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A $\boldsymbol{T}$-ψ FINITE ELEMENT METHOD FOR A NONLINEAR DEGENERATE EDDY CURRENT MODEL WITH FERROMAGNETIC MATERIALS

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Abstract. This paper is devoted to the study of a fully discrete $\boldsymbol{T}$-ψ finite element method based on $\boldsymbol{H}$-decomposition to solve a nonlinear degenerate transient eddy current problem with ferromagnetic materials. Here, the ferromagnetic properties are linked by a power material law. We first design a nonlinear time-discrete scheme for approximation in suitable function spaces. We show the well-posedness of the semidiscrete problem and prove the convergence of the nonlinear scheme by the Minty-Browder technique. Finally, we suggest a fully discrete scheme, derive its error estimate and give some numerical experiments to validate the theoretical result.

Key words. nonlinear degenerate eddy current problem, $\boldsymbol{T}$-ψ method, nodal elements, convergence, and error estimates.

1. Introduction

The growing industrial applications of superconducting materials increase the necessity for accurate numerical methods and their solid mathematical analysis. To derive a precise mathematical model, we usually use the eddy current approximation of Maxwell's equations by formally dropping the displacement currents:

$$\begin{cases} \dfrac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0, \\ \nabla \times \boldsymbol{H} = \boldsymbol{J}, \end{cases} \tag{1}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ stands for the magnetic induction, $\boldsymbol{H}$ denotes the magnetic field and $\boldsymbol{J}$ is the current density.

Let $\Omega \subset \mathbb{R}^3$ be a sufficiently large, bounded polyhedron with the connected boundary $\partial\Omega$. This domain consists of some simply-connected convex subdomains occupied by ferromagnetic materials, which are denoted by Ω_c with the boundary $\partial\Omega_c$. Let the complement $\Omega_e = \Omega \backslash \overline{\Omega}_c$ be the nonconducting domain. Taking into account Ohm's law,

$$\boldsymbol{J} = \sigma \boldsymbol{E},$$

where σ is the conductivity. We assume that σ is piecewise constants in Ω_c and vanishes outside Ω_c, and there exist two constants $\sigma_{\min}$ and $\sigma_{\max}$ such that $0 < \sigma_{\min} \le \sigma \le \sigma_{\max}$ in Ω_c. The time-dependent magnetic variables are related as follows:

$$\boldsymbol{B}(\boldsymbol{H}) = \begin{cases} \mu_0\big(\boldsymbol{H} + \boldsymbol{M}(\boldsymbol{H})\big) & \text{in } \Omega_c, \\ \mu_0 \boldsymbol{H} & \text{in } \Omega_e, \end{cases} \tag{2}$$

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where μ_0 denotes the magnetic permeability of free space and $\boldsymbol{M}$ stands for the magnetization vector. One can characterize the relationship between $\boldsymbol{M}$ and $\boldsymbol{H}$ by a material law (for some $0 < \alpha < 1$)

$$\boldsymbol{M}(\boldsymbol{H}) = \begin{cases} |\boldsymbol{H}|^{\alpha-1}\boldsymbol{H}, & \text{if } |\boldsymbol{H}| \leq 1, \\ |\boldsymbol{H}|^{-1}\boldsymbol{H}, & \text{if } |\boldsymbol{H}| > 1. \end{cases} \tag{3}$$

We consider the eddy current problem (1) with magnetic and anisotropic materials. Assume the boundary condition

$$\boldsymbol{H} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \times (0, T]$$

and the initial condition

$$\boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0(\boldsymbol{x}),$$

where $\boldsymbol{n}$ is regarded as the unit outward normal vector on $\partial\Omega$ or $\partial\Omega_c$. For physical reasons, it is supposed that $\nabla \cdot \boldsymbol{B}(\boldsymbol{H}_0) = 0$. Then we obtain the following initial boundary value problem:

$$\begin{cases} \dfrac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{H}) + \nabla \times \left(\dfrac{1}{\sigma}\nabla \times \boldsymbol{H}\right) = \boldsymbol{0} & \text{in } \Omega_c, \\ \nabla \cdot \dfrac{\partial}{\partial t}(\mu_0 \boldsymbol{H}) = 0 & \text{in } \Omega_e, \\ \boldsymbol{H} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega, \\ \boldsymbol{H}(\cdot, 0) = \boldsymbol{H}_0(\cdot) & \text{in } \Omega. \end{cases} \tag{4}$$

The nonlinear PDEs of the type (4) have some applications in superconductors (see [13, 14]). It is well known that high-field (hard) type-II superconductors are not ideal conductors of electric current and are usually treated as electrically nonlinear conductors. The process of electromagnetic field penetration in such devices is the process of nonlinear diffusion. The equations describing the process can degenerate. For an overview of models with some hierarchy structure we refer the readers to [4, 10]. The magnetization of type-II superconductors in a nonstationary external magnetic field can also be formulated in terms of a scalar p-Laplacian equation if the magnetic field lies only in one direction. This situation has been studied in many papers, e.g. [3, 29, 30]. Slodička applied the backward Euler scheme to this type of equations for discretization in time and derived the error estimates for a degenerate problem in [24] and an application in superconductors in [25]. These error estimates for the time-discretization in both papers are suboptimal. Some similar works can be also found in [8, 15, 19, 26].

To solve quasistationary Maxwell's equations by the finite element methods, various formulations different in the choices of the primary unknowns are suggested, such as, direct approaches based on the electric/magnetic field, and indirect approaches based on potential fields (e.g. the $\boldsymbol{A}$-ϕ method from $\boldsymbol{E}$-decomposition and the $\boldsymbol{T}$-ψ method from $\boldsymbol{H}$-decomposition). The main difficulty in application of nodal elements for direct approaches is that the normal component of the field is discontinuous on the interface between different materials due to the presence of inhomogeneous mediums, but indirect approaches can avoid it.

The $\boldsymbol{T}$-ψ method is to decompose the magnetic field into summation of a vector potential $\boldsymbol{T}$ and the gradient of a scalar potential ψ in the simply-connected conductors, and only the gradient of a scalar potential ψ outside the conductors [1, 2, 6, 16, 17, 21, 33], afterward to approximate both potential fields by piecewise polynomial functions. The $\boldsymbol{T}$-ψ method has some advantages: First, although introducing the vector and scalar potentials increases the number of unknowns and equations, this seeming complication is justified by a better way of dealing with possible discontinuities of mediums. Second, we only need to solve a scalar potential instead of a vector field in nonconducting domains. Third, it is shown in applications of engineering fields that this method has good numerical accuracy. Finally, many popular nodal finite element softwares and computational techniques can be applied directly. As far as we know, relevant theoretical works on the $\boldsymbol{T}$-ψ method for the system (4) have not been shown so far.

The aim of this paper is to study a fully discrete $\boldsymbol{T}$-ψ finite element scheme based on the backward Euler discretization in time and nodal finite elements in space to solve the system (4). The paper is organized as follows. In Section 2, we give some notations used in this paper and study the $\boldsymbol{T}$-ψ formulation for the problem (4). In Section 3, we present a $\boldsymbol{T}$-ψ scheme by discretization in time (Rothe's method). We prove the existence and uniqueness of these discrete fields by using the theory of monotone operators (see [27]). Based on the stability estimates from Section 4, we prove in Section 5 that the solution of the semidiscrete problem converges to the weak solution of the continuous problem. The monotonicity and α-Hölder continuity (see [28]) of the function $\boldsymbol{M}(s)$ with $0 < \alpha < 1$ is of vital importance such that the Minty-Browder technique (see [9]) can be used to obtain the convergence of the nonlinear term. The corresponding error estimate for the semidiscrete problem is given in Section 6. In Section 7, we propose a fully discrete $\boldsymbol{T}$-ψ scheme based on nodal elements and discuss the error estimate. Finally, we verify our scheme by some numerical experiments in Section 8 and give some conclusions in the last section.

2. Variational formulation

For convenience of presentation, we first give some notations that will be used throughout this paper. Let $L^2(\Omega)$ be the usual Hilbert space of square integrable functions equipped with the inner product and norm:

$$(u,v)_{L^2(\Omega)} := \int_\Omega u(\boldsymbol{x})v(\boldsymbol{x})d\boldsymbol{x} \text{ and } \|u\|_{L^2(\Omega)} := (u,u)^{1/2}_{L^2(\Omega)}.$$

Define $H^m(\Omega) := \{v \in L^2(\Omega) : D^\xi v \in L^2(\Omega), |\xi| \le m\}$ which is equipped with the following norm

$$\|u\|_{H^m(\Omega)} := \left(\sum_{|\xi|\le m} \|D^\xi u\|^2_{L^2(\Omega)} \right)^{1/2},$$

where m is a non-negative integer and ξ represents non-negative triple index. Throughout we use boldface notation to represent vector-valued quantities, such as $\boldsymbol{L}^2(\Omega) := \left(L^2(\Omega)\right)^3$. The above definitions for Ω are similarly defined for Ω_c and Ω_e.

Define the Hilbert spaces

$$\widehat{\boldsymbol{H}}^1_0(\Omega_c) := \left\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega_c) : \ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \partial\Omega_c \right\},$$
$$\boldsymbol{H}(\boldsymbol{curl}, \Omega_c) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega_c) : \nabla \times \boldsymbol{v} \in \boldsymbol{L}^2(\Omega_c)\},$$
$$\boldsymbol{H}_0(\boldsymbol{curl}, \Omega_c) := \left\{\boldsymbol{v} \in \boldsymbol{H}(\boldsymbol{curl}, \Omega_c) : \boldsymbol{n} \times \boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\Omega_c\right\}.$$

Further, we denote the space $\boldsymbol{V} := \widehat{\boldsymbol{H}}^1_0(\Omega_c) \times H^1(\Omega)/\mathbb{R}$ equipped with the inner product

$$\big((\boldsymbol{P},\varphi),(\boldsymbol{Q},\phi)\big)_{\boldsymbol{V}} := (\boldsymbol{P},\boldsymbol{Q})_{\boldsymbol{H}^1(\Omega_c)} + (\nabla\varphi, \nabla\phi)_{\boldsymbol{L}^2(\Omega)}$$

and the norm

$$\|(\boldsymbol{Q},\phi)\|_{\boldsymbol{V}} := \left(\|\boldsymbol{Q}\|^2_{\boldsymbol{H}^1(\Omega_c)} + \|\nabla\phi\|^2_{\boldsymbol{L}^2(\Omega)} \right)^{1/2}.$$

The dual space of $\boldsymbol{V}$ is denoted by $\boldsymbol{V}^*$.

Referring to [12], by definition of the space $\boldsymbol{X} := \nabla H^1(\Omega) + \boldsymbol{H}_0(\boldsymbol{curl}, \Omega_c)$, the weak formulation of (4) yields the variational equation: Given $\boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0(\boldsymbol{x})$, find $\boldsymbol{H} \in \boldsymbol{X}$ such that

$$\int_\Omega \frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{H}) \cdot \boldsymbol{v}\; d\boldsymbol{x} + \int_{\Omega_c} \frac{1}{\sigma}\nabla \times \boldsymbol{H} \cdot \nabla \times \boldsymbol{v}\; d\boldsymbol{x} = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{X}. \tag{5}$$

Here we adopt the convention that each function in $\boldsymbol{H}_0(\boldsymbol{curl}, \Omega_c)$ or $\widehat{\boldsymbol{H}}_0^1(\Omega_c)$ is extended by zero to Ω. Thus, to split the magnetic field $\boldsymbol{H}$ into summation of a vector potential $\boldsymbol{T}$ and the gradient of a scalar potential ψ, we need the following lemma borrowed from [12].

Lemma 2.1. Let the nonconductive region Ω_e be simply connected. Then, for any $\boldsymbol{v} \in \boldsymbol{X}$, there exist a unique $\boldsymbol{v}_c \in \boldsymbol{X}_c$ and a unique $\phi \in H^1(\Omega)/\mathbb{R}$ such that

$$\boldsymbol{v} = \boldsymbol{v}_c + \nabla\phi, \qquad \|\boldsymbol{v}_c\|_{\boldsymbol{H}(\boldsymbol{curl},\Omega_c)} + \|\phi\|_{H^1(\Omega)} \le C\,\|\boldsymbol{v}\|_{\boldsymbol{H}(\boldsymbol{curl},\Omega)},$$

where $\boldsymbol{X}_c := \{\boldsymbol{v} \in \boldsymbol{H}_0(\boldsymbol{curl}, \Omega_c) : \ \nabla \cdot \boldsymbol{v} = 0 \text{ in } \Omega_c\}$ and C is a positive constant depending on Ω_e.

A direct application of Lemma 2.1 yields the following result.

Lemma 2.2. $\boldsymbol{H} \in \boldsymbol{X}$ admits a unique decomposition

$$\boldsymbol{H} = \boldsymbol{T} + \nabla\psi, \qquad \boldsymbol{T} \in \boldsymbol{X}_c,\ \psi \in \boldsymbol{H}^1(\Omega)/\mathbb{R}.$$

It is clear that we can rewrite the equations of the problem (4) as follows:

$$\frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{T} + \nabla\psi) + \nabla \times \left(\frac{1}{\sigma}\nabla \times \boldsymbol{T}\right) = \boldsymbol{0} \quad \text{in } \Omega_c, \tag{6}$$

$$\nabla \cdot \frac{\partial}{\partial t}(\mu_0 \nabla\psi) = 0 \quad \text{in } \Omega_e, \tag{7}$$

and also take the divergence of (6) to obtain

$$\nabla \cdot \frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{T} + \nabla\psi) = 0 \text{ in } \Omega_c.$$

We remark that the previous equations and the following $\boldsymbol{T}$-ψ form (8) should be understood in a distribution sense.

For $\boldsymbol{T} \in \boldsymbol{X}_c$, we adopt the penalty function method [32] to deal with divergence-free of $\boldsymbol{T}$, that is, adding the penalty function term $-\nabla(\sigma^{-1}\nabla \cdot \boldsymbol{T})$ to the left-hand side of (6). Thus the problem (4) becomes the following $\boldsymbol{T}$-ψ formulation:

$$\begin{cases} \dfrac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{T} + \nabla\psi) + \nabla \times \left(\dfrac{1}{\sigma}\nabla \times \boldsymbol{T}\right) - \nabla\left(\dfrac{1}{\sigma}\nabla \cdot \boldsymbol{T}\right) = \boldsymbol{0} & \text{in } \Omega_c, \\ \nabla \cdot \dfrac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{T} + \nabla\psi) = 0 & \text{in } \Omega_c, \\ \nabla \cdot \dfrac{\partial}{\partial t}\mu_0(\nabla\psi) = 0 & \text{in } \Omega_e, \\ \boldsymbol{T} \times \boldsymbol{n} = \boldsymbol{0},\ \dfrac{1}{\sigma}\nabla \cdot \boldsymbol{T} = 0 & \text{on } \partial\Omega_c, \\ \boldsymbol{B}(\boldsymbol{T} + \nabla\psi) \cdot \boldsymbol{n} = \mu_0 \nabla\psi \cdot \boldsymbol{n},\ [\psi] = 0 & \text{on } \partial\Omega_c, \\ \nabla\psi \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

We denote the initial values of $\boldsymbol{T}(\cdot, t)$ and $\psi(\cdot, t)$ by $\boldsymbol{T}_0$ and ψ_0 satisfying $\boldsymbol{H}_0 = \boldsymbol{T}_0 + \nabla\psi_0$ with divergence-free of $\boldsymbol{T}_0$. Assume $(\boldsymbol{T}_0, \psi_0) \in \boldsymbol{V}$.

Definition 2.1. The potential field pair $(\boldsymbol{T},\psi)\in C\big([0,T],\boldsymbol{V}\big)$ is the weak solution of (8) if the following identity is satisfied

(9)

$$\begin{cases} \left(\dfrac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{T}+\nabla\psi),\boldsymbol{Q}+\nabla\phi\right)_{\boldsymbol{L}^2(\Omega_c)}+\left(\dfrac{\partial}{\partial t}(\mu_0\nabla\psi),\nabla\phi\right)_{\boldsymbol{L}^2(\Omega_e)} \\ \quad+\left(\dfrac{1}{\sigma}\nabla\times\boldsymbol{T},\nabla\times\boldsymbol{Q}\right)_{\boldsymbol{L}^2(\Omega_c)}+\left(\dfrac{1}{\sigma}\nabla\cdot\boldsymbol{T},\nabla\cdot\boldsymbol{Q}\right)_{L^2(\Omega_c)}=0,\quad\forall(\boldsymbol{Q},\phi)\in\boldsymbol{V}, \\ \boldsymbol{T}(\boldsymbol{x},0)=\boldsymbol{T}_0,\quad \boldsymbol{x}\in\Omega_c, \\ \psi(\boldsymbol{x},0)=\psi_0,\quad \boldsymbol{x}\in\Omega. \end{cases}$$

We end this section by the following two lemmas (see [18]) used in the consequent sections.

Lemma 2.3 (coercivity)**.** Let Ω_c be some convex bounded polyhedra. Then there exists a constant $C>0$ such that

$$C\big(\,\|\boldsymbol{Q}+\nabla\phi\|^2_{\boldsymbol{L}^2(\Omega_c)}+\|\nabla\times\boldsymbol{Q}\|^2_{\boldsymbol{L}^2(\Omega_c)}+\|\nabla\cdot\boldsymbol{Q}\|^2_{L^2(\Omega_c)}+\|\nabla\phi\|^2_{\boldsymbol{L}^2(\Omega_e)}\,\big)\geq\|(\boldsymbol{Q},\phi)\|^2_{\boldsymbol{V}}$$

for all $(\boldsymbol{Q},\phi)\in\boldsymbol{V}$.

Lemma 2.4 (continuity)**.** There exists a constant $C>0$ such that

$$\begin{aligned} &(\boldsymbol{P}+\nabla\varphi,\boldsymbol{Q}+\nabla\phi)_{\boldsymbol{L}^2(\Omega_c)}+(\nabla\times\boldsymbol{P},\nabla\times\boldsymbol{Q})_{\boldsymbol{L}^2(\Omega_c)} \\ &+(\nabla\cdot\boldsymbol{P},\nabla\cdot\boldsymbol{Q})_{L^2(\Omega_c)}+(\nabla\varphi,\nabla\phi)_{\boldsymbol{L}^2(\Omega_e)} \\ \leq&C\,\|(\boldsymbol{P},\varphi)\|_{\boldsymbol{V}}\cdot\|(\boldsymbol{Q},\phi)\|_{\boldsymbol{V}} \end{aligned}$$

for any $(\boldsymbol{P},\varphi),(\boldsymbol{Q},\phi)\in\boldsymbol{V}$.

3. Semidiscretization

In this section we use Rothe's method to study the weak solution. Let n be a positive integer and $\{t_i=i\tau:\ \ i=0,\cdots,n\}$ be a equidistant partition of $[0,T]$ with $\tau=T/n$. Now set

$$u_i=u(\boldsymbol{x}_i),\ \delta u_i=\frac{u_i-u_{i-1}}{\tau}.$$

The semidiscrete approximation to (9) reads: Given $(\boldsymbol{T}_0,\psi_0)$, find $(\boldsymbol{T}_i,\psi_i)\in\boldsymbol{V}$, $1\leq i\leq n$, such that

(10)

$$\begin{aligned} &(\delta\boldsymbol{B}_i,\boldsymbol{Q}+\nabla\phi)_{\boldsymbol{L}^2(\Omega_c)}+\left(\frac{1}{\sigma}\nabla\times\boldsymbol{T}_i,\nabla\times\boldsymbol{Q}\right)_{\boldsymbol{L}^2(\Omega_c)}+\left(\frac{1}{\sigma}\nabla\cdot\boldsymbol{T}_i,\nabla\cdot\boldsymbol{Q}\right)_{L^2(\Omega_c)} \\ &\quad+(\mu_0\nabla\delta\psi_i,\nabla\phi)_{\boldsymbol{L}^2(\Omega_e)}=0,\qquad\forall(\boldsymbol{Q},\phi)\in\boldsymbol{V}. \end{aligned}$$

Here we denote $\boldsymbol{B}_i=\boldsymbol{B}(\boldsymbol{T}_i+\nabla\psi_i)$ and $\boldsymbol{M}_i=\boldsymbol{M}(\boldsymbol{T}_i+\nabla\psi_i)$. In the following theorem we prove the existence and uniqueness of these fields.

Theorem 3.1. Assume $(\boldsymbol{T}_0,\psi_0)\in\boldsymbol{V}$. Then there exists a unique solution pair $(\boldsymbol{T}_i,\psi_i)$ to the variational problem (10) for each $1\leq i\leq n$.

Proof. Let the operator $\mathcal{L}$ from $\boldsymbol{V}$ to $\boldsymbol{V}^*$ be defined as

$$\begin{aligned} \big\langle\mathcal{L}(\boldsymbol{P},\varphi),(\boldsymbol{Q},\phi)\big\rangle=&\left(\frac{\boldsymbol{B}(\boldsymbol{P}+\nabla\varphi)}{\tau},\boldsymbol{Q}+\nabla\phi\right)_{\boldsymbol{L}^2(\Omega_c)}+\left(\frac{\mu_0\nabla\varphi}{\tau},\nabla\phi\right)_{\boldsymbol{L}^2(\Omega_e)} \\ &+\left(\frac{1}{\sigma}\nabla\times\boldsymbol{P},\nabla\times\boldsymbol{Q}\right)_{\boldsymbol{L}^2(\Omega_c)}+\left(\frac{1}{\sigma}\nabla\cdot\boldsymbol{P},\nabla\cdot\boldsymbol{Q}\right)_{L^2(\Omega_c)}, \end{aligned}$$

and the functional $\boldsymbol{f}_i \in \boldsymbol{V}^*$ as

$$\left\langle \boldsymbol{f}_i, (\boldsymbol{Q}, \phi) \right\rangle = \left(\frac{\boldsymbol{B}_{i-1}}{\tau}, \boldsymbol{Q} + \nabla\phi \right)_{\boldsymbol{L}^2(\Omega_c)} + \left(\frac{\mu_0 \nabla\psi_{i-1}}{\tau}, \nabla\phi \right)_{\boldsymbol{L}^2(\Omega_e)}$$

for all $(\boldsymbol{P}, \varphi),\ (\boldsymbol{Q}, \phi) \in \boldsymbol{V}$. Clearly (10) is equivalent to the operator equation

$$\mathcal{L}(\boldsymbol{T}_i, \psi_i) = \boldsymbol{f}_i \qquad \text{in } \boldsymbol{V}^*.$$

Using the relation

$$(\alpha+\beta)^2 (x^\alpha - y^\alpha)(x^\beta - y^\beta) \geq 4\alpha\beta \Big(x^{\frac{\alpha+\beta}{2}} - y^{\frac{\alpha+\beta}{2}} \Big)^2$$

for all positive α, β and all non-negative real numbers x and y, the monotonicity of the function $\boldsymbol{M}(s)$ follows from

$$\begin{aligned} (\boldsymbol{M}(\boldsymbol{u}) - \boldsymbol{M}(\boldsymbol{v}), \boldsymbol{u} - \boldsymbol{v})_{\boldsymbol{L}^2(\Omega_c)} &= \mu_0 \left(|\boldsymbol{u}|^{\alpha-1} \boldsymbol{u} - |\boldsymbol{v}|^{\alpha-1} \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v} \right)_{\boldsymbol{L}^2(\Omega_c)} \\ &\geq C \int_{\Omega_c} (|\boldsymbol{u}|^\alpha - |\boldsymbol{v}|^\alpha)(|\boldsymbol{u}| - |\boldsymbol{v}|)\, d\boldsymbol{x} \\ &\geq C \int_{\Omega_c} \left(|\boldsymbol{u}|^{\frac{\alpha+1}{2}} - |\boldsymbol{v}|^{\frac{\alpha+1}{2}} \right)^2 d\boldsymbol{x} \geq 0 \text{ for any } \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega_c). \end{aligned}$$

Then the strict monotonicity of $\mathcal{L}$ comes directly from the monotonicity of the function $\boldsymbol{M}$ and Lemma 2.3, that is, for any $(\boldsymbol{P}, \varphi), (\boldsymbol{Q}, \phi) \in \boldsymbol{V}$,

$$\begin{aligned} &\left\langle \mathcal{L}(\boldsymbol{P}, \varphi) - \mathcal{L}(\boldsymbol{Q}, \phi), (\boldsymbol{P}, \varphi) - (\boldsymbol{Q}, \phi) \right\rangle \\ &\geq \frac{\mu_0}{\tau} \|(\boldsymbol{P} + \nabla\varphi) - (\boldsymbol{Q} + \nabla\phi)\|^2_{\boldsymbol{L}^2(\Omega_c)} + \frac{\mu_0}{\tau} \|\nabla\varphi - \nabla\phi\|^2_{\boldsymbol{L}^2(\Omega_e)} \\ &\quad + \left\| \frac{1}{\sqrt{\sigma}} \nabla \times (\boldsymbol{P} - \boldsymbol{Q}) \right\|^2_{\boldsymbol{L}^2(\Omega_c)} + \left\| \frac{1}{\sqrt{\sigma}} \nabla \cdot (\boldsymbol{P} - \boldsymbol{Q}) \right\|^2_{L^2(\Omega_c)} \\ &\geq C(\tau) \|(\boldsymbol{P}, \varphi) - (\boldsymbol{Q}, \phi)\|^2_{\boldsymbol{V}}. \end{aligned}$$

In addition,

$$\begin{aligned} &\frac{\left\langle \mathcal{L}(\boldsymbol{P}, \varphi), (\boldsymbol{P}, \varphi) \right\rangle}{\|(\boldsymbol{P}, \varphi)\|_{\boldsymbol{V}}} \\ &\geq \frac{\frac{\mu_0}{\tau} \|\boldsymbol{P} + \nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega_c)} + \frac{\mu_0}{\tau} \|\boldsymbol{P} + \nabla\varphi\|^{\alpha+1}_{\boldsymbol{L}^{\alpha+1}(\Omega_c)} + \frac{\mu_0}{\tau} \|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega_e)}}{\|(\boldsymbol{P}, \varphi)\|_{\boldsymbol{V}}} \\ &+ \frac{\frac{1}{\sigma_{max}} \|\nabla \times \boldsymbol{P}\|^2_{\boldsymbol{L}^2(\Omega_c)} + \frac{1}{\sigma_{max}} \|\nabla \cdot \boldsymbol{P}\|^2_{L^2(\Omega_c)}}{\|(\boldsymbol{P}, \varphi)\|_{\boldsymbol{V}}} \\ &\geq \min\left\{ \frac{\mu_0}{\tau}, \frac{1}{\sigma_{max}} \right\} \\ &\times \frac{\|\boldsymbol{P} + \nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega_c)} + \|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega_e)} + \|\nabla \times \boldsymbol{P}\|^2_{\boldsymbol{L}^2(\Omega_c)} + \|\nabla \cdot \boldsymbol{P}\|^2_{L^2(\Omega_c)}}{\|(\boldsymbol{P}, \varphi)\|_{\boldsymbol{V}}} \\ &\geq C(\tau) \|(\boldsymbol{P}, \varphi)\|_{\boldsymbol{V}} \to \infty \qquad \text{as } \|(\boldsymbol{P}, \varphi)\|_{\boldsymbol{V}} \to \infty, \end{aligned}$$

which confirms that the operator $\mathcal{L}$ is coercive. Due to the continuity of the function $\boldsymbol{B}$, there exists a unique solution $(\boldsymbol{T}_i, \psi_i) \in \boldsymbol{V}$ to the equation $\mathcal{L}(\boldsymbol{T}_i, \psi_i) = \boldsymbol{f}_i$ since $\boldsymbol{f}_i \in \boldsymbol{V}^*$ (by Theorem 18.2 in [27]). □

4. Stability

In the following lemmas some stability estimates for the potential field $(\boldsymbol{T}_j, \psi_j)$ are given. We will see in later that these uniform bounds play important roles to prove convergence.

Lemma 4.1. For $j = 1, \ldots, n$, there are two positive real numbers τ_0 and C such that

(11)
$$\begin{aligned}&\sum_{i=1}^{j} \tau\mu_0 \|\delta\boldsymbol{T}_i + \nabla\delta\psi_i\|^2_{\boldsymbol{L}^2(\Omega_c)} + \sum_{i=1}^{j} \tau\mu_0 \|\nabla\delta\psi_i\|^2_{\boldsymbol{L}^2(\Omega_e)} + \sum_{i=1}^{j} \left\| \frac{\tau}{\sqrt{\sigma}} \nabla\times\delta\boldsymbol{T}_i \right\|^2_{\boldsymbol{L}^2(\Omega_c)} \\ &+ \left\| \frac{1}{\sqrt{\sigma}} \nabla\times\boldsymbol{T}_j \right\|^2_{\boldsymbol{L}^2(\Omega_c)} + \sum_{i=1}^{j} \left\| \frac{\tau}{\sqrt{\sigma}} \nabla\cdot\delta\boldsymbol{T}_i \right\|^2_{L^2(\Omega_c)} + \left\| \frac{1}{\sqrt{\sigma}} \nabla\cdot\boldsymbol{T}_j \right\|^2_{L^2(\Omega_c)} \\ &\le C \qquad \text{for any } 0 < \tau < \tau_0.\end{aligned}$$

Proof. Setting $(\boldsymbol{Q}, \psi) = (\tau\delta\boldsymbol{T}_i, \tau\delta\psi_i)$ in (10) and summing for $i = 1, \ldots, j$ yields

$$\begin{aligned}&\sum_{i=1}^{j} \tau\mu_0 \|\delta\boldsymbol{T}_i + \nabla\delta\psi_i\|^2_{\boldsymbol{L}^2(\Omega_c)} + \sum_{i=1}^{j} \tau\mu_0 \big(\delta\boldsymbol{M}_i, \delta\boldsymbol{T}_i + \nabla\delta\psi_i\big)_{\boldsymbol{L}^2(\Omega_c)} \\ &\quad + \sum_{i=1}^{j} \tau\mu_0 \|\nabla\delta\psi_i\|^2_{\boldsymbol{L}^2(\Omega_e)} + \sum_{i=1}^{j} \tau \left(\frac{1}{\sigma} \nabla\times\boldsymbol{T}_i, \nabla\times\delta\boldsymbol{T}_i \right)_{\boldsymbol{L}^2(\Omega_c)} \\ &\quad + \sum_{i=1}^{j} \tau \left(\frac{1}{\sigma} \nabla\cdot\boldsymbol{T}_i, \nabla\cdot\delta\boldsymbol{T}_i \right)_{L^2(\Omega_c)} = 0.\end{aligned}$$

The fourth and fifth term can be rewritten using Abel's summation rule as follows.

$$\begin{aligned}&\sum_{i=1}^{j} \tau \left(\frac{1}{\sigma} \nabla\times\boldsymbol{T}_i, \nabla\times\delta\boldsymbol{T}_i \right)_{\boldsymbol{L}^2(\Omega_c)} \\ &= \frac{1}{2} \sum_{i=1}^{j} \left\| \frac{\tau}{\sqrt{\sigma}} \nabla\times\delta\boldsymbol{T}_i \right\|^2_{\boldsymbol{L}^2(\Omega_c)} + \frac{1}{2} \left\| \frac{1}{\sqrt{\sigma}} \nabla\times\boldsymbol{T}_j \right\|^2_{\boldsymbol{L}^2(\Omega_c)} - \frac{1}{2} \left\| \frac{1}{\sqrt{\sigma}} \nabla\times\boldsymbol{T}_0 \right\|^2_{\boldsymbol{L}^2(\Omega_c)},\end{aligned}$$

$$\begin{aligned}&\sum_{i=1}^{j} \tau \left(\frac{1}{\sigma} \nabla\cdot\boldsymbol{T}_i, \nabla\cdot\delta\boldsymbol{T}_i \right)_{L^2(\Omega_c)} \\ &= \frac{1}{2} \sum_{i=1}^{j} \left\| \frac{\tau}{\sqrt{\sigma}} \nabla\cdot\delta\boldsymbol{T}_i \right\|^2_{L^2(\Omega_c)} + \frac{1}{2} \left\| \frac{1}{\sqrt{\sigma}} \nabla\cdot\boldsymbol{T}_j \right\|^2_{L^2(\Omega_c)} - \frac{1}{2} \left\| \frac{1}{\sqrt{\sigma}} \nabla\cdot\boldsymbol{T}_0 \right\|^2_{L^2(\Omega_c)}.\end{aligned}$$

From the monotonicity of $\boldsymbol{M}$, we know that $\big(\delta\boldsymbol{M}_i, \delta\boldsymbol{T}_i + \nabla\delta\psi_i\big)_{\boldsymbol{L}^2(\Omega_c)} \ge 0$. Thus we conclude the proof. □

Lemma 4.2. For $j = 1, \ldots, n$, there are two positive real numbers τ_0 and C such that

(12)
$$\|\boldsymbol{T}_j + \nabla\psi_j\|_{\boldsymbol{L}^2(\Omega_c)} + \|\nabla\psi_j\|_{\boldsymbol{L}^2(\Omega_e)} + \|\boldsymbol{M}(\boldsymbol{T}_j + \nabla\psi_j)\|_{\boldsymbol{L}^2(\Omega_c)} \le C \text{ for all } 0 < \tau < \tau_0.$$

Proof. The relation $\|\boldsymbol{M}(\boldsymbol{T}_j + \nabla\psi_j)\|_{\boldsymbol{L}^2(\Omega_c)} \le C$ follows from the definition of $\boldsymbol{M}$ satisfying $|\boldsymbol{M}(\boldsymbol{x})| \le 1$. The assertions $\|\boldsymbol{T}_j + \nabla\psi_j\|_{\boldsymbol{L}^2(\Omega_c)} \le C$ and $\|\nabla\psi_j\|_{\boldsymbol{L}^2(\Omega_e)} \le$

C respectively follow from Lemma 4.1 and the identities

$$\boldsymbol{T}_j + \nabla\psi_j = \boldsymbol{T}_0 + \nabla\psi_0 + \sum_{i=1}^{j} \tau(\delta \boldsymbol{T}_i + \nabla\delta\psi_i) \qquad \text{in } \Omega_c,$$

$$\nabla\psi_j = \nabla\psi_0 + \sum_{i=1}^{j} \tau\nabla\delta\psi_i \qquad \text{in } \Omega_e.$$

□

From now on, we define a dual inner product on $\boldsymbol{V}^* \times \boldsymbol{V} \to \mathbb{R}$ as follows:

$$\langle \boldsymbol{P}, (\boldsymbol{Q}, \psi)\rangle = (\boldsymbol{P}, \boldsymbol{Q} + \nabla\psi)_{\boldsymbol{L}^2(\Omega_c)}.$$

We can easily view from the definition that $\boldsymbol{L}^2(\Omega_c) \subset \boldsymbol{V}^*$.

Lemma 4.3. For $j = 1, \ldots, n$, there is a positive real numbers C such that

$$\sum_{i=1}^{j} \tau \left\|\delta \boldsymbol{M}(\boldsymbol{T}_i + \nabla\psi_i)\right\|^2_{\boldsymbol{V}^*} \le C. \tag{13}$$

Proof. From the weak formulation for $(\boldsymbol{T}_i, \psi_i)$ and Lemma 2.4, we obtain for all $(\boldsymbol{Q}, \phi) \in \boldsymbol{V}$,

$$\begin{aligned}
&\sum_{i=1}^{j} \tau \Big| \langle \delta \boldsymbol{M}(\boldsymbol{T}_i + \nabla\psi_i), (\boldsymbol{Q}, \phi)\rangle \Big| = \sum_{i=1}^{j} \tau \Big| (\delta \boldsymbol{M}(\boldsymbol{T}_i + \nabla\psi_i), \boldsymbol{Q} + \nabla\phi)_{\boldsymbol{L}^2(\Omega_c)} \Big| \\
&= \sum_{i=1}^{j} \tau \Big| (\delta(\boldsymbol{T}_i + \nabla\psi_i), \boldsymbol{Q} + \nabla\phi)_{\boldsymbol{L}^2(\Omega_c)} + (\delta\nabla\psi_i, \nabla\phi)_{\boldsymbol{L}^2(\Omega_e)} \\
&\quad + \frac{1}{\mu_0}\left(\frac{1}{\sigma}\nabla\times \boldsymbol{T}_i, \nabla\times \boldsymbol{Q}\right)_{\boldsymbol{L}^2(\Omega_c)} + \frac{1}{\mu_0}\left(\frac{1}{\sigma}\nabla\cdot \boldsymbol{T}_i, \nabla\cdot \boldsymbol{Q}\right)_{L^2(\Omega_c)} \Big| \\
&\le C\big(\sum_{i=1}^{j} \tau \left\|\delta(\boldsymbol{T}_i + \nabla\psi_i)\right\|^2_{\boldsymbol{L}^2(\Omega_c)} + \sum_{i=1}^{j} \tau \left\|\delta\nabla\psi_i\right\|^2_{\boldsymbol{L}^2(\Omega_e)} \\
&\quad + \sum_{i=1}^{j} \tau \left\|\nabla\times \boldsymbol{T}_i\right\|^2_{\boldsymbol{L}^2(\Omega_c)} + \sum_{i=1}^{j} \tau \left\|\nabla\cdot \boldsymbol{T}_i\right\|^2_{L^2(\Omega_c)} \big)^{1/2} \\
&\quad \times \big(\left\|\boldsymbol{Q} + \nabla\phi\right\|^2_{\boldsymbol{L}^2(\Omega_c)} + \left\|\nabla\phi\right\|^2_{\boldsymbol{L}^2(\Omega_e)} + \left\|\nabla\times \boldsymbol{Q}\right\|^2_{\boldsymbol{L}^2(\Omega_c)} + \left\|\nabla\cdot \boldsymbol{Q}\right\|^2_{L^2(\Omega_c)} \big)^{1/2} \\
&\le C \left\|(\boldsymbol{Q}, \phi)\right\|_{\boldsymbol{V}}.
\end{aligned}$$

By the definition of the operator norm in $\boldsymbol{V}^*$, we draw the inequality (13). □

5. Convergence

We first define some interpolations of the discrete fields in time by:

$$\begin{cases} \boldsymbol{T}_n(t) = \boldsymbol{T}_{i-1} + (t - t_{i-1})\delta \boldsymbol{T}_i, & t \in (t_{i-1}, t_i], \\ \boldsymbol{T}_n(0) = \boldsymbol{T}_0, \end{cases} \qquad \begin{cases} \overline{\boldsymbol{T}}_n(t) = \boldsymbol{T}_i, & t \in (t_{i-1}, t_i], \\ \overline{\boldsymbol{T}}_n(0) = \boldsymbol{T}_0, \end{cases}$$

$$\begin{cases} \psi_n(t) = \psi_{i-1} + (t - t_{i-1})\delta\psi_i, & t \in (t_{i-1}, t_i], \\ \psi_n(0) = \psi_0, \end{cases} \qquad \begin{cases} \overline{\psi}_n(t) = \psi_i, & t \in (t_{i-1}, t_i], \\ \overline{\psi}_n(0) = \psi_0, \end{cases}$$

$$\begin{cases} \boldsymbol{M}_n(t) = \boldsymbol{M}_{i-1} + (t - t_{i-1})\delta \boldsymbol{M}_i, & t \in (t_{i-1}, t_i], \\ \boldsymbol{M}_n(0) = \left|\boldsymbol{T}_0 + \nabla\psi_0\right|^{\alpha-1}(\boldsymbol{T}_0 + \nabla\psi_0), \end{cases}$$

$$\begin{cases}\overline{\boldsymbol{M}}_n(t) = \boldsymbol{M}_i, \quad t \in (t_{i-1}, t_i],\\ \overline{\boldsymbol{M}}_n(0) = |\boldsymbol{T}_0 + \nabla\psi_0|^{\alpha-1}(\boldsymbol{T}_0 + \nabla\psi_0),\end{cases}$$

$$\begin{cases}\boldsymbol{B}_n(t) = \boldsymbol{B}_{i-1} + (t - t_{i-1})\delta\boldsymbol{B}_i, \quad t \in (t_{i-1}, t_i],\\ \boldsymbol{B}_n(0) = \mu_0\big(\boldsymbol{T}_0 + \nabla\psi_0\big) + \mu_0\,|\boldsymbol{T}_0 + \nabla\psi_0|^{\alpha-1}(\boldsymbol{T}_0 + \nabla\psi_0),\end{cases}$$

$$\begin{cases}\overline{\boldsymbol{B}}_n(t) = \boldsymbol{B}_i, \quad t \in (t_{i-1}, t_i],\\ \overline{\boldsymbol{B}}_n(0) = \mu_0\big(\boldsymbol{T}_0 + \nabla\psi_0\big) + |\boldsymbol{T}_0 + \nabla\psi_0|^{\alpha-1}(\boldsymbol{T}_0 + \nabla\psi_0).\end{cases}$$

Obviously, we have $\overline{\boldsymbol{M}}_n(t) = \boldsymbol{M}(\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n)$ and $\overline{\boldsymbol{B}}_n(t) = \boldsymbol{B}(\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n)$. With these fields we can reformulate the semidiscrete problem (10) as follows:
(14)

$$(\partial_t\boldsymbol{B}_n, \boldsymbol{Q} + \nabla\phi)_{\boldsymbol{L}^2(\Omega_c)} + \left(\frac{1}{\sigma}\nabla\times\overline{\boldsymbol{T}}_n, \nabla\times\boldsymbol{Q}\right)_{\boldsymbol{L}^2(\Omega_c)} + \left(\frac{1}{\sigma}\nabla\cdot\overline{\boldsymbol{T}}_n, \nabla\cdot\boldsymbol{Q}\right)_{L^2(\Omega_c)}$$
$$+ (\mu_0\partial_t(\nabla\psi_n), \nabla\phi)_{\boldsymbol{L}^2(\Omega_e)} = 0,\ \forall(\boldsymbol{Q}, \phi) \in \boldsymbol{V}.$$

Now we are in a position to prove the convergence of an approximate solution of (14) to a weak solution of (9). The following theorem is the main result of this section and its proof refers to the framework of Theorem 5.1 in [8].

Theorem 5.1. There exists a pair $(\boldsymbol{T}, \psi) \in L^2((0,T), \boldsymbol{V})$ such that

(a) $(\overline{\boldsymbol{T}}_n, \overline{\psi}_n) \rightharpoonup (\boldsymbol{T}, \psi)$ in $L^2((0,T), \boldsymbol{V})$ and $\boldsymbol{T}_n + \nabla\psi_n \rightharpoonup \boldsymbol{T} + \nabla\psi$ in $L^2((0,T), \boldsymbol{L}^2(\Omega_c))$ and $\nabla\psi_n \rightharpoonup \nabla\psi$ in $L^2((0,T), \boldsymbol{L}^2(\Omega_e))$,
(b) $\overline{\boldsymbol{M}}_n \rightharpoonup \boldsymbol{D}$ in $L^{\frac{1+\alpha}{\alpha}}\Big((0,T), \boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Omega_c)\Big)$,
(c) $\boldsymbol{M}_n \rightharpoonup \boldsymbol{D}$ in $L^2\big((0,T), \boldsymbol{V}^*\big)$,
(d) $\boldsymbol{D} = \boldsymbol{M}(\boldsymbol{T} + \nabla\psi)$,
(e) $\big(\boldsymbol{M}_n(t), \boldsymbol{Q} + \nabla\phi\big)_{\boldsymbol{L}^2(\Omega_c)} \to \Big(\boldsymbol{M}(\boldsymbol{T} + \nabla\psi), \boldsymbol{Q} + \nabla\phi\Big)_{\boldsymbol{L}^2(\Omega_c)}$ for all $t \in [0,T]$ and $(\boldsymbol{Q}, \phi) \in \boldsymbol{V}$,
(f) $(\boldsymbol{T}, \psi)$ is the weak solution of (9).

Proof.

(a) From Lemma 4.1 and Lemma 4.2, the sequence $\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n$ is bounded in $L^2((0,T), \boldsymbol{L}^2(\Omega_c))$ and the sequence $\nabla\overline{\psi}_n$ is bounded in $L^2((0,T), \boldsymbol{L}^2(\Omega_e))$. We also conclude that the sequences $\nabla\times\overline{\boldsymbol{T}}_n$ and $\nabla\cdot\overline{\boldsymbol{T}}_n$ are bounded in $L^2((0,T), \boldsymbol{L}^2(\Omega_c))$. Thus by using Lemma 2.3, we have

$$\big\|(\overline{\boldsymbol{T}}_n, \overline{\psi}_n)\big\|_{\boldsymbol{V}} \le C\big(\,\big\|\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n\big\|_{\boldsymbol{L}^2(\Omega_c)} + \big\|\nabla\times\overline{\boldsymbol{T}}_n\big\|_{\boldsymbol{L}^2(\Omega_c)}$$
$$+ \big\|\nabla\cdot\overline{\boldsymbol{T}}_n\big\|_{L^2(\Omega_c)} + \big\|\nabla\overline{\psi}_n\big\|_{\boldsymbol{L}^2(\Omega_e)}\big) \le C,$$

which concludes that the sequence $(\overline{\boldsymbol{T}}_n, \overline{\psi}_n)$ is bounded in $L^2((0,T), \boldsymbol{V})$. Following the reflexivity of the space, $(\overline{\boldsymbol{T}}_n, \overline{\psi}_n) \rightharpoonup (\boldsymbol{T}, \psi)$ in $L^2((0,T), \boldsymbol{V})$. Then, $\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n \rightharpoonup \boldsymbol{T} + \nabla\psi$ in $L^2((0,T), \boldsymbol{L}^2(\Omega_c))$ and $\nabla\overline{\psi}_n \rightharpoonup \nabla\psi$ in $L^2((0,T), \boldsymbol{L}^2(\Omega_e))$. From Lemma 4.1, we have the boundedness of $\partial_t(\boldsymbol{T}_n + \nabla\psi_n)\big(= \delta(\boldsymbol{T}_i + \nabla\psi_i), t \in (t_{i-1}, t_i]\big)$ in $L^2((0,T), \boldsymbol{L}^2(\Omega_c))$ and $\partial_t(\nabla\psi_n)\big(= \delta\nabla\psi_i, t \in (t_{i-1}, t_i]\big)$ in $L^2((0,T), \boldsymbol{L}^2(\Omega_e))$, which leads to the inequalities

$$\int_0^T \big\|\boldsymbol{T}_n + \nabla\psi_n - (\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n)\big\|^2_{\boldsymbol{L}^2(\Omega_c)} \le C\tau^2\int_0^T \|\partial_t(\boldsymbol{T}_n + \nabla\psi_n)\|^2_{\boldsymbol{L}^2(\Omega_c)} \le C\tau^2,$$

$$\int_0^T \big\|\nabla\psi_n - \nabla\overline{\psi}_n\big\|^2_{\boldsymbol{L}^2(\Omega_e)} \le C\tau^2\int_0^T \|\partial_t(\nabla\psi_n)\|^2_{\boldsymbol{L}^2(\Omega_e)} \le C\tau^2.$$

Thus $\boldsymbol{T}_n + \nabla\psi_n \rightharpoonup \boldsymbol{T} + \nabla\psi$ in $L^2((0,T), \boldsymbol{L}^2(\Omega_c))$ and $\nabla\psi_n \rightharpoonup \nabla\psi$ in $L^2((0,T), \boldsymbol{L}^2(\Omega_e))$.

(b) From Lemma 4.2,

$$\int_0^T \|\overline{\boldsymbol{M}}_n\|^{\frac{1+\alpha}{\alpha}}_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Omega_c)} = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|\boldsymbol{M}(\boldsymbol{T}_i + \nabla\psi_i)\|^{\frac{1+\alpha}{\alpha}}_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Omega_c)}$$
$$\le \sum_{i=1}^n \tau \|\boldsymbol{M}(\boldsymbol{T}_i + \nabla\psi_i)\|^{\frac{1+\alpha}{\alpha}}_{\boldsymbol{L}^2(\Omega_c)} \le C.$$

Due to the reflexivity of the space $L^{\frac{1+\alpha}{\alpha}}\big((0,T), \boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Omega_c)\big)$, we obtain the existence of a weak limit.

(c) For $0 < \alpha < 1$,

$$L^{\frac{1+\alpha}{\alpha}}\Big((0,T), \boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Omega_c)\Big) \subset L^2\big((0,T), \boldsymbol{L}^2(\Omega_c)\big) \subset L^2\big((0,T), \boldsymbol{V}^*\big).$$

This yields $\overline{\boldsymbol{M}}_n \rightharpoonup \boldsymbol{D}$ in $L^2((0,T), \boldsymbol{V}^*)$, according to the point (b). By Lemma 4.3, we get

$$\int_0^T \big\|\boldsymbol{M}_n - \overline{\boldsymbol{M}}_n\big\|^2_{\boldsymbol{V}^*} \le C\tau^2 \int_0^T \|\partial_t \boldsymbol{M}_n\|^2_{\boldsymbol{V}^*} \le C\tau^2.$$

Then the sequence $\boldsymbol{M}_n$ shares the same weak limit with $\overline{\boldsymbol{M}}_n$ in $L^2((0,T), \boldsymbol{V}^*)$.

(d) Applying the Minty-Browder technique (see [9]), we will prove that the limit $\boldsymbol{D}$ of the sequence $\overline{\boldsymbol{M}}_n = \boldsymbol{M}(\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n)$ is equal to $\boldsymbol{M}(\boldsymbol{T} + \nabla\psi)$. Using the equality

$$\begin{aligned}
&\mu_0\Big((\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n) + \overline{\boldsymbol{M}}_n - \big[(\boldsymbol{Q} + \nabla\phi) + \boldsymbol{M}(\boldsymbol{Q} + \nabla\phi)\big], (\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n) \\
&\quad - (\boldsymbol{Q} + \nabla\phi)\Big)_{\boldsymbol{L}^2(\Omega_c)} + \mu_0\big(\nabla\overline{\psi}_n - \nabla\phi, \nabla\overline{\psi}_n - \nabla\phi\big)_{\boldsymbol{L}^2(\Omega_e)} \\
&= \mu_0\Big((\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n) - (\boldsymbol{Q} + \nabla\phi), (\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n) - (\boldsymbol{Q} + \nabla\phi)\Big)_{\boldsymbol{L}^2(\Omega_c)} \\
&\quad + \mu_0\Big(\overline{\boldsymbol{M}}_n - \boldsymbol{M}(\boldsymbol{Q} + \nabla\phi), (\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n) - (\boldsymbol{Q} + \nabla\phi)\Big)_{\boldsymbol{L}^2(\Omega_c)} \\
&\quad + \mu_0\big(\nabla\overline{\psi}_n - \nabla\phi, \nabla\overline{\psi}_n - \nabla\phi\big)_{\boldsymbol{L}^2(\Omega_e)}, \ \forall (\boldsymbol{Q}, \phi) \in H^1\big((0,T), \boldsymbol{L}^2(\Omega_c) \times H^1(\Omega)\big)
\end{aligned}$$

and the monotonicity of $\boldsymbol{M}$ yields

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^t \mu_0\Big((\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n) + \overline{\boldsymbol{M}}_n \\
&\quad - \big[(\boldsymbol{Q} + \nabla\phi) + \boldsymbol{M}(\boldsymbol{Q} + \nabla\phi)\big], (\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n) - (\boldsymbol{Q} + \nabla\phi)\Big)_{\boldsymbol{L}^2(\Omega_c)} \\
&\quad + \lim_{n\to\infty} \int_0^t \mu_0\big(\nabla\overline{\psi}_n - \nabla\phi, \nabla\overline{\psi}_n - \nabla\phi\big)_{\boldsymbol{L}^2(\Omega_e)} \ge 0.
\end{aligned} \tag{15}$$

From the first points in the proof, we already have

$$\int_0^T \Big((\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n) + \overline{\boldsymbol{M}}_n, \boldsymbol{Q} + \nabla\phi\Big)_{\boldsymbol{L}^2(\Omega_c)} \to \int_0^T \Big((\boldsymbol{T} + \nabla\psi) + \boldsymbol{D}, \boldsymbol{Q} + \nabla\phi\Big)_{\boldsymbol{L}^2(\Omega_c)},$$

$$\int_0^T \Big((\boldsymbol{Q} + \nabla\phi) + \boldsymbol{M}(\boldsymbol{Q} + \nabla\phi), \overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n\Big)_{\boldsymbol{L}^2(\Omega_c)}$$
$$\to \int_0^T \Big((\boldsymbol{Q} + \nabla\phi) + \boldsymbol{M}(\boldsymbol{Q} + \nabla\phi), \boldsymbol{T} + \nabla\psi\Big)_{\boldsymbol{L}^2(\Omega_c)},$$

$$\int_0^T \big(\nabla\overline{\psi}_n, \nabla\phi\big)_{\boldsymbol{L}^2(\Omega_e)} \to \int_0^T \big(\nabla\psi, \nabla\phi\big)_{\boldsymbol{L}^2(\Omega_e)}.$$

Next, we will determine the limit of the mixed term in (15). Multiplying (10) by τ and summing for $i = 1, \dots, j$ results in

$$(16)\quad \begin{aligned} &(\boldsymbol{B}_j, \boldsymbol{Q} + \nabla\phi)_{\boldsymbol{L}^2(\Omega_c)} + \left(\frac{1}{\sigma}\sum_{i=1}^{j} \tau\nabla\times\boldsymbol{T}_i, \nabla\times\boldsymbol{Q}\right)_{\boldsymbol{L}^2(\Omega_c)} \\ &+ \left(\frac{1}{\sigma}\sum_{i=1}^{j} \tau\nabla\cdot\boldsymbol{T}_i, \nabla\cdot\boldsymbol{Q}\right)_{L^2(\Omega_c)} + \mu_0(\nabla\psi_j, \nabla\phi)_{\boldsymbol{L}^2(\Omega_e)} \\ =&(\boldsymbol{B}_0, \boldsymbol{Q} + \nabla\phi)_{\boldsymbol{L}^2(\Omega_c)} + \mu_0(\nabla\psi_0, \nabla\phi)_{\boldsymbol{L}^2(\Omega_e)}. \end{aligned}$$

Set $(\boldsymbol{Q}, \phi) = (\boldsymbol{T}, \psi)$ and rewrite this equation in terms of the semidiscrete fields as following:

$$(17)\quad \begin{aligned} &\mu_0\big(\overline{\boldsymbol{T}}_n(s) + \nabla\overline{\psi}_n(s), \boldsymbol{T} + \nabla\psi\big)_{\boldsymbol{L}^2(\Omega_c)} \\ &+ \mu_0\big(\overline{\boldsymbol{M}}_n(s), \boldsymbol{T} + \nabla\psi\big)_{\boldsymbol{L}^2(\Omega_c)} + \mu_0\big(\nabla\overline{\psi}_n(s), \nabla\psi\big)_{\boldsymbol{L}^2(\Omega_e)} \\ =& \mu_0\big(\boldsymbol{T}_0 + \nabla\psi_0, \boldsymbol{T} + \nabla\psi\big)_{\boldsymbol{L}^2(\Omega_c)} + \mu_0(\boldsymbol{M}_0, \boldsymbol{T} + \nabla\psi)_{\boldsymbol{L}^2(\Omega_c)} + \mu_0\big(\nabla\psi_0, \nabla\psi\big)_{\boldsymbol{L}^2(\Omega_e)} \\ &- \left(\int_0^s \frac{1}{\sigma}\nabla\times\overline{\boldsymbol{T}}_n, \nabla\times\boldsymbol{T})\right)_{\boldsymbol{L}^2(\Omega_c)} + \left(\int_{t_j}^s \frac{1}{\sigma}\nabla\times\overline{\boldsymbol{T}}_n, \nabla\times\boldsymbol{T}\right)_{\boldsymbol{L}^2(\Omega_c)} \\ &- \left(\int_0^s \frac{1}{\sigma}\nabla\cdot\overline{\boldsymbol{T}}_n, \nabla\cdot\boldsymbol{T}\right)_{L^2(\Omega_c)} + \left(\int_{t_j}^s \frac{1}{\sigma}\nabla\cdot\overline{\boldsymbol{T}}_n, \nabla\cdot\boldsymbol{T}\right)_{L^2(\Omega_c)} \end{aligned}$$

for $t_{j-1} \le s \le t_j$. From the point (a), $\overline{\boldsymbol{T}}_n \rightharpoonup \boldsymbol{T}$ in $L^2((0,T), \widehat{\boldsymbol{H}}_0^1(\Omega_c))$, which leads to $\nabla\times\overline{\boldsymbol{T}}_n \rightharpoonup \nabla\times\boldsymbol{T}$ in $L^2((0,T), \boldsymbol{L}^2(\Omega_c))$ and $\nabla\cdot\overline{\boldsymbol{T}}_n \rightharpoonup \nabla\cdot\boldsymbol{T}$ in $L^2((0,T), L^2(\Omega_c))$. After time integration of (17) for $s = 0, \dots, t$ and in the limit $n \to \infty$, we obtain

$$(18)\quad \begin{aligned} &\int_0^t \mu_0\big(\boldsymbol{T} + \nabla\psi, \boldsymbol{T} + \nabla\psi\big)_{\boldsymbol{L}^2(\Omega_c)} + \int_0^t \mu_0(\boldsymbol{D}, \boldsymbol{T} + \nabla\psi)_{\boldsymbol{L}^2(\Omega_c)} + \int_0^t \mu_0\big(\nabla\psi, \nabla\psi\big)_{\boldsymbol{L}^2(\Omega_e)} \\ =& \int_0^t \mu_0\Big(\boldsymbol{T}_0 + \nabla\psi_0, \boldsymbol{T} + \nabla\psi\Big)_{\boldsymbol{L}^2(\Omega_c)} + \int_0^t \mu_0(\boldsymbol{M}_0, \boldsymbol{T} + \nabla\psi)_{\boldsymbol{L}^2(\Omega_c)} \\ &+ \int_0^t \mu_0\big(\nabla\psi_0, \nabla\psi\big)_{\boldsymbol{L}^2(\Omega_e)} - \int_0^t \left(\int_0^s \frac{1}{\sigma}\nabla\times\boldsymbol{T}, \nabla\times\boldsymbol{T}\right)_{\boldsymbol{L}^2(\Omega_c)} \\ &- \int_0^t \left(\int_0^s \frac{1}{\sigma}\nabla\cdot\boldsymbol{T}, \nabla\cdot\boldsymbol{T}\right)_{L^2(\Omega_c)}. \end{aligned}$$

Recalling (17) and setting $(\boldsymbol{Q}, \psi) = (\overline{\boldsymbol{T}}_n, \overline{\psi}_n)$ and integrating in time, we have

$$\begin{aligned} &\int_0^t \mu_0\big(\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n, \overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n\big)_{\boldsymbol{L}^2(\Omega_c)} + \int_0^t \mu_0\big(\overline{\boldsymbol{M}}_n, \overline{\boldsymbol{T}}_n \\ &+ \nabla\overline{\psi}_n\big)_{\boldsymbol{L}^2(\Omega_c)} + \int_0^t \mu_0\big(\nabla\overline{\psi}_n, \nabla\overline{\psi}_n\big)_{\boldsymbol{L}^2(\Omega_e)} \\ =& \int_0^t \mu_0\big(\boldsymbol{T}_0 + \nabla\psi_0, \overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n\big)_{\boldsymbol{L}^2(\Omega_c)} \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t \mu_0(\boldsymbol{M}_0, \overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n)_{\boldsymbol{L}^2(\Omega_c)} + \int_0^t \mu_0\big(\nabla\psi_0, \nabla\overline{\psi}_n\big)_{\boldsymbol{L}^2(\Omega_e)} \\
&- \frac{1}{2}\left\| \frac{1}{\sqrt{\sigma}} \int_0^t \nabla\times\overline{\boldsymbol{T}}_n \right\|^2_{\boldsymbol{L}^2(\Omega_c)} + \int_0^t \left(\int_{t_j}^s \frac{1}{\sigma}\nabla\times\overline{\boldsymbol{T}}_n, \nabla\times\overline{\boldsymbol{T}}_n \right)_{\boldsymbol{L}^2(\Omega_c)} \\
&- \frac{1}{2}\left\| \frac{1}{\sqrt{\sigma}} \int_0^t \nabla\cdot\overline{\boldsymbol{T}}_n \right\|^2_{L^2(\Omega_c)} + \int_0^t \left(\int_{t_j}^s \frac{1}{\sigma}\nabla\cdot\overline{\boldsymbol{T}}_n, \nabla\cdot\overline{\boldsymbol{T}}_n \right)_{L^2(\Omega_c)}.
\end{aligned}
$$

The following inequalities come from the result that if $u_n \rightharpoonup u$, then $\lim\limits_{n\to\infty} \|u_n\|^2 \geq \|u\|^2$ in Hilbert spaces:

$$
\lim_{n\to\infty} \left\| \frac{1}{\sqrt{\sigma}} \int_0^t \nabla\times\overline{\boldsymbol{T}}_n \right\|^2_{\boldsymbol{L}^2(\Omega_c)} \geq \left\| \frac{1}{\sqrt{\sigma}} \int_0^t \nabla\times\boldsymbol{T} \right\|^2_{\boldsymbol{L}^2(\Omega_c)},
$$

$$
\lim_{n\to\infty} \left\| \frac{1}{\sqrt{\sigma}} \int_0^t \nabla\cdot\overline{\boldsymbol{T}}_n \right\|^2_{L^2(\Omega_c)} \geq \left\| \frac{1}{\sqrt{\sigma}} \int_0^t \nabla\cdot\boldsymbol{T} \right\|^2_{L^2(\Omega_c)}.
$$

Thus we derive

$$
\begin{aligned}
&\lim_{n\to\infty} \int_0^t \mu_0\big(\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n, \overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n\big)^2_{\boldsymbol{L}^2(\Omega_c)} + \lim_{n\to\infty} \int_0^t \mu_0\big(\overline{\boldsymbol{M}}_n, \overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n\big)^2_{\boldsymbol{L}^2(\Omega_c)} \\
&\quad + \lim_{n\to\infty} \int_0^t \mu_0\big(\nabla\overline{\psi}_n, \nabla\overline{\psi}_n\big)_{\boldsymbol{L}^2(\Omega_e)} \\
&\leq \int_0^t \mu_0\big(\boldsymbol{T}_0 + \nabla\psi_0, \boldsymbol{T} + \nabla\psi\big)^2_{\boldsymbol{L}^2(\Omega_c)} + \int_0^t \mu_0(\boldsymbol{M}_0, \boldsymbol{T} + \nabla\psi)^2_{\boldsymbol{L}^2(\Omega_c)} \\
&\quad + \int_0^t \mu_0\big(\nabla\psi_0, \nabla\psi\big)_{\boldsymbol{L}^2(\Omega_e)} - \frac{1}{2}\left\| \frac{1}{\sqrt{\sigma}} \int_0^t \nabla\times\boldsymbol{T} \right\|^2_{\boldsymbol{L}^2(\Omega_c)} - \frac{1}{2}\left\| \frac{1}{\sqrt{\sigma}} \int_0^t \nabla\cdot\boldsymbol{T} \right\|^2_{L^2(\Omega_c)}.
\end{aligned}
$$

Obviously, we get the same right-hand side as in (18), which leads to

$$
\begin{aligned}
&\lim_{n\to\infty} \int_0^t \mu_0\big(\overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n, \overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n\big)_{\boldsymbol{L}^2(\Omega_c)} + \lim_{n\to\infty} \int_0^t \mu_0\big(\overline{\boldsymbol{M}}_n, \overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n\big)_{\boldsymbol{L}^2(\Omega_c)} \\
&\quad + \lim_{n\to\infty} \int_0^t \mu_0\big(\nabla\overline{\psi}_n, \nabla\overline{\psi}_n\big)_{\boldsymbol{L}^2(\Omega_e)} \\
&\leq \int_0^t \mu_0\big(\boldsymbol{T} + \nabla\psi, \boldsymbol{T} + \nabla\psi\big)_{\boldsymbol{L}^2(\Omega_c)} + \int_0^t \mu_0(\boldsymbol{D}, \boldsymbol{T} + \nabla\psi)_{\boldsymbol{L}^2(\Omega_c)} \\
&\quad + \int_0^t \mu_0\big(\nabla\psi, \nabla\psi\big)_{\boldsymbol{L}^2(\Omega_e)}.
\end{aligned}
$$

Combining (15) and (18), we obtain

$$
\begin{aligned}
&\int_0^t \mu_0\Big((\boldsymbol{T} + \nabla\psi) + \boldsymbol{D} - \big[(\boldsymbol{Q} + \nabla\phi) + \boldsymbol{M}(\boldsymbol{Q} + \nabla\phi)\big], (\boldsymbol{T} + \nabla\psi) - (\boldsymbol{Q} + \nabla\phi)\Big)_{\boldsymbol{L}^2(\Omega_c)} \\
&\quad + \int_0^t \mu_0\Big(\nabla\psi - \nabla\phi, \nabla\psi - \nabla\phi\Big)_{\boldsymbol{L}^2(\Omega_e)} \geq 0.
\end{aligned}
$$

Substituting $\boldsymbol{T} + \nabla\psi + s\boldsymbol{w}$ for $\boldsymbol{Q} + \nabla\phi$ in Ω_c and $\nabla\psi$ for $\nabla\phi$ in Ω_e, we get

$$
\begin{aligned}
&\int_0^t \mu_0\Big(\boldsymbol{D} - \big[s\boldsymbol{w} + \boldsymbol{M}(\boldsymbol{T} + \nabla\psi + s\boldsymbol{w})\big], -s\boldsymbol{w}\Big)_{\boldsymbol{L}^2(\Omega_c)} \\
&= \frac{\mu_0 s^2}{2} \|\boldsymbol{w}(t)\|^2_{\boldsymbol{L}^2(\Omega_c)} - \mu_0 s \int_0^t \Big(\boldsymbol{D} - \boldsymbol{M}(\boldsymbol{T} + \nabla\psi + s\boldsymbol{w}), \boldsymbol{w}\Big)_{\boldsymbol{L}^2(\Omega_c)} \geq 0,
\end{aligned}
$$

where s is any small positive number. Dividing both side by s, taking the limit as $s \to 0$ and replacing $\boldsymbol{w}$ with $\boldsymbol{D} - \boldsymbol{M}(\boldsymbol{T} + \nabla\psi)$, we finally obtain

$$\int_0^t \|\boldsymbol{D} - \boldsymbol{M}(\boldsymbol{T} + \nabla\psi)\|^2_{\boldsymbol{L}^2(\Omega_c)} \leq 0.$$

Therefore, we prove the conclusion of the point (d).

(e) Considering Lemma 4.3 and the reflexivity of $\boldsymbol{L}^2\big((0,T), \boldsymbol{V}^*\big)$, we conclude that there exits a certain function $\boldsymbol{w}$ such that

$$\big\langle \boldsymbol{M}_n(t) - \boldsymbol{M}_n(0), (\boldsymbol{Q}, \phi)\big\rangle = \int_0^t \langle \partial_t \boldsymbol{M}_n, (\boldsymbol{Q}, \phi)\rangle \to \langle \boldsymbol{w}, (\boldsymbol{Q}, \phi)\rangle\,,$$

for all $(\boldsymbol{Q}, \phi) \in \boldsymbol{V}$ and a.e. $t \in (0,T)$. Using the convergence result of the point (c), we have

$$\begin{aligned}
&\lim_{n\to\infty} \big\langle \boldsymbol{M}_n(t), (\boldsymbol{Q}, \phi)\big\rangle - \lim_{n\to\infty} \big\langle \boldsymbol{M}_n(0), (\boldsymbol{Q}, \phi)\big\rangle \\
&= \big\langle \boldsymbol{M}(\boldsymbol{T} + \nabla\psi)(t), (\boldsymbol{Q}, \phi)\big\rangle - \big\langle \boldsymbol{M}(\boldsymbol{T} + \nabla\psi)(0), (\boldsymbol{Q}, \phi)\big\rangle \\
&= \Big\langle \int_0^t \partial_t \boldsymbol{M}(\boldsymbol{T} + \nabla\psi), (\boldsymbol{Q}, \phi)\Big\rangle,
\end{aligned}$$

which means that $\boldsymbol{w} = \int_0^t \partial_t \boldsymbol{M}(\boldsymbol{T} + \nabla\psi)$ in Ω_c. This yields that

$$\begin{aligned}
&\big\langle \boldsymbol{M}_n(t), (\boldsymbol{Q}, \phi)\big\rangle \to \big\langle \boldsymbol{M}_n(0), (\boldsymbol{Q}, \phi)\big\rangle + \big\langle \boldsymbol{w}, (\boldsymbol{Q}, \phi)\big\rangle \\
&= \big\langle \boldsymbol{M}(\boldsymbol{T} + \nabla\psi), (\boldsymbol{Q}, \psi)\big\rangle \quad \text{for all } t \in (0,T].
\end{aligned}$$

Therefore, we conclude the existence of $\boldsymbol{M}(\boldsymbol{T} + \nabla\psi)$ in $\boldsymbol{V}^*$ everywhere in interval $(0,T]$.

(f) Integrating (14) in time, we get

$$\begin{aligned}
&\big(\boldsymbol{B}_n(t), \boldsymbol{Q} + \nabla\psi\big) - \big(\boldsymbol{B}_n(0), \boldsymbol{Q} + \nabla\phi\big)_{\boldsymbol{L}^2(\Omega_c)} + \int_0^t \left(\frac{1}{\sigma}\nabla \times \overline{\boldsymbol{T}}_n, \nabla \times \boldsymbol{Q}\right)_{\boldsymbol{L}^2(\Omega_c)} \\
&+ \int_0^t \left(\frac{1}{\sigma}\nabla \cdot \overline{\boldsymbol{T}}_n, \nabla \cdot \boldsymbol{Q}\right)_{L^2(\Omega_c)} + \mu_0\big(\nabla\psi_n(t), \nabla\phi\big)_{\boldsymbol{L}^2(\Omega_e)} \\
&- \mu_0\big(\nabla\psi_n(0), \nabla\phi\big)_{\boldsymbol{L}^2(\Omega_e)} = 0
\end{aligned}$$

for all $(\boldsymbol{Q}, \phi) \in \boldsymbol{V}$. Taking the limit as $n \to \infty$ and summarizing all the results from (a) and (c), we obtain

$$\begin{aligned}
&\int_0^t \big(\partial_t \boldsymbol{B}(\boldsymbol{T} + \nabla\psi), \boldsymbol{Q} + \nabla\phi\big)_{\boldsymbol{L}^2(\Omega_c)} + \int_0^t \Big(\partial_t(\mu_0\nabla\psi), \nabla\phi\Big)_{\boldsymbol{L}^2(\Omega_e)} \\
&+ \int_0^t \left(\frac{1}{\sigma}\nabla \times \boldsymbol{T}, \nabla \times \boldsymbol{Q}\right)_{\boldsymbol{L}^2(\Omega_c)} + \int_0^t \left(\frac{1}{\sigma}\nabla \cdot \boldsymbol{T}, \nabla \cdot \boldsymbol{Q}\right)_{L^2(\Omega_c)} = 0,\ \forall (\boldsymbol{Q}, \phi) \in \boldsymbol{V}.
\end{aligned}$$

Finally, differentiating the identity in time leads to the result of (f) on almost everywhere in $(0,T]$. Thus, we conclude that the function pair $(\boldsymbol{T}, \psi)$ solves the continuous problem (6) on a.e. $t \in (0,T]$. This completes the proof of the whole theorem.

□

To obtain the additional regularity for the limit $(\boldsymbol{T}, \psi)$, we need the following lemma (see [8]).

Lemma 5.1. Let the set of functions $u_n : (0,T) \to V$ be equibounded and equicontinous. Then there exists $u \in C\big([0,T], V\big)$ such that up to subsequences $u_n(t) \rightharpoonup u(t)$ in V for all $t \in (0,T)$.

Theorem 5.2. The limit $(\boldsymbol{T},\psi)$ from Theorem 5.1 satisfies

$$(\boldsymbol{T},\psi)\in C\big([0,T],\boldsymbol{V}\big).$$

Proof. From Lemma 2.3, Lemma 4.1 and Lemma 4.2, we have

$$\begin{aligned}\|(\boldsymbol{T}_n,\psi_n)\|_{\boldsymbol{V}} \le C(&\|\boldsymbol{T}_n+\nabla\psi_n\|_{\boldsymbol{L}^2(\Omega_c)}+\|\nabla\times\boldsymbol{T}_n\|_{\boldsymbol{L}^2(\Omega_c)}\\ &+\|\nabla\cdot\boldsymbol{T}_n\|_{L^2(\Omega_c)}+\|\nabla\psi_n\|_{\boldsymbol{L}^2(\Omega_e)})\le C,\end{aligned}$$

i.e. the sequence $(\boldsymbol{T}_n,\psi_n)$ is equibounded in $\boldsymbol{V}$. Moreover, we obtain by using Lemma 4.1 that

$$\begin{aligned}&\big\|\big(\boldsymbol{T}_n(t_2),\psi_n(t_2)\big)-\big(\boldsymbol{T}_n(t_1),\psi_n(t_1)\big)\big\|_{\boldsymbol{V}}\\
&\le C\big(\,\big\|\boldsymbol{T}_n(t_2)+\nabla\psi_n(t_2)-\big(\boldsymbol{T}_n(t_1)+\nabla\psi_n(t_1)\big)\big\|_{\boldsymbol{L}^2(\Omega_c)}\\
&\quad+\big\|\nabla\times\big(\boldsymbol{T}_n(t_2)-\boldsymbol{T}_n(t_1)\big)\big\|_{\boldsymbol{L}^2(\Omega_c)}\\
&\quad+\big\|\nabla\cdot\big(\boldsymbol{T}_n(t_2)-\boldsymbol{T}_n(t_1)\big)\big\|_{L^2(\Omega_c)}+\|\nabla\psi_n(t_2)-\nabla\psi_n(t_1)\|_{\boldsymbol{L}^2(\Omega_e)}\big)\\
&\le C\int_{t_1}^{t_2}\big(\,\|\partial_t(\boldsymbol{T}_n+\nabla\psi_n)\|_{\boldsymbol{L}^2(\Omega_c)}+\|\nabla\times\partial_t\boldsymbol{T}_n\|_{\boldsymbol{L}^2(\Omega_c)}\\
&\quad+\|\nabla\cdot\partial_t\boldsymbol{T}_n\|_{L^2(\Omega_c)}+\|\partial_t(\nabla\psi_n)\|_{\boldsymbol{L}^2(\Omega_e)}\big)\\
&\le C\left(\sqrt{\int_0^T\|\partial_t(\boldsymbol{T}_n+\nabla\psi_n)\|^2_{\boldsymbol{L}^2(\Omega_c)}}+\sqrt{\int_0^T\|\nabla\times\partial_t\boldsymbol{T}_n\|^2_{\boldsymbol{L}^2(\Omega_c)}}\right.\\
&\quad\left.+\sqrt{\int_0^T\|\nabla\cdot\partial_t\boldsymbol{T}_n\|^2_{L^2(\Omega_c)}}+\sqrt{\int_0^T\|\partial_t(\nabla\psi_n)\|^2_{L^2(\Omega_e)}}\right)|t_2-t_1|^{1/2}\\
&\le C\left(\sqrt{\sum_{i=1}^n\tau\mu_0\,\|\delta(\boldsymbol{T}_i+\nabla\psi_i)\|^2_{\boldsymbol{L}^2(\Omega_c)}}+\sqrt{\sum_{i=1}^n\left\|\frac{\sqrt{\tau}}{\sqrt{\sigma}}\nabla\times\delta\boldsymbol{T}_i\right\|^2_{\boldsymbol{L}^2(\Omega_c)}}\right.\\
&\quad\left.+\sqrt{\sum_{i=1}^n\left\|\frac{\sqrt{\tau}}{\sqrt{\sigma}}\nabla\cdot\delta\boldsymbol{T}_i\right\|^2_{L^2(\Omega_c)}}+\sqrt{\sum_{i=1}^n\tau\mu_0\,\|\delta(\nabla\psi_i)\|^2_{\boldsymbol{L}^2(\Omega_e)}}\right)|t_2-t_1|^{1/2}\\
&\le C(\tau)\,|t_2-t_1|^{1/2},\end{aligned}$$

which states that the sequence $(\boldsymbol{T}_n,\psi_n)$ is equicontinuous. From Lemma 5.1, we conclude that $(\boldsymbol{T},\psi)\in C\big([0,T],\boldsymbol{V}\big)$. □

6. Error estimate

Theorem 6.1. Let $(\boldsymbol{T}_0,\psi_0)\in\boldsymbol{V}$. Then there exist positive constants C and τ_0 such that

$$\begin{aligned}&\int_0^t\mu_0\,\big\|\overline{\boldsymbol{T}}_n+\nabla\overline{\psi}_n-(\boldsymbol{T}+\nabla\psi)\big\|^2_{\boldsymbol{L}^2(\Omega_c)}+\int_0^t\mu_0\,\big\|\nabla\overline{\psi}_n-\nabla\psi\big\|^2_{\boldsymbol{L}^2(\Omega_e)}\\
&\quad+\left\|\int_0^t\frac{1}{\sqrt{\sigma}}\nabla\times(\overline{\boldsymbol{T}}_n-\boldsymbol{T})\right\|^2_{\boldsymbol{L}^2(\Omega_c)}+\left\|\int_0^t\frac{1}{\sqrt{\sigma}}\nabla\cdot(\overline{\boldsymbol{T}}_n-\boldsymbol{T})\right\|^2_{L^2(\Omega_c)}\\
&\le C\tau\qquad\text{for all } 0<\tau<\tau_0 \text{ and } 0<t\le T.\end{aligned}$$

Proof. Subtract (14) from (9), integrate in time over $(0,t)$ and put $(\boldsymbol{Q},\psi)=(\overline{\boldsymbol{T}}_n-\boldsymbol{T},\overline{\psi}_n-\psi)$. This leaves

$$
\begin{aligned}
&\Big(\overline{\boldsymbol{B}}_n(t)-\boldsymbol{B}\big(\boldsymbol{T}(t)+\nabla\psi(t)\big),\overline{\boldsymbol{T}}_n+\nabla\overline{\psi}_n-(\boldsymbol{T}+\nabla\psi)\Big)_{\boldsymbol{L}^2(\Omega_c)}\\
&+\mu_0\left(\nabla\overline{\psi}_n-\nabla\psi,\nabla\overline{\psi}_n-\nabla\psi\right)_{\boldsymbol{L}^2(\Omega_e)}\\
&+\bigg(\int_0^t\frac{1}{\sigma}\nabla\times(\overline{\boldsymbol{T}}_n-\boldsymbol{T}),\nabla\times(\overline{\boldsymbol{T}}_n-\boldsymbol{T})\bigg)_{\boldsymbol{L}^2(\Omega_c)}\\
&+\bigg(\int_0^t\frac{1}{\sigma}\nabla\cdot(\overline{\boldsymbol{T}}_n-\boldsymbol{T}),\nabla\cdot(\overline{\boldsymbol{T}}_n-\boldsymbol{T})\bigg)_{\boldsymbol{L}^2(\Omega_c)}\\
=&\Big(\overline{\boldsymbol{B}}_n-\boldsymbol{B}_n,\overline{\boldsymbol{T}}_n+\nabla\overline{\psi}_n-(\boldsymbol{T}+\nabla\psi)\Big)_{\boldsymbol{L}^2(\Omega_c)}+\mu_0\left(\nabla\overline{\psi}_n-\nabla\psi_n,\nabla\overline{\psi}_n-\nabla\psi\right)_{\boldsymbol{L}^2(\Omega_e)}.
\end{aligned}
$$

After another time integration, we obtain

$$
\begin{aligned}
&\int_0^t\mu_0\left\|\overline{\boldsymbol{T}}_n+\nabla\overline{\psi}_n-(\boldsymbol{T}+\nabla\psi)\right\|^2_{\boldsymbol{L}^2(\Omega_c)}\\
&+\int_0^t\mu_0\Big(\overline{\boldsymbol{M}}_n-\boldsymbol{M}(\boldsymbol{T}+\nabla\psi),\overline{\boldsymbol{T}}_n+\nabla\overline{\psi}_n-(\boldsymbol{T}+\nabla\psi)\Big)_{\boldsymbol{L}^2(\Omega_c)}\\
&+\int_0^t\mu_0\left\|\nabla\overline{\psi}_n-\nabla\psi\right\|^2_{\boldsymbol{L}^2(\Omega_e)}+\frac{1}{2}\left\|\int_0^t\frac{1}{\sqrt{\sigma}}\nabla\times(\overline{\boldsymbol{T}}_n-\boldsymbol{T})\right\|^2_{\boldsymbol{L}^2(\Omega_c)}\\
&+\frac{1}{2}\left\|\int_0^t\frac{1}{\sqrt{\sigma}}\nabla\cdot(\overline{\boldsymbol{T}}_n-\boldsymbol{T})\right\|^2_{L^2(\Omega_c)}\\
\lesssim&\int_0^t\mu_0\Big(\overline{\boldsymbol{T}}_n+\nabla\overline{\psi}_n-(\boldsymbol{T}_n+\nabla\psi_n),\overline{\boldsymbol{T}}_n+\nabla\overline{\psi}_n-(\boldsymbol{T}+\nabla\psi)\Big)_{\boldsymbol{L}^2(\Omega_c)}\\
&+\int_0^t\mu_0\Big(\overline{\boldsymbol{M}}_n-\boldsymbol{M}_n,\overline{\boldsymbol{T}}_n+\nabla\overline{\psi}_n-(\boldsymbol{T}+\nabla\psi)\Big)_{\boldsymbol{L}^2(\Omega_c)}\\
&+\int_0^t\mu_0\big(\nabla\overline{\psi}_n-\nabla\psi_n,\nabla\overline{\psi}_n-\nabla\psi\big)_{\boldsymbol{L}^2(\Omega_e)}.
\end{aligned}
$$

Then the final estimate follows from the monotonicity of $\boldsymbol{M}$ together with the following bounds

$$
\begin{aligned}
&\int_0^t\mu_0\Big(\overline{\boldsymbol{T}}_n+\nabla\overline{\psi}_n-(\boldsymbol{T}_n+\nabla\psi_n),\overline{\boldsymbol{T}}_n+\nabla\overline{\psi}_n-(\boldsymbol{T}+\nabla\psi)\Big)_{\boldsymbol{L}^2(\Omega_c)}\\
&\le C\tau\sqrt{\int_0^t\|\partial_t(\boldsymbol{T}_n+\nabla\psi_n)\|^2_{\boldsymbol{L}^2(\Omega_c)}}\sqrt{\int_0^t\left\|\overline{\boldsymbol{T}}_n+\nabla\psi_n\right\|^2_{\boldsymbol{L}^2(\Omega_c)}+\int_0^t\|\boldsymbol{T}+\nabla\psi\|^2_{\boldsymbol{L}^2(\Omega_c)}}\\
&\le C\tau,
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^t\mu_0\Big(\overline{\boldsymbol{M}}_n-\boldsymbol{M}_n,\overline{\boldsymbol{T}}_n+\nabla\overline{\psi}_n-(\boldsymbol{T}+\nabla\psi)\Big)_{\boldsymbol{L}^2(\Omega_c)}\\
&\le C\tau\sqrt{\int_0^t\|\partial_t\boldsymbol{M}_n\|^2_{\boldsymbol{V}^*}}\sqrt{\int_0^t\left\|(\overline{\boldsymbol{T}}_n,\overline{\psi}_n)\right\|^2_{\boldsymbol{V}}+\int_0^t\|(\boldsymbol{T},\psi)\|^2_{\boldsymbol{V}}}\le C\tau,
\end{aligned}
$$

$$\int_0^t \mu_0\Big(\nabla\overline{\psi}_n - \nabla\psi_n, \nabla\overline{\psi}_n - \nabla\psi\Big)^2_{\boldsymbol{L}^2(\Omega_e)}$$
$$\le C\tau\sqrt{\int_0^t \|\partial_t(\nabla\psi_n)\|^2_{\boldsymbol{L}^2(\Omega_e)}}\sqrt{\int_0^t \|\nabla\psi_n\|^2_{\boldsymbol{L}^2(\Omega_e)} + \int_0^t \|\nabla\psi\|^2_{\boldsymbol{L}^2(\Omega_e)}} \le C\tau.$$

□

Remark 6.1. Using Theorem 6.1 and denoting $\overline{\boldsymbol{H}}_n = \overline{\boldsymbol{T}}_n + \nabla\overline{\psi}_n$ in Ω_c and $\overline{\boldsymbol{H}}_n = \nabla\overline{\psi}_n$ in Ω_e, we conclude that

$$\mu_0\int_0^T \|\overline{\boldsymbol{H}}_n - \boldsymbol{H}\|^2_{\boldsymbol{L}^2(\Omega)} + \left\|\int_0^T \frac{1}{\sqrt{\sigma}}\nabla\times(\overline{\boldsymbol{H}}_n - \boldsymbol{H})\right\|^2_{\boldsymbol{L}^2(\Omega_c)} \le C\tau. \tag{19}$$

7. Full discretization

Now we analyse the fully discretized problem. Let $\mathcal{T}^h$ be a standard tetrahedral triangulation of Ω with the mesh size h, which subdivides Ω_c and Ω_e into the union of tetrahedra. We define

$$\boldsymbol{Y}_h^0 := \Big\{\boldsymbol{Q} \in \widehat{\boldsymbol{H}}_0^1(\Omega_c) : \boldsymbol{Q}|_{\mathcal{K}} \in (\mathcal{P}_1)^3,\ \forall\mathcal{K}\in\mathcal{T}^h\Big\},$$
$$W_h := \big\{\psi \in H^1(\Omega) : \psi|_{\mathcal{K}} \in \mathcal{P}_1,\ \forall\mathcal{K}\in\mathcal{T}^h\big\},$$

where $\mathcal{P}_1$ is the space of linear polynomials. For simplicity, let $\boldsymbol{V}_h := \boldsymbol{Y}_h^0 \times W_h$.

We denote Π^h as the Lagrange interpolation operator for $\mathcal{T}^h$. Thus the interpolation error estimates (see [6]) are given by

$$\begin{cases} \|u - \Pi^h u\|_{L^2(\mathcal{D})} \le Ch\|u\|_{H^1(\mathcal{D})}, \\ \|u - \Pi^h u\|_{L^2(\mathcal{D})} + h\|u - \Pi^h u\|_{H^1(\mathcal{D})} \le Ch^2\|u\|_{H^2(\mathcal{D})} \end{cases} \tag{20}$$

for any $u \in H^2(\mathcal{D})$, where $\mathcal{D} = \Omega$, Ω_c or Ω_e.

With this finite element setting we can formulate the fully discretized problem as follows: Find $(\boldsymbol{T}_i^h, \psi_i^h) \in \boldsymbol{V}_h$, $1 \le i \le n$, such that

$$\begin{cases} \mu_0\big(\delta(\boldsymbol{T}_i^h + \nabla\psi_i^h), \boldsymbol{Q}^h + \nabla\phi^h\big)_{\boldsymbol{L}^2(\Omega_c)} \\ \quad + \mu_0\Big(\delta\big(\big|\boldsymbol{T}_i^h + \nabla\psi_i^h\big|^{\alpha-1}(\boldsymbol{T}_i^h + \nabla\psi_i^h)\big), \boldsymbol{Q}^h + \nabla\phi^h\Big)_{\boldsymbol{L}^2(\Omega_c)} \\ \quad + \left(\frac{1}{\sigma}\nabla\times\boldsymbol{T}_i^h, \nabla\times\boldsymbol{Q}^h\right)_{\boldsymbol{L}^2(\Omega_c)} + \left(\frac{1}{\sigma}\nabla\cdot\boldsymbol{T}_i^h, \nabla\cdot\boldsymbol{Q}^h\right)_{L^2(\Omega_c)} \\ \quad + \big(\mu_0\delta(\nabla\psi_i^h), \nabla\phi^h\big)_{\boldsymbol{L}^2(\Omega_e)} = 0,\ \forall(\boldsymbol{Q}^h, \phi^h) \in \boldsymbol{V}_h, \\ \boldsymbol{T}_0^h = \Pi^h\boldsymbol{T}_0,\ \psi_0^h = \Pi^h\psi_0. \end{cases} \tag{21}$$

In the next theorem we will prove the solvability of the discretized problem.

Theorem 7.1. The fully discretized system (21) has a solution $(\boldsymbol{T}_i^h, \psi_i^h) \in \boldsymbol{V}_h$ on every time step.

Proof. On every time step we need to solve the following problem to find $(\boldsymbol{T}^h, \psi^h) \in \boldsymbol{V}_h$:

$$\frac{\mu_0}{\tau}(\boldsymbol{T}^h + \nabla\psi^h, \boldsymbol{Q}^h)_{\boldsymbol{L}^2(\Omega_c)} + \frac{\mu_0}{\tau}\left(\big|\boldsymbol{T}^h + \nabla\psi^h\big|^{\alpha-1}(\boldsymbol{T}^h + \nabla\psi^h), \boldsymbol{Q}^h\right)_{\boldsymbol{L}^2(\Omega_c)}$$
$$+ \left(\frac{1}{\sigma}\nabla\times\boldsymbol{T}^h, \nabla\times\boldsymbol{Q}^h\right)_{\boldsymbol{L}^2(\Omega_c)} + \left(\frac{1}{\sigma}\nabla\cdot\boldsymbol{T}^h, \nabla\cdot\boldsymbol{Q}^h\right)_{L^2(\Omega_c)} = \big(\boldsymbol{f}, \boldsymbol{Q}^h\big)_{\boldsymbol{L}^2(\Omega_c)},$$

$$\frac{\mu_0}{\tau}(\boldsymbol{T}^h+\nabla\psi^h,\nabla\phi^h)_{\boldsymbol{L}^2(\Omega_c)}+\frac{\mu_0}{\tau}\left(\left|\boldsymbol{T}^h+\nabla\psi^h\right|^{\alpha-1}(\boldsymbol{T}^h+\nabla\psi^h),\nabla\phi^h\right)_{\boldsymbol{L}^2(\Omega_c)}$$
$$+\frac{\mu_0}{\tau}(\nabla\psi^h,\nabla\phi^h)_{\boldsymbol{L}^2(\Omega_e)}=\left(\boldsymbol{f},\nabla\phi^h\right)_{\boldsymbol{L}^2(\Omega)},\qquad\forall(\boldsymbol{Q}^h,\phi^h)\in\boldsymbol{V}_h.$$

Let $\varphi_1,\cdots,\varphi_{k_1}$ be the nodal basis functions of $\mathcal{T}^h(\Omega_c)$, $\varphi_{k_1+1},\cdots,\varphi_{k_1+k_2}$ be the nodal basis functions of $\mathcal{T}^h(\Gamma_c)$ and $\varphi_{k_1+k_2+1},\cdots,\varphi_{k_1+k_2+k_3}$ be the nodal basis functions of $\mathcal{T}^h(\Omega_e\cup\Gamma)$. Let $k=k_1+k_2+k_3$. If we write $\boldsymbol{T}^h=\sum_{i=1}^{k_1+k_2}\boldsymbol{\alpha}_i\varphi_i$ with $\boldsymbol{\alpha_i}=(\alpha_{ix},\alpha_{iy},\alpha_{iz})$ and $\phi=\sum_{i=1}^{k}\beta_i\varphi_i$, then we need to solve this problem to find $\boldsymbol{\gamma}=\left(\boldsymbol{\alpha}_1,\beta_1,\ldots,\boldsymbol{\alpha}_{k_1+k_2},\beta_{k_1+k_2},\beta_{k_1+k_2+1},\ldots,\beta_k\right)$. We define the nonlinear operators $\mathcal{L}$ from $\mathbb{R}^{3(k_1+k_2)+k}$ to $\mathbb{R}^{3(k_1+k_2)}$: $\mathcal{L}(\boldsymbol{\gamma})=\left(\mathcal{L}_1(\boldsymbol{\gamma}),\cdots,\mathcal{L}_{k_1+k_2}(\boldsymbol{\gamma})\right)$ with

$$\begin{aligned}\mathcal{L}_j(\boldsymbol{\gamma})=&\frac{\mu_0}{\tau}\left(\sum_{i=1}^{k_1+k_2}(\boldsymbol{\alpha}_i\varphi_i+\beta_i\nabla\varphi_i),\varphi_j\right)_{L^2(\Omega_c)}\\&+\frac{\mu_0}{\tau}\left(\left|\sum_{i=1}^{k_1+k_2}(\boldsymbol{\alpha}_i\varphi_i+\beta_i\nabla\varphi_i)\right|^{\alpha-1}\sum_{i=1}^{k_1+k_2}(\boldsymbol{\alpha}_i\varphi_i+\beta_i\nabla\varphi_i),\varphi_j\right)_{L^2(\Omega_c)}\\&+\left(\frac{1}{\sigma}\nabla\times\sum_{i=1}^{k_1+k_2}\boldsymbol{\alpha}_i\varphi_i,\boldsymbol{M_j}\right)_{\boldsymbol{L}^2(\Omega_c)}+\left(\frac{1}{\sigma}\nabla\cdot\sum_{i=1}^{k_1+k_2}\boldsymbol{\alpha}_i\varphi_i,\nabla\varphi_j\right)_{L^2(\Omega_c)}\\&-(\boldsymbol{f},\varphi_j)_{L^2(\Omega_c)},\ j=1,\ldots,k_1+k_2,\end{aligned}$$

where

$$\boldsymbol{M}_j=\begin{pmatrix}0&\frac{\partial\varphi_j}{\partial z}&-\frac{\partial\varphi_j}{\partial y}\\-\frac{\partial\varphi_j}{\partial z}&0&\frac{\partial\varphi_j}{\partial x}\\\frac{\partial\varphi_j}{\partial y}&-\frac{\partial\varphi_j}{\partial x}&0\end{pmatrix}$$

is a 3×3 coefficient matrix such that $\boldsymbol{\alpha}_j\boldsymbol{M}_j=\nabla\times(\boldsymbol{\alpha}_j\varphi_j)$, and ℓ from $\mathbb{R}^{3(k_1+k_2)+k}$ to $\mathbb{R}^k$: $\ell(\gamma)=\left(\ell_1(\boldsymbol{\gamma}),\cdots,\ell_k(\boldsymbol{\gamma})\right)$ with

$$\begin{aligned}\ell_j(\boldsymbol{\gamma})=&\frac{\mu_0}{\tau}\left(\sum_{i=1}^{k_1+k_2}(\boldsymbol{\alpha}_i\varphi_i+\beta_i\nabla\varphi_i),\nabla\varphi_j\right)_{\boldsymbol{L}^2(\Omega_c)}\\&+\frac{\mu_0}{\tau}\left(\left|\sum_{i=1}^{k_1+k_2}(\boldsymbol{\alpha}_i\varphi_i+\beta_i\nabla\varphi_i)\right|^{\alpha-1}\sum_{i=1}^{k_1+k_2}(\boldsymbol{\alpha}_i\varphi_i+\beta_i\nabla\varphi_i),\nabla\varphi_j\right)_{\boldsymbol{L}^2(\Omega_c)}\\&+\frac{\mu_0}{\tau}\left(\sum_{i=k_1+1}^{k}\beta_i\nabla\varphi_i,\nabla\varphi_j\right)_{\boldsymbol{L}^2(\Omega_e)}-(\boldsymbol{f},\nabla\varphi_j)_{\boldsymbol{L}^2(\Omega)},\ j=1,\ldots,k.\end{aligned}$$

Denote the operator $\boldsymbol{L}=(\mathcal{L}_1,\ell_1,\cdots,\mathcal{L}_{k_1+k_2},\ell_{k_1+k_2},\ell_{k_1+k_2+1},\cdots,\ell_k)$. The problem is now reduced to solving the nonlinear algebraic equation $\boldsymbol{L}(\boldsymbol{\gamma})=\boldsymbol{0}$. Then we obtain by using Lemma 2.3 that

$$\begin{aligned}&\boldsymbol{L}(\boldsymbol{\gamma})\cdot\boldsymbol{\gamma}\\&\geq\frac{\mu_0}{\tau}\left\|\sum_{i=1}^{k_1+k_2}(\boldsymbol{\alpha}_i\varphi_i+\beta_i\nabla\varphi_i)\right\|^2_{\boldsymbol{L}^2(\Omega_c)}+\frac{1}{\sigma_{max}}\left\|\sum_{i=1}^{k_1+k_2}\nabla\times(\boldsymbol{\alpha}_i\varphi_i)\right\|^2_{\boldsymbol{L}^2(\Omega_c)}\\&\quad+\frac{1}{\sigma_{max}}\left\|\sum_{i=1}^{k_1+k_2}\nabla\cdot(\boldsymbol{\alpha}_i\varphi_i)\right\|^2_{L^2(\Omega_c)}+\frac{\mu_0}{\tau}\left\|\sum_{i=1}^{k_1+k_2}(\boldsymbol{\alpha}_i\varphi_i+\beta_i\nabla\varphi_i)\right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Omega_c)}\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\mu_0}{\tau}\left\|\sum_{i=k_1+1}^{k}\beta_i\nabla\varphi_i\right\|^2_{\boldsymbol{L}^2(\Omega_e)} - \left(\boldsymbol{f},\ \sum_{i=1}^{k_1+k_2}(\boldsymbol{\alpha}_i\varphi_i+\beta_i\nabla\varphi_i)\right)_{\boldsymbol{L}^2(\Omega_c)} \\
&- \left(\boldsymbol{f}, \sum_{i=k_1}^{k}\beta_i\nabla\varphi_i\right)_{\boldsymbol{L}^2(\Omega_e)} \\
\geq{}& C\left(\left\|\sum_{i=1}^{k_1+k_2}(\boldsymbol{\alpha}_i\varphi_i+\beta_i\nabla\varphi_i)\right\|^2_{\boldsymbol{L}^2(\Omega_c)} + \left\|\sum_{i=1}^{k_1+k_2}\nabla\times(\boldsymbol{\alpha}_i\varphi_i)\right\|^2_{\boldsymbol{L}^2(\Omega_c)}\right. \\
&\left. + \left\|\sum_{i=1}^{k_1+k_2}\nabla\cdot(\boldsymbol{\alpha}_i\varphi_i)\right\|^2_{L^2(\Omega_c)} + \left\|\sum_{i=k_1+1}^{k}\beta_i\nabla\varphi_i\right\|^2_{\boldsymbol{L}^2(\Omega_e)}\right) - C_2\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} \\
\geq{}& C\left(\left\|\sum_{i=1}^{k_1+k_2}\boldsymbol{\alpha}_i\varphi_i\right\|^2_{\boldsymbol{H}^1(\Omega_c)} + \left\|\sum_{i=1}^{k}\beta_i\varphi_i\right\|^2_{H^1(\Omega)}\right) - C_2\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} \\
\geq{}& C\left(\left\|\sum_{i=1}^{k_1+k_2}\boldsymbol{\alpha}_i\varphi_i\right\|^2_{\boldsymbol{L}^2(\Omega_c)} + \left\|\sum_{i=1}^{k}\beta_i\varphi_i\right\|^2_{L^2(\Omega)}\right) - C_2\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} \\
\geq{}& C\Big(|\boldsymbol{\gamma}|^2 - \|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}\Big).
\end{aligned}
$$

Applying Lemma 18.2 in [27], we conclude that the equation $\boldsymbol{L}(\boldsymbol{\gamma}) = \boldsymbol{0}$ has at least one solution in the set $\left\{\boldsymbol{\gamma}\in\mathbb{R}^{3(k_1+k_2)+k} :\ |\boldsymbol{\gamma}|\leq r\right\}$ if $r > \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}$. □

It is easy to prove that the potential fields $\boldsymbol{T}_i^h$ and ψ_i^h satisfy the estimates similar to Lemma 4.1- 4.2. But the dual space estimate in Lemma 4.3 will not be valid for the fully discrete solutions since the weak formulation is formulated for test functions in the subspace $\boldsymbol{V}_h$ of $\boldsymbol{V}$ only. However, we still have

$$
\sum_{i=1}^{j}\tau\left|\left(\delta\boldsymbol{M}(\boldsymbol{T}_i^h+\nabla\psi_i^h), \boldsymbol{Q}_i^h+\nabla\phi_i^h\right)_{\boldsymbol{L}^2(\Omega_c)}\right| \leq C\sum_{i=1}^{j}\tau\left\|(\boldsymbol{Q}_i^h,\phi_i^h)\right\|^2_{\boldsymbol{V}}, \quad \forall(\boldsymbol{Q}_i^h,\phi_i^h)\in\boldsymbol{V}_h,\ j=1,\cdots,n. \tag{22}
$$

With the field pair $(\boldsymbol{T}_i^h,\psi_i^h)$, we can construct analogous interpolating fields as in Section 5

$$
\begin{cases}\boldsymbol{T}_n^h(t) = \boldsymbol{T}_{i-1}^h + (t-t_{i-1})\delta\boldsymbol{T}_i^h, & t\in(t_{i-1},t_i],\\ \boldsymbol{T}_n^h(0) = \Pi^h\boldsymbol{T}_0,\end{cases} \qquad \begin{cases}\overline{\boldsymbol{T}}_n^h(t) = \boldsymbol{T}_i^h, & t\in(t_{i-1},t_i],\\ \overline{\boldsymbol{T}}_n^h(0) = \Pi^h\boldsymbol{T}_0,\end{cases}
$$

$$
\begin{cases}\psi_n^h(t) = \psi_{i-1}^h + (t-t_{i-1})\delta\psi_i^h, & t\in(t_{i-1},t_i],\\ \psi_n^h(0) = \Pi^h\psi_0,\end{cases} \qquad \begin{cases}\overline{\psi}_n^h(t) = \psi_i^h, & t\in(t_{i-1},t_i],\\ \overline{\psi}_n^h(0) = \Pi^h\psi_0,\end{cases}
$$

$$
\begin{cases}\boldsymbol{M}_n^h(t) = \boldsymbol{M}_{i-1}^h + (t-t_{i-1})\delta\boldsymbol{M}_i^h, & t\in(t_{i-1},t_i],\\ \boldsymbol{M}_n^h(0) = \left|\Pi^h\boldsymbol{T}_0+\nabla\Pi^h\psi_0\right|^{\alpha-1}(\Pi^h\boldsymbol{T}_0+\nabla\Pi^h\psi_0),\end{cases}
$$

$$
\begin{cases}\overline{\boldsymbol{M}}_n^h(t) = \boldsymbol{M}_i^h, & t\in(t_{i-1},t_i],\\ \overline{\boldsymbol{M}}_n^h(0) = \left|\Pi^h\boldsymbol{T}_0+\nabla\Pi^h\psi_0\right|^{\alpha-1}(\Pi^h\boldsymbol{T}_0+\nabla\Pi^h\psi_0),\end{cases}
$$

$$
\begin{cases}\boldsymbol{B}_n^h(t) = \boldsymbol{B}_{i-1}^h + (t-t_{i-1})\delta\boldsymbol{B}_i^h, & t\in(t_{i-1},t_i],\\ \boldsymbol{B}_n^h(0) = \mu_0\big(\Pi^h\boldsymbol{T}_0+\nabla\Pi^h\psi_0\big) + \mu_0\left|\Pi^h\boldsymbol{T}_0+\nabla\Pi^h\psi_0\right|^{\alpha-1}(\Pi^h\boldsymbol{T}_0+\nabla\Pi^h\psi_0),\end{cases}
$$

$$\begin{cases}\overline{\boldsymbol{B}}_n^h(t) = \boldsymbol{B}_i^h, \quad t \in (t_{i-1}, t_i], \\ \overline{\boldsymbol{B}}_n^h(0) = \mu_0\big(\Pi^h \boldsymbol{T}_0 + \nabla \Pi^h \psi_0\big) + \big|\Pi^h \boldsymbol{T}_0 + \nabla \Pi^h \psi_0\big|^{\alpha-1} (\Pi^h \boldsymbol{T}_0 + \nabla \Pi^h \psi_0).\end{cases}$$

Note that $\overline{\boldsymbol{M}}_n^h(t) = \boldsymbol{M}(\overline{\boldsymbol{T}}_n^h + \nabla \overline{\psi}_n^h)$ and $\overline{\boldsymbol{B}}_n^h(t) = \boldsymbol{B}(\overline{\boldsymbol{T}}_n^h + \nabla \overline{\psi}_n^h)$. Thus the weak form (21) can be reformulated as follows:

(23)

$$\big(\partial_t \boldsymbol{B}_n^h, \boldsymbol{Q}^h + \nabla \phi^h\big)_{\boldsymbol{L}^2(\Omega_c)} + \left(\frac{1}{\sigma} \nabla \times \overline{\boldsymbol{T}}_n^h, \nabla \times \boldsymbol{Q}^h\right)_{\boldsymbol{L}^2(\Omega_c)} + \left(\frac{1}{\sigma} \nabla \cdot \overline{\boldsymbol{T}}_n^h, \nabla \cdot \boldsymbol{Q}^h\right)_{\boldsymbol{L}^2(\Omega_c)}$$
$$+ \big(\mu_0 \partial_t (\nabla \psi_n^h), \nabla \phi^h\big)_{\boldsymbol{L}^2(\Omega_e)} = 0, \ \forall (\boldsymbol{Q}^h, \phi^h) \in \boldsymbol{V}_h.$$

For these fully discrete potential fields, it is clear that the similar convergence results as Theorem 5.1 can be obtained as both τ and h go to zero. Moreover, we have the following result on the error estimate. Here, $a \lesssim b$ denotes $a \leq Cb$.

Theorem 7.2. Let the weak solution $(\boldsymbol{T}, \psi)$ and the initial conditions $(\boldsymbol{T}_0, \psi_0)$ of the problem (9) satisfy

$$(\boldsymbol{T}, \psi) \in L^2\big((0,T), \boldsymbol{H}^2(\Omega_c) \times H^2(\Omega)\big), \quad (\boldsymbol{T}_0, \psi_0) \in \boldsymbol{H}^1(\Omega_c) \times H^2(\Omega).$$

Then the fully discretized problem satisfies the following error estimate:

$$\begin{aligned} &mu_0 \int_0^T \Big\|\overline{\boldsymbol{T}}_n^h + \nabla \overline{\psi}_n^h - (\boldsymbol{T} + \nabla \psi)\Big\|^2_{\boldsymbol{L}^2(\Omega_c)} + \mu_0 \int_0^T \Big\|\nabla \overline{\psi}_n^h - \nabla \psi\Big\|^2_{\boldsymbol{L}^2(\Omega_e)} \\ &+ \left\|\int_0^T \frac{1}{\sqrt{\sigma}} \nabla \times (\overline{\boldsymbol{T}}_n^h - \boldsymbol{T})\right\|^2_{\boldsymbol{L}^2(\Omega_c)} + \left\|\int_0^T \frac{1}{\sqrt{\sigma}} \nabla \cdot (\overline{\boldsymbol{T}}_n^h - \boldsymbol{T})\right\|^2_{L^2(\Omega_c)} \\ &\lesssim \tau + h^{min\{1,2\alpha\}}. \end{aligned} \tag{24}$$

Proof. We now subtract (23) from (9), integrate in time and put $(\boldsymbol{Q}^h, \phi^h) = (\overline{\boldsymbol{T}}_n^h - \Pi^h \boldsymbol{T}, \overline{\psi}_n^h - \Pi^h \psi)$, leaving

$$\begin{aligned} &\left(\overline{\boldsymbol{B}}_n^h - \boldsymbol{B}(\boldsymbol{T} + \nabla \psi), \overline{\boldsymbol{T}}_n^h + \nabla \overline{\psi}_n^h - (\Pi^h \boldsymbol{T} + \nabla \Pi^h \psi)\right)_{\boldsymbol{L}^2(\Omega_c)} \\ &+ \left(\int_0^t \frac{1}{\sigma} \nabla \times (\overline{\boldsymbol{T}}_n^h - \boldsymbol{T}), \nabla \times (\overline{\boldsymbol{T}}_n^h - \Pi^h \boldsymbol{T})\right)_{\boldsymbol{L}^2(\Omega_c)} \\ &+ \left(\int_0^t \frac{1}{\sigma} \nabla \cdot (\overline{\boldsymbol{T}}_n^h - \boldsymbol{T}), \nabla \cdot (\overline{\boldsymbol{T}}_n^h - \Pi^h \boldsymbol{T})\right)_{\boldsymbol{L}^2(\Omega_c)} \\ &+ \mu_0 \big(\nabla \overline{\psi}_n^h - \nabla \psi, \nabla \overline{\psi}_n^h - \nabla \Pi^h \psi\big)_{\boldsymbol{L}^2(\Omega_e)} \\ = &\left(\overline{\boldsymbol{B}}_n^h - \boldsymbol{B}_n^h, \overline{\boldsymbol{T}}_n^h + \nabla \overline{\psi}_n^h - (\Pi^h \boldsymbol{T} + \nabla \Pi^h \psi)\right)_{\boldsymbol{L}^2(\Omega_c)} \\ &+ \mu_0 \big(\nabla \overline{\psi}_n^h - \nabla \psi_n^h, \nabla \overline{\psi}_n^h - \nabla \Pi^h \psi\big)_{\boldsymbol{L}^2(\Omega_e)} \\ &+ \mu_0 \left(\Pi^h \boldsymbol{T}_0 + \nabla \Pi^h \psi_0 - (\boldsymbol{T}_0 + \nabla \psi_0), \overline{\boldsymbol{T}}_n^h + \nabla \overline{\psi}_n^h - (\Pi^h \boldsymbol{T} + \nabla \Pi^h \psi)\right)_{\boldsymbol{L}^2(\Omega_c)} \\ &+ \mu_0 \left(\boldsymbol{M}\big(\Pi^h \boldsymbol{T}_0 + \nabla \Pi^h \psi_0\big) - \boldsymbol{M}(\boldsymbol{T}_0 + \nabla \psi_0), \overline{\boldsymbol{T}}_n^h + \nabla \overline{\psi}_n^h - (\Pi^h \boldsymbol{T} + \nabla \Pi^h \psi)\right)_{\boldsymbol{L}^2(\Omega_c)} \\ &+ \mu_0 \left(\nabla \Pi^h \psi_0 - \nabla \psi_0, \nabla \overline{\psi}_n^h - \nabla \Pi^h \psi\right)_{\boldsymbol{L}^2(\Omega_e)}. \end{aligned}$$

Applying the monotonicity of $\boldsymbol{M}$

$$\left(\overline{\boldsymbol{M}}_n^h - \boldsymbol{M}(\boldsymbol{T}+\nabla\psi), \overline{\boldsymbol{T}}_n^h + \nabla\overline{\psi}_n^h - (\boldsymbol{T}+\nabla\psi)\right)_{\boldsymbol{L}^2(\Omega_c)} \geq 0$$

yields

$$\begin{aligned}
&\mu_0 \int_0^\xi \left(\overline{\boldsymbol{T}}_n^h + \nabla\overline{\psi}_n^h - (\boldsymbol{T}+\nabla\psi), \overline{\boldsymbol{T}}_n^h + \nabla\overline{\psi}_n^h - (\boldsymbol{T}+\nabla\psi)\right)_{\boldsymbol{L}^2(\Omega_c)} \\
&+ \int_0^\xi \left(\int_0^t \frac{1}{\sigma}\nabla\times(\overline{\boldsymbol{T}}_n^h - \boldsymbol{T}), \nabla\times(\overline{\boldsymbol{T}}_n^h - \boldsymbol{T})\right)_{\boldsymbol{L}^2(\Omega_c)} \\
&+ \int_0^\xi \left(\int_0^t \frac{1}{\sigma}\nabla\cdot(\overline{\boldsymbol{T}}_n^h - \boldsymbol{T}), \nabla\cdot(\overline{\boldsymbol{T}}_n^h - \boldsymbol{T})\right)_{\boldsymbol{L}^2(\Omega_c)} \\
&+ \mu_0 \int_0^\xi (\nabla\overline{\psi}_n^h - \nabla\psi, \nabla\overline{\psi}_n^h - \nabla\psi)_{\boldsymbol{L}^2(\Omega_e)} \\
\leq\ &\mu_0 \int_0^\xi \left(\overline{\boldsymbol{T}}_n^h + \nabla\overline{\psi}_n^h - (\boldsymbol{T}_n^h + \nabla\psi_n^h), \overline{\boldsymbol{T}}_n^h + \nabla\overline{\psi}_n^h - (\Pi^h\boldsymbol{T} + \nabla\Pi^h\psi)\right)_{\boldsymbol{L}^2(\Omega_c)} \\
&+ \mu_0 \int_0^\xi \left(\overline{\boldsymbol{M}}_n^h - \boldsymbol{M}_n^h, \overline{\boldsymbol{T}}_n^h + \nabla\overline{\psi}_n^h - (\Pi^h\boldsymbol{T} + \nabla\Pi^h\psi)\right)_{\boldsymbol{L}^2(\Omega_c)} \\
&+ \mu_0 \int_0^\xi \left(\nabla\overline{\psi}_n^h - \nabla\psi_n^h, \nabla\overline{\psi}_n^h - \nabla\Pi^h\psi\right)_{\boldsymbol{L}^2(\Omega_e)} \\
&+ \mu_0 \int_0^\xi \left(\Pi^h\boldsymbol{T}_0 + \nabla\Pi^h\psi_0 - (\boldsymbol{T}_0 + \nabla\psi_0), \overline{\boldsymbol{T}}_n^h + \nabla\overline{\psi}_n^h - (\Pi^h\boldsymbol{T} + \nabla\Pi^h\psi)\right)_{\boldsymbol{L}^2(\Omega_c)} \\
&+ \mu_0 \int_0^\xi \left(\boldsymbol{M}(\Pi^h\boldsymbol{T}_0 + \nabla\Pi^h\psi_0) - \boldsymbol{M}(\boldsymbol{T}_0 + \nabla\psi_0), \overline{\boldsymbol{T}}_n^h + \nabla\overline{\psi}_n^h - (\Pi^h\boldsymbol{T} + \nabla\Pi^h\psi)\right)_{\boldsymbol{L}^2(\Omega_c)} \\
&+ \mu_0 \int_0^\xi \left(\nabla\Pi^h\psi_0 - \nabla\psi_0, \nabla\overline{\psi}_n^h - \nabla\Pi^h\psi\right)_{\boldsymbol{L}^2(\Omega_e)} \\
&+ \mu_0 \int_0^\xi \left(\overline{\boldsymbol{T}}_n^h + \nabla\overline{\psi}_n^h - (\boldsymbol{T}+\nabla\psi), \Pi^h\boldsymbol{T} + \nabla\Pi^h\psi - (\boldsymbol{T}+\nabla\psi)\right)_{\boldsymbol{L}^2(\Omega_c)} \\
&+ \mu_0 \int_0^\xi \left(\overline{\boldsymbol{M}}_n^h - \boldsymbol{M}(\boldsymbol{T}+\nabla\psi), \Pi^h\boldsymbol{T} + \nabla\Pi^h\psi - (\boldsymbol{T}+\nabla\psi)\right)_{\boldsymbol{L}^2(\Omega_c)} \\
&+ \int_0^\xi \left(\int_0^t \frac{1}{\sigma}\nabla\times(\overline{\boldsymbol{T}}_n^h - \boldsymbol{T}), \nabla\times(\Pi^h\boldsymbol{T} - \boldsymbol{T})\right)_{\boldsymbol{L}^2(\Omega_c)} \\
&+ \int_0^\xi \left(\int_0^t \frac{1}{\sigma}\nabla\cdot(\overline{\boldsymbol{T}}_n^h - \boldsymbol{T}), \nabla\cdot(\Pi^h\boldsymbol{T} - \boldsymbol{T})\right)_{\boldsymbol{L}^2(\Omega_c)} \\
&+ \mu_0 \int_0^\xi \left(\nabla\overline{\psi}_n^h - \nabla\psi, \nabla\Pi^h\psi - \nabla\psi\right)_{\boldsymbol{L}^2(\Omega_e)} \\
:=\ &\sum_{i=1}^{11} S_i.
\end{aligned} \tag{25}$$

We can now bound each of the eleven terms on the right-hand side as follows. We use Cauchy's inequality, Young's inequality, Lemma 2.3 and Lemma 4.1-4.2 to obtain

$$S_1 \leq \mu_0 \sqrt{\int_0^\xi \left\| (\overline{\boldsymbol{T}}_n^h + \nabla \overline{\psi}_n^h) - (\boldsymbol{T}_n^h + \nabla \psi_n^h) \right\|^2_{\boldsymbol{L}^2(\Omega_c)}}$$

$$\cdot \sqrt{\int_0^\xi \|\Pi^h \boldsymbol{T} + \nabla \Pi^h \psi\|^2_{\boldsymbol{L}^2(\Omega_c)} + \int_0^\xi \left\| \overline{\boldsymbol{T}}_n^h + \nabla \overline{\psi}_n^h \right\|^2_{\boldsymbol{L}^2(\Omega_c)}} \lesssim \tau,$$

$$S_2 \leq \mu_0 \tau \int_0^\xi \left| \left\langle \partial_t \boldsymbol{M}_n^h, \overline{\boldsymbol{T}}_n^h + \nabla \overline{\psi}_n^h - (\Pi^h \boldsymbol{T} + \nabla \Pi^h \psi) \right\rangle \right| \Big(\text{using (22)}\Big)$$

$$\lesssim \tau \sqrt{\int_0^\xi \|(\Pi^h \boldsymbol{T}, \Pi^h \psi)\|^2_V + \int_0^\xi \left\| (\overline{\boldsymbol{T}}_n^h, \overline{\psi}_n^h) \right\|^2_V} \lesssim \tau,$$

$$S_3 \leq \mu_0 \sqrt{\int_0^\xi \left\| \nabla \overline{\psi}_n^h - \nabla \psi_n^h) \right\|^2_{\boldsymbol{L}^2(\Omega_e)}} \cdot \sqrt{\int_0^\xi \left\| \nabla \overline{\psi}_n^h \right\|^2_{\boldsymbol{L}^2(\Omega_e)} + \int_0^\xi \|\nabla \Pi^h \psi\|^2_{\boldsymbol{L}^2(\Omega_e)}}$$

$$\lesssim \tau \sqrt{\int_0^\xi \|\partial_t (\nabla \psi_n^h)\|^2_{\boldsymbol{L}^2(\Omega_e)}} \lesssim \tau,$$

$$S_4 \leq \mu_0 \sqrt{\int_0^\xi \|\Pi^h \boldsymbol{T}_0 + \nabla \Pi^h \psi_0 - (\boldsymbol{T}_0 + \nabla \psi_0)\|^2_{\boldsymbol{L}^2(\Omega_c)}}$$

$$\times \sqrt{\int_0^\xi \left\| \overline{\boldsymbol{T}}_n^h + \nabla \overline{\psi}_n^h - (\Pi^h \boldsymbol{T} + \nabla \Pi^h \psi) \right\|^2_{\boldsymbol{L}^2(\Omega_c)}}$$

$$\lesssim \left\| \Pi^h \boldsymbol{T}_0 + \nabla \Pi^h \psi_0 - (\boldsymbol{T}_0 + \nabla \psi_0) \right\|^2_{\boldsymbol{L}^2(\Omega_c)} + \int_0^\xi \left\| \overline{\boldsymbol{T}}_n^h + \nabla \overline{\psi}_n^h - (\boldsymbol{T} + \nabla \psi) \right\|^2_{\boldsymbol{L}^2(\Omega_c)}$$

$$+ \int_0^\xi \left\| \boldsymbol{T} + \nabla \psi - (\Pi^h \boldsymbol{T} + \nabla \Pi^h \psi) \right\|^2_{\boldsymbol{L}^2(\Omega_c)}$$

$$\lesssim h^2 \Big(\|\boldsymbol{T}_0\|^2_{\boldsymbol{H}^1(\Omega_c)} + \|\psi_0\|^2_{H^2(\Omega_c)} \Big) + \int_0^\xi \left\| \overline{\boldsymbol{T}}_n^h + \nabla \overline{\psi}_n^h - (\boldsymbol{T} + \nabla \psi) \right\|^2_{\boldsymbol{L}^2(\Omega_c)}$$

$$+ h^2 \int_0^T \left(\|\boldsymbol{T}\|^2_{\boldsymbol{H}^1(\Omega_c)} + \|\psi\|^2_{H^2(\Omega_c)} \right),$$

$$S_5 \leq \mu_0 \sqrt{\int_0^\xi \|\Pi^h \boldsymbol{T}_0 + \nabla \Pi^h \psi_0 - (\boldsymbol{T}_0 + \nabla \psi_0)\|^{2\alpha}_{\boldsymbol{L}^2(\Omega_c)}}$$

$$\times \sqrt{\int_0^\xi \left\| \overline{\boldsymbol{T}}_n^h + \nabla \overline{\psi}_n^h - (\Pi^h \boldsymbol{T} + \nabla \Pi^h \psi) \right\|^2_{\boldsymbol{L}^2(\Omega_c)}}$$

$$\Big(\text{using the } \alpha\text{-Hölder continuity of the function } \boldsymbol{M}(s)\Big)$$

$$\lesssim \left\| \Pi^h \boldsymbol{T}_0 + \nabla \Pi^h \psi_0 - (\boldsymbol{T}_0 + \nabla \psi_0) \right\|^{2\alpha}_{\boldsymbol{L}^2(\Omega_c)} + \int_0^\xi \left\| \overline{\boldsymbol{T}}_n^h + \nabla \overline{\psi}_n^h - (\boldsymbol{T} + \nabla \psi) \right\|^2_{\boldsymbol{L}^2(\Omega_c)}$$

$$+ \int_0^\xi \left\| \boldsymbol{T} + \nabla \psi - (\Pi^h \boldsymbol{T} + \nabla \Pi^h \psi) \right\|^2_{\boldsymbol{L}^2(\Omega_c)}$$

$$\lesssim h^{2\alpha} \Big(\|\boldsymbol{T}_0\|^{2\alpha}_{\boldsymbol{H}^1(\Omega_c)} + \|\psi_0\|^{2\alpha}_{H^2(\Omega_c)} \Big) + \int_0^\xi \left\| \overline{\boldsymbol{T}}_n^h + \nabla \overline{\psi}_n^h - (\boldsymbol{T} + \nabla \psi) \right\|^2_{\boldsymbol{L}^2(\Omega_c)}$$

$$+ h^2 \int_0^T \left(\|\boldsymbol{T}\|^2_{\boldsymbol{H}^1(\Omega_c)} + \|\psi\|^2_{H^2(\Omega_c)} \right),$$

$$\begin{aligned} S_6 &\le \mu_0 \sqrt{\|\nabla \Pi^h \psi_0 - \nabla \psi_0\|^2_{\boldsymbol{L}^2(\Omega_e)}} \cdot \sqrt{\int_0^\xi \left\|\nabla \overline{\psi}^h_n - \nabla \Pi^h \psi\right\|^2_{\boldsymbol{L}^2(\Omega_e)}} \\ &\lesssim \left\|\nabla \Pi^h \psi_0 - \nabla \psi_0\right\|^2_{\boldsymbol{L}^2(\Omega_e)} + \int_0^\xi \left\|\nabla \overline{\psi}^h_n - \nabla \psi\right\|^2_{\boldsymbol{L}^2(\Omega_e)} + \int_0^\xi \left\|\nabla \psi - \nabla \Pi^h \psi\right\|^2_{\boldsymbol{L}^2(\Omega_e)} \\ &\lesssim h^2 \|\psi_0\|^2_{H^2(\Omega_e)} + \int_0^\xi \left\|\nabla \overline{\psi}^h_n - \nabla \psi\right\|^2_{\boldsymbol{L}^2(\Omega_e)} + h^2 \int_0^T \|\psi\|^2_{H^2(\Omega_e)}, \end{aligned}$$

$$\begin{aligned} S_7 &\lesssim \int_0^\xi \left\|\overline{\boldsymbol{T}}^h_n + \nabla \overline{\psi}^h_n - (\boldsymbol{T} + \nabla \psi)\right\|^2_{\boldsymbol{L}^2(\Omega_c)} + \int_0^\xi \left\|\Pi^h \boldsymbol{T} + \nabla \Pi^h \psi - (\boldsymbol{T} + \nabla \psi)\right\|^2_{\boldsymbol{L}^2(\Omega_c)} \\ &\lesssim \int_0^\xi \left\|\overline{\boldsymbol{T}}^h_n + \nabla \overline{\psi}^h_n - (\boldsymbol{T} + \nabla \psi)\right\|^2_{\boldsymbol{L}^2(\Omega_c)} + h^2 \int_0^T \left(\|\boldsymbol{T}\|^2_{\boldsymbol{H}^1(\Omega_c)} + \|\psi\|^2_{H^2(\Omega_c)} \right), \end{aligned}$$

$$\begin{aligned} S_8 &\le \mu_0 \int_0^\xi \left\|\Pi^h \boldsymbol{T} + \nabla \Pi^h \psi - (\boldsymbol{T} + \nabla \psi)\right\|_{\boldsymbol{L}^{1+\alpha}(\Omega_c)} \cdot \left\|\overline{\boldsymbol{M}}^h_n - \boldsymbol{M}(\boldsymbol{T} + \nabla \psi)\right\|_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Omega_c)} \\ &\le \mu_0 \left(\int_0^\xi \left\|\Pi^h \boldsymbol{T} + \nabla \Pi^h \psi - (\boldsymbol{T} + \nabla \psi)\right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Omega_c)} \right)^{\frac{1}{1+\alpha}} \\ &\quad \cdot \left(\int_0^\xi \left\|\overline{\boldsymbol{M}}^h_n - \boldsymbol{M}(\boldsymbol{T} + \nabla \psi)\right\|^{\frac{1+\alpha}{\alpha}}_{\boldsymbol{L}^{\frac{1+\alpha}{\alpha}}(\Omega_c)} \right)^{\frac{\alpha}{1+\alpha}} \\ &\le \mu_0 \left(\int_0^\xi \left\|\Pi^h \boldsymbol{T} + \nabla \Pi^h \psi - (\boldsymbol{T} + \nabla \psi)\right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Omega_c)} \right)^{\frac{1}{1+\alpha}} \\ &\quad \cdot \left(\sum_{i=1}^n \tau \left\|\boldsymbol{T}^h_i + \nabla \psi^h_i\right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Omega_c)} + \int_0^\xi \|\boldsymbol{T} + \nabla \psi\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Omega_c)} \right)^{\frac{\alpha}{1+\alpha}} \\ &\lesssim \left(\int_0^\xi \left\|\Pi^h \boldsymbol{T} + \nabla \Pi^h \psi - (\boldsymbol{T} + \nabla \psi)\right\|^{1+\alpha}_{\boldsymbol{L}^{1+\alpha}(\Omega_c)} \right)^{\frac{1}{1+\alpha}} \\ &\lesssim \left\|\Pi^h \boldsymbol{T} + \nabla \Pi^h \psi - (\boldsymbol{T} + \nabla \psi)\right\|_{L^2((0,T),\boldsymbol{L}^2(\Omega_c))} \lesssim h, \end{aligned}$$

$$\begin{aligned} S_9 &\lesssim \int_0^\xi \left\|\int_0^t \frac{1}{\sqrt{\sigma}} \nabla \times (\overline{\boldsymbol{T}}^h_n - \boldsymbol{T})\right\|^2_{\boldsymbol{L}^2(\Omega_c)} + \left\|\int_0^\xi \frac{1}{\sqrt{\sigma}} \nabla \times (\Pi^h \boldsymbol{T} - \boldsymbol{T})\right\|^2_{\boldsymbol{L}^2(\Omega_c)} \\ &\lesssim \int_0^\xi \left\|\int_0^t \frac{1}{\sqrt{\sigma}} \nabla \times (\overline{\boldsymbol{T}}^h_n - \boldsymbol{T})\right\|^2_{\boldsymbol{L}^2(\Omega_c)} + h^2 \int_0^T \|\boldsymbol{T}\|^2_{\boldsymbol{H}^2(\Omega_c)}, \end{aligned}$$

$$\begin{aligned} S_{10} &\lesssim \int_0^\xi \left\|\int_0^t \frac{1}{\sqrt{\sigma}} \nabla \cdot (\overline{\boldsymbol{T}}^h_n - \boldsymbol{T})\right\|^2_{L^2(\Omega_c)} + \left\|\int_0^\xi \frac{1}{\sqrt{\sigma}} \nabla \cdot (\Pi^h \boldsymbol{T} - \boldsymbol{T})\right\|^2_{L^2(\Omega_c)} \\ &\lesssim \int_0^\xi \left\|\int_0^t \frac{1}{\sqrt{\sigma}} \nabla \cdot (\overline{\boldsymbol{T}}^h_n - \boldsymbol{T})\right\|^2_{L^2(\Omega_c)} + h^2 \int_0^T \|\boldsymbol{T}\|^2_{\boldsymbol{H}^2(\Omega_c)}, \end{aligned}$$

$$S_{11} \lesssim \int_0^\xi \left\|\nabla\overline{\psi}_n^h - \nabla\psi\right\|^2_{\boldsymbol{L}^2(\Omega_e)} + \int_0^\xi \left\|\nabla\Pi^h\psi - \nabla\psi\right\|^2_{\boldsymbol{L}^2(\Omega_e)}$$
$$\lesssim \int_0^\xi \left\|\nabla\overline{\psi}_n^h - \nabla\psi\right\|^2_{\boldsymbol{L}^2(\Omega_e)} + h^2 \int_0^T \|\boldsymbol{T}\|^2_{\boldsymbol{H}^2(\Omega_e)}.$$

The left-hand side of (25) is

$$\mu_0 \int_0^\xi \left\|\overline{\boldsymbol{T}}_n^h + \nabla\overline{\psi}_n^h - (\boldsymbol{T} + \nabla\psi)\right\|^2_{\boldsymbol{L}^2(\Omega_c)} + \mu_0 \int_0^\xi \left\|\nabla\overline{\psi}_n^h - \nabla\psi\right\|^2_{\boldsymbol{L}^2(\Omega_e)}$$
$$+ \left\|\int_0^\xi \frac{1}{\sqrt{\sigma}}\nabla\times(\overline{\boldsymbol{T}}_n^h - \boldsymbol{T})\right\|^2_{\boldsymbol{L}^2(\Omega_c)} + \left\|\int_0^\xi \frac{1}{\sqrt{\sigma}}\nabla\cdot(\overline{\boldsymbol{T}}_n^h - \boldsymbol{T})\right\|^2_{L^2(\Omega_c)}.$$

We finally apply Grönwall's inequality to arrive at the error estimate for the fully discrete problem. □

Remark 7.1. Using Theorem 7.2 and denoting $\overline{\boldsymbol{H}}_n^h = \overline{\boldsymbol{T}}_n^h + \nabla\overline{\psi}_n^h$ in Ω_c and $\overline{\boldsymbol{H}}_n^h = \nabla\overline{\psi}_n^h$ in Ω_e, we conclude that

(26)
$$\mu_0 \int_0^T \left\|\overline{\boldsymbol{H}}_n^h - \boldsymbol{H}\right\|^2_{\boldsymbol{L}^2(\Omega)} + \left\|\int_0^T \frac{1}{\sqrt{\sigma}}\nabla\times(\overline{\boldsymbol{H}}_n^h - \boldsymbol{H})\right\|^2_{\boldsymbol{L}^2(\Omega_c)} \lesssim \tau + h^{min\{1,2\alpha\}}.$$

8. Numerical Experiments

The purpose of this section is to validate Theorem 7.2 with three numerical tests. For simplicity, we consider the system (3) with nonlinear magnetic permeability on the entire computational domain $[0,1]^3 \times [0,T]$ and solve it numerically using the proposed scheme (21) with $\sigma = 1.0$ and $\mu_0 = 1.0$. The nonlinearity is solved iteratively by the Newton's method. In order to compute the approximation error, we introduce a field $\boldsymbol{H} = \boldsymbol{H}_{ex}$ and calculate the source function $\boldsymbol{g}$ by the following equation

$$\partial_t \boldsymbol{B}(\boldsymbol{H}_{ex}) + \nabla\times\nabla\times\boldsymbol{H}_{ex} = \boldsymbol{g}. \tag{27}$$

Then we decompose $\boldsymbol{H}_{ex} = \boldsymbol{T}_{ex} + \nabla\psi_{ex}$ with $\nabla\cdot\boldsymbol{T}_{ex} = 0$ and solve the following problem to get $\boldsymbol{T}$ and ψ:

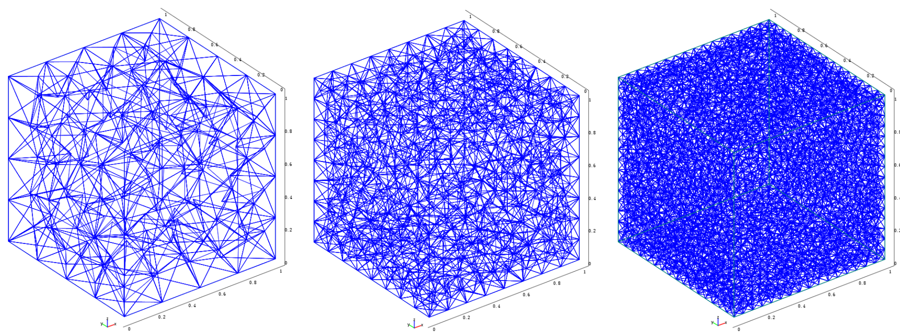
$$\begin{cases} \dfrac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{T}+\nabla\psi) + \nabla\times\Big(\dfrac{1}{\sigma}\nabla\times\boldsymbol{T}\Big) - \nabla\Big(\dfrac{1}{\sigma}\nabla\cdot\boldsymbol{T}\Big) = \boldsymbol{g} & \text{in } \Omega, \\ \nabla\cdot\dfrac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{T}+\nabla\psi) = \nabla\cdot\boldsymbol{g} & \text{in } \Omega, \\ \boldsymbol{T}\times\boldsymbol{n} = \boldsymbol{T}_{ex}\times\boldsymbol{n},\ \dfrac{1}{\sigma}\nabla\cdot\boldsymbol{T} = 0,\ \psi = \psi_{ex} & \text{on } \partial\Omega, \\ \boldsymbol{T}(\boldsymbol{x},0) = \boldsymbol{T}_{ex}(\boldsymbol{x},0),\ \psi(\boldsymbol{x},0) = \psi_{ex}(\boldsymbol{x},0) & \boldsymbol{x}\in\Omega, \end{cases} \tag{28}$$

by the fully discrete scheme (21) using the function $\boldsymbol{g}$ as a source and compute the error $\big(\boldsymbol{T}_n^h(t) + \nabla\psi_n^h(t)\big) - \boldsymbol{H}_{ex}$ in the appropriate function spaces. The theoretical results remain valid for the nonhomogeneous boundary condition provided a suitable lift function exists. Space discretization is based on the linear polynomial elements as implemented in the COMSOL software in accordance with the theory from Section 7. For every experiment the time step τ is varied as 2^{-n} with

$n = 2, \ldots, n_{\max}$ and the error

$$err := \int_0^T \left\| \overline{\boldsymbol{T}}_n^h(t) + \nabla \overline{\psi}_n^h(t) - \boldsymbol{H}_{ex}(t) \right\|^2_{\boldsymbol{L}^2(\Omega)} \tag{29}$$

is computed. This procedure is repeated for the three meshes in Figure 1 in order to evaluate the dependence of the error on the mesh parameter h. The mesh in Figure 1(b and c) are obtained by refinement (by splitting) of the first mesh.



(a) $h = 1/4$ (b) $h = 1/8$ (c) $h = 1/16$

FIGURE 1. Tetrahedral meshes on which the problem is solved, constructed in COMSOL. (a) 943 tetrahedra; (b) 8434 tetrahedra; (c) 70418 tetrahedra.

Experiment 8.1. We choose the following exact solution:

$$\boldsymbol{H}_{ex}(\boldsymbol{x}, t) = \left(t^3 + \frac{5}{8}\right) \begin{pmatrix} z - y + 1 \\ x - z + 1 \\ y - x + 1 \end{pmatrix}$$

with $t \in [0, 1]$. Decompose the field $\boldsymbol{H}_{ex}$ to obtain

$$\boldsymbol{T}_{ex}(\boldsymbol{x}, t) = \left(t^3 + \frac{5}{8}\right) \begin{pmatrix} z - y \\ x - z \\ y - x \end{pmatrix}$$

and $\psi_{ex}(\boldsymbol{x}, t) = (t^3 + 5/8)(x + y + z)$ with the form $\mathcal{P}_1$ in the spatial part, which is exactly represented by the finite element basis functions. Thus the error will only depend on the time step τ (the calculations can be done on the coarse mesh). We calculate the error err in (29) as a function of the time step τ for several values of the exponent α of the power law. The results are presented in Figure 2, where $\log_2(err)$ is plotted as a function of $\log_2 \tau$. Figure 2 gives the convergence rate for $0 < \alpha < 1$. We obtain the almost same convergence rate for all values of α, approaching the estimated $\mathcal{O}(\tau)$ rate at the previous iteration steps.

Experiment 8.2. As a second experiment we take

$$\boldsymbol{H}_{ex}(\boldsymbol{x}, t) = \left(t^2 + \frac{7}{8}\right) \begin{pmatrix} \frac{1}{3} cos(x) + \frac{1}{5} \\ \frac{1}{2} sin(y) + \frac{1}{10} \\ 2cos(z) - 1 \end{pmatrix}, \ t \in [0, 1]$$

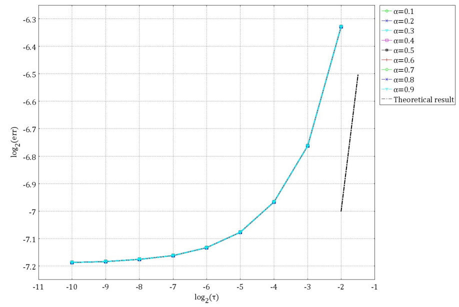
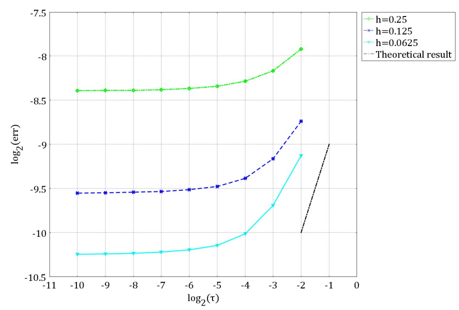
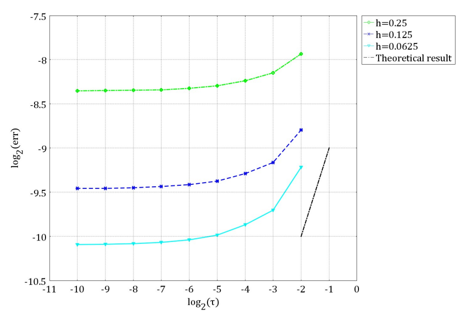


FIGURE 2. Convergence rate for Experiment 1 with α between 0 and 1. The slope of the solid line (theoretical result) equals to 1.



(a) $\alpha = 0.5$



(b) $\alpha = 0.8$

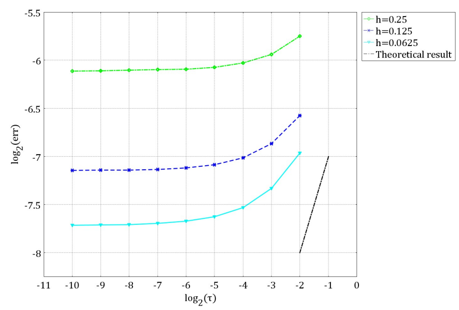
FIGURE 3. Convergence rate for Experiment 2. The slope of the solid line (theoretical result) equals to 1.

as an exact solution. Now the mesh parameter will play an important role in the error and the calculations are repeated for the three meshes in Figure 1. The results for $\alpha = 0.5$ and $\alpha = 0.8$ are shown in Figure 3 respectively. The error decreases with the time step, but eventually attains a constant value only dependent on the mesh parameter. For large time steps, the error due to space discretization is much smaller than the error of time discretization and reducing the time step will decrease the total error. However, for small time steps, the error of space discretization will dominate. Refining the mesh twice decreases the total error as are clearly shown in Figure 3. These are in accordance with the theoretical result of Theorem 7.2, where the error is bounded by two terms with respective to the space size and the time step. We also observe that the first refinement leads to a better convergence rate in space and the second refinement a less convergence rate than the predicted $\mathcal{O}(h)$ rate, while the convergence rate in time is less than the estimated $\mathcal{O}(\tau)$ rate.

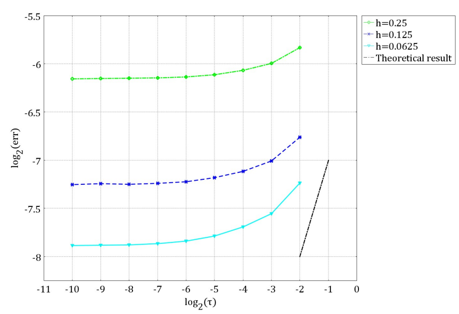
Experiment 8.3. Finally, we take the following exact solution

$$\boldsymbol{H}_{ex}(\boldsymbol{x},t) = (t^2 + \frac{7}{8})\begin{pmatrix} \frac{1}{4}x^4 + \frac{1}{8}yz \\ \frac{1}{3}y^3 + \frac{1}{9}xyz \\ z^2 + \frac{1}{2}xy \end{pmatrix}, \ t \in [0,1].$$

The results are shown in Figure 4 for $\alpha = 0.5$ and $\alpha = 0.8$ respectively and confirm the conclusions of Experiment 8.2.



(a) $\alpha = 0.5$



(b) $\alpha = 0.8$

FIGURE 4. Convergence rate for Experiment 3. The slope of the solid line (theoretical result) equals to 1.

9. Conclusions

We have studied a nonlinear degenerate eddy current problem with ferromagnetic materials by means of the $\boldsymbol{T}$-ψ method. We first use Rothe's method to discuss the weak solution of the continuous problem. We design a nonlinear time-discrete scheme for approximation in suitable function spaces, show the well-posedness of the approximate problem and prove the convergence of the semidiscrete scheme to the weak solution. Finally, we present a fully discrete $\boldsymbol{T}$-ψ scheme to solve nonlinear quasistationary Maxwell's equations based on the backward Euler discretization in time and nodal finite elements in space. We discuss its error estimate and support the theoretical result by some numerical experiments.

The obtained error estimate is suboptimal for both time and space discretization. As we know, the optimality of the backward Euler method for the presented problem cannot be expected a priori due to the combination of the parabolic character and the nonlinearity of the problem. The suboptimality of space discretization is due to the estimation of S_8. If the nonlinear function $\boldsymbol{M}$ is replaced by a Lipschitz continuous approximation, then we could use Young's inequality to estimate S_8 and obtain the optimality with respect to h.

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