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ANALYSIS OF A SECOND-ORDER, UNCONDITIONALLY STABLE, PARTITIONED METHOD FOR THE EVOLUTIONARY STOKES-DARCY MODEL

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Abstract. We propose and analyze a partitioned numerical method for the fully evolutionary Stokes-Darcy equations that model the coupling between surface and groundwater flows. The proposed method uncouples the surface from the groundwater flow by using the implicit-explicit combination of the Crank-Nicolson and Leapfrog methods for the discretization in time with added stabilization terms. We prove that the method is asymptotically, unconditionally stable—requiring no time step condition—and second-order accurate in time with optimal rates in space. We verify the method's unconditional stability and second-order accuracy numerically.

Key words. Stokes, Darcy, groundwater, surface water, partitioned, decoupled, second-order accuracy, unconditional stability, asymptotic stability

1. Introduction

One of the difficulties in solving the Stokes-Darcy problem arises from the coupling of two different physical processes in two adjacent domains. Using partitioned methods to uncouple the Stokes from the Darcy equations resolves this issue and allows one to leverage existing algorithms already optimized to solve the physical processes in each subdomain. The first partitioned methods (first-order accurate) for the evolutionary Stokes-Darcy equations were studied in [19]. Other first-order partitioned methods were analyzed in [17], and second-order, long-time accurate, partitioned methods in [6]. In [15], it was shown that the implicit-explicit combination of the Crank-Nicolson and Leapfrog methods (CNLF) results in a second-order partitioned method for the Stokes-Darcy system. However, the conditional stability of CNLF makes the method impractical when faced with certain small-value model parameters.

By adding appropriate stabilization terms to both the Stokes as well as the groundwater flow equation, the proposed numerical scheme, denoted CNLF-stab and introduced in Section 3, equations (19)-(21), is unconditionally, asymptotically stable, as well as second-order convergent. More specifically, we prove that the added stabilization terms eliminate the time step restriction without affecting the second-order accuracy of the method. Further, we show that CNLF-stab controls the unstable mode due to Leapfrog and is thus asymptotically stable.

We let Ω_f, Ω_p denote two regular, bounded domains, the fluid and porous media regions respectively, and assume they lie across an interface I (Figure 1). Suppose that an incompressible fluid flows both ways across I, described by time-dependent Stokes flow in Ω_f and the groundwater flow equation in Ω_p. The fluid velocity field $u = u(x,t)$ and pressure $p = p(x,t)$, defined in Ω_f, and porous media hydraulic

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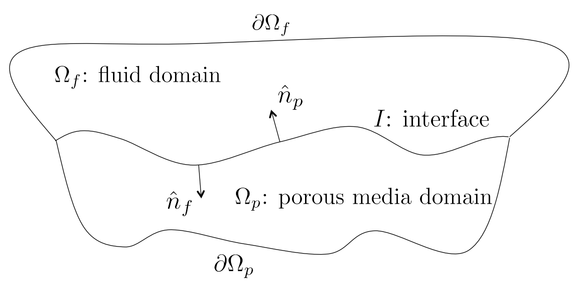

FIGURE 1. Fluid and porous media domains

head $\phi = \phi(x,t)$, defined in Ω_p, satisfy

$$\begin{gathered} u_t - \nu\Delta u + \nabla p = f_f(x,t), \nabla\cdot u = 0, \text{ in } \Omega_f, \\ S_0\phi_t - \nabla\cdot(\mathcal{K}\nabla\phi) = f_p(x,t), \text{ in } \Omega_p, \\ u(x,0) = u_0, \text{ in } \Omega_f, \ \phi(x,0) = \phi_0, \text{ in } \Omega_p, \\ u(x,t) = 0, \text{ in } \partial\Omega_f\backslash I, \phi(x,t) = 0, \text{ in } \partial\Omega_p\backslash I, \\ + \text{ coupling conditions across } I, \end{gathered} \tag{1}$$

where the pressure, p, and the body forces in the fluid region, f_f, have been normalized by the fluid density, ρ. Denoted by f_p are the sinks or sources in the porous media region, $\nu > 0$ is the kinematic viscosity of the fluid, and $\mathcal{K}$ is the hydraulic conductivity tensor, assumed to be symmetric, positive definite with spectrum$(\mathcal{K}) \in [k_{\min}, k_{\max}]$. We assume Dirichlet boundary conditions at the exterior boundaries of the two domains (not including the interface I). We discuss the assumed coupling conditions in Section 2.

In the aforementioned equations, S_0 is the specific storage, defined as the volume of water that a portion of a fully saturated porous medium releases from storage per unit volume and per unit drop in hydraulic head, see [9, 11]. Table 1 gives values of S_0 for different materials [8, 13]. The time step condition for stability in regular CNLF, derived in [15], is

$$\Delta t \leq C \max\{\min\{h^2, gS_0\}, \min\{h, gS_0h\}\},$$

where g is the gravitational acceleration constant, h the mesh size in the finite element discretization, and C a positive constant independent of both h and Δt. The time step condition is sensitive to values of S_0 and this can be computationally restrictive in certain cases. For instance, since $g = O(10^1)$, if $S_0 \leq \mathcal{O}(10^{-3})$ and $h = \mathcal{O}(10^{-1})$, then the time step condition implies that $\Delta t \leq \mathcal{O}(S_0)$. A small time step is prohibitive since studying flow in large aquifers with low conductivity necessitates accurate calculations over long-time periods.

Another important parameter in our analysis is the hydraulic conductivity tensor, $\mathcal{K}$. In exact arithmetic, stability of CNLF does not depend upon $\mathcal{K}$. Since the hydraulic conductivity is often very small (Table 2, [1]), and computations are required over long-time intervals, unconditional stability—independent of $\mathcal{K}$—of our numerical scheme is desirable.

In Section 2 we present necessary preliminaries and the equivalent weak formulation of the Stokes-Darcy problem. In Section 3 we introduce the CNLF-stab method for the evolutionary Stokes-Darcy model and present the proof for unconditional, asymptotic stability. We prove second-order convergence of the method in

TABLE 1. Specific storage (S_0) values for different materials.

Material	$\mathbf{S_0}$ (m^{-1})
Plastic clay	$2.0 \times 10^{-2} - 2.6 \times 10^{-3}$
Stiff clay	$2.6 \times 10^{-3} - 1.3 \times 10^{-3}$
Medium hard clay	$1.3 \times 10^{-3} - 9.2 \times 10^{-4}$
Loose sand	$1.0 \times 10^{-3} - 4.9 \times 10^{-4}$
Dense sand	$2.0 \times 10^{-4} - 1.3 \times 10^{-4}$
Dense sandy gravel	$1.0 \times 10^{-4} - 4.9 \times 10^{-5}$
Rock, fissured jointed	$6.9 \times 10^{-5} - 3.3 \times 10^{-6}$
Rock, sound	less than 3.3×10^{-6}

TABLE 2. Hydraulic conductivity ($k_{\min}$) values for different materials.

Material	$\mathbf{k}_{\min}$ (m/s)
Well sorted gravel	$10^{-1} - 10^{0}$
Highly fractured rocks	$10^{-3} - 10^{0}$
Well sorted sand or sand & gravel	$10^{-4} - 10^{-2}$
Oil reservoir rocks	$10^{-6} - 10^{-4}$
Very fine sand, silt, loess, loam	$10^{-8} - 10^{-5}$
Layered clay	$10^{-8} - 10^{-6}$
Fresh sandstone, limestone, dolomite, granite	$10^{-12} - 10^{-7}$
Fat/Unweathered clay	$10^{-12} - 10^{-9}$

Section 4. Section 5 demonstrates the method's unconditional, asymptotic stability and second-order accuracy through a series of numerical tests. Finally, we present conclusions in Section 6.

2. Preliminaries

Before discussing the CNLF-stab method, we present the equivalent variational formulation along with some inequalities relevant to our analysis. To couple the two flows modeled by the equations in (1), we must add appropriate conditions to describe the flow along the interface, I. The coupling conditions consist of conservation of mass across the interface

$$u \cdot \hat{n}_f - \mathcal{K}\nabla\phi \cdot \hat{n}_p = 0, \text{ on } I, \tag{2}$$

and balance of normal forces across the interface

$$p - \nu\ \hat{n}_f \cdot (\nabla u + \nabla u^\top) \cdot \hat{n}_f = g\phi, \text{ on } I, \tag{3}$$

where $\hat{n}_p = -\hat{n}_f$ are the outward pointing unit normal vectors on $\Omega_{f/p}$ (Figure 1). The last condition is a condition on the tangential velocity on I. Let $\hat{\tau}_i$, $i = 1, \ldots, d-1$, denote an orthonormal basis of tangent vectors on I, $d = 2$ or 3. We assume the Beavers-Joseph-Saffman condition, see [14, 21]:

$$-\nu\ \hat{\tau}_i \cdot (\nabla u + \nabla u^\top) \cdot \hat{n}_f = \frac{\alpha}{\sqrt{\hat{\tau}_i \cdot \mathcal{K} \cdot \hat{\tau}_i}}\ u \cdot \hat{\tau}_i, \text{ for } i = 1, \ldots, d-1, \text{ on } I, \tag{4}$$

which is a simplification of the original and more physically realistic Beavers-Joseph condition, see [2]. The parameter α in (4) is an experimentally determined constant. For more information on this condition see, e.g., [12, 20].

The equivalent variational formulation of equations (1)-(4) follows, see, e.g., [7]. Let the L^2 norm on $\Omega_{f/p}$ be denoted by $\|\cdot\|_{f/p}$ and the L^2 norm on I by $\|\cdot\|_I$; denote the corresponding inner products on $\Omega_{f/p}$ by $(\cdot,\cdot)_{f/p}$. Furthermore, denote the H^1 norm on $\Omega_{f/p}$ by $\|\cdot\|_{1,f/p}$. Define the spaces

$$\begin{aligned}
X_f &:= \left\{v \in (H^1(\Omega_f))^d : v = 0 \text{ on } \partial\Omega_f \backslash I\right\},\\
X_p &:= \left\{\psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \partial\Omega_p \backslash I\right\},\\
Q &:= L_0^2(\Omega_f),\\
V_f &:= \left\{v \in X_f : (\nabla\cdot v, q)_f = 0 \text{ for all } q \in Q\right\}.
\end{aligned}$$

The norms on the dual spaces X_f^* and X_p^* are given by

$$\|f\|_{-1,f/p} = \underset{0\neq v\in X_{f/p}}{sup} \frac{(f,v)_{f/p}}{\|\nabla v\|_{f/p}}.$$

In the analysis to follow we use some standard inequalities recalled next. The first is the Poincaré-Friedrichs inequality. The second is a trace inequality, see, e.g., [3, Chapter 1.6, p. 36-38]. The first and second inequalities hold for any function w that belongs to either X_f or X_p and the third inequality holds for any $u \in X_f$.

$$\|w\|_{f/p} \le C_{P_{f/p}} \|\nabla w\|_{f/p}, \text{ for some constants } C_{P_{f/p}} > 0, \tag{5}$$

$$\|w\|_{L^2(\partial\Omega_{f/p})} \le C_{\Omega_{f/p}} \|w\|_{f/p}^{\frac{1}{2}} \|\nabla w\|_{f/p}^{\frac{1}{2}}, \text{ for some constants } C_{\Omega_{f/p}} > 0, \tag{6}$$

$$\|\nabla\cdot u\|_f \le \sqrt{d}\|\nabla u\|_f, \text{ where } d = 2, \text{ or } 3. \tag{7}$$

Define the bilinear forms

$$\begin{aligned}
a_f(u,v) &= (\nu\nabla u, \nabla v)_f + \sum_{i=1}^{d-1} \int_I \frac{\alpha}{\sqrt{\hat\tau_i\cdot\mathcal{K}\cdot\hat\tau_i}} (u\cdot\hat\tau_i)(v\cdot\hat\tau_i)\ ds,\\
a_p(\phi,\psi) &= g\left(\mathcal{K}\nabla\phi, \nabla\psi\right)_p,\\
c_I(u,\phi) &= g\int_I \phi u\cdot\hat n_f\ ds.
\end{aligned}$$

The interface coupling term, $c_I(\cdot,\cdot)$, plays a key role in our analysis. The following inequalities hold for our bilinear forms.

Lemma 1. *The bilinear forms $a_f(\cdot,\cdot), a_p(\cdot,\cdot)$ and $c_I(\cdot,\cdot)$ satisfy*

$$a_f(u,v) \le \max\left\{\nu+1, 2^{-1}C(\Omega_f)\alpha k_{min}^{-1/2}\right\}\|u\|_{1,f}\|v\|_{1,f}, \tag{8}$$

$$a_f(u,u) \ge \nu\|\nabla u\|_f^2 + \alpha k_{max}^{-1/2}\sum_{i=1}^{d-1}\int_I (u\cdot\hat\tau_i)^2\ d\sigma \ge \nu\|\nabla u\|_f^2, \tag{9}$$

$$a_p(\phi,\psi) \le g k_{max}\|\nabla\phi\|_p\|\nabla\psi\|_p, \tag{10}$$

$$a_p(\phi,\phi) \ge g k_{min}\|\nabla\phi\|_p^2, \tag{11}$$

$$|c_I(u,\phi)| \le 2^{-1} g C(\Omega_f,\Omega_p)\|u\|_{1,f}\|\phi\|_{1,p}, \tag{12}$$

for all $u, v \in X_f$ and all $\phi,\psi \in X_p$.

Proof. The proofs are straightforward. For the first four inequalities, see, e.g., [15, Section II Lemma 2.3]. For the last, see, e.g., [18, Section 2 Lemma 2.2]. □

An additional inequality on the interface term is given below and holds under conditions on the domains Ω_f, Ω_p. The constant $C_{f,p}$ depends on $\Omega_{f/p}$ and in the special case of a flat interface I, with Ω_f and Ω_p being arbitrary domains, $C_{f,p}$ equals 1, see [18, Section 3 Lemmas 3.1 and 3.2].

$$|c_I(u,\phi)| \le gC_{f,p}\|u\|_{\mathrm{div},f}\|\phi\|_{1,p}, \text{ where } \|u\|^2_{\mathrm{div},f} := \|u\|^2_f + \|\nabla u\|^2_f. \tag{13}$$

The variational formulation of the Stokes-Darcy problem (1)-(4) reads: given $u(x,0) = u_0(x)$, $\phi(x,0) = \phi_0(x)$, find $u : [0,\infty) \to V_f$, $\phi : [0,\infty) \to X_p$ satisfying

$$(u_t, v)_f + a_f(u,v) + c_I(v,\phi) = (f_f, v)_f, \tag{14}$$

$$gS_0(\phi_t,\psi)_p + a_p(\phi,\psi) - c_I(u,\psi) = g(f_p,\psi)_p, \tag{15}$$

for all $v \in V_f$, and $\psi \in X_p$. The existence and uniqueness of a solution (u,ϕ) to the problem (14)-(15) follows by the theory of saddle point problems found in, e.g., [4, 5], established in, e.g., [16].

We discretize in space using the Finite Element Method (FEM). Select a quasi-uniform triangular mesh, $\mathcal{T}_h$, for the combined subdomains, $\Omega_f \cup \Omega_p$, where h denotes the maximum triangle diameter. Next, choose FEM spaces based on a conforming FEM triangulation:

$$\text{Fluid velocity: } X_f^h \subset X_f,$$
$$\text{Darcy Pressure: } X_p^h \subset X_p,$$
$$\text{Stokes Pressure: } Q_f^h \subset Q_f.$$

Additionally, we must select X_f^h and Q_f^h so that they satisfy the discrete inf-sup condition (LBB^h) (see, e.g., [10]) for stability of the discrete pressure. Notice that $V_f^h := \{v_h \in X_f^h : (q_h, \nabla\cdot v_h)_f = 0 \quad \forall q_h \in Q_f^h\}$ is not necessarily a subspace of V_f. Hence, we must include the incompressibility condition (17) in the semi-discretized formulation. Given $u_h(x,0) = u_0(x), \phi_h(x,0) = \phi_0(x)$, find $(u_h, p_h, \phi_h) : [0,\infty) \to X_f^h \times Q_f^h \times X_p^h$ such that

$$(u_{h,t}, v_h)_f + a_f(u_h,v_h) - (p_h, \nabla\cdot v_h)_f + c_I(v_h,\phi_h) = (f_f, v_h)_f, \tag{16}$$

$$(q_h, \nabla\cdot u_h)_f = 0, \tag{17}$$

$$gS_0(\phi_{h,t},\psi_h)_p + a_p(\phi_h,\psi_h) - c_I(u_h,\psi_h) = g(f_p,\psi_h)_p, \tag{18}$$

for all $(v_h, q_h, \psi_h) \in X_f^h \times Q_f^h \times X_p^h$.

3. CNLF-stab Method and Unconditional, Asymptotic Stability

The CNLF-stab method for the numerical solution of the evolutionary Stokes-Darcy problem given in (1)-(4) is introduced next.

Algorithm 2 (CNLF-stab Method)**.** *Let $t^n := n\Delta t$ and $v^n := v(x,t^n)$ for any function $v(x,t)$. CNLF with added stabilization for the evolutionary Stokes-Darcy equations is as follows.*

Given (u_h^k, p_h^k, ϕ_h^k), $(u_h^{k-1}, p_h^{k-1}, \phi_h^{k-1}) \in X_f^h \times Q_f^h \times X_p^h$, find $(u_h^{k+1}, p_h^{k+1}, \phi_h^{k+1}) \in X_f^h \times Q_f^h \times X_p^h$ satisfying for all $(v_h, q_h, \psi_h) \in X_f^h \times Q_f^h \times X_p^h$:

$$\left(\frac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}, v_h\right)_f + \left(\nabla\cdot\left(\frac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}\right), \nabla\cdot v_h\right)_f + a_f\left(\frac{u_h^{k+1}+u_h^{k-1}}{2}, v_h\right)$$
$$-\left(\frac{p_h^{k+1}+p_h^{k-1}}{2}, \nabla\cdot v_h\right)_f + c_I(v_h, \phi_h^k) = (f_f^k, v_h)_f, \tag{19}$$

$$\left(q_h, \nabla\cdot u_h^{k+1}\right)_f = 0, \tag{20}$$

$$gS_0\left(\frac{\phi_h^{n+1}-\phi_h^{k-1}}{2\Delta t}, \psi_h\right)_p + a_p\left(\frac{\phi_h^{k+1}+\phi_h^{k-1}}{2}, \psi_h\right) - c_I(u_h^k, \psi_h) \tag{21}$$
$$+\Delta t g^2 C_{f,p}^2\left\{\left(\phi_h^{k+1}-\phi_h^{k-1}, \psi_h\right)_p + \left(\nabla(\phi_h^{k+1}-\phi_h^{k-1}), \nabla\psi_h\right)_p\right\} = g(f_p^k, \psi_h)_p,$$

where $C_{f,p}$ is the constant from inequality (13).

CNLF-stab is a three-level method. The zeroth terms, (u_h^0, p_h^0, ϕ_h^0), come from the initial conditions of the problem. We must obtain the first terms, (u_h^1, p_h^1, ϕ_h^1), by a one-step method, for example Backward Euler Leapfrog (BELF). The errors in this first step will affect the overall convergence rate of the method. Notice that the added stability terms,

$$\left(\nabla\cdot\left(\frac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}\right), \nabla\cdot v_h\right)_f \text{ in (19) and}$$
$$\Delta t g^2 C_{f,p}^2\left\{\left(\phi_h^{k+1}-\phi_h^{k-1}, \psi_h\right)_p + \left(\nabla(\phi_h^{k+1}-\phi_h^{k-1}), \nabla\psi_h\right)_p\right\} \text{ in (21)},$$

are $\mathcal{O}(\Delta t^2)$. Similar to CNLF, CNLF-stab uncouples the Stokes-Darcy equations into two subdomain problems by treating the coupling terms explicitly with Leapfrog. By adding the above stabilization terms to CNLF, we eliminate the CFL type time step restriction for stability. The proofs of unconditional and asymptotic stability of CNLF-stab follow.

Theorem 3 (Unconditional Stability of CNLF-stab)**.** *CNLF-stab is unconditionally stable: for any $N > 1$, there holds*

$$\begin{aligned}
&\frac{1}{2}\left(\|u_h^N\|_{div,f}^2 + \|u_h^{N-1}\|_{div,f}^2\right) + gS_0\left(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2\right)\\
&+\frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\nu\|\nabla\left(u_h^{k+1}+u_h^{k-1}\right)\|_f^2 + gk_{min}\|\nabla\left(\phi_h^{k+1}+\phi_h^{k-1}\right)\|_p^2\right\}\\
&\le \|u_h^1\|_{div,f}^2 + \|u_h^0\|_{div,f}^2 + gS_0\left(\|\phi_h^1\|_p^2 + \|\phi_h^0\|_p^2\right)\\
&+2\Delta t^2 g^2 C_{f,p}^2\left(\|\phi_h^1\|_{1,p}^2 + \|\phi_h^0\|_{1,p}^2\right) + 2\Delta t\left\{c_I(\phi_h^0, u_h^1) - c_I(\phi_h^1, u_h^0)\right\}\\
&+\frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\frac{1}{\nu}\|f_f^k\|_{-1,f}^2 + \frac{g}{k_{min}}\|f_p^k\|_{-1,p}^2\right\}.
\end{aligned} \tag{22}$$

Proof. In (19), (21) set $v_h = u_h^{k+1} + u_h^{k-1}$, $\psi_h = \phi_h^{k+1} + \phi_h^{k-1}$. Then the pressure term in (19) cancels by (20). Adding the two equations together and multiplying both sides by $2\Delta t$ yields

$$\begin{aligned}
&\|u_h^{k+1}\|^2_{\mathrm{div},f} - \|u_h^{k-1}\|^2_{\mathrm{div},f} + gS_0\left(\|\phi_h^{k+1}\|_p^2 - \|\phi_h^{k-1}\|_p^2\right)\\
&+ 2\Delta t^2 g^2 C_{f,p}^2\left(\|\phi_h^{k+1}\|_{1,p}^2 - \|\phi_h^{k-1}\|_{1,p}^2\right)\\
&+ \Delta t\left\{a_f\left(u_h^{k+1}+u_h^{k-1}, u_h^{k+1}+u_h^{k-1}\right) + a_p\left(\phi_h^{k+1}+\phi_h^{k-1}, \phi_h^{k+1}+\phi_h^{k-1}\right)\right\}\\
&+ 2\Delta t\left(c_I(u_h^{k+1}+u_h^{k-1}, \phi_h^k) - c_I(u_h^k, \phi_h^{k+1}+\phi_h^{k-1})\right)\\
&\quad = 2\Delta t\left\{\left(f_f^k, u_h^{k+1}+u_h^{k-1}\right)_f + g\left(f_p^k, \phi_h^{k+1}+\phi_h^{k-1}\right)_p\right\}.
\end{aligned}$$

If we let

$$C^{k+1/2} = c_I(\phi_h^k, u_h^{k+1}) - c_I(\phi_h^{k+1}, u_h^k),$$

then the interface terms in the equation above become

$$c_I(u_h^{k+1}+u_h^{k-1}, \phi_h^k) - c_I(u_h^k, \phi_h^{k+1}+\phi_h^{k-1}) = C^{k+\frac{1}{2}} - C^{k-\frac{1}{2}}.$$

Using coercivity of the bilinear forms $a_{f/p}(\cdot,\cdot)$, the dual norms on X_p, X_f, and Young's inequality we obtain, after rearranging,

$$\begin{aligned}
&\|u_h^{k+1}\|^2_{\mathrm{div},f} - \|u_h^{k-1}\|^2_{\mathrm{div},f} + gS_0\left(\|\phi_h^{k+1}\|_p^2 - \|\phi_h^{k-1}\|_p^2\right)\\
&+ 2\Delta t^2 g^2 C_{f,p}^2\left(\|\phi_h^{k+1}\|_{1,p}^2 - \|\phi_h^{k-1}\|_{1,p}^2\right) + 2\Delta t\left\{C^{k+\frac{1}{2}} - C^{k-\frac{1}{2}}\right\}\\
&+ \frac{\Delta t}{2}\left\{\nu\|\nabla\left(u_h^{k+1}+u_h^{k-1}\right)\|_f^2 + gk_{\min}\|\nabla\left(\phi_h^{k+1}+\phi_h^{k-1}\right)\|_p^2\right\}\\
&\quad \le \frac{\Delta t}{2}\left\{\frac{1}{\nu}\|f_f^k\|^2_{-1,f} + \frac{g}{k_{\min}}\|f_p^k\|^2_{-1,p}\right\}.
\end{aligned} \tag{23}$$

Denote the energy terms by

$$\begin{aligned}
E^{k+1/2} = \|u_h^{k+1}\|^2_{\mathrm{div},f} + \|u_h^k\|^2_{\mathrm{div},f} + gS_0\left(\|\phi_h^{k+1}\|_p^2 + \|\phi_h^k\|_p^2\right)\\
+2\Delta t^2 g^2 C_{f,p}^2\left(\|\phi_h^{k+1}\|_{1,p}^2 + \|\phi_h^k\|_{1,p}^2\right).
\end{aligned}$$

Then (23) becomes

$$\begin{aligned}
E^{k+1/2} - E^{k-1/2} + \frac{\Delta t}{2}\left\{\nu\|\nabla\left(u_h^{k+1}+u_h^{k-1}\right)\|_f^2 + gk_{\min}\|\nabla\left(\phi_h^{k+1}+\phi_h^{k-1}\right)\|_p^2\right\}\\
+2\Delta t\left\{C^{k+1/2} - C^{k-1/2}\right\} \le \frac{\Delta t}{2}\left\{\frac{1}{\nu}\|f_f^k\|^2_{-1,f} + \frac{g}{k_{\min}}\|f_p^k\|^2_{-1,p}\right\}.
\end{aligned}$$

Sum up the inequality from $k = 1$ to $N-1$ to find

$$\begin{aligned}
E^{N-1/2} + \frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\nu\|\nabla\left(u_h^{k+1}+u_h^{k-1}\right)\|_f^2 + gk_{\min}\|\nabla\left(\phi_h^{k+1}+\phi_h^{k-1}\right)\|_p^2\right\}\\
+2\Delta t C^{N-1/2} \le E^{1/2} + C^{1/2} + \frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\frac{1}{\nu}\|f_f^k\|^2_{-1,f} + \frac{g}{k_{\min}}\|f_p^k\|^2_{-1,p}\right\}.
\end{aligned} \tag{24}$$

Applying inequality (13) to the interface terms involved in the term $C^{N-1/2}$ gives

$$\begin{aligned}
&|c_I(u_h^N, \phi_h^{N-1})| \le gC_{f,p}\|u_h^N\|_{\mathrm{div},f}\|\phi_h^{N-1}\|_{1,p} \text{ and}\\
&|c_I(u_h^{N-1}, \phi_h^N)| \le gC_{f,p}\|u_h^{N-1}\|_{\mathrm{div},f}\|\phi_h^N\|_{1,p}.
\end{aligned}$$

Therefore, we may bound the term $C^{N-1/2}$ by the Cauchy-Schwarz and Young inequalities as follows.

$$\begin{aligned}|2\Delta t C^{N-1/2}| \le \frac{1}{2}\left(\|u_h^N\|_{\mathrm{div},f}^2 + \|u_h^{N-1}\|_{\mathrm{div},f}^2\right) \\ + \ 2\Delta t^2 g^2 C_{f,p}^2 \left(\|\phi_h^{N-1}\|_{1,p}^2 + \|\phi_h^N\|_{1,p}^2\right).\end{aligned}$$

Thus,

$$\begin{aligned}E^{N-1/2} + 2\Delta t C^{N-1/2} \ge \frac{1}{2}\left(\|u_h^N\|_{\mathrm{div},f}^2 + \|u_h^{N-1}\|_{\mathrm{div},f}^2\right) \\ + gS_0\left(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2\right).\end{aligned} \tag{25}$$

After combining inequalities (24) and (25) we achieve the desired unconditional stability bound (22). □

The following corollary shows that the stable modes of Leapfrog, $(u_h^{k+1} + u_h^{k-1})$ and $(\phi_h^{k+1} + \phi_h^{k-1})$, are asymptotically stable in the CNLF-stab method.

Corollary 4 (Control of the Stable Modes). *The following inequality for the stable modes of CNLF-stab holds for all $\Delta t > 0$, $h > 0$, and $N > 1$:*

$$\begin{aligned}\frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\nu\|\nabla\left(u_h^{k+1} + u_h^{k-1}\right)\|_f^2 + gk_{min}\|\nabla\left(\phi_h^{k+1} + \phi_h^{k-1}\right)\|_p^2\right\} \\ \le E^{1/2} + C^{1/2} + \frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\frac{1}{\nu}\|f_f^k\|_{-1,f}^2 + \frac{g}{k_{min}}\|f_p^k\|_{-1,p}^2\right\}.\end{aligned} \tag{26}$$

Additionally, if $f_f = f_p \equiv 0$, then the stable modes converge to zero as $n \to \infty$.

Proof. Drop the positive term "$\frac{1}{2}(\|u_h^N\|_{\mathrm{div},f} + \|u^{N-1}\|_{\mathrm{div},f}) + gS_0(\|\phi_h^N\|_p^2 + \|\phi^{N-1}\|_p^2)$" from the left-hand side of (22) in the proof of Theorem 3. This proves control of the stable mode.

When $f_f \equiv 0$ and $f_p \equiv 0$, the bound (26) implies that $\sum_{n=1}^{\infty}\|\nabla(u_h^{n+1} + u_h^{n-1})\|_f^2$ converges. Thus, $\|\nabla(u_h^{n+1} + u_h^{n-1})\|_f \to 0$, as $n \to \infty$, and by the Poincaré-Friedrichs inequality, $\|u_h^{n+1} + u_h^{n-1}\|_f \to 0$ as well. (Similarly for $(\phi_h^{n+1} + \phi_h^{n-1})$.) □

In Theorem 5, we show that CNLF-stab also controls the unstable modes of Leapfrog, $(u_h^{n+1} - u_h^{n-1})$ and $(\phi_h^{n+1} - \phi_h^{n-1})$, and that they are asymptotically stable.

Theorem 5 (Control of the Stable and Unstable Modes). *The CNLF-stab method controls both the stable and unstable modes for all $\Delta t > 0$ and $h > 0$. That is, there exists $\mathcal{M} > 0$, such that for any $N > 1$,*

$$
\begin{aligned}
&\mathcal{M}\Bigg\{\Delta t\sum_{k=1}^{N-1}\{\|\nabla(u_h^{k+1}+u_h^{k-1})\|_f^2+\|\nabla(\phi_h^{k+1}+\phi_h^{k-1})\|_p^2\}\\
&+\sum_{k=1}^{N-1}\{\|u_h^{k+1}-u_h^{k-1}\|_{div,f}^2+\|\phi_h^{k+1}-\phi_h^{k-1}\|_p^2+\Delta t^2\|\phi_h^{k+1}-\phi_h^{k-1}\|_{1,p}^2\}\Bigg\}\\
&\le \Delta t\sum_{k=1}^{N-1}\{\|f_f^k\|_{-1,f}^2+\|f_p^k\|_{-1,p}^2+\Delta t\left(\|f_f^k\|_f^2+\|f_p^k\|_p^2\right)\}\\
&+\|u_h^1\|_{div,f}^2+\|u_h^0\|_{div,f}^2+\|\phi_h^1\|_p^2+\|\phi_h^0\|_p^2+\Delta t^2(\|\phi_h^1\|_{1,p}^2+\|\phi_h^0\|_{1,p}^2)\\
&+\Delta t\left(\|\nabla u_h^1\|_f^2+\|\nabla u_h^0\|_f^2+\|\nabla\phi_h^1\|_p^2+\|\nabla\phi_h^0\|_p^2\right)\\
&+\Delta t\left(c_I(u_h^1,\phi_h^0)-c_I(u_h^0,\phi_h^1)+c_I(u_h^2-u_h^0,\phi_h^1)-c_I(u_h^1,\phi_h^2-\phi_h^0)\right).
\end{aligned}
\tag{27}
$$

Proof. Set $v_h = 2\delta\Delta t(u_h^{k+1}-u_h^{k-1})$ and $\psi_h = 2\delta\Delta t(\phi_h^{k+1}-\phi_h^{k-1})$ in (19)-(21) where $\delta > 0$ will be determined later. After adding the equations, this produces

$$
\begin{aligned}
&\delta\left\{\|u_h^{k+1}-u_h^{k-1}\|_{\mathrm{div},f}^2+gS_0\|\phi_h^{k+1}-\phi_h^{k+1}\|_p^2\right\}+2\delta\Delta t^2g^2C_{f,p}^2\|\phi_h^{k+1}-\phi_h^{k+1}\|_{1,p}^2\\
&+\delta\Delta t\left\{\mathcal{A}^{k+1/2}-\mathcal{A}^{k-1/2}\right\}+2\delta\Delta t\left\{c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k)-c_I(u_h^k,\phi_h^{k+1}-\phi_h^{k-1})\right\}\\
&=2\delta\Delta t\left\{(f_f^k,u_h^{k+1}-u_h^{k-1})_f+g(f_p^k,\phi_h^{k+1}-\phi_h^{k-1})_p\right\},
\end{aligned}
$$

with $\mathcal{A}^{k+1/2} := a_f(u_h^{k+1},u_h^{k+1})+a_p(\phi_h^{k+1},\phi_h^{k+1})+a_f(u_h^k,u_h^k)+a_p(\phi_h^k,\phi_h^k)\ge 0.$

Next, use Cauchy-Schwarz and Young's inequality on the right-hand side, absorb the resulting unstable mode terms into the left-hand side, and sum from 1 to $N-1$:

$$
\begin{aligned}
&\delta\Delta t\mathcal{A}^{N-1/2}+\delta(1-\varepsilon)\sum_{k=1}^{N-1}\left\{\|u_h^{k+1}-u_h^{k-1}\|_{\mathrm{div},f}^2+gS_0\|\phi_h^{k+1}-\phi_h^{k+1}\|_p^2\right\}\\
&+\delta\varepsilon\sum_{k=1}^{N-1}\|\nabla\cdot(u_h^{k+1}-u_h^{k-1})\|_f^2+2\delta\Delta t^2g^2C_{f,p}^2\sum_{k=1}^{N-1}\|\phi_h^{k+1}-\phi_h^{k+1}\|_{1,p}^2\\
&+2\delta\Delta t\sum_{k=1}^{N-1}\left\{c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k)-c_I(u_h^k,\phi_h^{k+1}-\phi_h^{k-1})\right\}\\
&\le\delta\Delta t\mathcal{A}^{1/2}+\frac{\delta\Delta t^2}{\varepsilon}\sum_{k=1}^{N-1}\left\{\|f_f^k\|_f^2+\frac{g}{S_0}\|f_p^k\|_p^2\right\}.
\end{aligned}
$$

In the previous equation, $\varepsilon\in(0,1)$ is the constant arising from Young's inequality. We add the above inequality to (26) obtained in Corollary 4 to find

$$
\begin{aligned}
&\delta\Delta t\mathcal{A}^{N-1/2}+\delta(1-\varepsilon)\sum_{k=1}^{N-1}\left\{\|u_h^{k+1}-u_h^{k-1}\|_{\mathrm{div},f}^2+gS_0\|\phi_h^{k+1}-\phi_h^{k+1}\|_p^2\right\}\\
&+\delta\varepsilon\sum_{k=1}^{N-1}\|\nabla\cdot(u_h^{k+1}-u_h^{k-1})\|_f^2+2\delta\Delta t^2g^2C_{f,p}^2\sum_{k=1}^{N-1}\|\phi_h^{k+1}-\phi_h^{k+1}\|_{1,p}^2\\
&+2\delta\Delta t\sum_{k=1}^{N-1}\left\{c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k)-c_I(u_h^k,\phi_h^{k+1}-\phi_h^{k-1})\right\}\\
&+\frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\nu\|\nabla(u_h^{k+1}+u_h^{k-1})\|_f^2+gk_{\min}\|\nabla(\phi_h^{k+1}+\phi_h^{k-1})\|_p^2\right\}\\
&\le\delta\Delta t\mathcal{A}^{1/2}+E^{1/2}+2\Delta tC^{1/2}+2\Delta t\sum_{k=1}^{N-1}\left\{\frac{1}{\nu}\|f_f^k\|_{-1,f}^2+\frac{g}{k_{\min}}\|f_p^k\|_{-1,p}^2\right\}\\
&+\frac{\delta\Delta t^2}{\varepsilon}\sum_{k=1}^{N-1}\left\{\|f_f^k\|_f^2+\frac{g}{S_0}\|f_p^k\|_p^2\right\}.
\end{aligned}
\tag{28}
$$

To simplify notation, let $\mathcal{Q}=2\delta\Delta t\sum_{k=1}^{N-1}\{c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k)-c_I(u_h^k,\phi_h^{k+1}-\phi_h^{k-1})\}$. The next step is to bound and subsume the coupling terms in $\mathcal{Q}$. We first rewrite them in terms of the stable and unstable modes as follows. For $k\ge 2$,

$$
\begin{aligned}
&c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k)-c_I(u_h^k,\phi_h^{k+1}-\phi_h^{k-1})\\
&=\tfrac12 c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k-\phi_h^{k-2})+\tfrac12 c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k+\phi_h^{k-2})\\
&-\tfrac12 c_I(u_h^k-u_h^{k-2},\phi_h^{k+1}-\phi_h^{k-1})-\tfrac12 c_I(u_h^k+u_h^{k-2},\phi_h^{k+1}-\phi_h^{k-1}).
\end{aligned}
$$

By (13),

$$
\begin{aligned}
&\mathcal{Q}\le\delta\Delta tgC_{f,p}\sum_{k=2}^{N-1}\left\{\|u_h^{k+1}-u_h^{k-1}\|_{\mathrm{div},f}\left(\|\phi_h^k-\phi_h^{k-2}\|_{1,p}+\|\phi_h^k+\phi_h^{k-2}\|_{1,p}\right)\right.\\
&\left.+\left(\|u_h^k-u_h^{k-2}\|_{\mathrm{div},f}+\|u_h^k+u_h^{k-2}\|_{\mathrm{div},f}\right)\|\phi_h^{k+1}-\phi_h^{k-1}\|_p\right\}\\
&+2\delta\Delta t\left[c_I(u_h^2-u_h^0,\phi_h^1)-c_I(u_h^1,\phi_h^2-\phi_h^0)\right].
\end{aligned}
\tag{29}
$$

Bound the terms with the stable modes above by Poincaré inequality (5) and inequality (7):

$$
\|u_h^k+u_h^{k-2}\|_{\mathrm{div},f}\le\sqrt{C_{P,f}^2+d}\,\|\nabla(u_h^k+u_h^{k-2})\|_f,
$$

$$
\|\phi_h^k+\phi_h^{k-2}\|_{1,p}\le\sqrt{1+C_{P,p}^2}\,\|\nabla(\phi_h^k+\phi_h^{k-2})\|_p.
$$

Apply Young's inequality to all terms on the right-hand side of (29). Let $\mathcal{B}=\Delta tgC_{f,p}$. By Young's inequality, for any $\epsilon_{1,2,3}>0$, there holds

$$\mathcal{B}\|u_h^{k+1}-u_h^{k-1}\|_{\mathrm{div},f}\|\phi_h^k-\phi_h^{k-2}\|_{1,p} \le \frac{\epsilon_1}{2}\|u_h^{k+1}-u_h^{k-1}\|_{\mathrm{div},f}^2+\frac{\mathcal{B}^2}{2\epsilon_1}\|\phi_h^k-\phi_h^{k-2}\|_{1,p}^2,$$

$$\begin{aligned}\mathcal{B}\|u_h^{k+1}-u_h^{k-1}\|_{\mathrm{div},f}&\|\phi_h^k+\phi_h^{k-2}\|_{1,p}\\ &\le \frac{\epsilon_2}{2}\|u_h^{k+1}-u_h^{k-1}\|_{\mathrm{div},f}^2+\frac{n\mathcal{B}^2(1+C_{P,p}^2)}{2\epsilon_2}\|\nabla(\phi_h^k+\phi_h^{k-2})\|_p,\end{aligned}$$

$$\mathcal{B}\|u_h^{k}-u_h^{k-2}\|_{\mathrm{div},f}\|\phi_h^{k+1}-\phi_h^{k-1}\|_{p} \le \frac{\epsilon_1}{2}\|u_h^{k}-u_h^{k-2}\|_{\mathrm{div},f}^2+\frac{\mathcal{B}^2}{2\epsilon_1}\|\phi_h^{k+1}-\phi_h^{k-1}\|_{1,p}^2,$$

$$\begin{aligned}\mathcal{B}\|u_h^{k}+u_h^{k-2}\|_{\mathrm{div},f}&\|\phi_h^{k+1}-\phi_h^{k-1}\|_{p}\\ &\le \frac{\epsilon_3(C_{P,f}^2+d)}{2}\|\nabla(u_h^k+u_h^{k-2})\|_f^2+\frac{\mathcal{B}^2}{2\epsilon_3}\|\phi_h^{k+1}-\phi_h^{k-1}\|_{1,p}^2.\end{aligned}$$

Using the above bounds, simplify (29) and shift the index of the sums to obtain a bound for $\mathcal{Q}$ by the stable and unstable modes.

$$\begin{aligned}\mathcal{Q} \le{}& \delta\sum_{k=1}^{N-1}\left\{(\varepsilon_1+\tfrac{\varepsilon_2}{2})\|u_h^{k+1}-u_h^{k-1}\|_{\mathrm{div},f}^2+g^2\Delta t^2C_{f,p}^2(\tfrac{1}{\varepsilon_1}+\tfrac{1}{2\varepsilon_3})\|\phi_h^{k+1}-\phi_h^{k-1}\|_{1,p}^2\right\}\\ &+\frac{\delta}{2}\sum_{k=1}^{N-1}\left\{\begin{aligned}&(C_{P,f}^2+d)^2\varepsilon_3\|\nabla(u_h^{k+1}-u_h^{k-1})\|_f^2\\ &+\frac{g^2\Delta t^2C_{f,p}^2(1+C_{P,p}^2)}{\varepsilon_2}\|\nabla(\phi_h^{k+1}+\phi_h^{k-1})\|_p^2\end{aligned}\right\}\\ &+2\delta\Delta t\left[c_I(u_h^2-u_h^0,\phi_h^1)-c_I(u_h^1,\phi_h^2-\phi_h^0)\right].\end{aligned}$$

Incorporate the bound for $\mathcal{Q}$ so that (28) becomes

$$\begin{aligned}&\frac{1}{2}\sum_{k=1}^{N-1}\left\{\left(\nu\Delta t-\delta(d+C_{P,f}^2)\varepsilon_3\right)\|\nabla(u_h^{k+1}+u_h^{k-1})\|_f^2\right\}\\ &+\frac{1}{2}\sum_{k=1}^{N-1}\left\{\left(gk_{\min}\Delta t-\frac{\delta g^2\Delta t^2C_{f,p}^2(1+C_{P,p}^2)}{\epsilon_2}\right)\|\nabla(\phi_h^{k+1}+\phi_h^{k-1})\|_p^2\right\}\\ &+\delta\sum_{k=1}^{N-1}\left\{\left((1-\varepsilon)-(\varepsilon_1+\tfrac{\epsilon_2}{2})\right)\|u_h^{k+1}-u_h^{k-1}\|_{\mathrm{div},f}^2\right\}\\ &+\delta\sum_{k=1}^{N-1}\left\{(gS_0(1-\varepsilon)))\|\phi_h^{k+1}-\phi_h^{k+1}\|_p^2\right\}\\ &+\delta\varepsilon\sum_{k=1}^{N-1}\left\{\|\nabla\cdot(u_h^{k+1}-u_h^{k-1})\|_f^2\right\}\\ &+\delta g^2\Delta t^2C_{f,p}^2\sum_{k=1}^{N-1}\left\{\left(2-(\tfrac{1}{\varepsilon_1}+\tfrac{1}{2\varepsilon_3})\right)\|\phi_h^{k+1}-\phi_h^{k+1}\|_{1,p}^2\right\}\\ &\le\sum_{k=1}^{N-1}\left\{2\Delta t\left(\frac{1}{\nu}\|f_f^k\|_{-1,f}^2+\frac{g}{k_{\min}}\|f_p^k\|_{-1,p}^2\right)+\frac{\delta\Delta t^2}{\varepsilon}\left(\|f_f^k\|_f^2+\frac{g}{S_0}\|f_p^k\|_p^2\right)\right\}\\ &+\delta\Delta t\mathcal{A}^{1/2}+E^{1/2}+2\Delta tC^{1/2}+2\delta\Delta t\left[c_I(u_h^2-u_h^0,\phi_h^1)-c_I(u_h^1,\phi_h^2-\phi_h^0)\right].\end{aligned} \tag{30}$$

Control over the stable and unstable modes, $\|\nabla(w_h^{k+1}+w_h^{k-1})\|_{f/p}$, $\|w_h^{k+1}-w_h^{k-1}\|_{f/p}$ for $w=u,\phi$, respectively, as given in (27), is obtained provided the coefficients of the sums on the left-hand side of (30) are positive:

$$\begin{aligned}\nu\Delta t-\delta(C_{P,f}^2+d)\varepsilon_3&>0,\\ gk_{\min}\Delta t-\frac{\delta g^2\Delta t^2C_{f,p}^2(1+C_{P,p}^2)}{\epsilon_2}&>0,\\ (1-\varepsilon)-(\varepsilon_1+\tfrac{\epsilon_2}{2})&>0,\\ 2-(\tfrac{1}{\varepsilon_1}+\tfrac{1}{2\varepsilon_3})&>0.\end{aligned}$$

Rearrange the last two inequalities:

$$\frac{\varepsilon_1+\frac{\varepsilon_2}{2}}{1-\varepsilon}<1,\quad \frac{1}{\varepsilon_1}+\frac{1}{2\varepsilon_3}<2.$$

Many choices of ϵ and $\varepsilon_{1,2,3}$ will satisfy these requirements. For example, choose $\varepsilon=\varepsilon_2=\frac{1}{8}$, $\varepsilon_1=\frac{3}{4}$, and $\varepsilon_3=\frac{3}{2}$. Then

$$\frac{\varepsilon_1+\frac{\varepsilon_2}{2}}{1-\varepsilon}=\frac{\frac{3}{4}+\frac{1}{16}}{\frac{7}{8}}=\frac{13}{14}<1,\quad \frac{1}{\varepsilon_1}+\frac{1}{2\varepsilon_3}=\frac{4}{3}+\frac{1}{3}=\frac{5}{3}<2.$$

As for δ, choose

$$\delta=\min\left\{\frac{\nu\Delta t}{(d+C_{P,f}^2)\varepsilon_3},\frac{k_{\min}\varepsilon_2}{g\Delta tC_{f,p}^2(1+C_{P,p}^2)}\right\}>0.$$

Thus, with careful choice of δ,ϵ, and $\varepsilon_{1,2,3}$, one may find a positive constant $\mathcal{M}$, independent of mesh width, h, and time step, Δt, so that the inequality (27) holds. This in turn implies control over both the stable and unstable modes and hence asymptotic stability for CNLF-stab. □

4. Error Analysis of CNLF-stab

In this section, in Theorem 7, we establish the method's second-order convergence over long-time intervals. An essential feature of the error analysis is that no form of Grönwall's inequality is used.

We assume that the FEM spaces, X_f^h, X_p^h, and Q_f^h, satisfy approximation properties of piecewise polynomials of degree $r-1$, r, and $r+1$:

$$\begin{aligned}\inf_{u_h\in X_f^h}\|u-u_h\|_f&\le Ch^{r+1}\|u\|_{H^{r+1}(\Omega_f)}\\ \inf_{u_h\in X_f^h}\|u-u_h\|_{1,f}&\le Ch^{r}\|u\|_{H^{r+1}(\Omega_f)}\\ \inf_{\phi_h\in X_p^h}\|\phi-\phi_h\|_p&\le Ch^{r+1}\|\phi\|_{H^{r+1}(\Omega_p)}\\ \inf_{\phi_h\in X_p^h}\|\phi-\phi_h\|_{1,p}&\le Ch^{r}\|\phi\|_{H^{r+1}(\Omega_p)}\\ \inf_{p_h\in Q_f^h}\|p-p_h\|_f&\le Ch^{r+1}\|p\|_{H^{r+1}(\Omega_f)}.\end{aligned}\tag{31}$$

Moreover, we assume that the spaces X_f^h and Q_f^h satisfy the (LBB^h) condition. As a consequence, there exists some constant C such that if $u\in V_f$, then

$$\inf_{v_h\in V_h}\|u-v_h\|_{1,f}\le C\inf_{x_h\in X_f^h}\|u-x_h\|_{1,f},\tag{32}$$

(see, e.g., [10, Chapter II, Proof of Theorem 1.1, Equation (1.12)]). Let $N \in \mathbf{N}$ be given and denote $t^n = n\Delta t$ and $T = N\Delta t$. If $T = \infty$ then $N = \infty$. We introduce the following discrete norms.

$$|||v|||^2_{L^2(0,T;H^s(\Omega_{f,p}))} := \Delta t \sum_{k=0}^{N} \|v^k\|^2_{H^s(\Omega_{f,p})},$$

$$|||v|||_{L^\infty(0,T;H^s(\Omega_{f,p}))} := \max_{0\le k\le N} \|v^k\|_{H^s(\Omega_{f,p})}.$$

In the proof of convergence to follow, we will need the bounds of the next lemma.

Lemma 6 (Consistency Errors)**.** *The following inequalities hold:*

$$\Delta t \sum_{k=1}^{N-1} \left\| u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t} \right\|_f^2 \le \frac{(\Delta t)^4}{20} \|u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{33}$$

$$\Delta t \sum_{k=1}^{N-1} \left\| \phi_t^k - \frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t} \right\|_p^2 \le \frac{(\Delta t)^4}{20} \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))}, \tag{34}$$

$$\Delta t \sum_{k=1}^{N-1} \left\| \nabla \left(u^k - \frac{u^{k+1}+u^{k-1}}{2} \right) \right\|_f^2 \le \frac{7(\Delta t)^4}{6} \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))}, \tag{35}$$

$$\Delta t \sum_{k=1}^{N-1} \left\| \nabla \left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2} \right) \right\|_p^2 \le \frac{7(\Delta t)^4}{6} \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))}, \tag{36}$$

$$\Delta t \sum_{k=1}^{N-1} \left\| \nabla \left(u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t} \right) \right\|_f^2 \le \frac{(\Delta t)^4}{20} \|\nabla u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{37}$$

$$\Delta t \sum_{k=1}^{N-1} \|\phi^{k+1} - \phi^{k-1}\|^2_{1,p} \le 4\Delta t^2 \|\phi_t\|^2_{L^2(0,T;H^1(\Omega_p))}. \tag{38}$$

Proof. For (33)-(37) see [15, Section 3 Lemma 3.2]. For the proof of (38), we have

$$\begin{aligned}
\Delta t \sum_{k=1}^{N-1} \|\phi^{k+1} - \phi^{k-1}\|_p^2 &= \Delta t \sum_{k=1}^{N-1} \int_{\Omega_f} \left(\int_{t_{k-1}}^{t^{k+1}} \phi_t \; dt \right)^2 \; dx \\
&\le \Delta t \int_{\Omega_f} \sum_{k=1}^{N-1} \int_{t^{k-1}}^{t^{k+1}} \; dt \int_{t^{k-1}}^{t^{k+1}} \phi_t^2 \; dt \; dx \\
&= \Delta t \int_{\Omega_f} \sum_{k=1}^{N-1} 2\Delta t \int_{t^{k-1}}^{t^{k+1}} \phi_t^2 \; dt \; dx \\
&\le 2\Delta t^2 \int_{\Omega_f} 2 \sum_{k=1}^{N} \int_{t^{k-1}}^{t^k} \phi_t^2 \; dt \; dx \\
&= 4\Delta t^2 \|\phi_t\|^2_{L^2(0,T;L^2(\Omega_p))}.
\end{aligned} \tag{39}$$

Similarly,

$$\Delta t \sum_{k=1}^{N-1} \|\nabla \left(\phi^{k+1} - \phi^{k-1}\right)\|_p^2 \le 4\Delta t^2 \|\nabla \phi_t\|^2_{L^2(0,T;L^2(\Omega_p))}. \tag{40}$$

Inequalities (39) and (40) combined give (38). $\square$

Denote the errors by $e_f^n = u^n - u_h^n$ and $e_p^n = \phi^n - \phi_h^n$.

Theorem 7 (Second-order Convergence of CNLF-stab). *Consider the CNLF-stab method* (19)-(21). *For any $0 < t_N = T \leq \infty$, if u, p, and ϕ satisfy the regularity conditions*

$$u \in L^2(0,T;H^{r+2}(\Omega_f)) \cap L^\infty(0,T;H^{r+1}(\Omega_f)) \cap H^3(0,T;H^1(\Omega_f)),$$
$$p \in L^2(0,T;H^{r+1}(\Omega_f)),$$
$$\phi \in L^2(0,T;H^{r+2}(\Omega_p)) \cap L^\infty(0,T;H^{r+1}(\Omega_p)) \cap H^3(0,T;H^1(\Omega_p)),$$

then there exists a constant $\widehat{C} > 0$, independent of the mesh width h, time step Δt, and final time $t_N = T$, such that, for $N > 1$,

$$\begin{aligned}
&\frac{1}{2}(\|e_f^N\|^2_{div,f} + \|e_f^{N-1}\|^2_{div,f}) + gS_0(\|e_p^N\|_p^2 + \|e_p^{N-1}\|_p^2) \\
&+ \frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\nu\|\nabla(e_f^{k+1}+e_f^{k-1})\|_f^2 + gk_{min}\|\nabla(e_p^{k+1}+e_p^{k-1})\|_p^2\right\} \\
&\leq \widehat{C}\Big\{h^{2r}\Big(\|u_t\|^2_{L^2(0,T;H^{r+1}(\Omega_f))} + |||u|||^2_{L^2(0,T;H^{r+1}(\Omega_f))} + |||u|||^2_{L^\infty(0,T;H^{r+1}(\Omega_f))} \\
&\quad +\Delta t^4\|\phi_t\|^2_{L^2(0,T;H^{r+1}(\Omega_p))} + |||\phi|||^2_{L^\infty(0,T;H^{r+1}(\Omega_p))}\Big) \\
&\quad +h^{2r+2}\Big(|||p|||^2_{L^2(0,T;H^{r+1}(\Omega_p))} + \|\phi_t\|^2_{L^2(0,T;H^{r+1}(\Omega_p))} + |||\phi|||^2_{L^\infty(0,T;H^{r+1}(\Omega_p))}\Big) \\
&\quad +\Delta t^4\Big(\|u_{ttt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} \\
&\quad +\|\phi_t\|^2_{L^2(0,T;H^1(\Omega_p))} + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))}\Big) + \|e_f^1\|^2_{div,f} + \|e_p^1\|^2_{1,p}\Big\}.
\end{aligned} \tag{41}$$

Proof. Consider CNLF-stab (19)-(21) over the discretely divergence free space

$$V^h := \{v_h \in X_f^h : (q_h, \nabla\cdot v_h)_f = 0 \ \ \forall q_h \in Q_f^h\},$$

instead of X_f^h, so that the pressure term $\left((p_h^{k+1}+p_h^{k-1})/2, \nabla\cdot v_h\right)$ cancels out. Subtract (19) and (21) from (14) and (15) evaluated at time t^k to get:

$$\begin{aligned}
&\left(u_t^k - \frac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}, v_h\right)_f - \left(\nabla\cdot\left(\frac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}\right), \nabla\cdot v_h\right)_f \\
&+ a_f\left(u^k - \frac{u_h^{k+1}+u_h^{k-1}}{2}, v_h\right) - \left(p^k, \nabla\cdot v_h\right)_f + c_I\left(v_h, \phi^k - \phi_h^k\right) = 0, \\
&gS_0\left(\phi_t^k - \frac{\phi_h^{k+1}-\phi_h^{k-1}}{2\Delta t}, \psi_h\right)_p + a_p\left(\phi^k - \frac{\phi_h^{k+1}+\phi_h^{k-1}}{2}, \psi_h\right) \\
&- \Delta t g^2 C_{f,p}^2\left\{(\phi_h^{k+1}-\phi_h^{k-1}, \psi_h)_p + (\nabla(\phi_h^{k+1}-\phi_h^{k-1}), \nabla\psi_h)_p\right\} \\
&\qquad - c_I\left(u^k - u_h^k, \psi_h\right) = 0.
\end{aligned}$$

Since v_h is discretely divergence free, we have that

$$\left(p^k, \nabla\cdot v_h\right)_f = \left(p^k - \lambda_h^k, \nabla\cdot v_h\right)_f, \text{ for any } \lambda_h \in Q_f^h.$$

Further, $\left(\nabla \cdot u_t^k, v_h\right) = 0$. Thus, after rearranging we get:

$$\begin{aligned}
&\left(\frac{e_f^{k+1} - e_f^{k-1}}{2\Delta t},\, v_h\right)_f + \left(\nabla \cdot \left(\frac{e_f^{k+1} - e_f^{k-1}}{2\Delta t}\right),\, \nabla \cdot v_h\right)_f \\
&\quad + a_f\left(\frac{e_f^{k+1} + e_f^{k-1}}{2},\, v_h\right) + c_I\left(v_h, e_p^k\right) \\
&= -\left(u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t}, v_h\right)_f - \left(\nabla \cdot \left(u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t}\right),\, \nabla \cdot v_h\right)_f \\
&\quad - a_f\left(u^k - \frac{u^{k+1} + u^{k-1}}{2}, v_h\right) + \left(p^k - \lambda_h^k, \nabla \cdot v_h\right)_f,
\end{aligned}$$

$$\begin{aligned}
&gS_0\left(\frac{e_p^{k+1} - e_p^{k-1}}{2\Delta t}, \psi_h\right)_p + a_p\left(\frac{e_p^{k+1} + e_p^{k-1}}{2}, \psi_h\right) \\
&+\Delta t g^2 C_{f,p}^2\left\{(e_p^{k+1} - e_p^{k-1}, \psi_h)_p + (\nabla(e_p^{k+1} - e_p^{k-1}), \nabla\psi_h)_p\right\} - c_I\left(e_f^k, \psi_h\right) \\
&= -gS_0\left(\phi_t^k - \frac{\phi^{k+1} - \phi^{k-1}}{2\Delta t}, \psi_h\right)_p - a_p\left(\phi^k - \frac{\phi^{k+1} + \phi^{k-1}}{2}, \psi_h\right) \\
&\quad +\Delta t g^2 C_{f,p}^2\left\{(\phi^{k+1} - \phi^{k-1}, \psi_h)_p + (\nabla(\phi^{k+1} - \phi^{k-1}), \nabla\psi_h)_p\right\}.
\end{aligned}$$

Denote the consistency errors by:

$$\begin{aligned}
\varepsilon_f^k(v_h) &= -\left(u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t}, v_h\right)_f - \left(\nabla \cdot \left(u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t}\right), \nabla \cdot v_h\right)_f \\
&\quad - a_f\left(u^k - \frac{u^{k+1} + u^{k-1}}{2}, v_h\right), \\
\varepsilon_p^k(\psi_h) &= -gS_0\left(\phi_t^k - \frac{\phi^{k+1} - \phi^{k-1}}{2\Delta t}, \psi_h\right)_p + \Delta t g^2 C_{f,p}^2\left\{(\phi^{k+1} - \phi^{k-1}, \psi_h)_p\right. \\
&\quad \left.+(\nabla(\phi^{k+1} - \phi^{k-1}), \nabla\psi_h)_p\right\} - a_p\left(\phi^k - \frac{\phi^{k+1} + \phi^{k-1}}{2}, \psi_h\right).
\end{aligned}$$

Decompose the error terms into

$$\begin{aligned}
e_f^{k+1} &= u^{k+1} - u_h^{k+1} = (u^{k+1} - \tilde{u}^{k+1}) + (\tilde{u}^{k+1} - u_h^{k+1}) = \eta_f^{k+1} + \xi_f^{k+1}, \\
e_p^{k+1} &= \phi^{k+1} - \phi_h^{k+1} = (\phi^{k+1} - \tilde{\phi}^{k+1}) + (\tilde{\phi}^{k+1} - \phi_h^{k+1}) = \eta_p^{k+1} + \xi_p^{k+1},
\end{aligned}$$

and take $\tilde{u}^{k+1} \in V^h$ and $\tilde{\phi}^{k+1} \in X_p^h$, so that $\xi_f^{k+1} \in V^h$. Then the error equations become:

$$\begin{aligned}
&\left(\frac{\xi_f^{k+1} - \xi_f^{k-1}}{2\Delta t}, v_h\right)_f + \left(\nabla \cdot \left(\frac{\xi_f^{k+1} - \xi_f^{k-1}}{2\Delta t}\right), v_h\right)_f + a_f\left(\frac{\xi_f^{k+1} + \xi_f^{k-1}}{2}, v_h\right) \\
&\quad + c_I(v_h, \xi_p^k) = -\left(\frac{\eta_f^{k+1} - \eta_f^{k-1}}{2\Delta t}, v_h\right)_f - \left(\nabla \cdot \left(\frac{\eta_f^{k+1} - \eta_f^{k-1}}{2\Delta t}\right), \nabla \cdot v_h\right)_f \\
&\quad - a_f\left(\frac{\eta_f^{k+1} + \eta_f^{k-1}}{2}, v_h\right) - c_I(v_h, \eta_p^k) + \varepsilon_f^k(v_h) + \left(p^k - \lambda_h^k, \nabla \cdot v_h\right)_f,
\end{aligned}$$

$$
\begin{aligned}
&gS_0\left(\frac{\xi_p^{k+1}-\xi_p^{k-1}}{2\Delta t},\psi_h\right)_p + a_p\left(\frac{\xi_p^{k+1}+\xi_p^{k-1}}{2},\psi_h\right) + \Delta t g^2 C_{f,p}^2\left\{(\xi_p^{k+1}-\xi_p^{k-1},\psi_h)_p\right.\\
&\quad\left.+(\nabla(\xi_p^{k+1}-\xi_p^{k-1}),\nabla\psi_h)_p\right\} - c_I(\xi_f^k,\psi_h)\\
&\quad = -gS_0\left(\frac{\eta_p^{k+1}-\eta_p^{k-1}}{2\Delta t},\psi_h\right)_p - a_p\left(\frac{\eta_p^{k+1}+\eta_p^{k-1}}{2},\psi_h\right) + c_I(\eta_f^k,\psi_h)\\
&\quad - \Delta t g^2 C_{f,p}^2\left\{(\eta_p^{k+1}-\eta_p^{k-1},\psi_h)_p + (\nabla(\eta_p^{k+1}-\eta_p^{k-1}),\nabla\psi_h)_p\right\} + \varepsilon_p^k(\psi_h).
\end{aligned}
$$

Pick $v_h = \xi_f^{k+1}+\xi_f^{k-1} \in V^h$ and $\psi_h = \xi_p^{k+1}+\xi_p^{k-1} \in X_p^h$ in the equations above and add to obtain:

$$
\begin{aligned}
&\frac{1}{2\Delta t}\left(\|\xi_f^{k+1}\|_{\mathrm{div},f}^2 + gS_0\|\xi_p^{k+1}\|_p^2 + \Delta t^2 g^2 C_{f,p}^2\|\xi_p^{k+1}\|_{1,p}^2\right)\\
&\quad - \frac{1}{2\Delta t}\left(\|\xi_f^{k-1}\|_{\mathrm{div},f}^2 + gS_0\|\xi_p^{k-1}\|_p^2 + \Delta t^2 g^2 C_{f,p}^2\|\xi_p^{k-1}\|_{1,p}^2\right)\\
&\quad + \left[c_I(\xi_f^{k+1}+\xi_f^{k-1},\xi_p^k) - c_I(\xi_f^k,\xi_p^{k+1}+\xi_p^{k-1})\right]\\
&\quad + \frac{1}{2}\left[a_f(\xi_f^{k+1}+\xi_f^{k-1},\xi_f^{k+1}+\xi_f^{k-1}) + a_p(\xi_p^{k+1}+\xi_p^{k-1},\xi_p^{k+1}+\xi_p^{k-1})\right]\\
&= -\frac{1}{2\Delta t}\left[\left(\eta_f^{k+1}-\eta_f^{k-1},\xi_f^{k+1}+\xi_f^{k-1}\right)_f\right.\\
&\quad\left. + \left(\nabla\cdot(\eta_f^{k+1}-\eta_f^{k-1}),\nabla\cdot(\xi_f^{k+1}-\xi_f^{k+1})\right)_f\right]\\
&\quad - \frac{1}{2\Delta t}[gS_0\left(\eta_p^{k+1}-\eta_p^{k-1},\xi_p^{k+1}+\xi_p^{k-1}\right)_p\\
&+\Delta t g^2 C_{f,p}^2\left\{(\eta_p^{k+1}-\eta_p^{k-1},\xi_p^{k+1}+\xi_p^{k-1})_p + (\nabla(\eta_p^{k+1}-\eta_p^{k-1}),\nabla(\xi_p^{k+1}+\xi_p^{k-1}))_p\right\}]\\
&\quad - \frac{1}{2}\left[a_f\left(\eta_f^{k+1}+\eta_f^{k-1},\xi_f^{k+1}+\xi_f^{k-1}\right) + a_p\left(\eta_p^{k+1}+\eta_p^{k-1},\xi_p^{k+1}+\xi_p^{k-1}\right)\right]\\
&\quad - \left[c_I(\xi_f^{k+1}+\xi_f^{k-1},\eta_p^k) - c_I(\eta_f^k,\xi_p^{k+1}+\xi_p^{k-1})\right]\\
&\quad + \varepsilon_f^k(\xi_f^{k+1}+\xi_f^{k-1}) + \left(p^k-\lambda_h^k,\nabla\cdot(\xi_f^{k+1}+\xi_f^{k-1})\right)_f + \varepsilon_p^k(\xi_p^{k+1}+\xi_p^{k-1}).
\end{aligned}
$$

Rewrite the coupling terms on the left-hand side equivalently as follows:

$$
\begin{aligned}
&c_I(\xi_f^{k+1}+\xi_f^{k-1},\xi_p^k) - c_I(\xi_f^k,\xi_p^{k+1}+\xi_p^{k-1})\\
&\qquad = \left(c_I(\xi_f^{k+1},\xi_p^k) - c_I(\xi_f^k,\xi_p^{k+1})\right) - \left(c_I(\xi_f^k,\xi_p^{k-1}) - c_I(\xi_f^{k-1},\xi_p^k)\right)\\
&\qquad = C_\xi^{k+\frac{1}{2}} - C_\xi^{k-\frac{1}{2}}.
\end{aligned}
$$

If we denote the ξ energy terms by

$$
\begin{aligned}
E_\xi^{k+1/2} := &\|\xi_f^{k+1}\|_{\mathrm{div},f}^2 + \|\xi_f^k\|_{\mathrm{div},f}^2 + gS_0\left(\|\xi_p^{k+1}\|_p^2 + \|\xi_p^k\|_p^2\right)\\
&+ \Delta t^2 g^2 C_{f,p}^2\left(\|\xi_p^{k+1}\|_{1,p}^2 + \|\xi_p^k\|_{1,p}^2\right)
\end{aligned}
$$

and also apply the coercivity of the forms $a_f(\cdot,\cdot)$ and $a_p(\cdot,\cdot)$, the inequality becomes

$$
\begin{aligned}
&E_\xi^{k+1/2} + 2\Delta t C_\xi^{k+\frac{1}{2}} - E_\xi^{k-1/2} - 2\Delta t C_\xi^{k-\frac{1}{2}} \\
&\qquad + \Delta t \left(\nu \|\nabla(\xi_f^{k+1} + \xi_f^{k-1})\|_f^2 + g k_{\min} \|\nabla(\xi_p^{k+1} + \xi_p^{k-1}\|_p^2\right) \\
&\leq - \left[\left(\eta_f^{k+1} - \eta_f^{k-1}, \xi_f^{k+1} + \xi_f^{k-1}\right)_f + \left(\nabla \cdot \left(\eta_f^{k+1} - \eta_f^{k-1}\right), \nabla \cdot \left(\xi_f^{k+1} + \xi_f^{k-1}\right)\right)_f \right] \\
&- \Big[g S_0 \left(\eta_p^{k+1} - \eta_p^{k-1}, \xi_p^{k+1} + \xi_p^{k-1}\right)_p + 2\Delta t^2 g^2 C_{f,p}^2 \left\{ (\eta_p^{k+1} - \eta_p^{k-1}, \xi_p^{k+1} + \xi_p^{k-1})_p \right. \\
&\quad \left. + (\nabla(\eta_p^{k+1} - \eta_p^{k-1}), \nabla(\xi_p^{k+1} + \xi_p^{k-1}))_p \right\} \Big] \qquad (42) \\
&\quad - \Delta t \left[a_f \left(\eta_f^{k+1} + \eta_f^{k-1}, \xi_f^{k+1} + \xi_f^{k-1}\right) + a_p \left(\eta_p^{k+1} + \eta_p^{k-1}, \xi_p^{k+1} + \xi_p^{k-1}\right) \right] \\
&\quad - 2\Delta t \left[c_I(\xi_f^{k+1} + \xi_f^{k-1}, \eta_p^k) - c_I(\eta_f^k, \xi_p^{k+1} + \xi_p^{k-1}) \right] \\
&\quad + 2\Delta t \left[\varepsilon_f^k(\xi_f^{k+1} + \xi_f^{k-1}) + (p^k - \lambda_h^k, \nabla \cdot (\xi_f^{k+1} + \xi_f^{k-1}))_f + \varepsilon_p^k(\xi_p^{k+1} + \xi_p^{k-1}) \right],
\end{aligned}
$$

where we multiplied by $2\Delta t$. Next, we bound each term on the right-hand side of the above inequality. We bound the first two terms by the Cauchy-Schwarz and Young inequalities along with the Poincaré inequality (5) and inequality (7).

$$
\begin{aligned}
&\left(\eta_f^{k+1} - \eta_f^{k-1}, \xi_f^{k+1} + \xi_f^{k-1}\right)_f + \left(\nabla \cdot \left(\eta_f^{k+1} - \eta_f^{k-1}\right), \nabla \cdot \left(\xi_f^{k+1} + \xi_f^{k-1}\right)\right)_f \\
&\leq \frac{6 C_{P,f}^2}{\nu \Delta t} \|\eta_f^{k+1} - \eta_f^{k-1}\|_f^2 + \frac{6 d^2}{\nu \Delta t} \|\nabla(\eta_f^{k+1} - \eta_f^{k-1})\|_f^2 + \Delta t \frac{\nu}{12} \|\nabla(\xi_f^{k+1} + \xi_f^{k-1})\|_f^2 ,
\end{aligned}
$$

$$
\begin{aligned}
g S_0(\eta_p^{k+1} - \eta_p^{k-1}, \xi_p^{k+1} + \xi_p^{k-1})_p &+ 2\Delta t^2 g^2 C_{f,p}^2 \Big\{ \left(\eta_p^{k+1} - \eta_p^{k-1}, \xi_p^{k+1} + \xi_p^{k-1}\right)_p \\
&+ \left(\nabla\left(\eta_p^{k+1} - \eta_p^{k-1}\right), \nabla\left(\xi_p^{k+1} + \xi_p^{k-1}\right)\right)_p \Big\} \\
&\leq \frac{15 g C_{P,p}^2}{2 k_{\min} \Delta t} \left(S_0^2 + 4\Delta t^4 g^2 C_{f,p}^4\right) \|\eta_p^{k+1} - \eta_p^{k-1}\|_p^2 \\
+ \frac{30 \Delta t^3 g^3 C_{f,p}^4}{k_{\min}} \|\nabla\left(\eta_p^{k+1} - \eta_p^{k-1}\right)\|_p^2 &+ \Delta t \frac{g k_{\min}}{10} \|\nabla(\xi_p^{k+1} + \xi_p^{k-1})\|_p^2.
\end{aligned}
$$

To bound the second term, we apply the continuity of the bilinear forms $a_f(\cdot,\cdot)$ and $a_p(\cdot,\cdot)$. Letting $M = \nu + \alpha C k_{\min}^{-1/2}$ gives:

$$
\begin{aligned}
&a_f(\eta_f^{k+1} + \eta_f^{k-1}, \xi_f^{k+1} + \xi_f^{k-1}) + a_p(\eta_p^{k+1} + \eta_p^{k-1}, \xi_p^{k+1} + \xi_p^{k-1}) \\
&\leq M \|\nabla(\eta_f^{k+1} + \eta_f^{k-1})\|_f \|\nabla(\xi_f^{k+1} + \xi_f^{k-1})\|_f \\
&\quad + g k_{\max} \|\nabla(\eta_p^{k+1} + \eta_p^{k-1})\|_p \|\nabla(\xi_p^{k+1} + \xi_p^{k-1})\|_p \\
&\leq \frac{3M^2}{\nu} \|\nabla(\eta_f^{k+1} + \eta_f^{k-1})\|_f^2 + \frac{5 g k_{\max}^2}{2 k_{\min}} \|\nabla(\eta_p^{k+1} + \eta_p^{k-1})\|_p^2 \\
&\quad + \frac{\nu}{12} \|\nabla(\xi_f^{k+1} + \xi_f^{k-1})\|_f^2 + \frac{g k_{\min}}{10} \|\nabla(\xi_p^{k+1} + \xi_p^{k-1})\|_p^2.
\end{aligned}
$$

We bound the coupling terms on the right-hand side using the trace (6), Poincaré (5) and Young inequalities. Letting $C = C_{\Omega_f}^2 C_{\Omega_p}^2 C_{P,f} C_{P,p} g^2$, this yields

$$
\begin{aligned}
&c_I(\xi_f^{k+1}+\xi_f^{k-1},\eta_p^k)-c_I(\eta_f^k,\xi_p^{k+1}+\xi_p^{k-1})\\
&\quad\le g\left(\|(\xi_f^{k+1}+\xi_f^{k-1})\cdot\hat{n}_f\|_I\|\eta_p^k\|_I+\|\eta_f^k\cdot\hat{n}_f\|_I\|\xi_p^{k+1}+\xi_p^{k-1}\|_I\right)\\
&\quad\le C_{\Omega_f}C_{\Omega_p}g\left(\|\xi_f^{k+1}+\xi_f^{k-1}\|_f^{1/2}\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^{1/2}\|\eta_p^k\|_p^{1/2}\|\nabla\eta_p^k\|_p^{1/2}\right)\\
&\qquad+\left(\|\xi_p^{k+1}+\xi_p^{k-1}\|_p^{1/2}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p^{1/2}\|\eta_f^k\|_f^{1/2}\|\nabla\eta_f^k\|_f^{1/2}\right)\\
&\quad\le\sqrt{C}\left(\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f\|\nabla\eta_p^k\|_p+\|\nabla\eta_f^k\|_f\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p\right)\\
&\quad\le\frac{6C}{\nu}\|\nabla\eta_f^k\|_f^2+\frac{5C}{gk_{\min}}\|\nabla\eta_p^k\|_p^2+\frac{\nu}{24}\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^2\\
&\qquad+\frac{gk_{\min}}{20}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p^2.
\end{aligned}
$$

Finally, we bound the consistency errors, ε_f^k and ε_p^k, and the pressure term by using the Cauchy-Schwarz, Young, and Poincaré (5) inequalities as well as inequality (7):

$$
\begin{aligned}
\varepsilon_f^k(\xi_f^{k+1}+\xi_f^{k-1})&=-\left(u_t^k-\frac{u^{k+1}-u^{k-1}}{2\Delta t},\xi_f^{k+1}+\xi_f^{k-1}\right)\\
&\quad-a_f\left(u^k-\frac{u^{k+1}+u^{k-1}}{2},\xi_f^{k+1}+\xi_f^{k-1}\right)\\
&\quad-\left(\nabla\cdot\left(u_t^k-\frac{u^{k+1}-u^{k-1}}{2\Delta t}\right),\nabla\cdot\left(\xi_f^{k+1}+\xi_f^{k-1}\right)\right)\\
&\le C_{P,f}\left(\left\|u_t^k-\frac{u^{k+1}-u^{k-1}}{2\Delta t}\right\|_f+d\left\|\nabla\left(u_t^k-\frac{u^{k+1}-u^{k-1}}{2\Delta t}\right)\right\|_f\right.\\
&\quad\left.+M\left\|\nabla\left(u^k-\frac{u^{k+1}+u^{k-1}}{2}\right)\right\|_f\right)\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f\\
&\le\frac{9C_{P,f}^2}{\nu}\left\|u_t^k-\frac{u^{k+1}-u^{k-1}}{2\Delta t}\right\|_f^2+\frac{9M^2}{\nu}\left\|\nabla\left(u^k-\frac{u^{k+1}+u^{k-1}}{2}\right)\right\|_f^2\\
&\quad+\frac{9d^2}{\nu}\left\|\nabla\left(u_t^k-\frac{u^{k+1}+u^{k-1}}{2\Delta t}\right)\right\|_f^2+\frac{\nu}{12}\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^2,
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_p^k(\xi_p^{k+1}+\xi_p^{k-1})=-gS_0&\left(\phi_t^k-\frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t},\xi_p^{k+1}+\xi_p^{k-1}\right)_p\\
&+\Delta t g^2C_{f,p}^2\left\{(\phi^{k+1}-\phi^{k-1},\xi_p^{k+1}+\xi_p^{k-1})_p\right.\\
&\left.+(\nabla(\phi^{k+1}-\phi^{k-1}),\nabla(\xi_p^{k+1}+\xi_p^{k-1}))_p\right\}\\
&-a_p\left(\phi^k-\frac{\phi^{k+1}+\phi^{k-1}}{2},\xi_p^{k+1}+\xi_p^{k-1}\right)\le
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Bigg(C_{P,p} \left\| \phi_t^k - \frac{\phi^{k+1} - \phi^{k-1}}{2\Delta t} \right\|_p + g k_{\max} \left\| \nabla \left(\phi^k - \frac{\phi^{k+1} + \phi^{k-1}}{2} \right) \right\|_p \\
&\qquad + \Delta t g^2 C_{f,p}^2 \left(\|\nabla(\phi^{k+1} - \phi^{k-1})\|_p \right. \\
&\qquad \left. + C_{P,p} \|\phi^{k+1} - \phi^{k-1}\|_p \right) \Bigg) \|\nabla(\xi_p^{k+1} + \xi_p^{k-1})\|_p \\
&\leq \frac{10 g S_0^2 C_{P,p}^2}{k_{\min}} \left\| \phi_t^k - \frac{\phi^{k+1} - \phi^{k-1}}{2\Delta t} \right\|_p^2 + \frac{10 \Delta t^2 g^3 C_{f,p}^4 C_{P,p}^2}{k_{\min}} \|\phi^{k+1} - \phi^{k-1}\|_p^2 \\
&\quad + \frac{10 \Delta t^2 g^3 C_{f,p}^4}{k_{\min}} \|\nabla(\phi^{k+1} - \phi^{k-1})\|_p^2 + \frac{10 g k_{\max}}{k_{\min}} \left\| \nabla \left(\phi^k - \frac{\phi^{k+1} + \phi^{k-1}}{2} \right) \right\|_p^2 \\
&\qquad + \frac{g k_{\min}}{10} \|\nabla(\xi_p^{k+1} + \xi_p^{k-1})\|_p^2,
\end{aligned}
$$

$$
\begin{aligned}
\left(p^k - \lambda_h^k, \nabla \cdot (\xi_f^{k+1} + \xi_f^{k-1}) \right)_f &\leq \|p^k - \lambda_h^k\|_f \|\nabla \cdot (\xi_f^{k+1} + \xi_f^{k-1})\|_f \\
&\leq \frac{6d}{\nu} \|p^k - \lambda_h^k\|_f^2 + \frac{\nu}{24} \|\nabla(\xi_f^{k+1} + \xi_f^{k-1})\|_f^2.
\end{aligned}
$$

After absorbing all the resulting ξ terms into the left-hand side of inequality (42) and grouping together the remaining terms, the inequality becomes

$$
\begin{aligned}
&E_\xi^{k+\frac{1}{2}} + 2\Delta t C_\xi^{k+\frac{1}{2}} - E_\xi^{k-\frac{1}{2}} - 2\Delta t C_\xi^{k-\frac{1}{2}} \\
&\quad + \frac{\Delta t}{2} \left\{ \nu \|\nabla(\xi_f^{k+1} + \xi_f^{k-1})\|_f^2 + g k_{\min} \|\nabla(\xi_p^{k+1} + \xi_p^{k-1})\|_p^2 \right\} \\
&\leq (\Delta t)^{-1} \Bigg\{ \frac{6 C_{P,f}^2}{\nu} \|\eta_f^{k+1} - \eta_f^{k-1}\|_f^2 + \frac{15 g C_{P,p}^2}{2 k_{\min}} \left(S_0^2 + 4 \Delta t^4 g^2 C_{f,p}^4 \right) \|\eta_p^{k+1} - \eta_p^{k-1}\|_p^2 \\
&\qquad + \frac{6 d^2}{\nu} \|\nabla(\eta_f^{k+1} - \eta_f^{k-1})\|_f^2 \Bigg\} \\
&\quad + \Delta t \Bigg\{ \frac{30 \Delta t^2 g^3 C_{f,p}^4}{k_{\min}} \|\nabla \left(\eta_p^{k+1} - \eta_p^{k-1} \right)\|_p^2 + \frac{3 M^2}{\nu} \|\nabla(\eta_f^{k+1} + \eta_f^{k-1})\|_f^2 \\
&\qquad + \frac{5 g k_{\max}^2}{2 k_{\min}} \|\nabla(\eta_p^{k+1} + \eta_p^{k-1})\|_p^2 + \frac{12 C}{\nu} \|\nabla \eta_f^k\|_f^2 + \frac{10 C}{g k_{\min}} \|\nabla \eta_p^k\|_p^2 \\
&\quad + \frac{18 C_{P,f}^2}{\nu} \left\| u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t} \right\|_f^2 + \frac{18 M^2}{\nu} \left\| \nabla \left(u^k - \frac{u^{k+1} + u^{k-1}}{2} \right) \right\|_f^2 \\
&\quad + \frac{18 d^2}{\nu} \left\| \nabla \left(u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t} \right) \right\|_f^2 + \frac{12 d}{\nu} \|p^k - \lambda_h^k\|_f^2 \\
&\quad + \frac{20 g S_0^2 C_{P,p}^2}{k_{\min}} \left\| \phi_t^k - \frac{\phi^{k+1} - \phi^{k-1}}{2\Delta t} \right\|_p^2 + \frac{20 \Delta t^2 g^3 C_{f,p}^4 C_{P,p}^2}{k_{\min}} \|\phi^{k+1} - \phi^{k-1}\|_p^2 \\
&\quad + \frac{20 \Delta t^2 g^3 C_{f,p}^4}{k_{\min}} \|\nabla(\phi^{k+1} - \phi^{k-1})\|_p^2 + \frac{20 g k_{\max}^2}{k_{\min}} \left\| \nabla \left(\phi^k - \frac{\phi^{k+1} + \phi^{k-1}}{2} \right) \right\|_p^2 \Bigg\}.
\end{aligned}
$$

Now, we sum this inequality from $k = 1, ..., N-1$. This yields

$$
\begin{aligned}
&E_\xi^{N-\frac{1}{2}} + 2\Delta t C_\xi^{N-\frac{1}{2}} - E_\xi^{\frac{1}{2}} - 2\Delta t C_\xi^{\frac{1}{2}} \\
&\quad + \frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\nu\|\nabla\left(\xi_f^{k+1}+\xi_f^{k-1}\right)\|_f^2 + gk_{\min}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p^2\right\} \\
&\leq (\Delta t)^{-1}\sum_{k=1}^{N-1}\left\{\frac{6C_{P,f}^2}{\nu}\|\eta_f^{k+1}-\eta_f^{k-1}\|_f^2\right. \\
&\quad \left. + \frac{15gC_{P,p}^2}{2k_{\min}}\left(S_0^2 + 4\Delta t^4 g^2 C_{f,p}^4\right)\|\eta_p^{k+1}-\eta_p^{k-1}\|_p^2 + \frac{6d^2}{\nu}\|\nabla\left(\eta_f^{k+1}-\eta_f^{k-1}\right)\|_f^2\right\} \\
&\quad + \Delta t\sum_{k=1}^{N-1}\left\{\frac{30\Delta t^2 g^3 C_{f,p}^4}{k_{\min}}\|\nabla\left(\eta_p^{k+1}-\eta_p^{k-1}\right)\|_p^2 + \frac{3M^2}{\nu}\|\nabla(\eta_f^{k+1}+\eta_f^{k-1})\|_f^2\right. \\
&\qquad + \frac{5gk_{\max}^2}{2k_{\min}}\|\nabla(\eta_p^{k+1}+\eta_p^{k-1})\|_p^2 + \frac{12C}{\nu}\|\nabla\eta_f^k\|_f^2 + \frac{10C}{gk_{\min}}\|\nabla\eta_p^k\|_p^2 \\
&\qquad + \frac{18C_{P,f}^2}{\nu}\left\|u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}\right\|_f^2 + \frac{18M^2}{\nu}\left\|\nabla\left(u^k - \frac{u^{k+1}+u^{k-1}}{2}\right)\right\|_f^2 \\
&\qquad + \frac{18d^2}{\nu}\left\|\nabla\left(u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}\right)\right\|_f^2 + \frac{12d}{\nu}\|p^k-\lambda_h^k\|_f^2 \\
&\qquad + \frac{20gS_0^2C_{P,p}^2}{k_{\min}}\left\|\phi_t^k - \frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}\right\|_p^2 + \frac{20\Delta t^2 g^3 C_{f,p}^4 C_{P,p}^2}{k_{\min}}\|\phi^{k+1}-\phi^{k-1}\|_p^2 \\
&\qquad \left. + \frac{20\Delta t^2 g^3 C_{f,p}^4}{k_{\min}}\|\nabla(\phi^{k+1}-\phi^{k-1})\|_p^2 + \frac{20gk_{\max}^2}{k_{\min}}\left\|\nabla\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2}\right)\right\|_p^2\right\}.
\end{aligned}
$$

To obtain a bound involving norms instead of summations, we use the Cauchy-Schwarz and other basic inequalities to bound each term on the right-hand side as follows. For the first term, we have:

$$
\begin{aligned}
\sum_{k=1}^{N-1}\|\eta_f^{k+1}-\eta_f^{k-1}\|_f^2 &= \sum_{k=1}^{N-1}\left\|\int_{t^{k-1}}^{t^{k+1}}\eta_{f,t}\,dt\right\|_f^2 \\
&\leq \sum_{k=1}^{N-1}\int_{\Omega_f}(2\Delta t)\int_{t^{k-1}}^{t^{k+1}}|\eta_{f,t}|^2\,dt\,dx \\
&\leq 4\Delta t\|\eta_{f,t}\|_{L^2(0,T;L^2(\Omega_f))}^2. \qquad (43)
\end{aligned}
$$

Likewise, we treat the second term,

$$\sum_{k=1}^{N-1}\|\eta_p^{k+1}-\eta_p^{k-1}\|_f^2 \leq 4\Delta t\|\eta_{p,t}\|_{L^2(0,T;L^2(\Omega_p))}^2. \qquad (44)$$

In a similar manner we bound the third and fourth terms.

$$\sum_{k=1}^{N-1}\|\nabla(\eta_f^{k+1}-\eta_f^{k-1})\|_f^2 \leq 4\Delta t\|\nabla\eta_{f,t}\|_{L^2(0,T;L^2(\Omega_f))}^2, \qquad (45)$$

$$\sum_{k=1}^{N-1}\|\nabla(\eta_p^{k+1}-\eta_p^{k-1})\|_p^2 \leq 4\Delta t\|\nabla\eta_{p,t}\|_{L^2(0,T;L^2(\Omega_p))}^2. \qquad (46)$$

Inequalities (43) and (45) imply the following.

$$\sum_{k=1}^{N-1}\left\{\|\eta_f^{k+1}-\eta_f^{k-1}\|_f^2+\|\nabla(\eta_f^{k+1}-\eta_f^{k-1})\|_f^2\right\}\le 4\Delta t\|\eta_{f,t}\|^2_{L^2(0,T;H^1(\Omega_f))}. \tag{47}$$

The rest of the η terms are bounded using Cauchy-Schwarz and the discrete norms.

$$\begin{aligned}\sum_{k=1}^{N-1}\|\nabla(\eta_f^{k+1}+\eta_f^{k-1})\|_f^2 &\le 2\sum_{k=1}^{N-1}\left(\|\nabla\eta_f^{k+1}\|_f^2+\|\nabla\eta_f^{k-1}\|_f^2\right)\\ &\le 4\sum_{k=0}^{N}\|\nabla\eta_f^k\|_f^2\le 4(\Delta t)^{-1}|||\nabla\eta_f|||^2_{L^2(0,T;L^2(\Omega_f))},\end{aligned} \tag{48}$$

$$\sum_{k=1}^{N-1}\|\nabla(\eta_p^{k+1}+\eta_p^{k-1})\|_f^2\le 4(\Delta t)^{-1}|||\nabla\eta_p|||^2_{L^2(0,T;L^2(\Omega_p))}, \tag{49}$$

$$\sum_{k=1}^{N-1}\|\nabla\eta_f^k\|_f^2\le(\Delta t)^{-1}|||\nabla\eta_f|||^2_{L^2(0,T;L^2(\Omega_f))}, \tag{50}$$

$$\sum_{k=1}^{N-1}\|\nabla\eta_p^k\|_p^2\le(\Delta t)^{-1}|||\nabla\eta_p|||^2_{L^2(0,T;L^2(\Omega_p))}, \tag{51}$$

$$\sum_{k=1}^{N-1}\|p^k-\lambda_h^k\|_f^2\le(\Delta t)^{-1}|||p-\lambda_h|||^2_{L^2(0,T;L^2(\Omega_f))}. \tag{52}$$

After applying the bounds (43)-(52), along with (33)-(38), and the bound (25) from the stability proof, and after absorbing all the constants into one constant, $\widehat{C}_1$, the inequality becomes

$$\begin{aligned}&\frac{1}{2}(\|\xi_f^N\|^2_{\mathrm{div},f}+\|\xi_f^{N-1}\|^2_{\mathrm{div},f})+gS_0(\|\xi_p^N\|_p^2+\|\xi_p^{N-1}\|_p^2)\\ &\quad+\frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\nu\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^2+gk_{\min}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p^2\right\}\\ &\quad\le\widehat{C}_1\left\{\|\eta_{f,t}\|^2_{L^2(0,T;H^1(\Omega_f))}+\|\eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))}+\Delta t^4\|\nabla\eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))}\right.\\ &\qquad+|||\nabla\eta_f|||^2_{L^2(0,T;L^2(\Omega_f))}+|||\nabla\eta_p|||^2_{L^2(0,T;L^2(\Omega_p))}+\Delta t^4\left(\|u_{ttt}\|^2_{L^2(0,T;H^1(\Omega_f))}\right.\\ &\qquad+\|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))}+\|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))}+\|\phi_t\|^2_{L^2(0,T,H^1(\Omega_p))}\\ &\qquad\left.\left.+\|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))}\right)+|||p-\lambda_h|||^2_{L^2(0,T;L^2(\Omega_f))}\right\}+E_\xi^{1/2}+2\Delta t C_\xi^{1/2}.\end{aligned} \tag{53}$$

Recall that the error terms equal

$$e_f^N=u^N-u_h^N=\eta_f^N+\xi_f^N,\quad e_p^N=\phi^N-\phi_h^N=\eta_p^N+\xi_p^N.$$

Applying the triangle inequality we have

$$
\begin{aligned}
&\frac{1}{4}(\|e_f^N\|^2_{\mathrm{div},f} + \|e_f^{N-1}\|^2_{\mathrm{div},f}) + \frac{gS_0}{2}(\|e_p^N\|^2_p + \|e_p^{N-1}\|^2_p) \\
&\quad + \frac{\Delta t}{4}\sum_{k=1}^{N-1}\left\{\nu\|\nabla(e_f^{k+1}+e_f^{k-1})\|^2_f + gk_{\min}\|\nabla(e_p^{k+1}+e_p^{k-1})\|^2_p\right\} \\
&\quad \le \frac{1}{2}(\|\xi_f^N\|^2_{\mathrm{div},f} + \|\xi_f^{N-1}\|^2_{\mathrm{div},f}) + gS_0(\|\xi_p^N\|^2_p + \|\xi_p^{N-1}\|^2_p) \\
&\qquad + \frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\nu\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|^2_f + gk_{\min}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|^2_p\right\} \\
&\qquad + \frac{1}{2}(\|\eta_f^N\|^2_{\mathrm{div},f} + \|\eta_f^{N-1}\|^2_{\mathrm{div},f}) + gS_0(\|\eta_p^N\|^2_p + \|\eta_p^{N-1}\|^2_p) \\
&\qquad + \frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\nu\|\nabla(\eta_f^{k+1}+\eta_f^{k-1})\|^2_f + gk_{\min}\|\nabla(\eta_p^{k+1}+\eta_p^{k-1})\|^2_p\right\}.
\end{aligned}
$$

Notice that $\|\eta_{f,p}^N\|^2_{f,p}, \|\eta_{f,p}^{N-1}\|^2_{f,p} \le |||\eta_{f,p}|||^2_{L^\infty(0,T;L^2(\Omega_{f,p}))}$ and therefore

$$\|\eta_f^N\|^2_{\mathrm{div},f} \le d|||\eta_f|||^2_{L^\infty(0,T;H^1(\Omega_f))}.$$

This fact, together with the previous bounds for the η terms and inequality (53) result in

$$
\begin{aligned}
&\frac{1}{4}(\|e_f^N\|^2_{\mathrm{div},f} + \|e_f^{N-1}\|^2_{\mathrm{div},f}) + \frac{gS_0}{2}(\|e_p^N\|^2_p + \|e_p^{N-1}\|^2_p) \\
&\quad + \frac{\Delta t}{4}\sum_{k=1}^{N-1}\left\{\nu\|\nabla(e_f^{k+1}+e_f^{k-1})\|^2_f + gk_{\min}\|\nabla(e_p^{k+1}+e_p^{k-1})\|^2_p\right\} \\
&\quad \le \widehat{C}_2\left\{\|\eta_{f,t}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))}\right. \qquad (54)\\
&\qquad + \Delta t^4\|\nabla\eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))} + |||\nabla\eta_f|||^2_{L^2(0,T;L^2(\Omega_f))} + |||\nabla\eta_p|||^2_{L^2(0,T;L^2(\Omega_p))} \\
&\qquad + \Delta t^4\Big(\|u_{ttt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} \\
&\qquad + \|\phi_t\|^2_{L^2(0,T;H^1(\Omega_p))} + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))}\Big) + |||p-\lambda_h|||^2_{L^2(0,T;L^2(\Omega_f))} \\
&\qquad \left. + |||\eta_f|||^2_{L^\infty(0,T;H^1(\Omega_f))} + |||\eta_p|||^2_{L^\infty(0,T;L^2(\Omega_p))}\right\} + \|\xi_f^1\|^2_{\mathrm{div},f} + \|\xi_f^0\|^2_{\mathrm{div},f} \\
&\qquad + gS_0(\|\xi_p^1\|^2_p + \|\xi_p^0\|^2_p) + \Delta t^2 g^2 C^2_{f,p}(\|\xi_p^1\|^2_{1,p} + \|\xi_p^0\|^2_{1,p}) + 2\Delta t C_\xi^{1/2},
\end{aligned}
$$

where we absorbed all constants into a new constant, $\widehat{C}_2 > 0$. Now, we bound the coupling terms on the right-hand side as follows:

$$C_\xi^{1/2} \le \frac{C}{2}\left(\|\xi_p^0\|^2_{1,p} + \|\xi_p^1\|^2_{1,p} + \|\xi_f^0\|^2_{\mathrm{div},f} + \|\xi_f^1\|^2_{\mathrm{div},f}\right). \qquad (55)$$

Inequality (54) holds for any $\tilde{u} \in V^h$, $\lambda_h \in Q_f^h$, and $\tilde{\phi} \in X_p^h$. Taking the infimum over the spaces V^h, Q_f^h, and X_p^h, using (32) to bound the infimum over V^h by the infimum over X_f^h, and using bound (55), we have the following for some positive constant $\widehat{C}_3$:

$$
\begin{aligned}
&\frac{1}{2}(\|e_f^N\|^2_{\mathrm{div},f} + \|e_f^{N-1}\|^2_{\mathrm{div},f}) + gS_0(\|e_p^N\|_p^2 + \|e_p^{N-1}\|_p^2) \\
&\quad + \frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\nu\|\nabla(e_f^{k+1}+e_f^{k-1})\|_f^2 + gk_{\min}\|\nabla(e_p^{k+1}+e_p^{k-1})\|_p^2\right\} \\
&\quad \le \widehat{C}_3\Bigg\{\inf_{\tilde{u}\in X_f^h}\{\|\eta_{f,t}\|^2_{L^2(0,T;H^1(\Omega_f))} + |||\nabla\eta_f|||^2_{L^2(0,T;L^2(\Omega_f))} \\
&\quad +|||\eta_f|||^2_{L^\infty(0,T;H^1(\Omega_f))} + \|\xi_f^1\|^2_{\mathrm{div},f} + \|\xi_f^0\|^2_{\mathrm{div},f}\} \\
&\quad + \inf_{\lambda_h\in Q_f^h}|||p-\lambda_h|||^2_{L^2(0,T;L^2(\Omega_f))} + \inf_{\tilde{\phi}\in X_p^h}\{\|\eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))} \\
&\quad +\Delta t^4\|\nabla\eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))} + |||\nabla\eta_p|||^2_{L^2(0,T;L^2(\Omega_p))} + |||\eta_p|||^2_{L^\infty(0,T;L^2(\Omega_p))} \\
&\quad +\|\xi_p^1\|^2_{1,p} + \|\xi_p^0\|^2_{1,p}\} + \Delta t^4\Big(\|u_{ttt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} \\
&\quad + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} + \|\phi_t\|^2_{L^2(0,T;H^1(\Omega_p))} + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))}\Big)\Bigg\}.
\end{aligned}
$$

The result of the theorem now immediately follows by applying the approximation assumptions given in (31). □

Corollary 8. *Under the same regularity conditions as in Theorem 7, the temporal growth of the error is at most*

$$\|e_f^N\|_{div,f}, \|e_p^N\|_p = \mathcal{O}(\sqrt{t_N}).$$

Proof. For any function $v:[0,\infty)\to X$ and any spatial norm $\|\cdot\|_X$ we have:

$$\int_0^{t_N}\|v\|_X^2\,dt \le t_N\|v\|^2_{L^\infty(0,\infty;X)},$$

for any $0 < t_N \le \infty$. Similarly, we have for the discrete norms:

$$\sum_{k=1}^N\|v^k\|_X^2\Delta t \le \|v\|^2_{L^\infty(0,\infty;X)}\sum_{k=1}^N\Delta t = t_N\|v\|^2_{L^\infty(0,\infty;X)}.$$

Applying the above to the terms on the RHS of (41) gives the claim of the Corollary. □

5. Numerical tests

We verify the method's unconditional, asymptotic stability and rate of convergence in a series of numerical tests. For these experiments we use the exact solutions from [19] recalled next. All tests were conducted with FreeFEM++ [22]. The computational domains are

$$\Omega_f = (0,1)\times(1,2), \qquad \Omega_p = (0,1)\times(0,1), \qquad I = \{(x,1): x\in(0,1)\},$$

and the true solution in the first test problem is

$$
\begin{aligned}
u(x,y,t) &= \big([x^2(y-1)^2+y]\cos t, [\tfrac{2}{3}x(1-y)^3 + 2 - \pi\sin(\pi x)]\cos t\big) \\
p(x,y,t) &= [2-\pi\sin(\pi x)]\sin(\tfrac{\pi}{2}y)\cos t \\
\phi(x,y,t) &= [2-\pi\sin(\pi x)][1-y-\cos(\pi y)]\cos t,
\end{aligned}
\tag{56}
$$

where all model parameters are set to one, and in the second test problem

$$\begin{aligned} u(x,y,t) &= \big((y-1)^2 \cos t, [x^2 - x] \cos t\big) \\ p(x,y,t) &= [2\mu(x+y-1) + \tfrac{\rho g n}{3k_{\min}}] \cos t \\ \phi(x,y,t) &= [\tfrac{n}{k_{\min}} \left\{x(1-x)(y-1) + \tfrac{1}{3}y^3 - y^2 + y\right\} + \tfrac{2\nu}{g} x] \cos t, \end{aligned} \tag{57}$$

where all model parameters may vary. We confirm second-order accuracy of CNLF-stab with both test problems. To confirm unconditional stability of the method, independent of the model parameter values, we set the body force and source functions, f_f and f_p, equal to zero in the second test problem and check that the solution vanishes over long-time intervals. In all tests, we enforce non-homogeneous Dirichlet boundary conditions, except along the interface: $u_h = u$ on $\Omega_f \setminus I$, $\phi_h = \phi$ on $\Omega_p \setminus I$, and choose the initial conditions, as well as the first terms in the method, to match the exact solutions.

5.1. Convergence Rate Verification. To confirm second-order accuracy, we set $h = \Delta t$ and calculate the errors and convergence orders for the variables u, p, and ϕ. We define the discrete error norms, $E(u)$, $E(p)$, and $E(\phi)$, as follows.

$$\begin{aligned} E(u) &= |||u - u_h|||_{L^\infty(0,T;H^1_{div}(\Omega_f))}, \\ E(p) &= |||p - p_h|||_{L^\infty(0,T;L^2(\Omega_f))}, \\ E(\phi) &= |||\phi - \phi_h|||_{L^\infty(0,T;L^2(\Omega_p))}. \end{aligned}$$

We let $r_{u,\phi,p}$ denote the calculated order of convergence. Table 3 gives the calculated errors for the first test problem and Table 4 the errors for the second test problem, with $S_0 = 10^{-4}$ and $k_{\min} = 10^{-1}$, both over the time interval $[0, 1]$. As expected, in both cases we have second-order convergence for the Stokes velocity, u, Stokes pressure, p, and Darcy pressure, ϕ.

TABLE 3. Second-order convergence of CNLF-stab for test problem 1.

$h = \Delta t$	E(u)	r_u	E(p)	r_p	E(ϕ)	r_ϕ
1/4	0.0304013	-	1.10942	-	0.130579	-
1/8	0.0048835	2.64	0.272517	2.03	0.0347465	1.91
1/16	0.00105315	2.21	0.0649257	2.07	0.00878685	1.98
1/32	0.000264613	1.99	0.0163038	1.99	0.00220226	2.00
1/64	0.000064201	2.04	0.00453213	1.85	0.000550882	2.00

TABLE 4. Second-order convergence of CNLF-stab for test problem 2, with $S_0 = 10^{-4}$ and $k_{\min} = 10^{-1}$.

$h = \Delta t$	E(u)	r_u	E(p)	r_p	E(ϕ)	r_ϕ
1/8	0.00163737	-	0.214387	-	0.0819758	-
1/16	0.000464456	1.82	0.0700264	1.61	0.0264185	1.63
1/32	0.000115658	2.01	0.0187149	1.90	0.00691258	1.93
1/64	0.000029022	2.00	0.00486284	1.94	0.00174539	1.99
1/128	0.00000726908	2.00	0.00126994	1.94	0.000437368	2.00

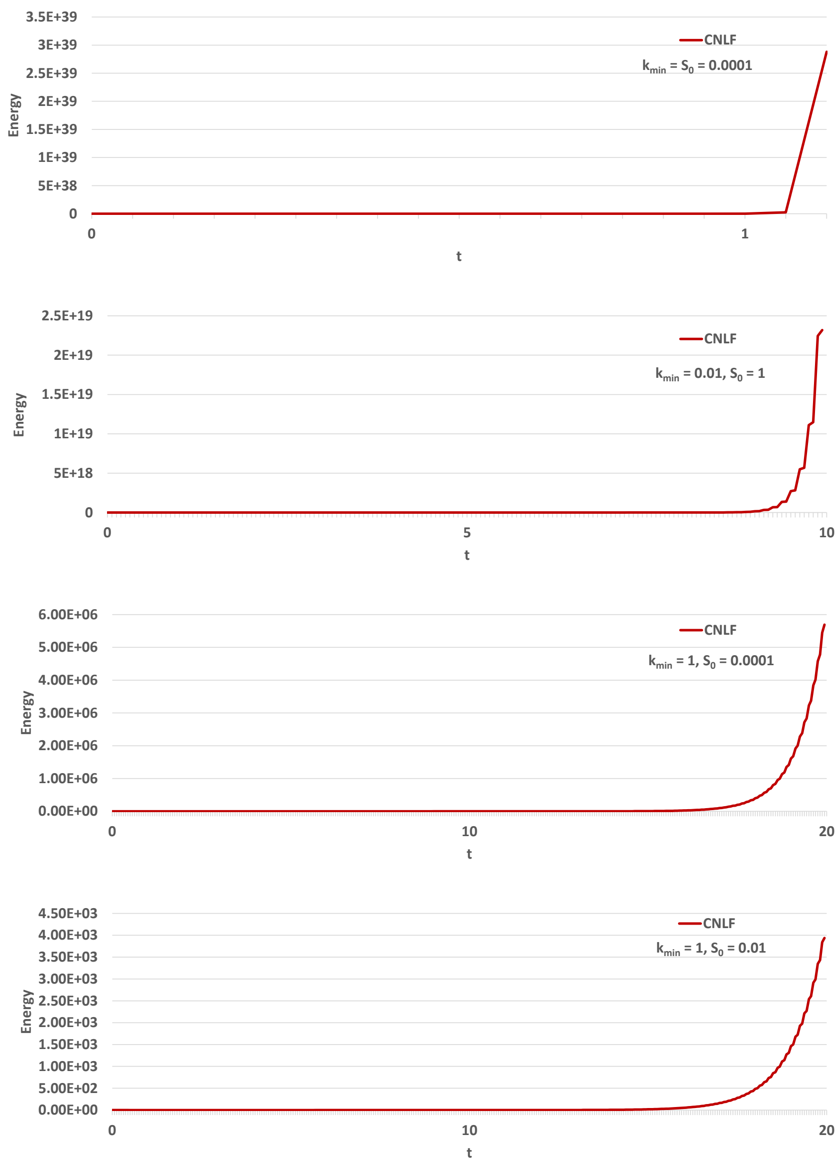

FIGURE 2. Instability of CNLF for test problem 2 with $\Delta t = h = 1/16$.

5.2. Unconditional, Asymptotic Stability. To check unconditional, asymptotic stability, we set $h = \Delta t = 1/16$ and calculate the discrete energy,

$$\text{Energy}(t^{n-1/2}) := \|\mathbf{u}_h^n\|_f^2 + \|\mathbf{u}_h^{n-1}\|_f^2 + S_0\left(\|\phi_h^n\|_p^2 + \|\phi_h^{n-1}\|_p^2\right),$$

over the time interval $[0, T]$, for T up to 40, and for varying S_0 and $k_{\min}$. The results of the computed energy for CNLF are shown in Figure 2 and the corresponding ones for CNLF-stab in Figure 3. The energy of CNLF-stab decays to zero over time, as expected, while CNLF blows up in all cases.

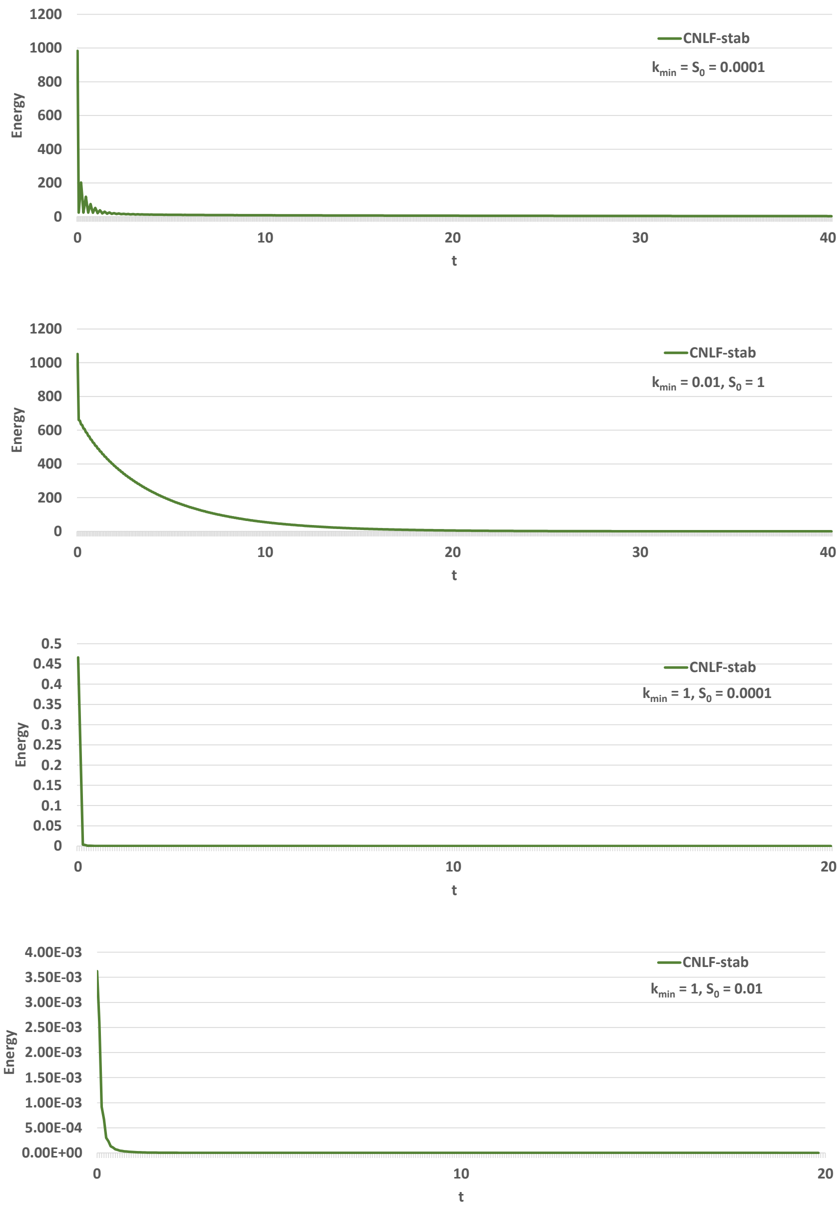

FIGURE 3. Unconditional stability of CNLF-stab for test problem 2 with $\Delta t = h = 1/16$.

6. Conclusions

The added stabilization terms in the proposed CNLF-stab method for the Stokes-Darcy model correct shortcomings of the original CNLF (Crank-Nicolson Leapfrog) method, namely conditional stability requiring small time step sizes and extreme sensitivity to small values of the specific storage parameter, S_0. Theoretical analysis of CNLF-stab showed that the method maintains second-order accuracy while eliminating all time step conditions for stability, and while effectively controlling both the stable and unstable modes of Leapfrog, resulting in unconditional, asymptotic stability. Numerical tests confirmed all theoretical results.

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