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VARIATIONAL FORMULATION FOR MAXWELL'S EQUATIONS WITH LORENZ GAUGE: EXISTENCE AND UNIQUENESS OF SOLUTION

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Abstract. The existence and uniqueness of a vector scalar potential representation with the Lorenz gauge (Schelkunoff potential) is proven for any vector field from H(curl). This representation holds for electric and magnetic fields in the case of a piecewise smooth conductivity, permittivity and permeability, for any frequency. A regularized formulation for the magnetic field is obtained for the case when the magnetic permeability μ is constant and thus the magnetic field is divergence free. In the case of a non divergence free electric field, an equation involving scalar and vector potentials is proposed. The solution to both electric and magnetic formulations may be approximated by the nodal shape functions in the finite element method with system matrices that remain well-conditioned for low frequencies. A numerical study of a forward problem of a computation of electromagnetic fields in the diffusive electromagnetic regime shows the efficiency of the proposed method.

Key words. Lorenz gauge, Schelkunoff potential, Maxwell's equations, Finite Element Method, Nodal shape functions, Regularization

1. Introduction

Fast and stable methods are needed for calculating electromagnetic (EM) fields in and over the Earth. Such a simulation has applications in imaging of subsurface electrical conductivity structures related to exploration for geothermal, mining, and hydrocarbon resources. Over commonly used frequencies, EM propagation in the Earth is diffusive since the conduction dominates over the dielectric displacement. The finite element method (FEM) is attractive for this simulation in comparison with other techniques in that it may be easily adapted to complex boundaries between regions of constant EM properties, including the topography or the bathymetry. The 3D interpretation of geophysical data is numerically expensive, as the forward problem needs to be computed many times [26, 3, 14].

For large scale simulation problems, iterative methods have been the ones of choice to solve linear systems resulting from FEM formulations [7, 16, 11, 34, 29]. The speed of iterative methods is strongly related to the properties of the variational problem used. Difficulties arise when the computational domain includes a high contrast, both the non-conducting air and a conducting medium in the Earth's subsurface, especially for low frequencies. Furthermore, the Earth's subsurface in general is characterized by the spatially changing conductivity, dielectric permittivity and magnetic permeability. This can slow or prevent iteration convergence [23, 31].

There have been multiple approaches to addressing the difficulties encountered with high physical property contrasts and potentially discontinuous EM field variables. One is to apply special finite elements, so-called edge elements, that have a discontinuous normal component of the vector field across elements, while keeping the tangential field component continuous [24, 18, 4]. The edge elements are also

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compatible with the curl operator and are a part of the de Rham diagram [6]. However, if the curl-curl equation for the electric field E is used, and if the conductivity is very small in a part of the domain (e.g., in the air) or if the frequency is very low, the problem becomes ill-posed and the system matrix has a very large near null space. This requires use of sophisticated preconditioners that handle the null space of the curl properly in order to use iterative solvers. Such preconditioners have been developed (see [38, 17, 19, 21, 2, 39]).

An alternative is to not solve directly for the EM fields themselves, but instead to initially solve a well conditioned equation for a quantity which is continuous across interfaces. Subsequently, the EM fields are obtained through a spatial differentiation with the field discontinuities defined by the property jumps. One such quantity is a vector potential with the Lorenz gauge, also called the Schelkunoff potential [37, 8, 33, 9], which we examine in this paper. In general, this potential has both scalar and vector components, and there are both electric and magnetic versions. Using the Lorenz gauge, the scalar potential can be expressed as a function of the vector potential, and as a result only the vector potential is needed to represent the EM field.

In this paper, we show that the Lorenz gauged vector potential representation exists for any member of $\mathcal{H}(\nabla\times)$. Thus one can use it to represent the electric field E as well as the magnetic field H. We prove that this representation exists for any frequency $\omega > 0$, if the permittivity ϵ is bounded and the magnetic permeability μ and the conductivity σ are bounded away from 0 and ∞. The electromagnetic properties ϵ, μ, σ are allowed to be discontinuous. We discuss an application of this potential for FEM approximation of the EM field. In principle, it is enough to use only the vector Lorenz gauged potential to represent the EM field. However, when the conductivity σ is not constant and the electric field is not divergence-free, it is difficult to find a weak equation involving only the vector potential. In particular, we show that the vector potential does not satisfy the weak form of the Helmholtz equation, sometimes erroneously used as a basis for FEM simulation [33]. For the general case of non divergence-free EM fields, we propose a mixed formulation involving the scalar and vector potentials.

We consider also the case of representing the magnetic field using a vector potential with the Lorenz gauge. If the magnetic permeability μ is constant, the magnetic field is divergence-free and the vector potential coincides with the magnetic field. We show that the Lorenz gauge approach leads to a regularized weak equation for the magnetic field involving a divergence term, and as a result the equation does not suffer from the large near null space.

We show that sesquilinear forms of the equations for both magnetic vector potential and electric scalar-vector formulations remain coercive at low frequencies. It makes iterative solvers fast even if only standard vector multigrid preconditioners [35] are used. Another advantage is that the considered vector potential is a member of $\mathcal{H}(\nabla\times) \cap \mathcal{H}(\nabla\cdot)$. This allows to use nodal elements, which have more widely available implementations than edge elements. The edge elements, due to a discontinuity of the shape functions across elements boundaries, require post processing to get a value of a field at a specific point within an element. In geophysical applications, the domain is a convex polygon, so nodal discretization is dense in $\mathcal{H}_0(\nabla\times) \cap \mathcal{H}(\nabla\cdot)$ or in $\mathcal{H}(\nabla\times) \cap \mathcal{H}_0(\nabla\cdot)$ [13, 6].

Regularization of the curl-curl equation using a divergence term has been also suggested in [1, 13]. The current paper extends these ideas to the case of nonconstant, complex valued electromagnetic properties and non divergence-free fields.

In [1], the authors consider the existence, the uniqueness and proper boundary conditions for a Lorenz gauged vector potential only for the case of constant electromagnetic properties. In [13], the authors consider non-constant properties; however, they seek a solution $E \in \mathcal{H}(\nabla\times)$ such that $\sigma E \in \mathcal{H}(\nabla\cdot)$. If σ is not constant, it is difficult to construct a compatible finite element discretization for the space of vector fields of the suggested kind.

In this paper, we consider a different approach. The vector potential term $-i\omega A$ and the vector electric field E differ by $\nabla\varphi$. The scalar potential φ satisfies the Poisson equation for which the source term is given by the jumps of the normal component of E across boundaries of regions with different EM properties. Representing the discontinuities of the electric field using $\nabla\varphi$ allows the vector potential to be continuous, or more precisely to lie in the space $\mathcal{H}(\nabla\times)\cap\mathcal{H}(\nabla\cdot)$, which allows to approximate it using the nodal elements.

A representation of the electric field related to our vector-scalar formulation was considered in ([9] Lorenz gauge #2), where the authors proved the uniqueness of the Schelkunoff potential continuous across interfaces for a nonlossy medium using a mixed formulation that involved both scalar and vector potentials. The mixed formulation involving scalar and vector potentials considered in the current paper (section 6) is a reformulation of this approach for a medium with losses. We prove not only the uniqueness, but also the existence of the solution (Theorem 6.1).

A closely related work was presented in [15], where the authors consider an eddy current problem, with $\epsilon = 0$ and $\sigma > 0$ in a part of the domain and $\epsilon = \sigma = 0$ in the rest of the domain. They show existence and uniqueness of the vector potential representation with the Lorenz gauge. They consider also a mixed formulation similar to ours. Here, we consider $\epsilon > 0$. Also in our equation we apply a scaling to the scalar potential, which makes a sesquilinear form coercive at $\omega \to 0$. Finally our proof of the coercivity is more general, it does not require a smallness of the coefficients used in the equation.

The structure of the paper is as follows. In section 2, a brief description of the vector-scalar representation of the electric field with the Lorenz gauge is given in the way it typically appears in the literature. We also show that it satisfies the Helmholtz equation if the electromagnetic properties are constant.

In the third section, a theorem of the existence and the uniqueness of a Lorenz gauged vector potential representation for any vector field in $\mathcal{H}(\nabla\times)$ is formulated and proven.

The purpose of section 4 is to build some intuition about the vector potential with the Lorenz gauge. We consider a representation of the electric field by the Schelkunoff potential. We present conditions that are satisfied on an interface between two regions with different conductivity. We show how a jump in the normal component of the electric field is represented by a jump of the normal derivative of the scalar potential, allowing the vector potential to be continuous.

In section 5, a difficulty in obtaining a weak equation involving only the vector electric Schelkunoff potential is presented.

In section 6, a mixed formulation involving a scalar and a vector potential is developed for the electric Schelkunoff potential.

In section 7, a different approach is suggested to avoid the difficulties with the electric potential. A magnetic Schelkunoff potential is defined and, in the situation where magnetic permeability μ is constant, an appealing weak form of the governing equation is derived.

The last section (8) shows results of numerical simulations. We use the developed magnetic Schelkunoff potential approach to calculate the electromagnetic field generated by a conductive brick in a resistive whole space with a plane-wave (magnetotelluric) source. A comparison of the results with calculations done by an independent Integral Equations code [36], is shown. A good agreement between the calculated fields provides a verification of the validity of the method.

2. Lorenz gauged formulation of Maxwell's equations

Let us consider the electromagnetic field satisfying Maxwell's equations in the frequency domain, with a time dependence $e^{i\omega t}$, with the electric source J^{imp}, in some bounded domain $\Omega \subset \mathbb{R}^3$:

$$\left\{ \begin{array}{lcl} \nabla \times E & = & -i\omega\mu H \\ \nabla \times H & = & \hat{\sigma} E + J^{imp} \end{array} \right. , \qquad \hat{\sigma} = \sigma + i\omega\epsilon \tag{1}$$

Here, σ and ϵ are the conductivity and the permittivity of the medium, μ is the magnetic permeability, and ω is the frequency.

The Schelkunoff potential, or the electric Schelkunoff potential, is a vector potential A used together with a scalar potential φ to represent the electric field E [37, 8, 33, 9] in a form:

$$E = -i\omega A + \nabla\varphi \tag{2}$$

A relationship between A and φ, called the Lorenz gauge, is imposed:

$$\nabla \left(\frac{\nabla \cdot A}{\hat{\sigma}\mu} \right) = \nabla\varphi \tag{3}$$

As a result the electric field is represented as:

$$E = -i\omega A + \nabla \left(\frac{\nabla \cdot A}{\hat{\sigma}\mu} \right) \tag{4}$$

Substituting the first equation to the second one in (1) and using (2) to represent the electric field E, in a region of *constant properties* $\hat{\sigma}, \mu$ we obtain:

$$\nabla \times \left(\nabla \times \frac{1}{\mu} A \right) = J^{imp} - \hat{\sigma} i\omega A + \hat{\sigma}\nabla\varphi$$

Application of the vector identity (51) results in:

$$\nabla \left(\nabla \cdot \frac{1}{\mu} A \right) - \nabla \cdot \left(\nabla \left(\frac{1}{\mu} A \right) \right) = J^{imp} - \hat{\sigma} i\omega A + \hat{\sigma}\nabla\varphi$$

If the equation is multiplied by $-\mu$ (it is assumed that $\hat{\sigma}, \mu$ are constant), the Lorenz gauge (3) is used, then the following vector Helmholtz equation is obtained:

$$\Delta A - i\hat{\sigma}\mu\omega A = -\mu J^{imp} \tag{5}$$

Yet the vector potential satisfies this equation only if the electromagnetic properties are constant. The weak form of the Helmholtz equation is a separate equation for each component A_k of the vector field, $k = 1, 2, 3$. For any test function $K_k \in \mathcal{H}^1(\Omega)$ the following is satisfied:

$$\int_\Omega \nabla K_k \cdot \nabla A_k + i\omega \int_\Omega \hat{\sigma}\mu A_k \cdot K_k = \int_\Omega \mu J_k^{imp} \cdot K_k \tag{6}$$

The equation above imposes conditions on interfaces between regions of different $\hat{\sigma}, \mu$ listed below:

1. A_k is continuous, $k = 1, 2, 3$ 2. $\frac{\partial}{\partial n} A_k$ is continuous, $k = 1, 2, 3$
where n is a vector normal to the interface. In section 3 the existence and the uniqueness of an electric Schelkunoff potential satisfying those conditions is investigated. As it turns out, with some reasonable assumptions when $\hat{\sigma}, \mu$ are not constant, an electric Schelkunoff potential continuous across interfaces (condition 1 is satisfied) exists, yet the condition 2 is not satisfied. As a result there is no electric Schelkunoff potential that satisfies the weak form of the Helmholtz equation (6), so it should not be used as a basis for finite element approximation if the electromagnetic properties are not constant.

3. Existence and uniqueness of the Schelkunoff potential

In this section we formulate and prove a theorem stating the existence and the uniqueness of the Schelkunoff potential. All is done in an abstract setting that uses the theory of the Sobolev spaces. Some physical interpretation, for the case of representation of the electric field E, is given in the following section.

Consider an open bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary. We use the following notation for the Sobolev spaces:

$$\begin{array}{lll} L^2 & = L^2(\Omega) = \left\{\psi : \Omega \to \mathbb{C} : \int_\Omega |\psi|^2 < \infty\right\} \\ \mathcal{H}^1 & = \mathcal{H}^1(\Omega) = \left\{\psi : \Omega \to \mathbb{C} : \int_\Omega |\nabla\psi|^2 + \int_\Omega |\psi|^2 < \infty\right\} \\ \mathcal{H}(\nabla\times) & = \mathcal{H}(\nabla\times, \Omega) = \left\{K : \Omega \to \mathbb{C}^3 : \int_\Omega |\nabla \times K|^2 + \int_\Omega |K|^2 < \infty\right\} \\ \mathcal{H}(\nabla\cdot) & = \mathcal{H}(\nabla\cdot, \Omega) = \left\{K : \Omega \to \mathbb{C}^3 : \int_\Omega |\nabla \cdot K|^2 + \int_\Omega |K|^2 < \infty\right\} \end{array} \tag{7}$$

If homogeneous boundary conditions are assumed, a subscript "0" is added. For $\mathcal{H}^1_0$, $\mathcal{H}_0(\nabla\times)$, $\mathcal{H}_0(\nabla\cdot)$, the value of the function, tangential and normal components of a vector field are fixed respectively. If n is a vector normal to the boundary $\partial\Omega$, then

$$\begin{array}{lllll} \mathcal{H}^1_0 & = & \mathcal{H}^1_0(\Omega) & = & \left\{\psi \in \mathcal{H}^1(\Omega) : \psi|_{\partial\Omega} = 0\right\} \\ \mathcal{H}_0(\nabla\times) & = & \mathcal{H}_0(\nabla\times, \Omega) & = & \{K \in \mathcal{H}(\nabla\times, \Omega) : n \times K|_{\partial\Omega} = 0\} \\ \mathcal{H}_0(\nabla\cdot) & = & \mathcal{H}_0(\nabla\cdot, \Omega) & = & \{K \in \mathcal{H}(\nabla\cdot, \Omega) : n \cdot K|_{\partial\Omega} = 0\} \end{array} \tag{8}$$

Additionally, the notation for norms is as follows:

$$\begin{array}{lll} \|\psi\|_0 & = & \sqrt{\int_\Omega |\psi|^2} \\ \|\psi\|_1 & = & \sqrt{\|\psi\|_0^2 + \|\nabla\psi\|_0^2} = \sqrt{\int_\Omega |\psi|^2 + \int_\Omega |\nabla\psi|^2} \end{array} \tag{9}$$

We use the following Poincare inequality (see Appendix A in [6]). There is a constant $c > 0$, dependent on the domain Ω, such that:

$$c\|\psi\|_0 \leq \|\nabla\psi\|_0 \quad \text{for } \psi \in \mathcal{H}^1_0 \tag{10}$$

Theorem 3.1. *For a vector field $G \in \mathcal{H}_0(\nabla\times)$ and a scalar complex valued function γ satisfying*

$$\begin{array}{rrl} & \gamma & = \gamma_R + i\gamma_I, \quad \gamma_R, \gamma_I : \Omega \to \mathbb{R} \\ & |\gamma_R| & \leq \gamma_{RM} < \infty \\ 0 < \gamma_{Im} \leq & |\gamma_I| & \leq \gamma_{IM} < \infty \\ \gamma_I > 0 \ in\ \Omega & or & \gamma_I < 0 \ in\ \Omega \end{array} \tag{11}$$

there is a unique $T \in \mathcal{H}_0(\nabla\times) \cap \mathcal{H}(\nabla\cdot)$ satisfying

$$\frac{\nabla \cdot T}{\gamma} \in H^1_0 \tag{12}$$

$$G = T + \nabla\left(\frac{\nabla \cdot T}{\gamma}\right) \tag{13}$$

Proof. Consider an equation for $\varphi \in \mathcal{H}_0^1$:

$$\int_\Omega \nabla\varphi \cdot \nabla\overline{\psi} - \int_\Omega \gamma\varphi\overline{\psi} = \int_\Omega G \cdot \nabla\overline{\psi} \tag{14}$$

satisfied for any $\psi \in \mathcal{H}_0^1$. We will prove that there is a unique solution φ to this equation, $\varphi = \frac{\nabla\cdot T}{\gamma}$. It is obvious that with assumptions (11), the sesquilinear form

$$\mathcal{B}(\varphi,\psi) = \int_\Omega \nabla\varphi \cdot \nabla\overline{\psi} - \int_\Omega \gamma\varphi\overline{\psi} \tag{15}$$

is bounded with respect to the norm $\|.\|_1$, defined in (9). We will prove that $\mathcal{B}$ is also coercive.

$$|\mathcal{B}(\psi,\psi)| = \left|\int_\Omega |\nabla\psi|^2 - \int_\Omega \gamma|\psi|^2\right| = \left|\left(\int_\Omega |\nabla\psi|^2 - \int_\Omega \gamma_R|\psi|^2\right) - i\int_\Omega \gamma_I|\psi|^2\right|$$

If the real part of a complex number is decreased, then the modulus is decreased, so we can write that for any $\alpha \in (0,1]$

$$\begin{aligned} |\mathcal{B}(\psi,\psi)| &\geq \left|\alpha\left(\int_\Omega |\nabla\psi|^2 - \int_\Omega \gamma_R|\psi|^2\right) - i\int_\Omega \gamma_I|\psi|^2\right| \\ &\geq \tfrac{1}{\sqrt{2}}\left[\left|\alpha\left(\int_\Omega |\nabla\psi|^2 - \int_\Omega \gamma_R|\psi|^2\right)\right| + \left|\int_\Omega \gamma_I|\psi|^2\right|\right] \\ &\geq \tfrac{1}{\sqrt{2}}\left[\alpha\left(\int_\Omega |\nabla\psi|^2 - \int_\Omega |\gamma_R||\psi|^2\right) + \int_\Omega |\gamma_I||\psi|^2\right] \\ &\geq \tfrac{1}{\sqrt{2}}\left[\alpha\left(\|\nabla\psi\|_0^2 - \gamma_{RM}\|\psi\|_0^2\right) + \gamma_{Im}\|\psi\|_0^2\right] \\ &\geq \min\left(\tfrac{\alpha}{\sqrt{2}}, \tfrac{\gamma_{Im}-\alpha\gamma_{RM}}{\sqrt{2}}\right)\left(\|\nabla\psi\|_0^2 + \|\psi\|_0^2\right) \end{aligned} \tag{16}$$

This proves the coercivity of $\mathcal{B}$ if only α is taken such that $\frac{\gamma_{Im}}{\gamma_{RM}} > \alpha > 0$.

As $G \in \mathcal{H}_0(\nabla\times) \subset (L^2)^3$, the right hand side of (14) is a bounded linear functional on H_0^1, thus from the Lax-Milgram theorem there is a unique $\varphi \in \mathcal{H}_0^1$ satisfying (14).

Define

$$T = G - \nabla\varphi \tag{17}$$

As $\varphi \in \mathcal{H}_0^1$, then $\nabla\varphi \in \mathcal{H}_0(\nabla\times)$. As $G \in \mathcal{H}_0(\nabla\times)$, we conclude that $T \in \mathcal{H}_0(\nabla\times)$.

Take any smooth function with a compact support in Ω, $\psi \in \mathcal{C}_c^\infty(\Omega)$. Such a function is also in $\mathcal{H}_0^1$, so it satisfies (14). Evaluation of $\nabla\cdot T$ at ψ gives

$$\langle\nabla\cdot T,\psi\rangle = -\int_\Omega T\cdot\nabla\overline{\psi} \overset{(17)}{=} -\int_\Omega (G-\nabla\varphi)\cdot\nabla\overline{\psi} \overset{(14)}{=} \int_\Omega \gamma\varphi\overline{\psi}$$

This shows that $\nabla\cdot T$ is a function and

$$\nabla\cdot T = \gamma\varphi$$

As $|\gamma| \leq \sqrt{\gamma_{RM}^2 + \gamma_{IM}^2} < \infty$ and $\varphi \in L^2$, then $\nabla\cdot T \in L^2$, which proves that $T \in \mathcal{H}(\nabla\cdot)$. Moreover as $\gamma \neq 0$, we have

$$\frac{\nabla\cdot T}{\gamma} = \varphi \tag{18}$$

which proves (12). Definition (17) of T, together with (18) proves (13). □

Remark 3.2.

- *One could consider non-homogeneous Dirichlet boundary conditions. For any $G \in \mathcal{H}(\nabla\times)$ the same proof would give a vector potential $T \in \mathcal{H}(\nabla\times) \cap \mathcal{H}(\nabla\cdot)$ such that $n \times T = n \times G$ on $\partial\Omega$.*

- *One could consider $G \in \mathcal{H}(\nabla\times)$ and a different potential T, satisfying different boundary conditions. If equation (14) is considered for $\phi, \psi \in \mathcal{H}^1$, it will lead to $T \in \mathcal{H}(\nabla\times) \cap \mathcal{H}_0(\nabla\cdot)$. To prove that in this case T has the normal component equal to 0 on $\partial\Omega$, one can take any $\psi \in \mathcal{H}^1$ and evaluate*

$$\int_{\partial\Omega}(T\cdot n)\overline{\psi} = \int_\Omega T\cdot\nabla\overline{\psi} + \int_\Omega(\nabla\cdot T)\overline{\psi} = \int_\Omega(G-\nabla\varphi)\cdot\nabla\overline{\psi} + \int_\Omega \gamma\varphi\overline{\psi} = 0$$

- *In the case of (14) for $\varphi, \psi \in \mathcal{H}^1_0$ assumption $|\gamma_I| > 0$. may be weakened. Even if $\gamma_I = 0$ the theorem holds as long as $\gamma_{RM} \neq 0$ and $\gamma_{RM} < c$, where c is the constant in Poincare inequality (10). The proof of the coercivity has to be adapted as follows. Continuing with the calculation (16) for $\alpha = 1$ we obtain for some β, such that $1 > \beta > \frac{\gamma_{RM}}{c} > 0$:*

$$\begin{aligned}\sqrt{2}|\mathcal{B}(\psi,\psi)| \quad &\geq \|\nabla\psi\|^2 - \gamma_{RM}\|\psi\|^2 = (1-\beta)\|\nabla\psi\|^2 + \beta\|\nabla\psi\|^2 - \gamma_{RM}\|\psi\|^2 \\ &\geq min(1-\beta, \beta c - \gamma_{RM})(\|\nabla\psi\|^2 + \|\psi\|^2)\end{aligned}$$

Corollary 3.3. *To obtain the Schelkunoff potential representation (4) of the electric field E, one has to set $G = E$, $T = -i\omega A$ and $\gamma = -i\omega\mu\hat{\sigma} = \omega^2\epsilon\mu - i\omega\sigma\mu$. The assumptions (11) of Theorem 3.1 will be satisfied for any $\omega > 0$ if there exist constants $\mu_m, \mu_M, \sigma_m, \sigma_M, \epsilon_M$ such that*

$$\begin{array}{rcl} & |\epsilon| & \leq \epsilon_M < \infty \\ 0 < \sigma_m \leq & \sigma & \leq \sigma_M < \infty \\ 0 < \mu_m \leq & \mu & \leq \mu_M < \infty \end{array} \qquad (19)$$

4. Interface conditions

In this section, we discuss interface conditions of the Schelkunoff potential for the electric field E. Consider a fragment of the domain Ω with two subsets V_1, V_2 and the interface $\partial V_1 \cap \partial V_2$ between them (see Figure 1). For simplicity, we assume that all considered vector and scalar fields are smooth in V_1 as well as in V_2, and have limits of values and derivatives on the interface $\partial V_1 \cap \partial V_2$, yet the limit if one approaches the interface from V_1 may be different from the limit if one approaches the interface from V_2. With this assumption, the members of $\mathcal{H}^1$, such as the scalar potential φ, are continuous across the interface. The members of $\mathcal{H}(\nabla\times)$, such as the electric field E and $\nabla\varphi$ have continuous tangential components across the interface, but may have discontinuous normal components. Members of $\mathcal{H}(\nabla\times) \cap \mathcal{H}(\nabla\cdot)$, such as A and T have continuous both tangential and normal components across the interface. The fields in the subdomain V_j are denoted by a subscript "j". The vector normal to the surface and pointing out of V_1, towards V_2 (see Figure 1), is denoted by n.

Let us consider representation (2) of the electric field E, with $\varphi = \frac{\nabla\cdot A}{\hat{\sigma}\mu}$. In this representation, all the components, E, A, $\nabla\varphi$ are members of $\mathcal{H}(\nabla\times)$, so they have continuous tangential components across the interface. Analysis of the normal components is more interesting. Using the fact that the normal component of A has to be continuous, we obtain the condition on the jump of the normal derivative of φ:

$$\begin{aligned} -i\omega\,(n\cdot A_1) &= -i\omega\,(n\cdot A_2) \\ n\cdot(E_1 - \nabla\varphi_1) &= n\cdot(E_2 - \nabla\varphi_2) \\ n\cdot(\nabla\varphi_2 - \nabla\varphi_1) &= n\cdot(E_2 - E_1) \end{aligned} \qquad (20)$$

We will show that this is exactly the condition imposed by equation (14). Integrating equation (14) by parts for a test function ψ with the support in the interior

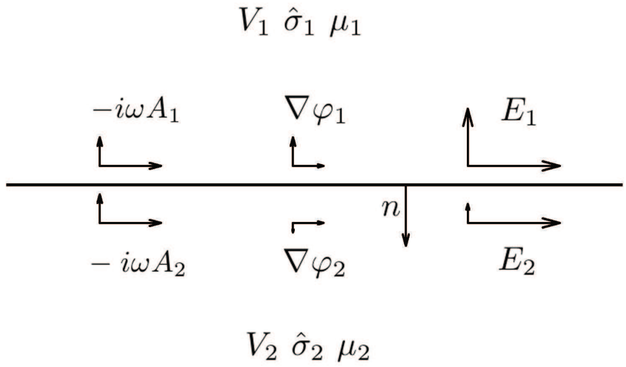

FIGURE 1. The properties $\hat{\sigma}$, μ experience a jump on $\partial V_1 \cap \partial V_2$. As a result the normal component of E has a jump. The field $\nabla\varphi$ is chosen in such a way, that its normal component jump allows $-i\omega A$ to be continuous.

of $V_1 \cup V_2$ and using $G = E$, we obtain the following:

$$\begin{aligned}\int_{V_1\cup V_2}[-\nabla\cdot\nabla\varphi - \gamma\varphi]\overline{\psi} + \int_{\partial V_1\cap\partial V_2} n\cdot(\nabla\varphi_1 - \nabla\varphi_2)\overline{\psi} = \\ -\int_{V_1\cup V_2}(\nabla\cdot E)\overline{\psi} + \int_{\partial V_1\cap\partial V_2} n\cdot(E_1 - E_2)\overline{\psi}\end{aligned} \tag{21}$$

For a test function with the support entirely in V_1 or entirely in V_2, the interface terms are 0, hence

$$\nabla\cdot\nabla\varphi + \gamma\varphi = \nabla\cdot E \tag{22}$$

almost everywhere in $V_1 \cup V_2$. Using this result in (21), for a test function non-zero on the interface, one gets only the boundary terms and subsequently one obtains condition (20) for the jump in the normal derivatives of φ.

Notice that in many applications, the source term J^{imp} in (1) is divergence free. If additionally $\hat{\sigma} = \text{const}$ in V_1 and in V_2, then taking divergence of the second equation in (1), one obtains that

$$\nabla\cdot E = 0$$

in V_1 as well as in V_2. In this case, the strong equation (22) has the right hand side equal to zero. As a result the source term in (14) is related only to the jump of the normal component of E. More precisely if E has a jump in the normal component, then its divergence is a distribution. This distribution is the source term in equation (14).

If the electromagnetic properties have corners or edges, then the electric field has singularities [12], which can be represented by a gradient of a scalar function $\nabla\varphi$. The Lorenz gauged vector potential that we consider, exploits exactly this property. It allows to represent the electromagnetic field, which is a member of $\mathcal{H}(\nabla\times)$, with a more regular field A, which is in $\mathcal{H}(\nabla\cdot)\cap\mathcal{H}(\nabla\times)$. The singularity is contained in the term $\nabla\left(\frac{\nabla\cdot A}{\hat{\sigma}\mu}\right)$.

5. A difficulty in obtaining a weak form of the governing equation for the vector potential representation of the electric field E

To be able to use the finite element method for a calculation of the EM field, a weak form of a governing equation satisfied by the electric Schelkunoff potential is needed.

In order to obtain a weak equation, one starts from Maxwell's equations (1). Dividing the first equation by $-i\omega\mu$, taking curl and substituting into the second equation, one obtains

$$\nabla \times \frac{1}{-i\omega\mu}\nabla \times E - \hat{\sigma}E = J^{imp} \tag{23}$$

Next $-i\omega A + \nabla\left(\frac{\nabla\cdot A}{\hat{\sigma}\mu}\right)$ is substituted for E and the equation is multiplied by a test vector field K. The result is

$$\int_\Omega \left(\nabla \times \frac{1}{\mu}\nabla \times A\right)\cdot K - \int_\Omega \nabla\left(\frac{\nabla\cdot A}{\hat{\sigma}\mu}\right)\cdot(\hat{\sigma}K) + \int_\Omega i\omega\hat{\sigma}A\cdot K = \int_\Omega J^{imp}\cdot K$$

In order to integrate by parts the first term in the above equation, one uses continuity of the tangential component of $\frac{1}{\mu}\nabla\times A$, which is equivalent to continuity of the tangential component of the magnetic field H and one needs the tangential components of K to be continuous across interfaces (like the interface $\partial V_1 \cap \partial V_2$ considered in section 4).

On the other hand, in order to integrate by parts the second term, one would use a continuity of $\frac{\nabla\cdot A}{\hat{\sigma}\mu}$, and one needs the normal components of $\hat{\sigma}K$ to be continuous across interfaces. So if $\hat{\sigma}$ is discontinuous, so is the normal component of K. This is the essence of the problem in obtaining a proper weak form of the equation for A. A family of vector finite element shape functions with continuous tangential components and normal components experiencing specific jumps is difficult to build. One may consider a mixed formulation involving scalar and vector potentials (see section 6), but that increases the number of coefficients needed to represent the field.

It turns out that, assuming that μ is constant, it is possible to obtain an equation involving only the vector potential, but for a vector potential representation of the magnetic field H. This idea is presented in section 7.

6. A formulation with both scalar and vector potentials

If the original field is not divergence free, an equation involving both scalar and vector components must be considered. Although the number of coefficients per point in space increases from 3 to 4, the obtained equation is valid for non-constant electromagnetic properties and a non divergence free field. Also, if the boundary of the domain Ω is connected, a sesquilinear form of the equation remains coercive as $\omega = 0$.

In [9], in the case of the Lorenz gauge #2, the authors proved the uniqueness of the Schelkunoff potential given as a solution of an equation (see (58) in [9]) that has the following bilinear form in the left hand side:

$$\begin{aligned}\mathcal{G}\left((A,\rho),\,(K,\psi)\right) &= \int_\Omega [\nabla\times A\cdot\tfrac{1}{\mu}\nabla\times K + \nabla\cdot A\tfrac{1}{\mu}\nabla\cdot K - \omega^2\epsilon A\cdot K \\ &\quad -i\omega\epsilon\nabla\cdot\psi A + \epsilon\nabla\rho\cdot\nabla\psi - \omega^2\epsilon^2\mu\rho\psi - i\omega\epsilon\nabla\cdot\rho K]\end{aligned}$$

This bilinear form, considered for a purely imaginary frequency $\omega = i\tilde{\omega}$, $\tilde{\omega} > 0$, may be rewritten as

$$\mathcal{G} = \int_\Omega \frac{1}{\mu}(\nabla\cdot A+\mu\tilde{\omega}\epsilon\rho)(\nabla\cdot K+\mu\tilde{\omega}\epsilon\psi)+\int_\Omega \epsilon(\tilde{\omega}A+\nabla\rho)\cdot(\tilde{\omega}K+\nabla\psi)+\int_\Omega \frac{1}{\mu}(\nabla\times A)\cdot(\nabla\times K)$$

Using this form we can prove the boundedness and the coercivity of $\mathcal{G}$ for $A,\ K \in \mathcal{H}_0(\nabla\times,\Omega)\cap\mathcal{H}(\nabla\cdot,\Omega)$, $\rho,\psi \in \mathcal{H}_0^1(\Omega)$. So from the Lax-Milgram theorem, there exists a unique solution to the equation for the Lorenz gauged vector and scalar potentials that is considered in [9]. This formulation may be adapted to a lossy medium, which is expressed in Theorem 6.1.

Theorem 6.1. *Let the assumptions (19) be satisfied. The unique electric Schelkunoff potential A, together with a scalar field*

$$\phi = \frac{\nabla\cdot A}{\sqrt{\omega}\hat{\sigma}\mu} \tag{24}$$

satisfy the following equation

$$\begin{aligned} &\textstyle\int_\Omega \frac{1}{\mu}(\nabla\times A)\cdot\overline{(\nabla\times K)} + \int_\Omega \frac{1}{\mu}(\nabla\cdot A - \sqrt{\omega}\mu\hat{\sigma}\phi)\overline{(\nabla\cdot K - \sqrt{\omega}\mu\hat{\sigma}\psi)} \\ &\quad + i\textstyle\int_\Omega \hat{\sigma}(\sqrt{\omega}A + i\nabla\phi)\cdot\overline{(\sqrt{\omega}K + i\nabla\psi)} = \int_\Omega J^{imp}\cdot\overline{(K + i\frac{\nabla\psi}{\sqrt{\omega}})} \end{aligned} \tag{25}$$

$$\forall K \in \mathcal{H}_0(\nabla\times)\cap\mathcal{H}(\nabla\cdot) \text{ and } \psi \in \mathcal{H}_0^1$$

$$A \in \mathcal{H}(\nabla\times)\cap\mathcal{H}(\nabla\cdot), \qquad n\times(-i\omega A) = n\times E \text{ on } \partial\Omega,\ \phi\in\mathcal{H}_0^1. \tag{26}$$

The sesquilinear form associated with equation (25) is bounded and coercive with respect to the norm

$$\|(K,\psi)\|_{\mathcal{B}} = \sqrt{\|K\|_0^2 + \|\nabla\times K\|_0^2 + \|\nabla\cdot K\|_0^2 + \|\nabla\psi\|_0^2 + \|\psi\|_0^2} \tag{27}$$

Hence if $J^{imp} \in (L^2)^3$, then the solution to this equation exists and is unique.

Remark 6.2.

- *If the domain is a convex polygon, or if the domain has C^2 boundary, then one may use nodal shape functions to approximate both A and ϕ.*
- *In order to obtain the electric field E, one has to calculate*

$$E = -i\omega A + \sqrt{\omega}\nabla\phi \tag{28}$$

- *If one drops all the terms multiplied by ω, the resulting sesquilinear form remains coercive. To prove this, one has to use the Poincare inequality for $H_0(\nabla\times)\cap\mathcal{H}(\nabla\cdot)$ (see* [1]*, Corollary 3.19). The proof of this result is easier than the proof of coercivity of the original sesquilinear form, so it is omitted.*
- *In* [15]*, the authors present a similar equation to ours. In our formulation, we apply a scaling $\frac{1}{\sqrt{w}}$ on the scalar function φ. As a result our sesquilinear form remains coercive for $\omega = 0$. Instead of $\frac{1}{\mu}$, one can consider an arbitrary weight in the middle term of the sesquilinear form, the term containing the divergences. The authors of* [15] *denoted this weight by $\frac{1}{\mu_*}$ and their proof of the coercivity of the sesquilinear form depends on the smallness of the upper bound of μ_*. The proof we present is not dependent on such a bound, thus is valid as long as μ_* is bounded away from 0 and from ∞. Also in our formulation and in the proof, we consider the case of non zero ϵ and an arbitrarily large frequency ω, thus an arbitrarily large term $i\omega\epsilon$.*

Proof. The fact that the vector potential A of Corollary 3.3 and ϕ defined in (24) satisfy equation (25) is straightforward and is explained as follows. A consequence of (24) is that the middle term on the right hand side of (25) vanishes. The definition of ϕ implies (28). If (28) is used, then equation (25) simplifies to

$$\int_\Omega \frac{1}{\mu}(\nabla\times E)\cdot\overline{(\nabla\times K)} + i\omega\int_\Omega \hat{\sigma}E\cdot\overline{\left(K + i\frac{\nabla\psi}{\sqrt{\omega}}\right)} = -i\omega\int_\Omega J^{imp}\cdot\overline{\left(K+i\frac{\nabla\psi}{\sqrt{\omega}}\right)}$$

Since $K\in\mathcal{H}_0(\nabla\times)\cap\mathcal{H}(\nabla\cdot)$ and $\psi\in\mathcal{H}^1_0$, then $\tilde{K} = \overline{K+i\frac{\nabla\psi}{\sqrt{\omega}}}\in\mathcal{H}_0(\nabla\times)$ and $\nabla\times\tilde{K} = \nabla\times\overline{K}$, so it remains to show that for any $\tilde{K}\in\mathcal{H}_0(\nabla\times)$ the following equation is satisfied:

$$\int_\Omega \frac{1}{\mu}(\nabla\times E)\cdot(\nabla\times\tilde{K}) + i\omega\int_\Omega\hat{\sigma}E\cdot\tilde{K} = -i\omega\int_\Omega J^{imp}\cdot\tilde{K}$$

This is a standard equation satisfied by the electric field E which satisfy Maxwell's equations (1). The equation is satisfied for all $\tilde{K}\in\mathcal{H}_0(\nabla\times)$. This concludes the proof that A and ϕ defined in (24) satisfy equation (25).

Let us now focus on a proof of the boundedness and the coercivity of the sesquilinear form $\mathcal{B}((A,\phi),(K,\psi))$ defined as the left hand side of the equation (25).

Denote $\hat{\sigma}_M = (\sigma_M + \omega\epsilon_M)$. The boundedness of $\mathcal{B}$ is straightforward, as from the Cauchy-Schwartz inequality, it follows that:

$$\begin{aligned}
&|\mathcal{B}((A,\phi),(K,\psi))| = \\
&= \Bigg|\int_\Omega\frac{1}{\mu}(\nabla\times A)\cdot\overline{(\nabla\times K)} + \int_\Omega\frac{1}{\mu}(\nabla\cdot A - \sqrt{\omega}\mu\hat{\sigma}\phi)\overline{(\nabla\cdot K - \sqrt{\omega}\mu\hat{\sigma}\psi)} \\
&\qquad + i\int_\Omega \hat{\sigma}(\sqrt{\omega}A + i\nabla\phi)\cdot\overline{(\sqrt{\omega}K + i\nabla\psi)}\Bigg| \\
&\le \frac{1}{\mu_m}\int_\Omega|\nabla\times A|\,|\nabla\times K| + \int_\Omega\frac{1}{\mu_m}|\nabla\cdot A - \sqrt{\omega}\mu\hat{\sigma}\phi|\,|\nabla\cdot K - \sqrt{\omega}\mu\hat{\sigma}\psi| \\
&\qquad + \int_\Omega\hat{\sigma}_M|\sqrt{\omega}A + i\nabla\phi|\,|\sqrt{\omega}K + i\nabla\psi| \\
&\le \frac{1}{\mu_m}\|\nabla\times A\|_0\|\nabla\times K\|_0 + \frac{1}{\mu_m}\|\nabla\cdot A - \sqrt{\omega}\mu\hat{\sigma}\phi\|_0\|\nabla\cdot K - \sqrt{\omega}\mu\hat{\sigma}\psi\|_0 \\
&\qquad + \hat{\sigma}_M\|\sqrt{\omega}A + i\nabla\phi\|_0\|\sqrt{\omega}K + i\nabla\psi\|_0 \\
&\le \frac{1}{\mu_m}\|\nabla\times A\|_0\|\nabla\times K\|_0 + \frac{1}{\mu_m}(\|\nabla\cdot A\|_0 + \sqrt{\omega}\mu_M\hat{\sigma}_M\|\phi\|_0)(\|\nabla\cdot K\|_0 + \sqrt{\omega}\mu_M\hat{\sigma}_M\|\psi\|_0) \\
&\qquad + \hat{\sigma}_M(\sqrt{\omega}\|A\|_0 + \|\nabla\phi\|_0)(\sqrt{\omega}\|K\|_0 + \|\nabla\psi\|_0) \\
&\le \max\left(\frac{1}{\mu_m}, \frac{\sqrt{\omega}\mu_M}{\mu_m}\hat{\sigma}_M, \frac{\omega\mu_M^2}{\mu_m}\hat{\sigma}_M^2, \hat{\sigma}_M, \hat{\sigma}_M\sqrt{\omega}, \hat{\sigma}_M\omega\right)\|(A,\phi)\|_{\mathcal{B}}\|(K,\psi)\|_{\mathcal{B}}
\end{aligned}$$

To prove the coercivity, we have to prove that there exists a constant $d>0$ such that for any $(K,\psi)\in(\mathcal{H}_0(\nabla\times,\Omega)\cap\mathcal{H}(\nabla\cdot,\Omega))\times\mathcal{H}^1_0(\Omega)$

$$|\mathcal{B}((K,\psi),(K,\psi))| \ge d\|(K,\psi)\|^2_{\mathcal{B}}$$

It is enough to prove that it is not possible to have a sequence of $(K_n,\psi_n)_{n=1}^\infty$ such that

$$\begin{aligned}
1 &= \|(K_n,\psi_n)\|^2_{\mathcal{B}} \\
&= \|K_n\|_0^2 + \|\nabla\times K_n\|_0^2 + \|\nabla\cdot K_n\|_0^2 + \|\nabla\psi_n\|_0^2 + \|\psi_n\|_0^2 \qquad (29)
\end{aligned}$$

and

$$\mathcal{B}((K_n,\psi_n),(K_n,\psi_n)) \xrightarrow[n\to\infty]{} 0 \qquad (30)$$

For a proof by contradiction, assume that there is a sequence (K_n, ψ_n) satisfying (29) and (30). We will prove that there is a subsequence of (K_n, ϕ_n) convergent to 0 in $\|.\|_{\mathcal{B}}$. Using the compact embedding of $\mathcal{H}_0(\nabla\times) \cap \mathcal{H}(\nabla\cdot)$ in $(L^2)^3$ (Maxwell's compactness property, [25]) and the compact embedding of H_0^1 in L^2 (Rellich's theorem), there is a subsequence (K_{n_k}, ψ_{n_k}) convergent to (K, ψ) in $(L^2)^4$. To simplify the notation we will write n instead of n_k, thus replacing the original sequence with its subsequence. We obtain that

$$\|K_n - K\|_0 \xrightarrow[n\to\infty]{} 0 \tag{31}$$

$$\|\psi_n - \psi\|_0 \xrightarrow[n\to\infty]{} 0 \tag{32}$$

We will prove that K_n converges to K in $\mathcal{H}_0(\nabla\times) \cap \mathcal{H}(\nabla\cdot)$, ψ_n converges to ψ in H_0^1, and that $\psi = 0$ and $K = 0$.

Consider the imaginary part of $\mathcal{B}((K_n, \psi_n), (K_n, \psi_n))$. Using the fact that $\hat{\sigma} = \sigma + i\omega\epsilon$, we obtain:

$$\mathrm{Im}[\mathcal{B}((K_n, \psi_n), (K_n, \psi_n))] = \int_\Omega \sigma|\sqrt{\omega}K_n + i\nabla\psi_n|^2 \xrightarrow[n\to\infty]{} 0 \tag{33}$$

Similarly, taking the real part, we have:

$$\mathrm{Re}[\mathcal{B}((K_n, \psi_n), (K_n, \psi_n))] =$$
$$\int_\Omega \frac{1}{\mu}|\nabla \times K_n|^2 + \int_\Omega \frac{1}{\mu}|\nabla \cdot K_n - \sqrt{\omega}\mu\hat{\sigma}\psi_n|^2 - \int_\Omega \omega\epsilon|\sqrt{\omega}K_n + i\nabla\psi_n|^2$$

Using (33) and the bounds (19) for σ and ϵ, we conclude that the third term in the above approaches 0. As the remaining two terms are nonnegative, we conclude that:

$$\int_\Omega \frac{1}{\mu}|\nabla \times K_n|^2 \xrightarrow[n\to\infty]{} 0 \tag{34}$$

$$\int_\Omega \frac{1}{\mu}|\nabla \cdot K_n - \sqrt{\omega}\mu\hat{\sigma}\psi_n|^2 \xrightarrow[n\to\infty]{} 0 \tag{35}$$

Using the bounds (19) on σ and μ, we conclude that (33), (34), (35) imply:

$$\|\sqrt{\omega}K_n + i\nabla\psi_n\|_0 \xrightarrow[n\to\infty]{} 0 \tag{36}$$

$$\|\nabla \times K_n\|_0^2 \xrightarrow[n\to\infty]{} 0 \tag{37}$$

$$\|\nabla \cdot K_n - \sqrt{\omega}\mu\hat{\sigma}\psi_n\|_0^2 \xrightarrow[n\to\infty]{} 0 \tag{38}$$

Taking any smooth vector field Z with a compact support in Ω, $Z \in (\mathcal{C}_c^\infty(\Omega))^3$, using (31) and (36), we obtain:

$$\left|\langle i\nabla\psi, Z\rangle + \int_\Omega \sqrt{w}K \cdot \overline{Z}\right| = \left|-\int_\Omega i\psi(\nabla \cdot \overline{Z}) + \int_\Omega \sqrt{w}K \cdot \overline{Z}\right|$$
$$\leq \left|\int_\Omega i(\psi_n - \psi)(\nabla \cdot \overline{Z}) + \int_\Omega (i\nabla\psi_n + \sqrt{w}K_n) \cdot \overline{Z} + \int_\Omega \sqrt{w}(K - K_n) \cdot \overline{Z}\right|$$
$$\leq \|\psi_n - \psi\|_0\|\nabla \cdot Z\|_0 + \|i\nabla\psi_n + \sqrt{w}K_n\|_0\|Z\|_0 + \sqrt{w}\|K - K_n\|_0\|Z\|_0 \xrightarrow[n\to\infty]{} 0$$

which implies that

$$i\nabla\psi = -\sqrt{\omega}K \in (L^2)^3 \tag{39}$$

Moreover as a consequence of (31) and (36)

$$\begin{aligned} &\|\nabla\psi_n - \nabla\psi\|_0 = \|i\nabla\psi_n - i\nabla\psi\|_0 \\ &\le \|\sqrt{\omega}K - \sqrt{\omega}K_n\|_0 + \|\sqrt{w}K_n + i\nabla\psi_n\|_0 \xrightarrow[n\to\infty]{} 0 \end{aligned} \tag{40}$$

Thus ψ_n converges to ψ in $\|.\|_1$, and as $\psi_n \in \mathcal{H}_0^1$ and $\mathcal{H}_0^1$ is a closed subspace of $\mathcal{H}^1$, then $\psi \in \mathcal{H}_0^1$.

Similarly, taking any $Z \in (\mathcal{C}_c^\infty(\Omega))^3$, using (31) and (37), we obtain

$$\begin{aligned} &|\langle\nabla\times K, Z\rangle| = |\langle\nabla\times(K-K_n), Z\rangle + \langle\nabla\times K_n, Z\rangle| \\ &= \left|\int_\Omega (K-K_n)\cdot(\nabla\times\overline{Z}) + \int_\Omega(\nabla\times K_n)\cdot\overline{Z}\right| \\ &\le \|K-K_n\|_0\|\nabla\times Z\|_0 + \|\nabla\times K_n\|_0\|Z\|_0 \xrightarrow[n\to\infty]{} 0 \end{aligned} \tag{41}$$

Thus $\nabla\times K = 0$. This, together with (37) implies that

$$\|\nabla\times K - \nabla\times K_n\|_0 = \|\nabla\times K_n\|_0 \xrightarrow[n\to\infty]{} 0 \tag{42}$$

In a similar way, using (38) and (32) one shows that

$$\nabla\cdot K = \sqrt{\omega}\mu\hat{\sigma}\psi \in L^2 \tag{43}$$

$$\|\nabla\cdot K_n - \nabla\cdot K\|_0 \xrightarrow[n\to\infty]{} 0 \tag{44}$$

We have proven that (K_n, ψ_n) converges to (K,ψ) in $\|.\|_\mathcal{B}$. To prove that $K = 0$ and $\psi = 0$, notice that (43) and (39) imply

$$-\nabla\cdot\nabla\psi + i\omega\mu\hat{\sigma}\psi = 0 \tag{45}$$

which rewritten in a weak form says:

$$\int_\Omega \nabla\psi\cdot\nabla\overline{\nu} - \int_\Omega(-i\omega\mu\hat{\sigma})\psi\overline{\nu} = 0 \tag{46}$$

for any test function $\nu \in \mathcal{H}_0^1$. The sesquilinear form of this equation is a bounded and coercive sesquilinear form, which has been shown in the proof of Theorem 3.1 for $\gamma = -i\omega\mu\hat{\sigma}$. Thus from the Lax-Milgram theorem the equation admits a unique solution $\psi = 0$. This and (39) imply $K = 0$. We have obtained a contradiction with (29). Hence the sesquilinear form $\mathcal{B}$ is coercive.

If $J^{imp} \in (L^2)^3$, then the right hand side of (25) is a bounded linear functional on the space $(\mathcal{H}_0(\nabla\times)\cap\mathcal{H}(\nabla\cdot))\times\mathcal{H}_0^1$ with the norm $\|.\|_\mathcal{B}$, thus from the Lax-Milgram theorem, there exists a unique solution to equation (25). □

The vector-scalar formulation of Theorem 6.1 forms a basis for a general finite element simulation scheme for non divergence-free EM fields.

7. A representation of the magnetic field H

If the original field is divergence-free, a simpler weak equation involving only the vector potential may be obtained. This approach is presented for a representation of the magnetic field H. This representation is mentioned in [37],

$$H = F - \nabla\left(\frac{\nabla\cdot F}{i\omega\hat{\sigma}\mu}\right) \tag{47}$$

Existence of this representation follows from Theorem 3.1 if assumptions (19) are satisfied. Although in a geophysical setting it cannot be assumed that the conductivity is constant, most of the rocks have magnetic permeability $\mu = \mu_0$. In this case the magnetic field H is divergence free:

$$\nabla\cdot H = 0$$

In this situation, the vector potential for which $\frac{\nabla \cdot F}{i\omega\hat{\sigma}\mu} = 0$ on $\partial\Omega$, coincides with the magnetic field:

$$F = H$$

We start with the standard curl-curl equation for the magnetic field H:

$$\int_\Omega \frac{1}{\hat{\sigma}}(\nabla \times H) \cdot (\overline{\nabla \times K}) + i\omega \int_\Omega \mu_0 H \cdot \overline{K} = \int_\Omega \frac{1}{\hat{\sigma}} J^{imp} \cdot (\overline{\nabla \times K}) \tag{48}$$

Substitution of $\left(H - \nabla\left(\frac{\nabla \cdot H}{i\omega\hat{\sigma}\mu}\right)\right)$ instead of H, results in the equation presented below:

$$\begin{aligned} \int_\Omega \frac{1}{\hat{\sigma}}(\nabla \times H) \cdot (\overline{\nabla \times K}) + \int_\Omega \frac{1}{\hat{\sigma}}(\nabla \cdot H)(\overline{\nabla \cdot K}) + i\omega \int_\Omega \mu_0 H \cdot \overline{K} \\ = \int_\Omega \frac{1}{\hat{\sigma}} J^{imp} \cdot (\overline{\nabla \times K}) \end{aligned} \tag{49}$$

$$\forall\ K \in \mathcal{H}(\nabla\times) \cap \mathcal{H}(\nabla\cdot), \quad n \times K|_{\partial\Omega} = 0$$

$$H \in \mathcal{H}(\nabla\times) \cap \mathcal{H}(\nabla\cdot), \quad n \times H|_{\partial\Omega} = n \times \hat{H}|_{\partial\Omega}$$

where $n \times \hat{H}$ denote tangential boundary values for H.

The sesquilinear form of this equation for $\hat{\sigma} \in \mathbb{R}$ and $0 < \sigma_m \leq \hat{\sigma} \leq \sigma_M < \infty$ is coercive and bounded with respect to the norm

$$\|K\|_{\nabla\cdot,\nabla\times} = \sqrt{\|\nabla \times K\|_0^2 + \|\nabla \cdot K\|_0^2 + \|K\|_0^2}$$

So the equation admits a unique solution, which is the magnetic field H.

The advantage of the equation (49) is that the sesquilinear form is coercive, even if the term $i\omega \int_\Omega \mu_0 H \cdot K$ is not present, as long as the boundary of the domain Ω is connected. This situation happens when the frequency $w = 0$. As a result the system matrix is well conditioned for small frequencies. If there is a jump in conductivity, the condition number of the system matrix increases, yet the situation is similar to the case of a discontinuous coefficient in the Poisson equation. Even for a high contrast in conductivity, it should be sufficient to use standard vector multigrid preconditioners [35] for an iterative solver to converge.

This kind of regularization has been studied in the literature (see [1, 13]) without introducing the notion of the Schelkunoff potential. Indeed, if the original field is divergence free, then the Schelkunoff potential of Theorem 3.1 coincides with the original field. An interesting eigenvalue analysis for the equation with and without the divergence term is presented in [30].

In geophysical applications a computational domain is usually a convex polygon (in magnetotellurics it is a cuboid). In this situation $(\mathcal{H}^1)^3 \cap \mathcal{H}_0(\nabla\times)$ is dense in $\mathcal{H}(\nabla\cdot) \cap \mathcal{H}_0(\nabla\times)$, so the use of the nodal shape functions leads to a convergent discretization. Caution is needed when applying this method to other problems as $(\mathcal{H}^1)^3 \cap \mathcal{H}_0(\nabla\times)$ is not always dense in $\mathcal{H}(\nabla\cdot) \cap \mathcal{H}_0(\nabla\times)$ (see [13] or Appendix B in [6]). In application to magnetotellurics, numerical tests involving equation (49) are presented in section 8.

8. Numerical results

In this section, the magnetic field H for a plane-wave (magnetotelluric) source is calculated using equation (49) and compared with a field calculated by an independent integral equation code of [36].

The considered model is a conductive brick of resistivity $1\Omega m$ and dimensions 1km x 2km x 2km in the whole space of resistivity $100\Omega m$. The field is calculated

500m above the brick, along a line going in the y-direction. The second order nodal shape functions for a hexahedral mesh (Q2) are used for each component of the field. A sketch of the model and the hexahedral mesh is presented in Figure 2.

The solution H is approximated by

$$H = \sum_{j=1}^{n} x_j N_j \tag{50}$$

where n is the number of degrees of freedom, N_j are shape functions. Inserting (50) into equation (49) gives

$$\int_\Omega \tfrac{1}{\hat{\sigma}} \left(\nabla \times \textstyle\sum_{j=1}^n x_j N_j\right) \cdot (\nabla \times N_k) + \int_\Omega \tfrac{1}{\hat{\sigma}} \left(\nabla \cdot \textstyle\sum_{j=1}^n x_j N_j\right) (\nabla \cdot N_k) \\ + i\omega \int_\Omega \mu_0 \textstyle\sum_{j=1}^n x_j N_j \cdot N_k = \int_\Omega \tfrac{1}{\hat{\sigma}} J^{imp} (\nabla \times N_k)$$

which produces a linear system $Ax = b$ to be solved, where

$$A_{kj} = \int_\Omega \frac{1}{\hat{\sigma}} (\nabla \times N_j) \cdot (\nabla \times N_k) + \int_\Omega \frac{1}{\hat{\sigma}} (\nabla \cdot N_j)(\nabla \cdot N_k) + i\omega \int_\Omega \mu_0 N_j \cdot N_k$$

$$b_k = \int_\Omega \frac{1}{\hat{\sigma}} J^{imp} \cdot (\nabla \times N_k)$$

The total field generated by a plane wave in the whole space with the brick, is decomposed into a primary electromagnetic field (H_p, E_p) and a secondary electromagnetic field (H_s, E_s)

$$H_t = H_p + H_s, \quad E_t = E_p + E_s$$

The primary field is a plane wave traveling in increasing z direction in the 100Ωm whole space with H field purely in the y direction. The secondary field is the change of the field due to the presence of the brick. The code solves for the secondary field H_s, with $n \times H_s = 0$ on $\partial\Omega$. It is assumed that $\sigma = \sigma_t$ is the conductivity of a conducting brick in a whole-space, with the source $J^{imp} = E_p \sigma_s$, where $\sigma_s = \sigma_t - \sigma_p$ is the difference between the conductivity of the whole-space with the conducting brick and the conductivity of the whole space. Two frequencies were considered: 0.001Hz and 10Hz. The mesh consisted of 15x15x20 hexahedral elements and extended more than 20km from the brick. The inner part of the mesh is presented in Figure 2. The linear system had $98,397$ unknowns. QMR with incomplete LU preconditioner converged to the relative residual norm of 10^{-7} in 28 iterations for the frequency 10Hz and in 54 iterations for 0.001Hz.

Figure 3 presents a ratio of the secondary field to the primary field. The fields calculated by an Integral Equation code [36] and FEM code that uses (49), are similar for both frequencies. The proposed method gives proper values of the magnetic field H.

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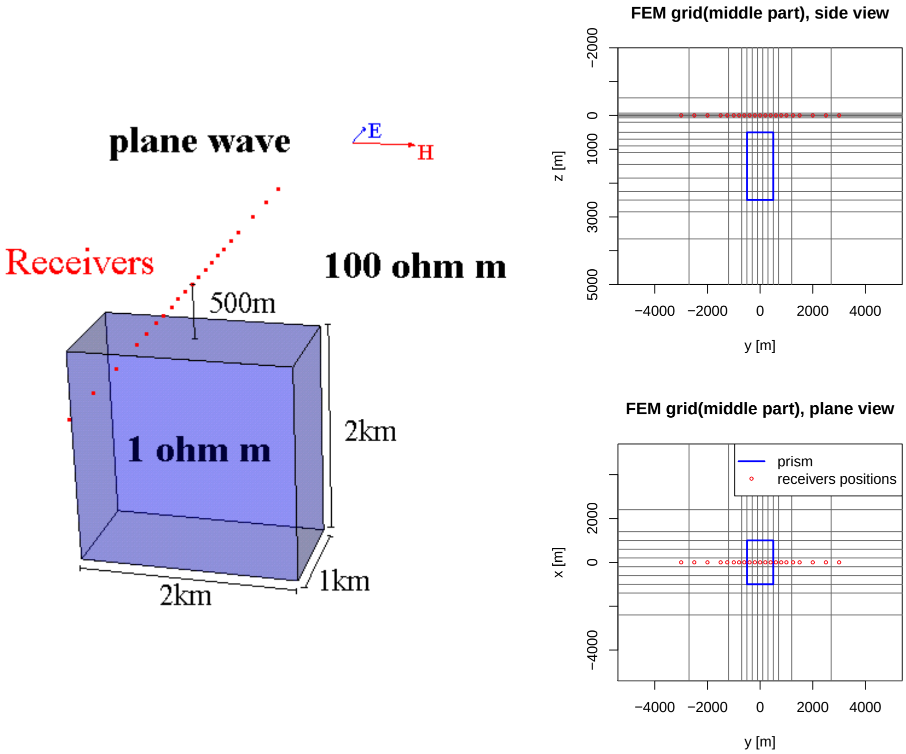

FIGURE 2. Sketch of a considered model for numerical simulation(left); Hexahedral mesh cross-sections(right).

9. Appendix

Three vector identities are used. For $K, L : \mathbb{R}^3 \to \mathbb{C}^3, u : \mathbb{R}^3 \to \mathbb{C}$, which are at least $\mathcal{C}^2$ regular in $\overline{\Omega}$, we have:

$$\nabla \times \nabla \times K = \nabla(\nabla \cdot K) - \nabla \cdot (\nabla K) \tag{51}$$

$$\int_\Omega (\nabla \times K) \cdot L = \int_\Omega K \cdot (\nabla \times L) + \int_{\partial\Omega} (n \times K) \cdot L \tag{52}$$

$$\int_\Omega \nabla u \cdot K = -\int_\Omega u \nabla \cdot K + \int_{\partial\Omega} u(K \cdot n) \tag{53}$$

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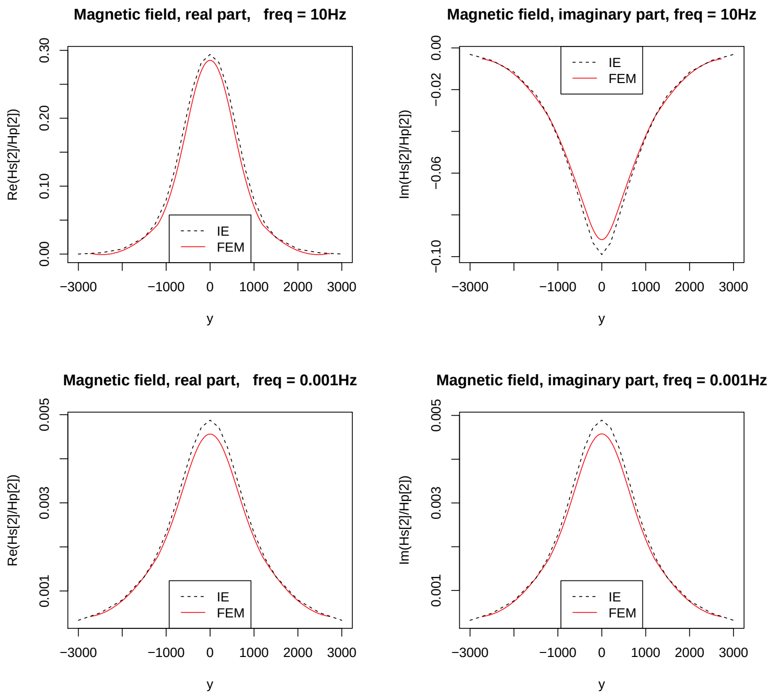

FIGURE 3. Ratio of the secondary field to the primary field for frequency 10Hz(top) and for frequency 0.001Hz(bottom).

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