is given by $\mathfrak{K}(\psi_v^{(0)}) = 1 + \frac{m}{2}$, $\mathfrak{K}(\psi_v^{(r)}) = 1$ for $r = 1, \ldots, m$ and $\mathfrak{K}(\psi_v^{\gamma}) = 0$ for $l(\gamma) > 1$,

In the following of the proof, we are going to solve the partial differential equations (31).

First consider the partial differential equations below:

$$\begin{aligned} &\frac{\partial \bar{b}}{\partial y} = \frac{\partial \bar{b}}{\partial x} + g(y)\bar{b}(x,y) \quad \text{on} \quad [0,+\infty)\times[0,+\infty), \\ &\bar{b}(x,0) = nN(2+m). \end{aligned} \tag{32}$$

where $g(y)$ is a function on $[0,+\infty)$.

Then, we have the solution of (32),

$$\bar{b}(x,y) = nN(2+m)e^{\int_0^y g(s)ds}. \tag{33}$$

Let

$$\bar{b}(0,y) = nN(2+m)e^{\int_0^y g(s)ds} = g(y). \tag{34}$$

It implies the ordinary differential equation

$$\frac{\partial g}{\partial y} = g^2, \quad g(0) = nN(2+m),$$

which has the solution

$$g(y) = \frac{nN(2+m)}{1 - nN(2+m)y}. \tag{35}$$

Due to (32) and (34), we solve the equations (31) by substituting (35) into (33). So

$$u_{1,N} = \left.\frac{\partial^N b}{\partial y^N}\right|_{0,0} = \left.\frac{\partial^N g}{\partial y^N}\right|_{0} = N![nN(2+m)]^{N+1}.$$

For $l(\alpha) = N-1$ and $l(\gamma) = 1$, (29) yields that

$$\begin{aligned} \mathfrak{K}(\psi^{\gamma}_{\alpha*(j)}) &\leq (2nN)^{-1} u_{1,l(\alpha)+1} \leq (2nN)^{-1} N![nN(2+m)]^{N+1} \\ &\leq (\frac{m}{2}+1)N!(nN(m+2))^N. \end{aligned}$$

□

5. Errors of ME

In general, ME do not converge, but we can find an optimized truncation for a sufficient small time step. Let ME match with EM up to mean square order N, i.e., $\psi_v^{\gamma*(r)} = d_{\gamma*(r)}$ for $p(\gamma*(r)) \leq N$ and $r = 0, \ldots, m$. Then the coefficients $\widetilde{f}_{r,\alpha}$ of ME are uniquely defined by the recurrence relation (20) for $p(\alpha*(r)) \leq N$. So, we consider ME truncated by the form of

(36)
$$\widetilde{X}_{t,N} = X_0 + \int_0^t \sum_{r=0}^m \widetilde{f}_{r,N}(\widetilde{X}_{s,N}) \circ dW_s^r = X_0 + \int_0^t \sum_{p(\alpha*(r))\leq N} \psi_\alpha^{(r)}(\widetilde{X}_{s,N}) Y_{\alpha,s} \circ dW_s^r,$$

which means that $\psi_\alpha^{(r)} = 0$, when $p(\alpha*(r)) > N$.

5.1. Uniqueness and Existence of Stochastic ME. By property 2.6 and theorem 4.2, we show that $\widetilde{f}_r$ is linearly growth,

$$\begin{aligned} E\|\widetilde{f}_{r,N}(X)\|^2 &= E\|\sum_{p(\alpha*(r))\leq N} \psi_\alpha^{(r)}(X) Y_{\alpha,s}\|^2 \\ &\leq CN \sum_{k=1}^N \sum_{p(\alpha*(r))=k} E\|\psi_\alpha^{(r)}(X)\|^2 E[Y_{\alpha,s}^2] \\ &\leq L_1(h,N)(1+E\|X\|^2) \quad \text{for any } X \in \mathbb{R}^n, \end{aligned} \tag{37}$$

where

$$L_1(h,N) = CN \sum_{k=1}^N k![n(m+1)(m+2)kK]^{2k}[2h]^k.$$

Since $\frac{\partial \psi_\alpha^{(r)}}{\partial x_i} \in B_{p(\alpha*(r)),p(\alpha*(r)),0}$ for $i = 1, \ldots, n$, similarly, we get

$$E\|\frac{\partial \widetilde{f}_{r,N}}{\partial x_i}\|^2 = E\|\sum_{p(\alpha*(r))\leq N} \frac{\partial \psi_\alpha^{(r)}}{\partial x_i} Y_{\alpha,s}\|^2 \leq L_2(h,N),$$

where

$$L_2(h,N) = CK^2N \sum_{k=1}^N k!k^2[n(m+2)(m+1)kK]^{2k}(2h)^k.$$

So

$$E\|\widetilde{f}_{r,N}(X_1) - \widetilde{f}_{r,N}(X_2)\|^2 \leq L_2(h,N) E\|X_1 - X_2\|^2. \tag{38}$$

Then we have an existence and uniqueness results on the stochastic ME.

Theorem 5.1. *If the Hypothesis [H1] and [H2] are satisfied, then the truncated modified equation (36) for EM with fixed h and N has a unique continuous solution $\widetilde{X}_t$ with the property that*

$$E\|\widetilde{X}_{h,N}\|^2 \leq C(1+3\|X_0\|^2)e^{C_1 L_1(h,N)h},$$

where C and C_1 are positive constants independent on h and N.

Proof: The proof of theorem 5.1 is similar to that of Theorem 2.3.1 and Lemma 2.3.2 in [12], and the detail is omitted here. ∎

Corollary 5.2. *Under the Hypothesis [H1] and [H2], if $h_1 N^3 h \leq 1/9$ and $h_1 = 2[n(m+2)(m+1)K]^2$, then*

$$L_1(h,N) < \infty, \quad L_2(h,N) < \infty$$

and

$$E||\widetilde{X}_{h,N}||^2 < \infty.$$

Proof: When $h_1 N^3 h \leq 1/9$, motivated by the fact that $\sum_{k=1}^{N}(\frac{k}{N})^k < \infty$, we know

$$\begin{aligned} L_1(h,N) &= CN\sum_{k=1}^{N} k![n(m+2)(m+1)kK]^{2k}(2h)^k \\ &\leq C\sum_{k=1}^{N}\frac{k!}{N^{N-1}}(\frac{k}{3N})^{2k} \\ &\leq C\sum_{k=1}^{N}(\frac{k}{3N})^{2k} < \infty. \end{aligned}$$

Moreover

$$\begin{aligned} L_2(h,N) &= CK^2 N\sum_{k=1}^{N} k!k^2[n(m+2)(m+1)kK]^{2k}(2h)^k \\ &\leq CK^2\sum_{k=1}^{N}\frac{k!}{N^{N-1}}k^2(\frac{k}{3N})^{2k} \\ &\leq CK^2\sum_{k=1}^{N}\frac{k!}{N^{N-1}}(\frac{k}{N})^{2k} < \infty. \end{aligned}$$

Then the corollary follows. □

5.2. Optimized Truncation of ME. If ME are truncated in the form of (36), we have the following estimation on ψ_α^γ

Lemma 5.3. *If the Hypothesis [H1] and [H2] are satisfied, ψ_α^γ in the truncated ME (36) for EM have the following properties:*

1) When $p(\alpha) + p(\gamma) \leq N$, and $l(\gamma) > 1$,

$$\psi_\alpha^\gamma = 0.$$

2) When $l(\gamma) = 2$,

$$\mathfrak{K}(\psi_\alpha^\gamma) \leq C[n(m+2)]^{p(\alpha)} p(\alpha)!(q-1)^{q-1} N^{N+1},$$

where $p(\alpha) + p(\gamma) = N + q$ with $1 \leq q \leq N$.

3) When $\gamma = (j_2, j_2, j_1)$ and $j_2 \neq 0$

$$\mathfrak{K}(\psi_\alpha^\gamma) \leq C[n(m+2)]^{p(\alpha)} p(\alpha)!(q-1)^{q-1} N^{N+2},$$

where $p(\alpha) + p(\gamma) = N + q$ with $1 \leq q \leq 2N$.

4) When $p(\alpha) + p(\gamma) > N \cdot l(\gamma)$,

$$\psi_\alpha^\gamma = 0.$$

Proof: See Appendix. □

Then the second moment of $\mathcal{L}^{(j_2)}\widetilde{f}_{j_1,N}(\widetilde{X}_{s,N})$ and $\mathcal{L}^{(j_2,j_2)}\widetilde{f}_{j_1,N}(\widetilde{X}_{s,N})$ can be estimated by Lemma 5.3.

Lemma 5.4. *Under the Hypothesis [H1] and [H2], if* $h_1 N^3 h \leq 1/9$ *where* $h_1 = 2[n(m+2)(m+1)K]^2$*, and* $\gamma = (j_1, j_2)$ *then*

1)
$$E||\mathcal{L}^{j_2}\widetilde{f}_{j_1,N}(\widetilde{X}_{s,N})||^2 \leq CN^{3N+2-p(\gamma)}(h_1 h)^{N-p(\gamma)}$$

2)
$$E||\mathcal{L}^{(j_2,j_2)}\widetilde{f}_{j_1,N}(\widetilde{X}_{s,N})||^2 \leq CN^{3N+3-p(\gamma)}(h_1 h)^{N-p(\gamma)}$$

for $s \leq h$

Proof: See Appendix. □

Let $\widetilde{\Phi}_{t,N}$ denote the stochastic flow associated to the truncated ME (36). We can find an optimal truncation for the one step difference between $\widetilde{\Phi}_{h,N}$ and EM Ψ_h.

Theorem 5.5. *Under the Hypothesis [H1] and [H2], for a sufficiently small h, there exists* $N^* = \lfloor e^{-1}(h_1 h)^{-1/3} \rfloor$ *and* $h_1 = 2[n(m+2)(m+1)K]^2$ *such that the difference between the ME and EM is bounded by*

$$\sqrt{E[||\widetilde{\Phi}_{h,N^*} - \Psi_h||^2]} \leq Ce^{-h_0/(h^{\frac{1}{3}})},$$

where h_0 *is a positive constant.*

Proof: Applying the stochastic Taylor expansion into the modified equations (see Lemma 5.6.4 in [10]), we have

$$\widetilde{\Phi}_{h,N}(X_0) - X_0 = \sum_{r=0}^{m} \widetilde{f}_r(\widetilde{X}_0) J_{(r),h} + R,$$

where R is the remainder term of the stochastic Taylor expansion, which is expressed as

$$R = \sum_{j_1=0}^{m}\sum_{j_2=0}^{m}\int_0^h\int_0^{s_1} \mathcal{L}^{(j_2)}\widetilde{f}_{j_1}(\widetilde{X}_{s_2}) \circ dw_{s_2}^{j_2} \circ dw_{s_1}^{j_1}. \tag{39}$$

By (26) and the definition of ψ_α^γ (18), the difference of ME and EM is remainder R, i.e.

$$\widetilde{\Phi}_{h,N}(X_0) - \Psi_h(X_0) = R.$$

Then we estimate the second moment of the remainder term R

By the relation of multiple Stratonovich integrals and multiple Ito integrals (see Chapter 5 in [10]), the remainder term is expressed in the form of Ito as

$$R = \sum_{j_1=0}^{m}\sum_{j_2=0}^{m}\int_0^h\int_0^{s_1} \mathcal{L}^{j_2}\widetilde{f}_{j_1}(\widetilde{X}_{s_2}) dw_{s_2}^{j_2} dw_{s_1}^{j_1} + \frac{m}{2}\int_0^h\int_0^{s_1} \mathcal{L}^{j_2}\widetilde{f}_{j_1}(\widetilde{X}_{s_2}) ds_2 ds_1$$
$$+ \frac{1}{2}\sum_{j_1=0}^{m}\sum_{j_2=1}^{m}\int_0^h\int_0^{s_1} \mathcal{L}^{(j_2,j_2)}\widetilde{f}_{j_1}(\widetilde{X}_{s_2}) ds_2 dw_{s_1}^{j_1}.$$

Thanks to Ito isometry, the Cauchy-Schwarz inequality and Lemma 5.4, we have

$$E[R^2] \leq C(h_1 N^3 h)^N,$$

Motivated by the fact that $(\epsilon x^3)^x$ reaches it minimum on $x = (e^3\epsilon)^{-1/3}$, we take N^* as the integer part of $e^{-1}(h_1 h)^{-1/3}$ such that $h_1(N^*)^3 h \leq e^{-3} \leq 1/9$, then

$$E[R^2] \leq Ce^{-3N^*}. \tag{40}$$

Since $e^{-1}(h_1 h)^{-1/3} - 1 \leq N^*$, for h is sufficiently small,

$$\sqrt{E[R^2]} \leq Ce^{-1/[2(h_1 h)^{\frac{1}{3}}]} \leq Ce^{-h_0/h^{\frac{1}{3}}}.$$

for $h_0 = \frac{1}{2}(h_1)^{-\frac{1}{3}}$ □

Moreover, the global error of ME is resulted as follows

Theorem 5.6. *Let $\widetilde{\Phi}_{t,N^*}$ be the flow associated by the optimal truncated ME stated in Theorem 5.5, and Ψ_t be EM, then over the time interval of length $T = O(h^{-\frac{1}{3}})$,*

$$\sqrt{E[||\widetilde{\Phi}_{nh,N^*} - \Psi_h^n||^2]} \leq Ce^{-h_0/(h^{\frac{1}{3}})}, \quad \textit{for } nh < T,$$

where h_0 is a positive constant, and C does not depends on T.

Proof: The proof of theorem 5.6 is similar to that of Theorem 1.1 by taking advantage of Corollary 5.2, and the detail is omitted here. □

6. Error growth of EM on Kubo oscillator

As a first application of Theorem 5.6, we study the error growth of Kubo oscillator(23). First, the ME of EM is constructed as

$$\begin{aligned}\widetilde{X}_t = X_0 + \int_0^t &-(aJ + \frac{\sigma^2}{2}I)\widetilde{X}_s - [(\frac{\sigma^4}{4} - a^2)I + a\sigma^2 J]\widetilde{X}_s Y_{(0),s} \\ &- [\frac{\sigma^3}{2}J - a\sigma I]\widetilde{X}_s Y_{(1),s} + \cdots ds + \int_0^t -\sigma J\widetilde{X}_s + \sigma^2 \widetilde{X}_s Y_{(1),s} + 2\sigma^3 J\widetilde{X}_s Y_{(1,1),s} \\ &- 6\sigma^4 \widetilde{X}_s Y_{(1,1,1),s} - [\frac{\sigma^3}{2}J - a\sigma I]\widetilde{X}_s Y_{(0),s} + \cdots \circ dW_s^1.\end{aligned}$$

The exact solution of ME can be expressed in the following form using the equal-distance time discretization $0 = t_0 < t_1 < \cdots < t_N = T$, where the time step h is a small positive number:

$$\widetilde{X}_{t_{k+1}} = e^{\lambda_k} F(\widetilde{\theta}_k)\widetilde{X}_{t_k}, \quad \widetilde{X}_{t_0} = X_0, \quad k = 0, 1, \ldots, N-1$$

where

$$\lambda_k = -\frac{\sigma^2}{2}h + \sigma^2 Y_{(1,1),t_{k+1}} + a\sigma Y_{(1),t_{k+1}}h - (\frac{\sigma^4}{4} - a^2)\frac{h^2}{2} - 6\sigma^4 Y_{(1,1,1,1),t_{k+1}} + \cdots,$$

$$F(\widetilde{\theta}_k) = \begin{bmatrix} \cos\widetilde{\theta} & \sin\widetilde{\theta} \\ -\sin\widetilde{\theta} & \cos\widetilde{\theta} \end{bmatrix}, \quad \widetilde{\theta}_k = ah + \sigma Y_{(1),t_{k+1}} - 2\sigma^3 Y_{(1,1,1),t_{k+1}} + \frac{\sigma^3}{2}Y_{(1),t_{k+1}}h + \cdots.$$

Moreover, the exact solution of Kubo oscillation is

$$X_{t_{k+1}} = F(\theta_k)X_{t_k}, \quad \widetilde{X}_{t_0} = X_0, \quad k = 0, 1, \ldots, N-1,$$

where

$$\theta_k = ah + \sigma Y_{(1),t_{k+1}}.$$

We get

$$E[e^{\lambda_k}] \sim h^{\frac{3}{2}}, \quad \text{and} \quad E[(\theta_k - \widetilde{\theta}_k)^2] \sim h^3, \quad k = 0, 1, \ldots, N-1.$$

Hence if Th is sufficiently small, then

$$
\begin{aligned}
E||\widetilde{X}_{t_N} - X_{t_N}||^2 &\le E||e^{\sum_{k=0}^{N-1}\lambda_k} F(\sum_{k=0}^{N-1}\widetilde{\theta}_k) - F(\sum_{k=0}^{N-1}\theta_k)||^2 \\
&\le E[(e^{\sum_{k=0}^{N-1}\lambda_k} - 1)^2] + E||F(\sum_{k=0}^{N-1}\widetilde{\theta}_k) - F(\sum_{k=0}^{N-1}\theta_k)||^2 \\
&= \mathcal{O}(n^2h^3) = \mathcal{O}(T^2h)
\end{aligned}
$$

Due to Theorem 5.6, the mean square error of the Euler-Maruyama method is $\mathcal{O}(T\sqrt{h})$ when $Th^{\frac{1}{3}}$ is sufficiently small.

Acknowledgments

I am indebted to Dr. Y. S. Wong and Dr. C. Anton for advices that both influenced the course of this research and improved its presentation.

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Appendix A. Proofs of Lemmas 5.3 and 5.4

Proof of Lemma 5.3: 1) Let γ be a multi-index satisfying $p(\gamma) \leq N$ and $l(\gamma) > 1$. Since $\psi_v^\gamma = d_\gamma = 0$ for $l(\gamma) > 1$, (20) implies that

$$\psi_{(j)}^\gamma = \psi_v^{(j)*\gamma} - \psi_v^{(j)} \cdot \bigtriangledown \psi_v^\gamma = 0.$$

It shows that $\psi_\alpha^\gamma = 0$, when $p(\alpha) + p(\gamma) \leq N$, $l(\alpha) = 1$ and $l(\gamma) > 1$.

For a fixed integer i, assume $\psi_\alpha^\gamma = 0$, when $p(\alpha) + p(\gamma) \leq N$, $l(\alpha) < i$ and $l(\gamma) > 1$. Let $\hat{\alpha}$ be the multi-index with $l(\hat{\alpha}) = i - 1$. by applying the recurrence relation (20) to $\psi_{\alpha*(j)}^\gamma$, we have

$$\psi_{\hat{\alpha}*(j)}^\gamma = \psi_{\hat{\alpha}}^{(j)*\gamma} - \sum_{\xi_1 \times \xi_2 = \hat{\alpha}} \psi_{\xi_1}^{(j)} \cdot \bigtriangledown \psi_{\xi_2}^\gamma = 0,$$

where $l(\hat{\alpha}) < i$ and $l(\xi_1) < i$. So $\psi_\alpha^\gamma = 0$, when $p(\alpha) + p(\gamma) \leq N$, $l(\alpha) = i$ and $l(\gamma) > 1$. Hence 1) is proved by induction.

2) For $N + 1 \leq p(\alpha) + p(\gamma) \leq 2N$, let $\gamma = (j_1, j_2)$, then (20) implies

$$\psi_\alpha^{(j_1,j_2)} = \psi_{\alpha*(j_1)}^{(j_2)} + \sum_{\xi_1 \times \xi_2 = \alpha} \psi_{\xi_1}^{(j_1)} \cdot \bigtriangledown \psi_{\xi_2}^{(j_2)}.$$

Here $p(\alpha * (j_2)) + p((j_1)) = p(\alpha) + p(\gamma) > N$ implies $\psi_{\alpha*(j_2)}^{(j_1)} = 0$.

Since $p(\xi_2) + p((j_2)) \leq p(\alpha) + p(\gamma) \leq 2N$ and $\psi_{\alpha*(j_2)}^{(j_1)} = 0$, from Property 2.6 it follows that

$$\mathfrak{K}(\psi_\alpha^{(j_1,j_2)}) \leq 2nN \sum_{\xi_1 \times \xi_2 = \alpha} \mathfrak{K}(\psi_{\xi_1}^{(j_1)}) \mathfrak{K}(\psi_{\xi_2}^{(j_2)}). \tag{A.1}$$

Let $i = p(\xi_1)$, then $p(\xi_2) = p(\alpha) - p(\xi_1) = N + q - p(\gamma) - i$. Because $\psi_\alpha^{(j)} = 0$ for $p(\alpha) + p((j)) > N$, $\psi_{\xi_1}^{(j_1)}$ and $\psi_{\xi_2}^{(j_2)}$ are zero if $i = p(\xi_1) > N - p((j_1))$ and $N + q - p(\gamma) - i = p(\xi_2) > N - p((j_2))$, i.e. $i < q - p((j_1))$, respectively. So applying Property 2.2 and Theorem 4.2 into (A.1), we obtain

$$\begin{aligned} &\mathfrak{K}(\psi_\alpha^{(j_1,j_2)}) \\ &\leq CN[n(m+2)]^{N+q-p(\gamma)}(N - p(\gamma) + q)! \sum_{i=q-p((j_1))}^{N-p((j_1))} (N - p(\gamma) + q - i)^{N-p(\gamma)+q-i} i^i. \end{aligned}$$

Since the function $(b-x)^{d-x}(a+x)^{c+x}$ is concave up on $[0, b-a]$, when $a+d = b+c$ and $a \geq c$, it achieves its maximum on $x = 0$ or $b - a$. Then,

$$\begin{aligned} &\mathfrak{K}(\psi_\alpha^{(j_1,j_2)}) \\ &\leq CN[2n(m+2)]^{N+q-p(\gamma)}(N - p(\gamma) + q)!(N - q + 1)(q-1)^{q-1}(N-1)^{N-1} \\ &\leq C[2n(m+2)]^{p(\alpha)} p(\alpha)!(q-1)^{q-1} N^{N+1}. \end{aligned}$$

3) Denote

$$\phi_\alpha^{(j_2,j_2,j_1)} = \sum_{\xi_1 \times \xi_2 = \alpha} \sum_k \psi_{k,\xi_1}^{(j_2)} \frac{\partial \psi_{\xi_2}^{(j_2,j_1)}}{\partial x_k}.$$

Since $p(\xi_2) + p((j_2, j_1)) \leq p(\alpha) + p(\gamma) \leq 3N$, it follows that

$$\mathfrak{K}(\phi_\alpha^{(j_2,j_2,j_1)}) \leq 3nN \sum_{\xi_1 \times \xi_2 = \alpha} \mathfrak{K}(\psi_{\xi_1}^{(j_2)}) \mathfrak{K}(\psi_{\xi_2}^{(j_2,j_1)}). \tag{A.2}$$

Because $\psi_{\xi_2}^{(j_2,j_1)} = 0$ with $p((j_2,j_1)) + p(\xi_2) \le N$, $\phi_\alpha^{(j_2,j_2,j_1)} = 0$ if $p((j_2,j_1)) + p(\xi_2) < p(\alpha) + p((j_2,j_2,j_1)) \le N+1$

Suppose $p(\alpha) + p(\gamma) = N + q$ with $2 \le q \le 2N$. Let $p(\xi_1) = i$, then $p(\xi_2) = N + q - p(\gamma) - i$. So $\psi_{\xi_1}^{(j_2)} = 0$ when $i+1 > N$; on the other hand $\psi_{\xi_2}^{(j_2,j_1)} = 0$ when $p(\xi_2) + p((j_2,j_1)) < N+1$ or $p(\xi_2) + p((j_2,j_1)) > 2N$. It means that $\phi_\alpha^\gamma \ne 0$ when $0 \le i \le N-1$ and $q-1-N \le i \le q-2$. So When $q \le N+1$, by Theorem 4.2 and the results in 3), we have

$$\begin{aligned}&\mathfrak{K}(\phi_\alpha^{(j_2,j_2,j_1)})\\ &\le CN^{N+2}[n(m+2)]^{N+q-p(\gamma)}(N+q-p(\gamma))!\sum_{i=0}^{q-2}(q-2-i)^{q-2-i}i^i\\ &\le C[n(m+2)]^{p(\alpha)}p(\alpha)!(q-1)^{q-1}N^{N+2}.\end{aligned}$$

When $1 + N < q \le 2N$,

$$\begin{aligned}&\mathfrak{K}(\phi_\alpha^{(j_2,j_2,j_1)})\\ &\le CN^{N+2}[2n(m+2)]^{N+q-p(\gamma)}(N+q-p(\gamma))!\sum_{i=q-1-N}^{N-1}(q-2-i)^{q-2-i}i^i\\ &\le CN^{N+2}[2n(m+2)]^{p(\alpha)}p(\alpha)!(q-1-N)^{q-1-N}N^{N-1}\\ &\le CN^{N+2}[2n(m+2)]^{p(\alpha)}p(\alpha)!(q-1)^{q-1}.\end{aligned}$$

Then 3) is follows from

$$\psi_\alpha^{(j_2,j_2,j_1)} = \psi_{(j_2)}^{(j_2,j_1)} + \phi_\alpha^{(j_2,j_2,j_1)} = \psi_{(j_2)}^{(j_2,j_1)} + \sum_{\xi_1\times\xi_2=\alpha}\sum_k \psi_{k,\xi_1}^{(j_2)}\frac{\partial\psi_{\xi_2}^{(j_2,j_1)}}{\partial x_k}.$$

4) Since $\psi_\alpha^\gamma = 0$ for $l(\gamma) = 1$ and $p(\gamma) + p(\alpha) > N$, the recurrence relation (20) yields to the result 3). ∎

Proof of Lemma 5.4: Let $\gamma = (j_1, j_2)$, by (18) and (5.3), we can show that $\mathcal{L}^{j_2}\widetilde{f}_{j_1,N}$ is expressed in the form of summation $\psi_\alpha^{(j_1,j_2)}$ as follow,

$$\mathcal{L}^{j_2}\widetilde{f}_{j_1,N}(\widetilde{X}_{s,N}) = \sum_{p(\alpha)+p(\gamma)=N+1}^{2N}\psi_\alpha^\gamma(\widetilde{X}_{s,N})Y_{\alpha,s}. \tag{A.3}$$

Since $\#\{\alpha,\gamma|p(\alpha)+p(\gamma)=k\} \le (m+1)^k$ and $(\sum_{i=1}^N a_i)^2 \le N\sum_{i=1}^N(a_i^2)$, we obtain

$$\begin{aligned}E[(\mathcal{L}^{j_2}\widetilde{f}_{j_1,N}(\widetilde{X}_{s,N}))^2] &\le N\sum_{k=N+1}^{2N}E[(\sum_{p(\alpha)+p(\gamma)=k}\psi_\alpha^\gamma(\widetilde{X}_{s,N})Y_{\alpha,s})^2]\\ &\le N\sum_{k=N+1}^{2N}(m+1)^k\sum_{p(\alpha)+p(\gamma)=k}E||\psi_\alpha^\gamma(\widetilde{X}_{s,N})||^2E[Y_{\alpha,s}^2.]\end{aligned}$$

We also know that $\psi_\alpha^\gamma \in B_{p(\alpha)+p(\gamma)-1,p(\alpha)+p(\gamma)}$, then by Property 2.6 and Corollary 5.2,

$$E||\psi_\alpha^\gamma(\widetilde{X}_{s,N})||^2 \le CK^{2(p(\alpha)+p(\gamma)-1)}(\mathfrak{K}(\psi_\alpha^\gamma))^2 \quad \text{for } s \le h.$$

Therefore

(A.4)

$$E[(\mathcal{L}^{j_2}\widetilde{f}_{j_1,N}(\widetilde{X}_{s,N}))^2] \leq N\sum_{k=N+1}^{2N}(m+1)^k\sum_{p(\alpha)+p(\gamma)=k}E||\psi_\alpha^\gamma(\widetilde{X}_{s,N})||^2E[Y_{\alpha,s}^2]$$
$$\leq CN\sum_{k=N+1}^{2N}[(m+1)K^2]^k\sum_{p(\alpha)+p(\gamma)=k}[\mathfrak{K}(\psi_\alpha^\gamma)]^2E[Y_{\alpha,s}^2].$$

Let $p(\alpha)+p(\gamma)=N+q$ for $1\leq q\leq N$, and $s\leq h$. Applying Corollary 2.5 and Proposition 5.3 into (A.4), we thus obtain

$$E[(\mathcal{L}^{j_2}\widetilde{f}_{j_1,N}(\widetilde{X}_{s,N}))^2] \leq$$
$$CN^{2N+3}\sum_{q=1}^{N}[(m+1)K^2]^{N+q}\sum_{\substack{p(\alpha)+p(\gamma)\\=N+q}}(2n^2(m+2)^2h)^{p(\alpha)}p(\alpha)!(q-1)^{2(q-1)}.$$

Then $p(\gamma)\leq 4$ implies

$$E[(\mathcal{L}^{j_2}\widetilde{f}_{j_1,N}(\widetilde{X}_{s,N}))^2] \leq CN^{2N+3}\sum_{q=1}^{N}[n(m+2)(m+1)K]^{2p(\alpha)}(2h)^{p(\alpha)}p(\alpha)!(q-1)^{2(q-1)}.$$

Let $h_1=2n^2(m+2)^2(m+1)^2K^2$,

(A.5)

$$E[(\mathcal{L}^{j_2}\widetilde{f}_{j_1,N}(\widetilde{X}_{s,N}))^2] \leq CN^{2N+3}\sum_{q=1}^{N}(h_1h)^{p(\alpha)}p(\alpha)!(q-1)^{2(q-1)}$$
$$= CN^{2N+3}(h_1h)^{N-p(\gamma)}\sum_{q=1}^{N}p(\alpha)!(h_1(q-1)^2h)^qq^{-2},$$

For $h_1N^3h\leq 1/9<1$, the summation term in (A.5) is bounded by

(A.6)

$$\sum_{q=1}^{N}p(\alpha)!(h_1(q-1)^2h)^qq^{-2}$$
$$\leq\sum_{q=1}^{N}\frac{(N+q-p(\gamma))!}{N^{q+1}}(\frac{q}{N})^{2q}q^{-2}$$
$$\leq\sum_{q=1}^{N}\frac{((N+q-p(\gamma)+1)/2)^{N+q-p(\gamma)}}{N^{q+1}}(\frac{q}{N})^{2q}$$
$$\leq\sum_{q=1}^{N}\frac{N^{N+q-p(\gamma)}}{N^{q+1}}(\frac{q}{N})^{2q}$$
$$\leq CN^{N-p(\gamma)-1}.$$

It can be verified that the summation $\sum_{q=1}^{N}(\frac{q}{N})^{2q}$ in (A.6) is bounded by a constant. Then the proof 1) is completed.

2) Similarly, let $\gamma = (j_2, j_2, j_1)$ and $j_2 \neq 0$, we can show that

$$\begin{aligned} & E[(\mathcal{L}^{(j_2,j_2)}\widetilde{f}_{j_1,N}(\widetilde{X}_{s,N}))^2] \\ & = 2N \sum_{q=N+1}^{3N} (m+1)^k \sum_{p(\alpha)+p(\gamma)=q} E||\psi_\alpha^{(j_2,j_2,j_1)}(\widetilde{X}_{s,N})||^2 E[Y_{\alpha,s}^2] \\ & \le CN^{2N+4} \sum_{q=N+1}^{3N} p(\alpha)!(h_1 h)^{p(\alpha)}(q-1)^{q-1} \\ & \le CN^{2N+4}(h_1 h)^{N-p(\gamma)} \sum_{q=N+1}^{3N} p(\alpha)!(h_1(q-1)^2 h)^q q^{-2}. \end{aligned} \tag{A.7}$$

When $9h_1N^3h \le 1$, the summation term in (A.7) is bounded by

$$\sum_{q=N+1}^{3N} p(\alpha)!(h_1(q-1)^2h)^q q^{-2} \le \sum_{q=N+1}^{3N} \frac{N^{N+q-p(\gamma)}}{N^{q+1}}(\frac{q}{3N})^{2q} \le CN^{N-p(\gamma)-1}.$$

□

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1 CANADA

E-mail: deng2@ualberta.ca