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A STABILIZED CHARACTERISTIC FINITE ELEMENT METHOD FOR THE VISCOELASTIC OLDROYD FLUID MOTION PROBLEM

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Abstract. In this article, a characteristic scheme is considered for the viscoelastic Oldroyd fluid flows based on the lowest equal-order finite element pair. The diffusion term in these equations is discretized by using finite element method, the temporal differentiation and advection terms are treated by characteristic scheme and the integral term is handled by applying right rectangle rule. Unconditionally stability and optimal error estimates for the velocity and pressure are derived. Finally, some numerical results are provided to verify the performance of the proposed method.

Key words. viscoelastic Oldroyd fluid motion problem, characteristic scheme, stabilized method, stability, error estimate.

1. Introduction

In this work, let Ω be an open bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Consider the following viscoelastic Oldroyd fluid flows

$$u_t - \nu\Delta u + \nabla p + (u\cdot\nabla)u - \int_0^t \rho e^{-\delta(t-s)}\Delta u ds = f, \tag{1}$$

with $x \in \Omega,\ t \in (0,T]$ and incompressible condition

$$\text{div } u(t,x) = 0 \qquad \forall\ t \in (0,T], \quad x \in \Omega, \tag{2}$$

and the initial and boundary conditions

$$u(x,0) = u_0(x) \quad x \in \Omega; \quad u|_{\partial\Omega} = 0 \quad \text{for all } t \in (0,T], \tag{3}$$

where $\rho \geq 0,\ 1/\delta$ is the relaxation time, u represents the velocity, p the pressure, f the prescribed external force, $u_0(x)$ the initial velocity, ν is the viscosity and $T > 0$ is a finite time.

From the expressions of equations (1)-(3), we know that (1)-(3) are the generalization of the initial boundary value problem for the Navier-Stokes equations, and equations (1)-(3) are used as model in viscoelastic Oldroyd flows [15, 18]. The importance of ensuring the compatibility of the approximations of velocity and pressure by satisfying the *inf-sup* condition is widely known in [7]. Although stable mixed finite element pairs have been studied over the years [13, 16, 19], the low order finite element pairs not satisfying the *inf-sup* condition may work well not only in theoretical filed but also in computation (see [2, 3, 8, 14, 24] and the reference therein). In these stabilized techniques, polynomial pressure projection method which developed in [2] is the most attractive due to the following reasons: (i) The method does not require the approximation of the pressure derivatives and

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the mesh-dependent parameters. (ii) The method is unconditionally stable. (iii) The method can be applied to the existing codes with a little additional effort. Therefore, much attention has been attracted to solve various kinds of problems by using this stabilized technique, for example, we can refer to [14, 20, 22, 26].

On the other hand, characteristic scheme is designed to deal with convection-diffusion problems. This scheme can treat the convection-dominated equations efficiently [1, 3, 6]. Characteristic scheme has also been applied to solve the incompressible flow problems. For example, Pironneau analyzed the Navier-Stokes equations and obtain the suboptimal convergence results in [17], Suli in [19] improved the results of [17], Zhang et al. considered incompressible flows by combining stabilized method with characteristic scheme in [23, 25] and [4, 6, 21] for convection-dominated problems.

In this work, we try to combine the modified method of characteristics with stabilized method to treat the viscoelastic Oldroyd flows. The combination is efficient and keeps the advantages of two methods and avoids their deficits. The main contribution of this article is to establish the stability and convergence of the stabilized characteristic finite element solutions based on the uniqueness condition.

The rest of this paper is organized as follows. In Section 2, the notations and some basic results for equations (1)-(3) are recalled. In Section 3, we provide the boundedness for the numerical solutions based on some regularity conditions. Section 4 is devoted to derive the optimal error estimates for the discrete variational formulation of equations (1)-(3). Finally, some numerical experiments are tested to confirm the established theoretical results and explore the effect of varying stabilized parameters to the errors. In this work, the letter c denotes the general positive constant, which depends on the smallest angle in the triangulation $\mathcal{T}_h$ and domain Ω, independents of the mesh size h and time-step Δt.

2. Preliminaries

2.1. Basic results. In order to present the weak formulation for equations (1)-(3), we need to introduce some Sobolev spaces:

$$X = H_0^1(\Omega)^2, \quad Y = L^2(\Omega)^2, \quad M = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}.$$

The spaces $L^2(\Omega)^m$ $(m = 1, 2)$ are endowed with the standard L^2-scalar product $(\cdot, \cdot)$ and norm $\| \cdot \|_0$, the spaces $H_0^1(\Omega)$ and X are equipped with the scalar product $(\nabla u, \nabla v)$ and norm $\|u\|_1$, $\forall\, u, v \in H_0^1(\Omega)$ or X.

Let H^{-1} be a dual, with respect to L^2-duality, space to H_0^1 with the corresponding norm:

$$\|f\|_{-1} = \sup_{0 \neq u \in H_0^1} \frac{(f, u)}{\|u\|_1}, \quad f \in H^{-1}.$$

Set

$$Au = -\Delta u, \quad \forall\, u \in D(A) = H^2(\Omega)^2 \cap X.$$

In particular, $D(A^{\frac{1}{2}}) = X$, $D(A^0) = Y$. It is known [3, 13] that

$$\| v \|_0^2 \leq \gamma_0 \|v\|_1^2, \quad \forall\, v \in X; \quad \|v\|_1^2 \leq \gamma_0 \|Av\|_0^2, \quad \forall\, v \in D(A),$$

where γ_0 is a positive constant only depending on Ω.

We usually make the following assumption about the prescribed data for problem (1)-(3) (see [9, 13]).

(**A1**). Assume that $u_0 \in D(A)$ with $\text{div} u_0 = 0$ and $f, f_t \in L^2(0,T;Y)$. Moreover

$$\|u_0\|_2 + \sup_{t\in[0,T]} \{\|f\|_0 + \|f_t\|_0\} \le c.$$

Furthermore, we define the continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X\times X$ and $X\times M$, respectively, by

$$a(u,v) = \nu(\nabla u, \nabla v), \quad d(v,q) = (q, \text{div } v), \quad \forall\, v\in X, \quad q\in M.$$

The trilinear form on $X\times X\times X$ is defined by

$$b(u,v,w) = ((u\cdot\nabla)v, w) + \frac{1}{2}((\text{div } u)v, w).$$

It is easy to verify that $b(\cdot,\cdot,\cdot)$ satisfies the following important properties for all $u, v, w \in X$ (see [7, 8, 9, 13]):

$$b(u,v,w) = -b(u,w,v), \quad b(u,v,v) = 0,$$
$$|b(u,v,w)| \le N\|\nabla u\|_0 \|\nabla v\|_0 \|\nabla w\|_0.$$

With above notations, the weak form of equations (1)-(3) reads as

$$(4) \quad \begin{cases} \text{Find } (u,p)\in(X,M), \forall\, t>0, \text{ for all } (v,q)\in(X,M), \text{ such that} \\ (u_t, v) + B((u,p),(v,q)) + b(u,u,v) + J(t,u,v) = (f,v), \\ u(0) = u_0, \end{cases}$$

where

$$B((u,p),(v,q)) = a(u,v) - d(p,v) + d(q,u),$$
$$J(t,u,v) = \rho e^{-\delta t}\int_0^t e^{\delta s}(Au(s), v)ds = \rho e^{-\delta t}\int_0^t e^{\delta s}(\nabla u(s), \nabla v)ds.$$

The following results about problem (4) can been found in [11, 16].

Theorem 2.1. Under the conditions of (**A1**), assume

$$(5) \qquad \nu^{-2}N\|f_\infty\|_{-1} \le 1, \text{ where } N = \sup_{u,v,w\in H_0^1(\Omega)^2} \frac{b(u,v,w)}{\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0}.$$

Then, for all $s\ge 0$, the solution (u,p) of problem (4) satisfies

$$\|u(t)\|_0^2 + \|\nabla u\|_0^2 + e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}(\|Au\|_0^2 + \|p\|_1^2 + \|u_t\|_0^2)dt \le c,$$
$$\sigma(s)(\|Au\|_0^2 + \|u_t\|_0^2 + \|p\|_1^2) + e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}(\sigma(t)\|\nabla u_t\|_0^2 + \|\nabla u\|_0^2)dt \le c,$$
$$\sigma^2(s)\|\nabla u_t\|_0^2 + e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}\sigma^2(t)(\|Au_t\|_0^2 + \|u_{tt}\|_0^2 + \|p_t\|_1^2)dt \le c,$$

where $\sigma(\cdot) = \min\{t,1\}$ and $0 < \delta_0 < \frac{1}{2}\min\{\delta, \nu/\gamma_0\}$.

2.2. Finite element approximation. Let $h \ge 0$ be a real positive parameter. Suppose that the finite element subspace (X_h, M_h) of (X,M) is characterized by $\mathcal{T}_h = \{K\}$, a partitioning of $\overline{\Omega}$ into triangles, assumed to be regular in the usual sense (see [5]). In this work, we consider the following mixed finite element spaces

$$X_h = \{v\in X : v_i|_K \in P_1(K), \forall K\in\mathcal{T}_h, i=1,2\},$$
$$M_h = \{q\in M : q|_K \in P_1(K), \forall K\in\mathcal{T}_h\},$$

where $P_1(K)$ is the 1st-order polynomials on K. Note that this pair is unstable in standard Babuška-Brezzi sense (see [7]). Let $\pi_h : M \to W_h$ be the standard L^2-projection with the following properties:

$$(p - \pi_h p, q_h) = 0 \qquad \forall\, p \in L^2(\Omega), q_h \in W_h, \tag{6}$$

$$\|\pi_h p\|_0 \le C_1 \|p\|_0 \qquad \forall\, p \in L^2(\Omega), \tag{7}$$

$$\|p - \pi_h p\|_0 \le C_2 h \|p\|_1 \qquad \forall\, p \in H^1(\Omega), \tag{8}$$

where $W_h \subset L^2(\Omega)$ denotes the piecewise constant space associated with $\mathcal{T}_h$. We define the following difference operator [2, 14]

$$G_h(p, q) = \alpha(p - \pi_h p, q - \pi_h q), \tag{9}$$

where α is the stabilized parameter. Set

$$p_i = [p_0, p_1, \cdots, p_{N-1}], \quad q_j = [q_0, q_1, \cdots, q_{N-1}],$$
$$M_{ij} = (\phi_i, \phi_j), \quad p_h = \sum_{i=0}^{N-1} p_i \phi_i,$$
$$p_i = p_h(x_i) \;\; \forall p_h \in M_h, \;\; i, j = 0, 1, \cdots, N-1,$$

where ϕ_i is the basis function for the pressure on the domain Ω such that its value is one at the node x_i and zero at other nodes. $M_k,\ k \ge 2$ and M_1 are symmetric and positive-definite pressure mass matrices computed by using k-order and 1-order Gauss integrations in each direction, respectively. Furthermore, p_i and $q_i, i = 0, 1, \cdots, N-1$ are the values of p_h and q_h at the node x_i. Due to (6) and (9), it is valid that

$$\begin{aligned} G_h(p_h, q_h) &= \alpha(p_h - \pi_h p_h, q_h - \pi_h q_h) \\ &= \alpha\Big[(p_h, q_h) - (\pi_h p_h, q_h) - (p_h, \pi_h q_h) + (\pi_h p_h, \pi_h q_h)\Big] \\ &= \alpha\Big[(p_h, q_h) - (\pi_h p_h, q_h) - (\pi_h p_h, \pi_h q_h) + (\pi_h p_h, \pi_h q_h)\Big] \\ &= \alpha(p_h, q_h) - \alpha(\pi_h p_h, q_h). \end{aligned}$$

Since $p_h \in M_h \subset M, \pi_h p_h \in W_h$, it follows that

$$(p_h, q_h) = p_i^T M_k q_j, \quad (\pi_h p_h, q_h) = p_i^T M_1 q_j,$$

where p_i^T is the transpose of the vector p_i. Then we have the following local difference operator between a consistent and an under-integrated mass matrices stabilized formulation

$$G_h(p_h, q_h) = \alpha p_i^T (M_k - M_1) q_j = \alpha p_i^T M_k q_j - \alpha p_i^T M_1 q_j.$$

With the help of (9), the discrete finite element formulation for equations (1)-(3) reads as:

$$\begin{cases} \text{Find } (u_h, p_h) \in X_h \times M_h \;\forall\, (v_h, q_h) \in (X_h, M_h), t \in (0, T] \text{ such that} \\ (u_{ht}, v_h) + \mathcal{B}((u_h, p_h), (v_h, q_h)) + b(u_h, u_h, v_h) + J(t, u_h, v_h) = (f, v_h), \\ u_h(0) = u_{0h}, \end{cases} \tag{10}$$

where u_{0h} is an approximation of u_0, and $\mathcal{B}((\cdot,\cdot),(\cdot,\cdot))$ defined by

$$\mathcal{B}((u_h, p_h), (v_h, q_h)) = a(u_h, v_h) - d(p_h, v_h) + d(u_h, q_h) + G_h(p_h, q_h).$$

The following theorem establishes the continuous and weak coercivity properties for $\mathcal{B}((u_h, p_h); (v_h, q_h))$ (see [2, 14]).

Theorem 2.2. There exist two positive constants c and β, independent of h, for all $(u_h,p_h),(v_h,q_h)\in X_h\times M_h$ such that

$$|\mathcal{B}((u_h,p_h);(v_h,q_h))|\le c(\|u_h\|_1+\|p_h\|_0)(\|v_h\|_1+\|q_h\|_0),$$
$$\sup_{0\neq(v_h,q_h)\in(X_h,M_h)}\frac{|\mathcal{B}((u_h,p_h);(v_h,q_h))|}{\|v_h\|_1+\|q_h\|_0}\ge\beta(\|u_h\|_1+\|p_h\|_0).$$

In order to obtain the error of numerical solution, we define the projection operator $(R_h,Q_h):(X,M)\to(X_h,M_h)$ by

$$\mathcal{B}((R_h(v,q),Q_h(v,q));(v_h,q_h))=B((v,q);(v_h,q_h)),\quad \forall\ (v,q)\in(X,M),(v_h,q_h)\in(X_h,M_h). \tag{11}$$

According to Theorem 2.2, (R_h,Q_h) is well defined and satisfies the following approximation properties (see [14]).

Lemma 2.3. Under the conditions of (**A1**) and (5), the projection (R_h,Q_h) satisfies

$$\|\nabla(v-R_h(v,q))\|_0+\|q-Q_h(v,q)\|_0\le c(\|v\|_1+\|q\|_0)$$

for all $(v,q)\in(X,M)$ and

$$\|v-R_h(v,q)\|_0+h(\|\nabla(v-R_h(v,q))\|_0+\|q-Q_h(v,q)\|_0)\le Ch^2(\|v\|_2+\|q\|_1)$$

for all $(v,q)\in D(A)\times(H^1(\Omega)\cap M)$.

Owing to $u_0\in D(A)$, we can define $p_0\in H^1(\Omega)\cap M$ (see [13]), following [12, 13] we set $(u_h(x,0),p_h(x,0))=(R_h(u_0,p_0),Q_h(u_0,p_0))$. Furthermore we denote $(e_h,\eta_h)=(R_h(u,p)-u_h,Q_h(u,p)-p_h)$. Combining Theorem 2.1 with Lemma 2.3, one finds

$$e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}\sigma^2(t)\|\nabla(u_t-R_{ht}(u,p))\|_0^2dt\le ch^2\qquad s>0. \tag{12}$$

Following the guidelines in [11, 12, 22], we can obtain the following results.

Theorem 2.4. Under the assumption of (**A1**) and condition (5), for all $s>0$, the solutions (u_h,p_h) of problem (10) satisfies

$$\|u_h\|_0^2+e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}(\frac{\nu}{2}\|\nabla u_h\|_0^2+G_h(p_h,p_h))dt\le c,$$
$$\nu\|\nabla u_h\|_0^2+G_h(p_h,p_h)+e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}\|u_{ht}(t)\|_0^2dt\le c,$$
$$\sigma(s)\|u_{ht}\|_0^2+e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}\sigma(t)\Big(\nu\|\nabla u_{ht}\|_0^2+G_h(p_{ht},p_{ht})\Big)dt\le c,$$
$$\sigma(s)\|\nabla(u-u_h)\|_0^2+e^{-2\delta_0 s}\int_0^s e^{2\delta_0 t}\sigma(t)\|u_t-u_{ht}\|_0^2dt\le ch^2.$$

3. Stabilized characteristic finite element method

The characteristic scheme is based on the approximation of the material derivative term, that is, the time derivative term plus the convection term, works well for convection-dominant problems (see [1, 6, 25]).

Let

$$\psi(x,t)=(1+|u|^2)^{\frac{1}{2}},$$

where $|u|^2=u_1^2(x,t)+u_2^2(x,t)$. The characteristic direction corresponding to the hyperbolic part of (1), $u_t+(u\cdot\nabla)u$, be denoted by τ, so (see [3, 17])

$$\frac{\partial}{\partial\tau}=\frac{1}{\psi(x,t)}\frac{\partial}{\partial t}+\frac{1}{\psi(x,t)}u\cdot\nabla.$$

Then the characteristic variational form of equations (1)-(3) reads as:

$$(13)\quad \begin{cases} \text{Find } (u,p)\in(X,M), \text{ for all } t\in(0,T],\ (v,q)\in X\times M, \text{ such that} \\ (\psi(x,t)\frac{\partial u}{\partial\tau},v)+B((u,p);(v,q))+J(t,u,v)=(f,v), \\ u(0)=u_0. \end{cases}$$

Now, we consider the backward difference along the τ characteristic tangent as the approximation of $\psi(x,t)\frac{\partial u}{\partial\tau}$. We choose the time step Δt and denote discrete times $t^n=n\Delta t,\ n=0,1,...,N=\frac{T}{\Delta t}$. Furthermore, we have (see [3, 4, 6, 17])

$$(\psi(x,t)\frac{\partial u}{\partial\tau})^n\approx\psi(x,t)\frac{u(x,t^n)-u(\overline{x},t^{n-1})}{\sqrt{(x-\overline{x})^2+\Delta t^2}}=\frac{u^n-\overline{u}^{n-1}}{\Delta t},$$

where $\overline{x}=x-u(x,t^{n-1})\triangle t$, u^n stands for $u(x,t^n)$ and $\overline{u}^{n-1}$ denotes $u(\overline{x},t^{n-1})$.

In this work, we combine the modified method of characteristic with stabilized method to treat the viscoelastic Oldroyd fluid flows of(1)-(3). For the time discretization of the integral term, we make analysis as in [16] and apply right rectangle rule to the integral term:

$$M^n(\phi)=\Delta t\rho\sum_{i=1}^{n}e^{-\delta(t^n-t^i)}\phi^i\approx\rho\int_0^{t^n}e^{-\delta(t^n-t)}\phi(t)dt.$$

Due to Theorem 2.4 and the fact that $1+\delta\triangle t\le e^{\delta\triangle t}\le c$, we have

$$\begin{aligned} |M^n(\nabla u_h)| &\le \rho\triangle te^{-\delta t^n}\sum_{i=1}^{n}e^{\delta t^i}\|\nabla u_h(t^i)\|_0\le c\triangle te^{-\delta t^n}\sum_{i=1}^{n}e^{\delta t^i} \\ &\le c\triangle te^{\delta\triangle t}\frac{1-e^{\delta t^n}}{e^{\delta\triangle t}-1}\le c. \end{aligned}\tag{14}$$

The stabilized characteristic finite element algorithm for (1)-(3) reads as: Find $(u_h^n,p_h^n)\in(X_h,M_h)$, for all $(v_h,q_h)\in(X_h,M_h)$ such that

$$\begin{aligned} &(\frac{u_h^n-\overline{u}_h^{n-1}}{\Delta t},v_h)+a(u_h^n,v_h)-d(v_h,p_h^n)+d(u_h^n,q_h) \\ &\qquad+G_h(p_h^n,q_h)+(M^n(\nabla u_h),\nabla v_h)=(f,v_h). \end{aligned}\tag{15}$$

Next, we will present the stability of scheme (15) and then provide the optimal error estimates for the numerical solution (u_h^n,p_h^n). Firstly, we recall the following lemma that plays an important role in the numerical analysis.

Lemma 3.1. It holds that

$$(\overline{u},\overline{u})-(u,u)\le C\Delta t\quad\forall\ u\in X,$$

where $\overline{u}=u(x-u(x,t)\Delta t)$.

Proof. We can refer to the proof of Lemma 4.2 in [25].

Theorem 3.2. Under the assumptions of (**A1**) and the uniqueness condition of (5), the solution (u_h^i,p_h^i) of problem (15) satisfies

$$\|u_h^n\|_0^2+e^{-2\delta_0t^n}\sum_{i=1}^{n}e^{2\delta_0t^i}\Big(\nu\|\nabla u_h^i\|_0^2+\|p_h^i\|_0^2\Big)\Delta t\le c\qquad(0<i\le n\le N).\tag{16}$$

Proof. Choosing $(v_h,q_h)=(u_h^i,p_h^i)$ in (15) and multiplying by Δt yields

$$\begin{aligned} (u_h^i-\overline{u}_h^{i-1},u_h^i) &+ \Big[\nu\|\nabla u_h^i\|_0^2+G_h(p_h^i,p_h^i)\Big]\Delta t \\ &+(M^i(\nabla u_h),\nabla u_h^i)\Delta t=(f^i,u_h^i)\Delta t. \end{aligned}\tag{17}$$

Applying the Cauchy inequality and (14), we arrive at

$$\|u_h^i\|_0^2+\Big[\nu\|\nabla u_h^i\|_0^2+G({}_hp_h^i,p_h^i)\Big]\Delta t\le c(\|\overline{u}_h^{i-1}\|_0^2+\Delta t+\|f^i\|_0^2\Delta t). \tag{18}$$

We prove the boundedness of $\|u_h^i\|_0$ by using the induction method. For $i=1$, one finds

$$\|u_h^1\|_0^2+\Big[\nu\|\nabla u_h^1\|_0^2+G_h(p_h^1,p_h^1)\Big]\Delta t\le c(\|\overline{u}_h(0)\|_0^2+\Delta t+\|f^1\|_0^2\Delta t). \tag{19}$$

With Lemma 2.3, we know that (see [8, 9, 10, 12])

$$\begin{aligned}&\|\overline{u}_h(0)\|_0=\|u_h(\overline{x},0)\|_0=\|R_h(u_0,p_0)\|_0\\ \le\ &\|u_0\|_1+\|\nabla(u_0-R_h(u_0,p_0))\|_0\le c(\|u_0\|_1+\|p_0\|_0).\end{aligned}$$

Combining above inequality with (19), using assumption (**A1**) yields

$$\|u_h^1\|_0^2+\Big[\nu\|\nabla u_h^1\|_0^2+G_h(p_h^1,p_h^1)\Big]\Delta t\le c.$$

For $i=m-1$, if there holds

$$\|u_h^{m-1}\|_0^2+\Big[\nu\|\nabla u_h^{m-1}\|_0^2+G_h(p_h^{m-1},p_h^{m-1})\Big]\Delta t\le c\quad\forall\ x\in\Omega. \tag{20}$$

Then, for $i=m$, combining (18) with (20), one finds

$$\begin{aligned}\|u_h^m\|_0^2\quad+\quad&\Big[\nu\|\nabla u_h^m\|_0^2+G_h(p_h^m,p_h^m)\Big]\Delta t\\ &\le\Big(\|u_h^{m-1}(\overline{x})\|_0^2+\Delta t+\|f^m\|_0^2\Delta t\Big)\le c.\end{aligned} \tag{21}$$

Noting that $ab\le\frac{1}{2}(a^2+b^2)$ we obtain

$$\begin{aligned}&(u_h^i-\overline{u}_h^{i-1},u_h^i)=(u_h^i,u_h^i)-(\overline{u}_h^{i-1},u_h^i)\ge\frac{1}{2}\Big[(u_h^i,u_h^i)-(\overline{u}_h^{i-1},\overline{u}_h^{i-1})\Big]\\ &=\frac{1}{2}\Big\{[(u_h^i,u_h^i)-(u_h^{i-1},u_h^{i-1})]+[(u_h^{i-1},u_h^{i-1})-(\overline{u}_h^{i-1},\overline{u}_h^{i-1})]\Big\}.\end{aligned}$$

Thanks to (14), we have

$$\begin{aligned}&(M^i(\nabla u_h),\nabla u_h^i)\Delta t\le\|M^i(\nabla u_h)\|_0\|\nabla u_h^i\|_0\Delta t\\ \le\ &\frac{\nu}{4}\|\nabla u_h^i\|_0^2\Delta t+\|M^i(\nabla u_h)\|_0^2\Delta t\le\frac{\nu}{4}\|\nabla u_h^i\|_0^2\Delta t+c\Delta t.\end{aligned}$$

Combing above estimates with (17), using the Lemma 3.1, Taylor's formula and multiplying $e^{2\delta_0 t^i}$, we arrive at (see [25])

$$\begin{aligned}&e^{2\delta_0 t^i}\|u_h^i\|_0^2-e^{2\delta_0 t^{i-1}}\|u_h^{i-1}\|_0^2+e^{2\delta_0 t^i}\nu\|\nabla u_h^i\|_0^2\Delta t\\ &\le ce^{2\delta_0 t^i}\Delta t+c\Delta te^{2\delta_0 t^i}\|u_h^{i-1}\|_0^2.\end{aligned} \tag{22}$$

Summing (22) with respect to i from 1 to n, multiplying by $e^{-2\delta_0 t^n}$ and using **(A1)**, (21), we have

$$\begin{aligned}&\|u_h^n\|_0^2+e^{-2\delta_0 t^n}\sum_{i=1}^n e^{2\delta_0 t^i}\nu\|\nabla u_h^i\|_0^2\Delta t\\ &\le e^{-2\delta_0 t_n}\|u_h^0\|_0^2+c\Delta te^{-2\delta_0 t^n}\sum_{i=1}^n e^{2\delta_0 t^i}(1+\|u_h^{i-1}\|_0^2)\le c\quad\forall\ 0<n\le N.\end{aligned} \tag{23}$$

Furthermore, combining Theorem 2.2 with (15), using the Cauchy and Poincaré inequalities, we get

$$\beta e^{\delta_0 t^i}(\|u_h^i\|_1+\|p_h^i\|_0) \le \sup_{0\neq(v_h,q_h)\in(X_h,M_h)} \frac{|e^{\delta_0 t^i}\mathcal{B}((u_h^i,p_h^i);(v_h,q_h))|}{\|v_h\|_1+\|q_h\|_0}$$

$$\le \sup_{0\neq(v_h,q_h)\in(X_h,M_h)} \frac{e^{\delta_0 t^i}\Big(|(\frac{u_h^i-\overline{u}_h^{i-1}}{\Delta t},v_h)|+|(f^i,v_h)|+|(M^i(\nabla u_h),\nabla v_h)|\Big)}{\|v_h\|_1+\|q_h\|_0}. \tag{24}$$

For $u_h^i-\overline{u}_h^{i-1}$, we write it as a sum of two terms $(u_h^i-u_h^{i-1})+(u_h^{i-1}-\overline{u}_h^{i-1})$, and use the results provided in [17] to obtain

$$e^{\delta_0 t^i}(\frac{u_h^i-\overline{u}_h^{i-1}}{\Delta t},v_h) \le (\frac{e^{\delta_0 t^i}u_h^i-e^{\delta_0 t^{i-1}}u_h^{i-1}}{\Delta t},v_h)+e^{\delta_0 t^i}(\frac{u_h^{i-1}-\overline{u}_h^{i-1}}{\Delta t},v_h)$$

$$\le \frac{c}{\Delta t}\|v_h\|_0\int_{t^{i-1}}^{t^i}\Big(e^{\delta_0 t}\|u_{ht}\|_0+\delta_0 e^{\delta_0 s}\|u_h\|_0\Big)dt$$

$$+\frac{ce^{\delta_0 t^i}}{\Delta t}\|\nabla v_h\|_0\|u_h^{i-1}-\overline{u}_h^{i-1}\|_{H^{-1}(\Omega)}$$

$$\le \frac{c}{\Delta t^{1/2}}\|\nabla v_h\|_0\Big(\int_{t^{i-1}}^{t^i}e^{2\delta_0 t}(\|u_{ht}\|_0^2+\|u_h\|_0^2)dt\Big)^{\frac{1}{2}}+ce^{\delta_0 t^i}\|\nabla v_h\|_0\|u_h^{i-1}\|_0.$$

With the help of (14), one finds

$$e^{\delta_0 t^i}|(M^i(\nabla u_h),\nabla v_h)| \le e^{\delta_0 t^i}\|M^i(\nabla u_h)\|_0\|\nabla v_h\|_0 \le ce^{\delta_0 t^i}\|\nabla v_h\|_0.$$

Combining above estimates with (21)-(22), (24), squaring and summing (24) for i from 1 to n, after multiplying by $e^{-2\delta_0 t^n}\triangle t$, we have

$$e^{-2\delta_0 t^n}\sum_{i=1}^n e^{2\delta_0 t^i}\|p_h^i\|_0^2\Delta t$$

$$\le ce^{-2\delta_0 t^n}\sum_{i=1}^n e^{2\delta_0 t^i}\Delta t+e^{-2\delta_0 t^n}\int_0^{t^n}e^{2\delta_0 t}(\|u_{ht}\|_0^2+\|u_h\|_0^2)dt$$

$$+ce^{-2\delta_0 t^n}\sum_{i=1}^n e^{2\delta_0 t^i}\|u_h^{i-1}\|_0^2\Delta t \le c. \tag{25}$$

We end the proof of (16) by combining (23) with (25).

4. Error estimates.

In this section, our aim is devoted to establish the convergence of the solution (u_h^n,p_h^n) for problem (15). Firstly, we borrow the definition $\theta^n(\phi)$ from [16]

$$\theta^n(\phi)=\int_0^{t^n}\rho e^{-\delta(t^n-t)}\phi(t)dt-M^n(\phi). \tag{26}$$

Combining the relationship

$$\phi(t^n)-\frac{1}{\triangle t}\int_{t^{n-1}}^{t^n}\phi(t)ds=\frac{1}{\triangle t}\int_{t^{n-1}}^{t^n}(t-t^{n-1})\phi_t(t)ds,$$

and Theorem 2.4, one finds

$$\begin{aligned}
&\|\theta^n(\nabla u_h)\|_0^2 \le \rho^2 \|\sum_{i=1}^n \int_{t^{i-1}}^{t^i} e^{-\delta t^n}(t-t^{i-1})\frac{\partial}{\partial t}(e^{\delta t}\nabla u_h)dt\|_0^2 \\
\le\ & c\|\sum_{i=1}^n \int_{t^{i-1}}^{t^i} e^{\delta(t-t^n)}(t-t^{i-1})(\delta\nabla u_h + \nabla u_{ht})dt\|_0^2 \\
\le\ & c\Big(\sum_{i=1}^n \int_{t^{i-1}}^{t^i} e^{2\alpha_0(t-t^n)}|t-t^{i-1}|^2 dt\Big)\cdot \\
& \Big(e^{-2\delta_0 t^n}\sum_{i=1}^n \int_{t^{i-1}}^{t^i} e^{2\delta_0 t}(\|\nabla u_h\|_0^2 + \|\nabla u_{ht}\|_0^2)dt\Big) \\
\le\ & c\triangle t^2 \int_0^{t^n} e^{2\alpha_0(t-t^n)}dt \cdot e^{-2\delta_0 t^n}\int_0^{t^n} e^{2\delta_0 t}(\|\nabla u_h\|_0^2 + \|\nabla u_{ht}\|_0^2)dt \\
\le\ & c\Delta t^2 \qquad (\text{where } \alpha_0 = \delta - \delta_0). \qquad (27)
\end{aligned}$$

Theorem 4.1. Let (u,p) be the solution of (4), under the conditions of (**A1**) and the uniqueness condition of (5), for all $0 < n \le m \le N$, the solution (u_h^n, p_h^n) of (15) satisfies

$$\sigma^2(t^n)\|u^n - u_h^n\|_0^2 \le c(\Delta t^2 + h^4),$$
$$e^{-2\delta_0 t^m}\nu\Delta t\sum_{n=1}^m e^{2\delta_0 t^n}\sigma^2(t^n)\Big(\|\nabla(u^n - u_h^n)\|_0^2 + \|p^n - p_h^n\|_0^2\Big) \le c(\Delta t^2 + h^2).$$

Proof. The proof consists of Theorem 2.4 and Lemmas 4.2-4.3.

Lemma 4.2. Under the assumptions of Theorem 4.1, the following error estimate holds for all $0 < m \le N$.

$$\sigma^2(t^m)\|e_h^m\|_0^2 + e^{-2\delta_0 t^m}\Delta t\sum_{n=1}^m e^{2\delta_0 t^n}\sigma^2(t^n)\nu\|\nabla e_h^n\|_0^2 \le c(\Delta t^2 + h^4). \qquad (28)$$

Proof. Thanks to (26) and (11), we can obtain the following error equation from (13) and (15) at $t = t^n$

$$\begin{aligned}
&(\frac{e_h^n - \overline{e}_h^{n-1}}{\Delta t}, v_h) + \mathcal{B}((e_h^n, \eta_h^n), (v_h, q_h)) + (\theta^n(\nabla u_h), \nabla v_h) \\
=\ & -(\psi(x,t^n)\frac{\partial u^n}{\partial \tau} - \frac{u^n - \overline{u}^{n-1}}{\Delta t}, v_h) \\
& +(\frac{(\overline{u}^{n-1} - R_h(\overline{u}^{n-1}, p^{n-1})) - (u^{n-1} - R_h(u^{n-1}, p^{n-1}))}{\Delta t}, v_h) \\
& +(\frac{(u^{n-1} - R_h(u^{n-1}, p^{n-1})) - (u^n - R_h(u^n, p^n))}{\Delta t}, v_h) \equiv \sum_{i=1}^3 T_i. \qquad (29)
\end{aligned}$$

Taking $(v_h, q_h) = (e_h^n, \eta_h^n)$ in (29) and using $a(a-b) \ge \frac{1}{2}(a^2 - b^2)$ to yield

$$\begin{aligned}
&(\frac{e_h^n - \overline{e}_h^{n-1}}{\Delta t}, e_h^n) + \mathcal{B}_h((e_h^n, \eta_h^n), (e_h^n, \eta_h^n)) \\
\ge\ & \frac{1}{2\Delta t}\Big[\|e_h^n\|_0^2 - \|e_h^{n-1}\|_0^2 + (\|e_h^{n-1}\|_0^2 - \|\overline{e}_h^{n-1}\|_0^2)\Big] + \nu\|\nabla e_h^n\|_0^2 + G(\eta_h^n, \eta_h^n). \\
& |(\theta^n(\nabla u_h), \nabla e_h^n)| \le c\|\theta^n(\nabla u_h)\|_0^2 + \frac{\nu}{16}\|\nabla e_h^n\|_0^2.
\end{aligned}$$

Applying the results provided in [6, 19], we can treat the terms T_1-T_3 as follows

$$
\begin{aligned}
|T_1| &\leq c\|\psi(x,t^n)\frac{\partial u^n}{\partial \tau} - \frac{u^n - \overline{u}^{n-1}}{\Delta t}\|_0\|e_h^n\|_0 \\
&\leq c\Delta t\int_{t^{n-1}}^{t^n}\|\frac{\partial^2 u}{\partial \tau^2}\|_0^2 dt + \frac{\nu}{16}\|\nabla e_h^n\|_0^2, \\
|T_2| &\leq \frac{c}{\Delta t}\|(\overline{u}^{n-1} - R_h(\overline{u}^{n-1},p^{n-1})) - (u^{n-1} - R_h(u^{n-1},p^{n-1}))\|_{-1}\cdot\|\nabla e_h^n\|_0 \\
&\leq c\|u^{n-1} - R_h(u^{n-1},p^{n-1})\|_0\cdot\|\nabla e_h^n\|_0 \\
&\leq c\|u^{n-1} - R_h(u^{n-1},p^{n-1})\|_0^2 + \frac{\nu}{16}\|\nabla e_h^n\|_0^2, \\
|T_3| &\leq \frac{ce^{-\delta_0 t^n}}{\Delta t}\|e^{\delta_0 t^n}(u^n - R_h(u^n,p^n)) - e^{\delta_0 t^n}(u^{n-1} - R_h(u^{n-1},p^{n-1}))\|_0\|e_h^n\|_0 \\
&\leq \frac{ce^{-\delta_0 t^n}}{\Delta t\sigma(t^n)}\Big[\|\sigma(t^n)e^{\delta_0 t^n}(u^n - R_h(u^n,p^n)) \\
&\quad -\sigma(t^{n-1})e^{\delta_0 t^{n-1}}(u^{n-1} - R_h(u^{n-1},p^{n-1}))\|_0 \\
&\quad +\|\sigma(t^n)e^{\delta_0 t^n}(u^{n-1} - R_h(u^{n-1},p^{n-1})) \\
&\quad\ -\sigma(t^{n-1})e^{\delta_0 t^{n-1}}(u^{n-1} - R_h(u^{n-1},p^{n-1}))\|_0\Big]\cdot\|\nabla e_h^n\|_0 \\
&\leq \frac{ce^{-\delta_0 t^n}}{\Delta t\sigma(t^n)}\Big[\int_{t^{n-1}}^{t^n}\|\frac{\partial(\sigma(t)e^{\delta_0 t}(R_h(u,p) - u))}{\partial t}\|_0 dt \\
&\quad +(\sigma(t^n) - \sigma(t^{n-1}))e^{\delta_0 t^{n-1}}\|u^{n-1} - R_h(u^{n-1},p^{n-1})\|_0 \\
&\quad +\sigma(t^n)(e^{\delta_0 t^n} - e^{\delta_0 t^{n-1}})\|u^{n-1} - R_h(u^{n-1},p^{n-1})\|_0\Big]\cdot\|\nabla e_h^n\|_0 \\
&\leq \frac{ce^{-\delta_0 t^n}}{\Delta t\sigma(t^n)}\Big[\int_{t^{n-1}}^{t^n}\Big(e^{\delta_0 t}\sigma(t)\|u_t - R_{ht}(u,p)\|_0 + \delta_0 e^{\delta_0 t}\sigma(t)\|u - R_h(u,p)\|_0 \\
&\quad +e^{\delta_0 t}\|u - R_h(u,p)\|_0\frac{d\sigma(t)}{dt}\Big)dt \\
&\quad +(\sigma(t^n) - \sigma(t^{n-1}))e^{\delta_0 t^{n-1}}\|u^{n-1} - R_h(u^{n-1},p^{n-1})\|_0 \\
&\quad +\sigma(t^n)(e^{\delta_0 t^n} - e^{\delta_0 t^{n-1}})\|u^{n-1} - R_h(u^{n-1},p^{n-1})\|_0\Big]\cdot\|\nabla e_h^n\|_0.
\end{aligned}
$$

Noting that $0 < \sigma(t) < t$, $\frac{d\sigma(t)}{dt} \leq 1(\forall t \geq 0)$ and $\sigma(t^n) \leq \sigma(t^{n-1}) + \Delta t$, using Taylor formula and the fact that $e^{-\delta_0\Delta t} \leq 1$, the above estimate can be rewritten as

$$
\begin{aligned}
|T_3| \leq& \frac{ce^{-\delta_0 t^n}}{\Delta t\sigma(t^n)}\Big[(\int_{t^{n-1}}^{t^n} e^{2\delta_0 t}\sigma^2(t)\|u_t - R_{ht}(u,p)\|_0^2 dt)^{\frac{1}{2}}(\int_{t^{n-1}}^{t^n} 1dt)^{\frac{1}{2}} \\
&+(\int_{t^{n-1}}^{t^n} e^{2\delta_0 t}\|u - R_h(u,p)\|_0^2 dt)^{\frac{1}{2}}(\int_{t^{n-1}}^{t^n}(1+\sigma^2(t))dt)^{\frac{1}{2}}\Big]\cdot\|\nabla e_h^n\|_0 \\
&+(1+\delta_0)\|u^{n-1} - R_h(u^{n-1},p^{n-1})\|_0\cdot\|\nabla e_h^n\|_0 \\
\leq& \frac{ce^{-\delta_0 t^n}}{\Delta t^{\frac{1}{2}}\sigma(t^n)}\Big[(\int_{t^{n-1}}^{t^n} e^{2\delta_0 t}\sigma^2(t)\|u_t - R_{ht}(u,p)\|_0^2 dt)^{\frac{1}{2}} \\
&+(\int_{t^{n-1}}^{t^n} e^{2\delta_0 t}\|u - R_h(u,p)\|_0^2 dt)^{\frac{1}{2}}\Big]\cdot\|\nabla e_h^n\|_0 \\
&+(1+\delta_0)\|u^{n-1} - R_h(u^{n-1},p^{n-1})\|_0\cdot\|\nabla e_h^n\|_0
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{ce^{-2\delta_0 t^n}}{\Delta t\sigma^2(t^n)}\Big[\Big(\int_{t^{n-1}}^{t^n} e^{2\delta_0 t}\sigma^2(t)\|u_t - R_{ht}(u,p)\|_0^2 dt\Big)\\
&+\Big(\int_{t^{n-1}}^{t^n} e^{2\delta_0 t}\|u - R_h(u,p)\|_0^2 dt\Big)\Big] + c\|u^{n-1} - R_h(u^{n-1},p^{n-1})\|_0^2 + \frac{\nu}{16}\|\nabla e_h^n\|_0^2,
\end{aligned}
$$

Combining above estimates with (29), using Lemma 2.3 and (27), multiplying by $e^{2\delta_0 t^n}\triangle t\sigma^2(t^n)$ yields

$$
\begin{aligned}
& e^{2\delta_0 t^n}\sigma^2(t^n)\|e_h^n\|_0^2 + \nu\Delta t e^{2\delta_0 t^n}\sigma^2(t^n)\|\nabla e_h^n\|_0^2 + \Delta t e^{2\delta_0 t^n}\sigma^2(t^n)G_h(\eta_h^n,\eta_h^n)\\
\leq\ & c\Big[(1+\Delta t)e^{2\delta_0 t^n}\sigma^2(t^n)\|e_h^{n-1}\|_0^2 + \int_{t^{n-1}}^{t^n} e^{2\delta_0 t}\sigma^2(t)\|u_t - R_{ht}(u,p)\|_0^2 dt\\
& + \int_{t^{n-1}}^{t^n} e^{2\delta_0 t}\|u - R_h(u,p)\|_0^2 dt + \Delta t e^{2\delta_0 t^n}\sigma^2(t^n)\|u^{n-1} - R_h(u^{n-1},p^{n-1})\|_0^2\\
& + \Delta t^2\Delta t e^{2\delta_0 t_n}\sigma^2(t^n) + \Delta t^2 e^{2\delta_0 t^n}\sigma^2(t^n)\int_{t^{n-1}}^{t^n}\|\frac{\partial^2 u}{\partial\tau^2}\|_0^2 dt\Big]. \qquad (30)
\end{aligned}
$$

Multiplying (30) by $e^{-2\delta_0 t^n}$, for $n=1$, noting that

$$e_h^0 = R_h(u_0,p_0) - u_h(x,0) = 0.$$

Thanks to (12), (21) and Theorem 2.4, we get

$$
\begin{aligned}
& \sigma^2(t^1)\|e_h^1\|_0^2 + \nu\Delta t\sigma^2(t^1)\|\nabla e_h^1\|_0^2 + \Delta t\sigma^2(t^1)G_h(\eta_h^1,\eta_h^1)\\
\leq\ & c\Big[e^{-2\delta_0 t^1}\int_0^{t^1} e^{2\delta_0 t}\sigma^2(t)\|u_t - R_{ht}(u,p)\|_0^2 dt + \Delta t^2\sigma^2(t^1)\int_0^{t^1}\|\frac{\partial^2 u}{\partial\tau^2}\|_0^2 dt\\
& + \sigma^2(t^1)\Delta t^3 + e^{-2\delta_0 t_1}\int_0^{t^1} e^{2\delta_0 t}\|u - R_h(u,p)\|_0^2 dt\Big]\\
& + \sigma^2(t^1)\Delta t\|u_0 - R_h(u_0,p_0)\|_0^2\\
\leq\ & c(h^4 + \Delta t^2).
\end{aligned}
$$

If $n = m-1$, we still have

$$\sigma^2(t^{m-1})\|e_h^{m-1}\|_0^2 \leq c(h^4 + \Delta t^2), \qquad (31)$$

then, as $n=m$, by using (12), (21), (31) and Theorem 2.4 yields

$$
\begin{aligned}
& \sigma^2(t^n)\|e_h^m\|_0^2 + \nu\Delta t\sigma^2(t^n)\|\nabla e_h^m\|_0^2 + \Delta t\sigma^2(t^n)G_h(\eta_h^m,\eta_h^m)\\
\leq\ & c\Big[(1+\Delta t)\sigma^2(t^n)\|e_h^{m-1}\|_0^2 + e^{-2\delta_0 t^m}\int_{t^{m-1}}^{t^m} e^{2\delta_0 t}\sigma^2(t)\|u_t - R_{ht}(u,p)\|_0^2 dt\\
& + \Delta t^2\sigma^2(t^n)\int_{t^{m-1}}^{t^m}\|\frac{\partial^2 u}{\partial\tau^2}\|_0^2 dt + e^{-2\delta_0 t^m}\int_{t^{m-1}}^{t^m} e^{2\delta_0 t}\|u - R_h(u,p)\|_0^2 dt\\
& + \sigma^2(t^n)\Delta t^3 + \Delta t\sigma^2(t^n)\|u^{m-1} - R_h(u^{m-1},p^{m-1})\|_0^2\Big]\\
\leq\ & c(h^4 + \Delta t^2). \qquad (32)
\end{aligned}
$$

Using the fact that

$$
\begin{aligned}
e^{\delta_0 t^n} = e^{\delta_0 t^{n-1}}e^{\delta_0\triangle t} &\leq e^{\delta_0 t^{n-1}}\Big(1 + \delta_0\triangle t + \frac{e^{\delta_0\triangle t}(\delta_0\triangle t)^2}{2}\Big)\\
&\leq e^{\delta_0 t^{n-1}}(1 + c\delta_0\triangle t) \qquad (33)
\end{aligned}
$$

and summing (30) for n from 1 to m, with (12), (16), (32), (33), Theorem 2.4, after multiplying $e^{-2\delta_0 t^m}$, we have

$$\begin{aligned}
&\sigma^2(t^m)\|e_h^m\|_0^2 + e^{-2\delta_0 t^m}\Delta t\sum_{n=1}^{m} e^{2\delta_0 t^n}\sigma^2(t^n)\Big(\nu\|\nabla e_h^n\|_0^2 + G_h(\eta_h^n,\eta_h^n)\Big)\\
\leq\ & c\Big[e^{-2\delta_0 t^m}\Delta t\sum_{n=1}^{m} e^{2\delta_0 t^n}\sigma^2(t^n)\|e_h^{n-1}\|_0^2\\
&+e^{-2\delta_0 t^m}\int_0^{t^m} e^{2\delta_0 t}\sigma^2(t)\|u_t - R_{ht}(u,p)\|_0^2 dt\\
&+e^{-2\delta_0 t^m}\int_0^{t^m} e^{2\delta_0 t}\|u - R_h(u,p)\|_0^2 dt + \Delta t^2 e^{-2\delta_0 t^m}\sum_{n=1}^{m} e^{2\delta_0 t^n}\Delta t\sigma^2(t^n)\\
&+\Delta t e^{-2\delta_0 t^m}\sum_{n=1}^{m} e^{2\delta_0 t^n}\sigma^2(t^n)\|u^{n-1} - R_h(u^{n-1},p^{n-1})\|_0^2\\
&+\Delta t^2\sigma^2(t^n)\int_0^{t^m}\|\frac{\partial^2 u}{\partial\tau^2}\|_0^2 dt\Big]\\
\leq\ & c(h^4+\Delta t^2) \quad \forall\ 0 < m \leq N.
\end{aligned}$$

Lemma 4.5. Under the conditions of Theorem 4.1, the solution (u_h^n, p_h^n) of problem (15) satisfies

$$e^{-2\delta_0 t^m}\Delta t\sum_{n=1}^{m} e^{2\delta_0 t^n}\sigma^2(t^n)\|\eta_h^n\|_0^2 \leq c(h^2+\Delta t^2). \tag{34}$$

Proof. From Theorem 2.2 and (29), we find that

$$\begin{aligned}
&\beta\sigma(t^n)(\|e_h^n\|_1 + \|\eta_h^n\|_0) \leq \sup_{0\neq(v_h,q_h)\in(X_h,M_h)}\frac{|\sigma(t^n)\mathcal{B}((e_h^n,\eta_h^n);(v_h,q_h))|}{\|v_h\|_1+\|q_h\|_0}\\
\leq\ & \sup_{0\neq(v_h,q_h)\in(X_h,M_h)}\frac{1}{\|v_h\|_1+\|q_h\|_0}\Big[|\sigma(t^n)(\psi(x,t^n)\frac{\partial u^n}{\partial\tau} - \frac{u^n-\overline{u}^{n-1}}{\Delta t}, v_h)|\\
&+|\sigma(t^n)(\frac{(u^n-u_h^n)-(\overline{u}^{n-1}-\overline{u}_h^{n-1})}{\Delta t}, v_h)| + |\sigma(t^n)(\theta^n(\nabla u_h^0),\nabla v_h)|\Big].
\end{aligned}\tag{35}$$

Now, we estimate the right hand terms of (35). Using the results provided in [19] and (27) we get

$$\begin{aligned}
\Big|\sigma(t^n)(\psi(x,t^n)\frac{\partial u^n}{\partial\tau} - \frac{u^n-\overline{u}^{n-1}}{\Delta t}, v_h)\Big| &\leq c\sigma(t^n)\Delta t^{\frac{1}{2}}\Big(\int_{t^{n-1}}^{t^n}\|\frac{\partial^2 u}{\partial\tau^2}\|_0^2 dt\Big)^{\frac{1}{2}}\|\nabla v_h\|_0,\\
|\sigma(t^n)(\theta^n(\nabla u_h),\nabla v_h)| &\leq c\sigma(t^n)\|\theta^n(\nabla u_h)\|_0\|\nabla v_h\|_0\\
&\leq c\sigma(t^n)\Delta t\|\nabla v_h\|_0.
\end{aligned}$$

From the definitions of $\sigma(t)$, we know that $\sigma(t^n) \leq \sigma(t^{n-1}) + \Delta t$, with this relationship in mind, thanks to the help of triangular inequality, we arrive at

$$\begin{aligned}
& \Big|\sigma(t^n)(\frac{(u^n - u_h^n) - (\overline{u}^{n-1} - \overline{u}_h^{n-1})}{\Delta t}, v_h)\Big| \\
\leq\ & \Big|(\frac{\sigma(t^n)[(u^n - u_h^n) - (u^{n-1} - u_h^{n-1})]}{\Delta t}, v_h)\Big| \\
& + \Big|(\frac{\sigma(t^n)[(u^{n-1} - u_h^{n-1}) - (\overline{u}^{n-1} - \overline{u}_h^{n-1})]}{\Delta t}, v_h)\Big| \\
\leq\ & \Big|(\frac{\sigma(t^n)(u^n - u_h^n) - \sigma(t^{n-1})(u^{n-1} - u_h^{n-1})}{\Delta t}, v_h)\Big| \\
& + \Big|((u^{n-1} - u_h^{n-1}) - (\overline{u}^{n-1} - \overline{u}_h^{n-1}), v_h)\Big| \\
& + \Big|(\frac{\sigma(t^{n-1})[(u^{n-1} - u_h^{n-1}) - (\overline{u}^{n-1} - \overline{u}_h^{n-1})]}{\Delta t}, v_h)\Big| \\
& + \Big|((u^{n-1} - u_h^{n-1}), v_h)\Big| \\
\equiv\ & \sum_{i=1}^{4} T_i.
\end{aligned}$$

For T_1, due to $\sigma(t^n) \leq \sigma(t^{n-1}) + \Delta t$, $0 < \sigma(t) \leq t \Rightarrow \frac{d\sigma(t)}{dt} \leq 1 (\forall t > 0)$ and $e^{-\delta_0 \Delta t} \leq 1$, using Taylor formula to obtain

$$\begin{aligned}
& e^{-\delta_0 t^n}\Big|(\frac{e^{\delta_0 t^n}\sigma(t^n)(u^n - u_h^n) - e^{\delta_0 t^n}\sigma(t^{n-1})(u^{n-1} - u_h^{n-1})}{\Delta t}, v_h)\Big| \\
\leq\ & e^{-\delta_0 t^n}\Big|(\frac{e^{\delta_0 t^n}\sigma(t^n)(u^n - u_h^n) - e^{\delta_0 t^{n-1}}\sigma(t^{n-1})(u^{n-1} - u_h^{n-1})}{\Delta t}, v_h)\Big| \\
& + e^{-\delta_0 t^n} \cdot e^{\delta_0 t^\epsilon}\delta_0 \sigma(t^{n-1})(u^{n-1} - u_h^{n-1}, v_h) \quad \text{where } t^\epsilon \in [t^{n-1}, t^n] \\
\leq\ & \frac{ce^{-\delta_0 t^n}}{\Delta t}\int_{t^{n-1}}^{t^n} \Big\|\frac{\partial(e^{\delta_0 t}\sigma(t)(u - u_h))}{\partial t}\Big\|_0 dt \cdot \|v_h\|_0 + c\|u^{n-1} - u_h^{n-1}\|_0\|v_h\|_0 \\
\leq\ & \frac{ce^{-\delta_0 t^n}}{\Delta t}\int_{t^{n-1}}^{t^n} (e^{\delta_0 t}\sigma(t)\|u_t - u_{ht}\|_0 + (1+\delta_0)e^{\delta_0 t}\|u - u_h\|_0)dt \cdot \|v_h\|_0 \\
& + c\|u^{n-1} - u_h^{n-1}\|_0\|v_h\|_0 \\
\leq\ & \frac{ce^{-\delta_0 t^n}}{\Delta t^{\frac{1}{2}}}\Big[(\int_{t^{n-1}}^{t^n} e^{2\delta_0 t}\sigma(t)\|u_t - u_{ht}\|_0^2 dt)^{\frac{1}{2}} + (\int_{t^{n-1}}^{t^n} e^{2\delta_0 t}\|u - u_h\|_0^2 dt)^{\frac{1}{2}}\Big] \cdot \|v_h\|_0 \\
& + c\|u^{n-1} - u_h^{n-1}\|_0\|v_h\|_0.
\end{aligned}$$

For T_2, T_3 and T_4, using Lemma 3.1 to obtain

$$\begin{aligned}
T_2 &= |((u^{n-1} - u_h^{n-1}) - (\overline{u}^{n-1} - \overline{u}_h^{n-1}), v_h)| \\
&\leq c\|(u^{n-1} - u_h^{n-1}) - (\overline{u}^{n-1} - \overline{u}_h^{n-1})\|_{H^{-1}(\Omega)}\|\nabla v_h\|_0 \\
&\leq c\Delta t\|u^{n-1} - u_h^{n-1}\|_0\|\nabla v_h\|_0, \\
T_3 &= |(\frac{\sigma(t^{n-1})[(u^{n-1} - u_h^{n-1}) - (\overline{u}^{n-1} - \overline{u}_h^{n-1})]}{\Delta t}, v_h)| \\
&\leq c\sigma(t^{n-1})\|\frac{(u^{n-1} - u_h^{n-1}) - (\overline{u}^{n-1} - \overline{u}_h^{n-1})}{\Delta t}\|_{H^{-1}(\Omega)}\|\nabla v_h\|_0 \\
&\leq c\|u^{n-1} - u_h^{n-1}\|_0\|\nabla v_h\|_0, \\
T_4 &= |((u^{n-1} - u_h^{n-1}), v_h)| \leq c\|u^{n-1} - u_h^{n-1}\|_0\|\nabla v_h\|_0.
\end{aligned}$$

Combining above inequalities with (35), squaring and multiplying by $e^{2\delta_0 t^n}\Delta t$, summing for n from 1 to m, using Theorem 2.4 and (14), (16), after multiplying by $e^{-2\delta_0 t^m}$, we get

$$\begin{aligned}
& e^{-2\delta_0 t^m}\Delta t\sum_{n=1}^{m} e^{2\delta_0 t^n}\sigma^2(t^n)(\|\nabla e_h^n\|_0^2+\|\eta_h^n\|_0^2)\\
\leq\ & c\Big[\Delta t^2\int_0^{t^m}\|\frac{\partial^2 u}{\partial\tau^2}\|_0^2 dt + h^2 e^{-2\delta_0 t^m}\Delta t\sum_{n=1}^{m} e^{2\delta_0 t^n}\|\nabla u_h^n\|_0^2\\
& +\Delta t^2 e^{-2\delta_0 t^m}\Delta t\sum_{n=1}^{m} e^{2\delta_0 t^n} + e^{-2\delta_0 t^m}\int_0^{t^m} e^{2\delta_0 t}\sigma(t)\|u_t-u_{ht}\|_0^2 ds\\
& +e^{-2\delta_0 t^m}\int_0^{t^m} e^{2\delta_0 t}\|u-u_h\|_0^2 ds + \Delta t e^{-2\delta_0 t^m}\sum_{n=1}^{m} e^{2\delta_0 t^n}\|u^{n-1}-u_h^{n-1}\|_0^2\Big]\\
\leq\ & c(h^2+\Delta t^2)\quad \forall\ 0<m\leq N.
\end{aligned}$$

Hence, we finish the proof of (34).

5. Numerical examples

In this section, we present some numerical results to illustrate the effectiveness of stabilized characteristic finite element method (SCFEM) for the viscoelastic Oldroyd fluid flows. We consider the problem (1)-(3) on the unit square $\Omega = [0,1]^2$ in all experiments.

TABLE 1. Numerical results by using SCFEM with $\nu = 0.1$: P_1-P_1 element.

1/h	$\frac{\|p_h-p\|_0}{\|p\|_0}$	$\frac{\|u_h-u\|_0}{\|u\|_0}$	$\frac{\|u_h-u\|_1}{\|u\|_1}$	p_{L^2} rate	u_{L^2} rate	u_{H^1} rate
10	0.0349235	0.0733102	0.305472			
20	0.00879073	0.0179178	0.150022	1.9901	2.0326	1.0259
30	0.00413155	0.00791444	0.0978188	1.8622	2.0152	1.0548
40	0.00246407	0.00451106	0.072288	1.7966	1.9541	1.0514
50	0.00166013	0.002973	0.057313	1.7698	1.8686	1.0403

5.1. An analytical solution: Convergence validation. Firstly, we set $\nu = 0.1,\ \rho = \nu,\ \Delta t = 0.0001,\ \delta = 100$. The exact solution for the velocity and pressure are

$$\begin{aligned}
u_1 &= 10x^2(x-1)^2 y(y-1)(2y-1)e^{-2\nu\pi^2 t},\\
u_2 &= -10x(x-1)(2x-1)y^2(y-1)^2 e^{-2\nu\pi^2 t},\\
p &= 20(2x-1)(2y-1)e^{-4\nu\pi^2 t}.
\end{aligned}$$

We present the numerical solution at the number of the iterations $n = 25$ with stabilized parameter $\alpha = 0.1$. Compared with the characteristic finite element method (CFEM) and standard Galerkin finite element method (GFEM) with stable MINI element, the errors of velocity among three methods are consistent, while the errors of pressure obtained by P_1-P_1 element bigger than MINI element's, see Tables 1-3. Of course, our method takes less CPU time than others, see Table 4.

Secondly, we consider the effect of varying parameter α on the errors with a fixed mesh size $h = \frac{1}{40}$. Figure 1 indicates the effect of α to the error of both

TABLE 2. Numerical results by using CFEM with $\nu = 0.1$: MINI element.

1/h	$\frac{\|p_h-p\|_0}{\|p\|_0}$	$\frac{\|u_h-u\|_0}{\|u\|_0}$	$\frac{\|u_h-u\|_1}{\|u\|_1}$	p_{L^2} rate	u_{L^2} rate	u_{H^1} rate
10	0.0102549	0.0897054	0.867995			
20	0.00197681	0.0194421	0.370599	2.3751	2.2060	1.2278
30	0.000873554	0.00662801	0.182261	2.0142	2.6541	1.7503
40	0.0004945	0.00320231	0.111201	1.9780	2.5286	1.7175
50	0.000320948	0.00191897	0.077685	1.9372	2.2949	1.6074

TABLE 3. Numerical results by using GFEM with $\nu = 0.1$: MINI element.

1/h	$\frac{\|p_h-p\|_0}{\|p\|_0}$	$\frac{\|u_h-u\|_0}{\|u\|_0}$	$\frac{\|u_h-u\|_1}{\|u\|_1}$	p_{L^2} rate	u_{L^2} rate	u_{H^1} rate
10	0.0102731	0.0896931	0.867844			
20	0.00197971	0.0194401	0.370536	2.3755	2.2060	1.2278
30	0.000872481	0.00662683	0.182232	2.0208	2.6543	1.7503
40	0.000491257	0.00319926	0.111185	1.9966	2.5313	1.7175
50	0.000315096	0.00191276	0.0776747	1.9902	2.3051	1.6073

TABLE 4. CPU(s) time of different methods.

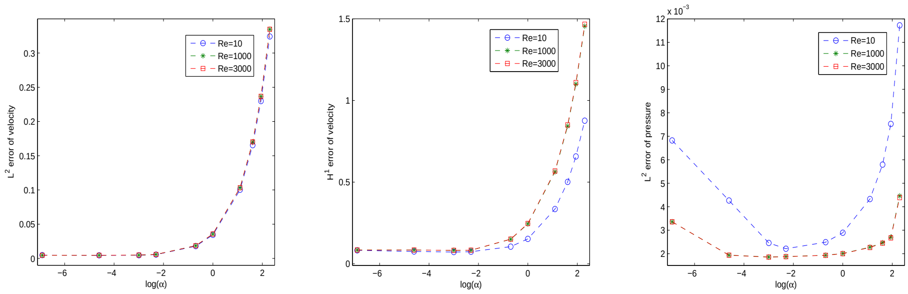
1/h	10	20	30	40	50
SCFEM P_1-P_1 element	1.5	6.453	14.765	26.406	41.719
CFEM MINI element	1.938	8.266	18.954	34.062	54.204
GFEM MINI element	3.594	14.594	33.25	59.515	94.953

velocity and pressure at different Reynolds numbers. From Fig.1 (a)-(b), we can see that the error of velocity becomes bigger as the parameter α increasing, while the error of pressure arrives the minimum when $\alpha = 0.1$, see Fig.1(c). Figure 2 shows the errors of velocity and pressure at different Reynolds numbers with some fixed parameters α. From these figures, we can see that the errors of velocity and pressure become larger as the parameter increasing. It seems that we can get good results by choosing suitable parameter.

5.2. Lid-Driven cavity problem. Lid-driven cavity flow serves as a standard test case in computational fluid dynamics. We set $f = 0$ and the boundary condition $u = 0$ on $[\{0\}\times(0,1)]\cup[(0,1)\times\{0\}]\cup[\{1\}\times(0,1)]$ and $u = (1,0)^T$ on $(0,1)\times\{1\}$, see Figure 3. The mesh size $h = \frac{1}{40}$, time step size $\triangle t = 0.01$, the final time $T = 100$.

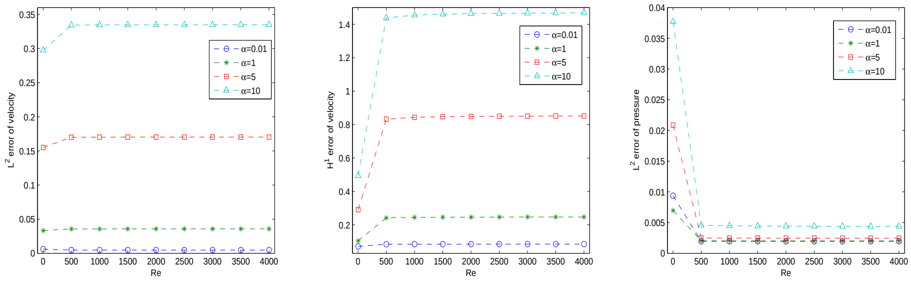
Figure 4 displays the velocity profiles passing through the geometric center of the cavity with different parameters. In order to verify the performance of SCFEM, we compute the lid-driven cavity problem by using characteristic method with P_2-P_1 element in $h = \frac{1}{100}$ and compare these results with our algorithm's. From these figures, we can see that the velocity profiles are in better agreement with the P_2-P_1's results as the parameter increases. The reason may be that the condition number of the coefficient matrix which is obtained by the variational formulation becomes smaller and smaller. Generally, the following linear algebra equations can be obtained from the discrete system from system (15)

$$\begin{pmatrix} A & -D \\ D^T & G \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix},$$



(a) (b) (c)

FIGURE 1. Effects of varying α at different Reynold numbers. (a) L^2 error of velocity, (b) H^1 error of velocity, (c) L^2 error of pressure.



(a) (b) (c)

FIGURE 2. Effects of varying α at different Reynold numbers. (a) L^2 error of velocity, (b) H^1 error of velocity, (c) L^2 error of pressure.

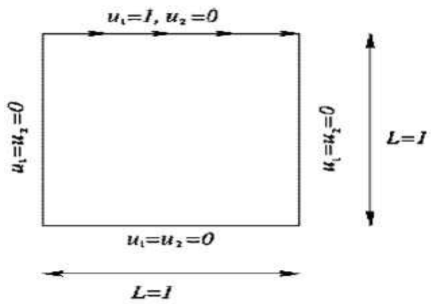
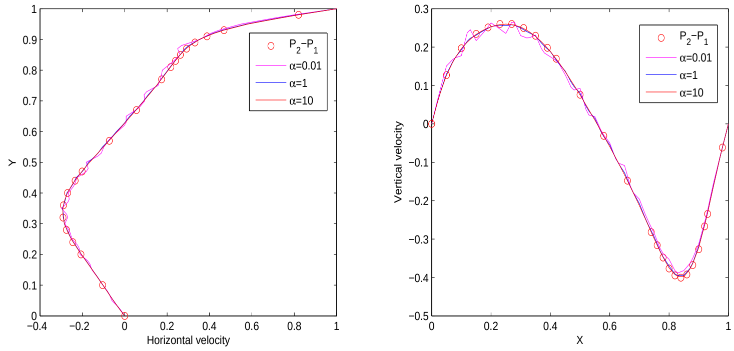


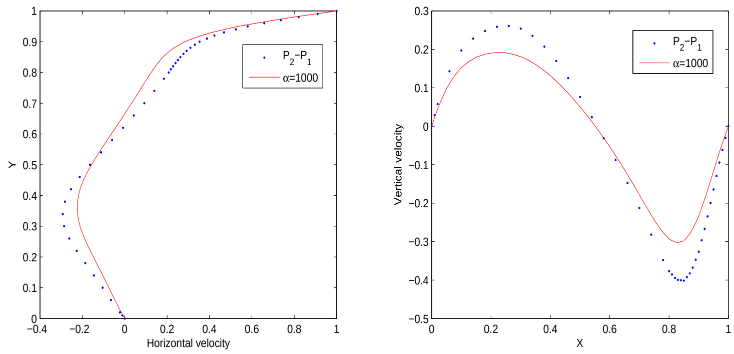
FIGURE 3. Lid-driven cavity flow.

where the matrices A, D and G are deduced in the usual manner from the bilinear forms $a(\cdot,\cdot), M(\cdot,\cdot), d(\cdot,\cdot)$ and $G(\cdot,\cdot)$, F is the variation of the source term. The norm of matrix A gets smaller as the Reynolds number increases. In order to obtain a good behavior of matrix $\begin{pmatrix} A & -D \\ D^T & G \end{pmatrix}$, we should choose a proper G, i.e., we should choose appropriate size of stabilized parameter α. This may explain why the numerical results with $\alpha = 10$ are better than the data obtained with $\alpha = 0.01$.



(a) (b)

FIGURE 4. The computed velocity profiles passing through the geometric center at T=100 at $Re = 3000$ with different α, (a) horizontal velocity, (b) vertical velocity.



(a) (b)

FIGURE 5. The computed velocity profiles passing through the geometric center at T=100 at $Re = 3000$ with $\alpha = 1000$, (a) horizontal velocity, (b) vertical velocity.

Nevertheless, it does not mean the larger the parameter, the better the numerical results, see Figure 5. Therefore, we can say that the stabilized characteristic finite element method is stable and efficient for the viscoelastic fluid problem with the suitable parameter.

6. Conclusion

In this paper, we combine the characteristic scheme with stabilized method to solve the viscoelastic Oldroyd fluid flows. This combination is efficient and retains the advantages of both algorithms and avoids their deficits. Unconditional stability and optimal error estimates are derived, Finally, some numerical results are presented to illustrate the performance of our method.

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