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A FINITE ELEMENT DUAL SINGULAR FUNCTION METHOD TO SOLVE THE STOKES EQUATIONS INCLUDING CORNER SINGULARITIES

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Abstract. The finite element dual singular function method [FE-DSFM] has been constructed and analyzed accuracy by Z. Cai and S. Kim to solve the Laplace equation on a polygonal domain with one reentrant corner. In this paper, we impose FE-DSFM to solve the Stokes equations via the mixed finite element method. To do this, we compute the singular and the dual singular functions analytically at a non-convex corner. We prove well-posedness by using the contraction mapping theorem and then estimate errors of the algorithm. We obtain optimal accuracy $O(h)$ for velocity in $\mathbf{H}^1(\Omega)$ and pressure in $L^2(\Omega)$, but we are able to prove only $O(h^{1+\lambda})$ error bounds for velocity in $\mathbf{L}^2(\Omega)$ and stress intensity factor, where λ is the eigenvalue (solution of (4)). However, we get optimal accuracy results in numerical experiments.

Key words. Stokes equations, dual singular function method, corner singularity, incompressible fluids.

1. Introduction

Solutions of elliptic boundary value problems on a domain with corners have singular behaviors near the corners. This occurs even when the given data of the governed equations is very smooth. Such singular behavior affects the accuracy of the finite element method throughout the whole domain. In order to overcome the singularity problem, the finite element dual singular function method [FE-DSFM] has been constructed in [3] to solve the Laplace equation and performed numerical tests in [4]. And then it is extended to solve the Helmholtz equation in [9] and the interface problem in [8]. The goal of this paper is to reconstruct FE-DSFM to solve Stokes equations:

$$
\begin{aligned}
-\mu\triangle\mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega,\\
\nabla\cdot\mathbf{u} &= 0, \quad \text{in } \Omega,\\
\mathbf{u} &= \mathbf{0}, \quad \text{on } \partial\Omega,
\end{aligned}
\tag{1}
$$

with $\mathbf{f}$ is a given function in $\mathbf{H}^{-1}(\Omega)$, Ω is a computational domain, and $\mu = Re^{-1}$ is the reciprocal of the Reynolds number. Here the unknowns are the (vector) velocity field $\mathbf{u} \in \mathbf{H}^1_0(\Omega)$ and the (scalar) pressure $p \in L^2_0(\Omega)$.

If the solution of (1) is smooth enough, namely $(\mathbf{u}, p) \in \mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$ with $s \geq 1$, and if a suitable finite element pair is imposed for velocity and pressure, then the finite element solution $(\mathbf{u}_h, p_h)$ using the standard mixed method has optimal error bounds as shown in [1, 6]:

$$
\|\mathbf{u}-\mathbf{u}_h\|_0 + h\|\mathbf{u}-\mathbf{u}_h\|_1 + h\|p-p_h\|_0 \leq Ch^{s+1}\left(\|\mathbf{u}\|_{s+1} + \|p\|_s\right), \tag{2}
$$

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where h is the biggest mesh size. However, if $s < 1$, then the error bounds of the method become only

$$\|\mathbf{u}-\mathbf{u}_h\|_0 + h^s\|\mathbf{u}-\mathbf{u}_h\|_1 + h^s\|p-p_h\|_0 \le Ch^{2s}\left(\|\mathbf{u}\|_{s+1} + \|p\|_s\right). \tag{3}$$

So we call $(\mathbf{u}, p)$ a singular solution for the case $s < 1$, otherwise a regular solution. Because the singularity is due to reentrant corners of computational domain Ω, we assume that Ω is an open and bounded polygonal domain in $\mathbb{R}^2$ with one reentrant corner. Extension to the domain with a finite number of reentrant corners is straightforward.

Let ω be the internal angle. Without the loss of generality, we assume that the corresponding vertex is at the origin and that the internal angle ω is spanned by the two half-lines $\theta = 0$ and $\theta = \omega$. We denote Γ_{in} for 2 edges on the boundary including the reentrant corner and Γ_{out} for other parts of the boundary. Even though the singular functions are already computed in [10], we will derive those again in §6 to get more advanced properties of the singular functions and newly find out the dual singular functions in (8) below.

The singular function $(\mathbf{u}_s, p_s)$, where $\mathbf{u}_s = (u_s, v_s)$, can be summarized with the eigenvalue $\lambda(> 0)$ which is the solution of

$$\sin^2(\lambda\omega) = \lambda^2 \sin^2(\omega), \tag{4}$$

by

$$\begin{pmatrix} u_d \\ v_d \\ p_d \end{pmatrix} = d_1 \begin{pmatrix} -r^{-\lambda}\dfrac{\lambda}{\mu}\sin(\theta)\sin((1+\lambda)\theta) \\ -r^{-\lambda}\dfrac{1}{\mu}\left(\sin(\lambda\theta) - \lambda\sin(\theta)\cos((1+\lambda)\theta)\right) \\ 2r^{-\lambda-1}\lambda\cos((1+\lambda)\theta) \end{pmatrix} + d_2 \begin{pmatrix} r^{-\lambda}\dfrac{1}{\mu}\left(\sin(\lambda\theta) + \lambda\sin(\theta)\cos((1+\lambda)\theta)\right) \\ r^{-\lambda}\dfrac{\lambda}{\mu}\sin(\theta)\sin((1+\lambda)\theta) \\ 2r^{-\lambda-1}\lambda\sin((1+\lambda)\theta) \end{pmatrix}, \tag{5}$$

where

$$C_1 = \sin(\lambda\omega) + \lambda\sin(\omega)\cos((1-\lambda)\omega) \quad \text{and} \quad C_2 = \lambda\sin(\omega)\sin((1-\lambda)\omega).$$

We note that the singular function $(\mathbf{u}_s, p_s)$ is the solution of homogeneous Stokes equations with vanishing Dirichlet boundary condition at Γ_{in}. And λ has to be a positive real number and $(\mathbf{u}_s, p_s) \in \mathbf{H}^{1+\lambda}(\Omega) \times H^\lambda(\Omega)$. As the conclusion in Lemma 6.1 below, $\lambda = 1$ for any $\omega \le \pi$, so $(\mathbf{u}_s, p_s) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$ is a regular solution and it becomes a singular solution for the case $\lambda < 1$, namely $\omega > \pi$. Moreover (4) has a unique non-trivial solution $\lambda \in \mathbb{R}$ for the case $\pi < \omega \le \beta\pi$. where $\beta :\approx 1.430296653124203$. And (4) has 2 non-trivial real solutions $0.5 < \lambda_1 < \lambda_2 < 1$ for the case $\omega \in (\beta\pi, 2\pi)$. In addition, $\lambda = 0.5$ is the unique non-trivial solution for $\omega = 2\pi$.

Let η be a smooth cut-off function which is equal one identically in neighborhood of origin, and the support of η is small enough so that the functions $\eta\mathbf{u}_s$ vanishes identically on $\partial\Omega$. Then, in general, the solution $(\mathbf{u}, p)$ including singular parts of

(1) can be rewritten with 2 singular functions as

$$\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{w} \\ q \end{pmatrix} + \alpha_1 \begin{pmatrix} \eta_1 \mathbf{u}_{s,1} \\ \eta_1 p_{s,1} \end{pmatrix} + \alpha_2 \begin{pmatrix} \eta_2 \mathbf{u}_{s,2} \\ \eta_2 p_{s,2} \end{pmatrix}, \tag{6}$$

where α_1 and α_2 are the stress intensity factors and $(\mathbf{w}, q) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$. However, for a simpler explanation, we consider $(\mathbf{u}, p)$ including only one singular function, namely,

$$\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{w} \\ q \end{pmatrix} + \alpha \begin{pmatrix} \eta \mathbf{u}_s \\ \eta p_s \end{pmatrix}, \tag{7}$$

because FE-DSFM can be extended to problems with 2 singular solutions of the form (6) by applying the adjusted Sherman-Morrison-Woodbury formula without any additional skill, as mentioned in Remark 2.4 below.

The goal of this paper is to construct FE-DSFM to have optimal error decay (2) for the Stokes equations (1) in a non-convex domain with $\omega > \pi$. The main strategy of FE-DSFM is to compute the regular solution $(\mathbf{w}, q)$ and the stress intensity factors α by applying the finite element method. To built FE-DSFM, we need to use the following dual singular functions $(\mathbf{u}_d, p_d)$, where $\mathbf{u}_d = (u_d, v_d)$, which is derived in §6.2:

$$\begin{pmatrix} u_d \\ v_d \\ p_d \end{pmatrix} = d_1 \begin{pmatrix} -r^{-\lambda} \dfrac{\lambda}{\mu} \sin(\theta) \sin((1+\lambda)\theta) \\ -r^{-\lambda} \dfrac{1}{\mu} \left(\sin(\lambda\theta) - \lambda \sin(\theta) \cos((1+\lambda)\theta)\right) \\ 2r^{-\lambda-1} \cos((1+\lambda)\theta) \end{pmatrix} + d_2 \begin{pmatrix} r^{-\lambda} \dfrac{1}{\mu} \left(\sin(\lambda\theta) + \lambda \sin(\theta) \cos((1+\lambda)\theta)\right) \\ r^{-\lambda} \dfrac{\lambda}{\mu} \sin(\theta) \sin((1+\lambda)\theta) \\ 2r^{-\lambda-1} \sin((1+\lambda)\theta) \end{pmatrix}, \tag{8}$$

where

$$d_1 = \sin(\lambda\omega) + \lambda \sin(\omega) \cos((1+\lambda)\omega) \quad \text{and} \quad d_2 = \lambda \sin(\omega) \sin((1+\lambda)\omega).$$

The paper is organized as follows. FE-DSFM to find the smooth part $(\mathbf{w}_h, q_h)$ of the solution and stress intensity factor α_h is constructed in (20) and (21) in §2. And we establish the well-posedness of the variational form of FE-DSFM in §3. We carry out the following error estimates in §4 by proving several lemmas.

Theorem 1 (Main theorem). *Let Assumption 1 below hold. If h is small enough, then we have*

$$|\alpha - \alpha_h| + \|\mathbf{w} - \mathbf{w}_h\|_0 \le C h^{1+\lambda} \quad \text{and} \quad \|\mathbf{w} - \mathbf{w}_h\|_1 + \|q - q_h\|_0 \le C h. \tag{9}$$

The sub-optimality for $|\alpha - \alpha_h| + \|\mathbf{w} - \mathbf{w}_h\|_0$ is due to the weak regularity of the functions in duality argument, as discussed in Remark 4.4. In §5, we perform numerical tests with known solution to compare to theoretical results in the Theorem 1. In these tests, we obtain the optimal accuracy (2) which is better than (9). Finally, we present the singular and the dual singular functions in §6.

2. The finite element dual singular function method

In this section, we build a new variational formulation to find the regular part of solution $(\mathbf{w}, q)$ and the stress intensity factor α. We start this section with introducing the following lemma for the properties of the singular and the dual singular functions.

Lemma 2.1 (Properties of singular and dual singular functions)**.** *The singular function* $(\mathbf{u}_s, p_s) \in \mathbf{H}^{1+\lambda}(\Omega) \times H^{\lambda}(\Omega)$ *and the dual singular function* $(\mathbf{u}_d, p_d) \notin \mathbf{H}^1(\Omega) \times L^2(\Omega)$ *are solutions of*

$$\begin{aligned} -\mu\triangle\mathbf{u}_s + \nabla p_s &= \mathbf{0}, \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u}_s &= 0, \quad \text{in } \Omega, \end{aligned} \tag{10}$$

and

$$\begin{aligned} -\mu\triangle\mathbf{u}_d + \nabla p_d &= \mathbf{0}, \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u}_d &= 0, \quad \text{in } \Omega, \end{aligned} \tag{11}$$

respectively. The boundary conditions of $\mathbf{u}_s$ *and* $\mathbf{u}_d$ *vanish on* Γ_{in}*, but the boundary value of* $\mathbf{u}_d$ *is not defined at the origin. Both of* $\mathbf{u}_s$ *and* $\mathbf{u}_d$ *are not* $\mathbf{0}$ *on* Γ_{out}*.*

In order to derive a explicit form of the singular functions, we set

$$B(r_1; r_2) = \{(r,\theta) : r_1 < r < r_2 \text{ and } 0 < \theta < \omega\} \cap \Omega$$

and

$$B(r_1) = B(0; r_1),$$

and define a smooth enough cut-off function of r, η_ρ, as follows:

$$\eta_\rho = \begin{cases} 1, & \text{in } B(\frac{1}{2}\rho R), \\ \text{very smooth function}, & \text{in } B(\frac{1}{2}\rho R; \rho R), \\ 0, & \text{in } \Omega \setminus \bar{B}(\rho R), \end{cases}$$

where ρ is a parameter in $(0, 2]$ and R is a fixed real number which will be determined later so that the singular part of $\eta_{2\rho}\mathbf{u}_s$ has $\mathbf{0}$ on whole $\partial\Omega$. Here and thereafter, we choose that $\eta = \eta_\rho$ in (7) and assume that $0 < \rho < 1$. That is, the singular function representation of the solution of problem (1) has the form

$$\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{w} \\ q \end{pmatrix} + \alpha \begin{pmatrix} \eta_\rho \mathbf{u}_s \\ \eta_\rho p_s \end{pmatrix},$$

where $\mathbf{w} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$ and $q \in \mathbf{L}_0^2(\Omega) \cap H^1(\Omega)$ satisfy

$$\begin{aligned} (-\mu\triangle\mathbf{w} + \nabla q) + \alpha\left(-\mu\triangle(\eta_\rho\mathbf{u}_s) + \nabla(\eta_\rho p_s)\right) &= \mathbf{f}, \quad \text{in } \Omega, \\ \nabla\cdot(\mathbf{w} + \alpha\eta_\rho\mathbf{u}_s) &= 0, \quad \text{in } \Omega. \end{aligned} \tag{12}$$

For the sake of a clear explanation, we note that the inner product of vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ is

$$\langle \mathbf{a}\,,\, \mathbf{b}\rangle = \langle a_1\,,\, b_1\rangle + \langle a_2\,,\, b_2\rangle$$

and

$$\langle \nabla\mathbf{a}\,,\, \nabla\mathbf{b}\rangle = \langle \partial_x a_1\,,\, \partial_x b_1\rangle + \langle \partial_x a_2\,,\, \partial_x b_2\rangle + \langle \partial_y a_1\,,\, \partial_y b_1\rangle + \langle \partial_y a_2\,,\, \partial_y b_2\rangle\,.$$

Then we can obtain the weak form for (12) by the standard finite element technique: find $(\mathbf{w}, q) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)$ satisfying, for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ and $\phi \in L^2(\Omega)$,

$$\begin{aligned} \mu\,\langle \nabla\mathbf{w}\,,\, \nabla\mathbf{v}\rangle + \langle \nabla q\,,\, \mathbf{v}\rangle + \alpha\,\langle -\mu\triangle(\eta_\rho\mathbf{u}_s) + \nabla(\eta_\rho p_s)\,,\, \mathbf{v}\rangle &= \langle \mathbf{f}\,,\, \mathbf{v}\rangle\,, \\ \langle \nabla\cdot\mathbf{w}\,,\, \phi\rangle + \alpha\,\langle \nabla\cdot(\eta_\rho\mathbf{u}_s)\,,\, \phi\rangle &= 0. \end{aligned} \tag{13}$$

Because 3 unknown variables $(\mathbf{w}, q)$ and α are involved in 2 equations of (13), we need one more linearly independent equation with (13). We test $\eta_{2\rho}\mathbf{u}_d \notin \mathbf{H}^1(\Omega)$ and $\eta_{2\rho}p_d \notin L^2(\Omega)$ with the first and the second equations in (12), respectively, to get the additional equation

$$(14)\qquad \begin{aligned} \langle -\mu\triangle\mathbf{w} + \nabla q\,,\, \eta_{2\rho}\mathbf{u}_d\rangle + \alpha\,\langle -\mu\triangle(\eta_\rho\mathbf{u}_s) + \nabla(\eta_\rho p_s)\,,\, \eta_{2\rho}\mathbf{u}_d\rangle &= \langle \mathbf{f}\,,\, \eta_{2\rho}\mathbf{u}_d\rangle\,, \\ \langle \nabla\cdot(\mathbf{w} + \alpha\eta_\rho\mathbf{u}_s)\,,\, \eta_{2\rho}p_d\rangle &= 0. \end{aligned}$$

Because the dual singular functions are not smooth enough to apply the integration by parts directly in (14), the following lemma is crucial.

Lemma 2.2 (Integration by parts for dual singular functions)**.** *For $\rho \in (0, 1]$, we have that*

$$(15)\qquad -\mu\,\langle \triangle\mathbf{w}\,,\, \eta_{2\rho}\mathbf{u}_d\rangle - \langle \nabla\cdot\mathbf{w}\,,\, \eta_{2\rho}p_d\rangle = \langle \mathbf{w}\,,\, -\mu\triangle(\eta_{2\rho}\mathbf{u}_d) + \nabla(\eta_{2\rho}p_d)\rangle$$

and

$$(16)\qquad \langle \nabla q\,,\, \eta_{2\rho}\mathbf{u}_d\rangle = -\,\langle q\,,\, \nabla\cdot(\eta_{2\rho}\mathbf{u}_d)\rangle\,.$$

Proof. We can obtain (15) by integration by parts, if all functions inside the inner products are smooth enough. So, in light of density argument of Hilbert space, we need to show boundedness of both sides. Since $\mathbf{w} \in \mathbf{H}^1_0(\Omega) \cap \mathbf{H}^2(\Omega)$ and $\eta_{2\rho}\mathbf{u}_d \in \mathbf{L}^2(\Omega)$ and since $\nabla\cdot\mathbf{w} = 0$ in $B(\frac{1}{2}\rho)$, we can get the boundedness of the left hand side in (15). On the other hand, the right hand side in (15) is also bounded, because of (11) and the definition of $\eta_{2\rho}$. So we conclude (15). By the same manner, the properties $q \in H^1(\Omega)$, $\eta_{2\rho}\mathbf{u}_d \in \mathbf{L}^2(\Omega)$, and $\nabla\cdot(\eta_{2\rho}\mathbf{u}_d) = 0$ in $B(\rho)$ yield (16). ■

We subtract the second equation from the first equation in (14) and then apply Lemma 2.2 to obtain

$$(17)\qquad \begin{aligned} \alpha(\beta_m - \beta_p) &= \beta_{\mathbf{f}} - \langle -\mu\triangle\mathbf{w} + \nabla q\,,\, \eta_{2\rho}\mathbf{u}_d\rangle + \langle \nabla\cdot\mathbf{w}\,,\, \eta_{2\rho}p_d\rangle \\ &= \beta_{\mathbf{f}} - \langle \mathbf{w}\,,\, -\mu\triangle(\eta_{2\rho}\mathbf{u}_d) + \nabla(\eta_{2\rho}p_d)\rangle + \langle q\,,\, \nabla\cdot(\eta_{2\rho}\mathbf{u}_d)\rangle\,, \end{aligned}$$

where

$$\begin{aligned} \beta_{\mathbf{f}} &:= \langle \mathbf{f}\,,\, \eta_{2\rho}\mathbf{u}_d\rangle\,, \\ \beta_m &:= \langle -\mu\triangle(\eta_\rho\mathbf{u}_s) + \nabla(\eta_\rho p_s)\,,\, \eta_{2\rho}\mathbf{u}_d\rangle\,, \\ \beta_p &:= \langle \nabla\cdot(\eta_\rho\mathbf{u}_s)\,,\, \eta_{2\rho}p_d\rangle\,. \end{aligned}$$

Finally we arrive at the following lemma to compute the stress intensity factor α.

Lemma 2.3 (Formula for the stress intensity factor α)**.** *The values $|\beta_{\mathbf{f}}|$, $|\beta_m|$, and $|\beta_p|$ are bounded and the stress intensity factor α can be expressed in terms of $(\mathbf{w}, q)$ by the following extraction formula:*

$$(18)\qquad \alpha = \frac{1}{\beta_m - \beta_p}\left(\beta_{\mathbf{f}} - \langle \mathbf{w}\,,\, -\mu\triangle(\eta_{2\rho}\mathbf{u}_d) + \nabla(\eta_{2\rho}p_d)\rangle + \langle q\,,\, \nabla\cdot(\eta_{2\rho}\mathbf{u}_d)\rangle\right).$$

Proof. Because (18) comes directly from (17), it is enough to show boundedness of 3 values $\beta_{\mathbf{f}}$, β_m, and β_p. First, $|\beta_{\mathbf{f}}| < \infty$ is trivial, because of both $\mathbf{f}$ and $\eta_{2\rho}\mathbf{u}_d \in \mathbf{L}^2(\Omega)$. In light of the definition of η_ρ, (10) yields $-\mu\triangle(\eta_\rho\mathbf{u}_s) + \nabla(\eta_\rho p_s) = \mathbf{0}$ on $B(\frac{1}{2}\rho) \cup (\Omega - B(\rho))$, or $-\mu\triangle(\eta_\rho\mathbf{u}_s) + \nabla(\eta_\rho p_s)$ is a smooth enough function. So we readily get $|\beta_m| < \infty$. By the same manner, we can readily obtain $|\beta_p| < \infty$ from $\nabla\cdot(\eta_\rho\mathbf{u}_s) = 0$ in $B(\frac{1}{2}\rho)$. ■

We note that both $-\mu\triangle(\eta_{2\rho}\mathbf{u}_d) + \nabla(\eta_{2\rho}p_d)$ and $\nabla\cdot(\eta_{2\rho}\mathbf{u}_d)$ are smooth enough functions, because the singular parts are removed by the properties $-\mu\triangle(\eta_{2\rho}\mathbf{u}_d) + \nabla(\eta_{2\rho}p_d) = \mathbf{0}$ and $\nabla\cdot(\eta_{2\rho}\mathbf{u}_d) = 0$ in $B(\rho)$. So we can compute (18) without any special technique to treat singular functions. In order to solve the coupled system (13) and (18) to find 3 unknown variables $(\mathbf{w}, q)$ and α, we insert α in (18) into (13), and then we obtain a mixed formula

$$
\begin{aligned}
\mu \langle \nabla \mathbf{w} \, , \, \nabla \mathbf{v} \rangle + \langle \nabla q \, , \, \mathbf{v} \rangle + (a(\mathbf{w}) + b(q))\, c(\mathbf{v}) &= \langle \mathbf{f} \, , \, \mathbf{v} \rangle - \frac{\beta_{\mathbf{f}}}{\beta_m - \beta_p} c(\mathbf{v}), \\
\langle \nabla \cdot \mathbf{w} \, , \, \phi \rangle + (a(\mathbf{w}) + b(q))\, d(\phi) &= -\frac{\beta_{\mathbf{f}}}{\beta_m - \beta_p} d(\phi),
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
a(\mathbf{w}) &= \frac{-1}{\beta_m - \beta_p} \langle \mathbf{w} \, , \, -\mu \triangle(\eta_{2\rho} \mathbf{u}_d) + \nabla(\eta_{2\rho} p_d) \rangle \, , \quad & b(q) &= \frac{1}{\beta_m - \beta_p} \langle q \, , \, \nabla \cdot (\eta_{2\rho} \mathbf{u}_d) \rangle \, , \\
c(\mathbf{v}) &= \langle -\mu \triangle(\eta_\rho \mathbf{u}_s) + \nabla(\eta_\rho p_s) \, , \, \mathbf{v} \rangle \, , & d(\phi) &= \langle \nabla \cdot (\eta_\rho \mathbf{u}_s) \, , \, \phi \rangle \, .
\end{aligned}
$$

In order to introduce the finite element discretization, we need further notations. Let $\mathfrak{T} = \{K\}$ be a shape-regular quasi-uniform partition of Ω of meshsize h into closed elements K [1, 2, 6]. The vector and scalar finite element spaces are:

$$
\begin{aligned}
&\mathbb{W}_h := \{\mathbf{w}_h \in \mathbf{L}^2(\Omega) : \mathbf{w}_h|_K \in \mathcal{P}(K) \quad \forall K \in \mathfrak{T}\}, \quad \mathbb{V}_h := \mathbb{W}_h \cap \mathbf{H}^1_0(\Omega), \\
&\mathbb{P}_h := \{q_h \in L^2_0(\Omega) \cap C^0(\Omega) : q_h|_K \in \mathcal{Q}(K) \quad \forall K \in \mathfrak{T}\},
\end{aligned}
$$

where $\mathcal{P}(K)$ and $\mathcal{Q}(K)$ are spaces of polynomials with degree bounded uniformly with respect to $K \in \mathfrak{T}$. We stress that the space $\mathbb{P}_h$ is composed of continuous functions to use integration by parts: for all $q_h \in \mathbb{P}_h$

$$
\langle \nabla \cdot \mathbf{v}_h \, , \, q_h \rangle = - \langle \mathbf{v}_h \, , \, \nabla q_h \rangle \, , \quad \forall \mathbf{v}_h \in \mathbb{V}_h.
$$

Finally, we arrive at FE-DSFM from (18) and (19): find $(\mathbf{w}_h, q_h) \in \mathbb{V}_h \times \mathbb{P}_h$ as the solution of, for all $\mathbf{v}_h \in \mathbb{V}_h$ and for all $\phi_h \in \mathbb{P}_h$,

(20)

$$
\begin{aligned}
\mu \langle \nabla \mathbf{w}_h \, , \, \nabla \mathbf{v}_h \rangle + \langle \nabla q_h \, , \, \mathbf{v}_h \rangle + (a(\mathbf{w}_h) + b(q_h))\, c(\mathbf{v}_h) &= \langle \mathbf{f} \, , \, \mathbf{v}_h \rangle - \frac{\beta_{\mathbf{f}}}{\beta_m - \beta_p} c(\mathbf{v}_h), \\
\langle \nabla \cdot \mathbf{w}_h \, , \, \phi_h \rangle + (a(\mathbf{w}_h) + b(q_h))\, d(\phi_h) &= -\frac{\beta_{\mathbf{f}}}{\beta_m - \beta_p} d(\phi_h),
\end{aligned}
$$

and then find $\alpha_h \in \mathbb{R}$ by computing

$$
\alpha_h = \frac{\beta_{\mathbf{f}}}{\beta_m - \beta_p} + a(\mathbf{w}_h) + b(q_h). \tag{21}
$$

We note that $a(\cdot)$, $b(\cdot)$, $c(\cdot)$, and $d(\cdot)$ are the same functionals within (19).

The matrix form of the coupled system (20) becomes

$$
\left[\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} a, b \end{pmatrix} \right] \begin{pmatrix} \mathbf{w} \\ q \end{pmatrix} = \begin{pmatrix} \mathbf{L} \\ l \end{pmatrix}, \tag{22}
$$

and it is solvable by the Sherman-Morrison-Woodbury formula in [7].

$$
(M + U \cdot V^T)^{-1} = M^{-1} - \frac{M^{-1} U V^T M^{-1}}{1 + V^T M^{-1} U}
$$

and then we find α_h by computing (21). So we conclude that FE-DSFM (20) and (21) is an applicable algorithm.

Even though we consider the solution including only one singular part in this paper, FE-DSFM can be applied to find a solution including 2 singular parts by the following remark:

Remark 2.4 (The solution with 2 singular parts). *If the solution has 2 stress intensity factors, then the solution is expressed by the form* (6) *and the matrix form* (22) *becomes*

$$\left[\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} + \sum_{i=1}^{2} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \begin{pmatrix} a_i, b_i \end{pmatrix} \right] \begin{pmatrix} \mathbf{w} \\ q \end{pmatrix} = \begin{pmatrix} \mathbf{L} \\ l \end{pmatrix}.$$

It is also solvable by using the generalized Sherman-Morrison-Woodbury formula:

$$(M + U_1 \cdot V_1^T + U_2 \cdot V_2^T)^{-1} = M^{-1} - M^{-1}[U1, U_2]Q^{-1}[V_1, V_2]^T M^{-1},$$

where Q *is* 2×2 *matrix given by*

$$Q = \begin{bmatrix} 1 + V_1^T M^{-1} U_1, & V_1^T M^{-1} U_2 \\ V_2^T M^{-1} U_1, & 1 + V_2^T M^{-1} U_2 \end{bmatrix}.$$

3. Well-posedness

In this section, we establish the well-posedness of the coupled system (19) by the use of the contraction mapping theorem. Because solving the system (19) is equivalent to solving equations of (13) and (18), we will prove the well-posedness of the system (13) and (18). To do this, we first check the compatibility condition of (13) as

$$\int_\Omega \nabla \cdot (\eta_\rho \mathbf{u}_s) dx = \int_{\partial\Omega} \eta_\rho \mathbf{u}_s \cdot \boldsymbol{\nu} ds = 0,$$

where $\boldsymbol{\nu}$ is the outward unit normal vector. The system (13) is a standard saddle point problem and has a unique solution for all given $\alpha \in \mathbb{R}$ and for all $\mathbf{f} \in \mathbf{L}^2(\Omega)$ in [1, 2, 6]. So we can define a mapping $T_\mathbf{f}$ from $\mathbb{R}$ to $\mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ by the unique solution of (13) for any given $\mathbf{f} \in \mathbf{L}^2(\Omega)$. It means that $T_\mathbf{f}(\alpha) := (\mathbf{w}_\alpha, q_\alpha)$ is the solution of (13) with $\alpha \in \mathbb{R}$. Also we define a mapping T_α from $\mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ to $\mathbb{R}$ by using (18) as

(23)

$$T_\alpha(\mathbf{w}, q) := \frac{1}{\beta_m - \beta_p} \left(\beta_\mathbf{f} - \langle \mathbf{w}\,,\, -\mu \triangle(\eta_{2\rho} \mathbf{u}_d) + \nabla(\eta_{2\rho} p_d) \rangle + \langle q\,,\, \nabla \cdot (\eta_{2\rho} \mathbf{u}_d) \rangle \right).$$

Then the composition $T_\alpha \circ T_\mathbf{f}$ is a mapping from $\mathbb{R}$ to $\mathbb{R}$. To prove the well-posedness of the problem (19) is equivalent to prove existence of the unique fixed point of $T_\alpha \circ T_\mathbf{f}$, and it is equivalent to prove $\|T_\alpha \circ T_\mathbf{f}\| < 1$. We now start to prove $\|T_\alpha \circ T_\mathbf{f}\| < 1$ by applying the contraction mapping theorem [11]. To do this, we choose arbitrary real numbers α_1 and α_2, and let

$$T_\mathbf{f}(\alpha_1) = (\mathbf{w}_{\alpha_1}, q_{\alpha_1}) \quad \text{and} \quad T_\mathbf{f}(\alpha_2) = (\mathbf{w}_{\alpha_2}, q_{\alpha_2}). \tag{24}$$

Then we can get from (13),

$$\begin{aligned} &\mu \langle \nabla (\mathbf{w}_{\alpha_1} - \mathbf{w}_{\alpha_2}), \nabla \mathbf{v} \rangle + \langle \nabla (q_{\alpha_1} - q_{\alpha_2}), \mathbf{v} \rangle \\ &\qquad\qquad = -(\alpha_1 - \alpha_2) \langle -\mu \triangle(\eta_\rho \mathbf{u}_s) + \nabla(\eta_\rho p_s), \mathbf{v} \rangle, \\ &\langle \nabla \cdot (\mathbf{w}_{\alpha_1} - \mathbf{w}_{\alpha_2}), \phi \rangle = -(\alpha_1 - \alpha_2) \langle \nabla \cdot (\eta_\rho \mathbf{u}_s), \phi \rangle. \end{aligned} \tag{25}$$

And we define the following Stokes type problem:

$$\begin{aligned} -\mu \triangle \mathbf{x} + \nabla k &= -\mu \triangle(\eta_{2\rho} \mathbf{u}_d) + \nabla(\eta_{2\rho} p_d), && \text{in } \Omega, \\ \nabla \cdot \mathbf{x} &= \nabla \cdot (\eta_{2\rho} \mathbf{u}_d), && \text{in } \Omega, \\ \mathbf{x} &= \mathbf{0}, && \text{on } \partial\Omega. \end{aligned} \tag{26}$$

We note here that the right hand side terms are smooth functions which come from Lemma 2.1. In order to assert the existence and uniqueness of the solution of (26), we need to check the compatibility condition. Since we have

$$\int_\Omega \nabla \cdot \mathbf{x} dx = \int_{\partial\Omega} \mathbf{x} \cdot \boldsymbol{\nu} ds = 0,$$

it is enough to prove $\int_\Omega \nabla \cdot (\eta_{2\rho}\mathbf{u}_d) = 0$. Because $\nabla \cdot (\eta_{2\rho}\mathbf{u}_d) = 0$ in $B(\rho)$ and $\eta_{2\rho}\mathbf{u}_d = \mathbf{0}$ on $\partial\Omega$ except the origin, there exists a function $\mathbf{y} \in H_0^1(\Omega)$ satisfying $\langle \nabla \cdot \mathbf{y}\, , \phi \rangle = \langle \nabla \cdot (\eta_{2\rho}\mathbf{u}_d)\, , \phi \rangle$, for all $\phi \in L^2(\Omega)$ and $\mathbf{y} = 0$ on $\partial\Omega$. So we can get

$$\int_\Omega \nabla \cdot (\eta_{2\rho}\mathbf{u}_d) dx = \int_\Omega \nabla \cdot \mathbf{y} dx = \int_{\partial\Omega} \mathbf{y} \cdot \boldsymbol{\nu} ds = 0 \tag{27}$$

and we obtain $\int_\Omega \nabla \cdot \mathbf{x} dx = \int_\Omega \nabla \cdot (\eta_{2\rho}\mathbf{u}_d) dx = 0$. Thus we conclude that there exist a unique solution $(\mathbf{x}, k) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ of (26).

We apply (24), (23) and (26) as a sequence and then use the integration by parts to obtain

$$\begin{aligned}
&\Big| T_\alpha \circ T_{\mathbf{f}}(\alpha_1) - T_\alpha \circ T_{\mathbf{f}}(\alpha_2) \Big| = \Big| T_\alpha \left(\mathbf{w}_{\alpha_1}, q_{\alpha_1} \right) - T_\alpha \left(\mathbf{w}_{\alpha_2}, q_{\alpha_2} \right) \Big| \\
=& \Big| \frac{1}{\beta_m - \beta_p} \big(- \langle \mathbf{w}_{\alpha_1} - \mathbf{w}_{\alpha_2}\, , -\mu \triangle (\eta_{2\rho}\mathbf{u}_d) + \nabla(\eta_{2\rho} p_d) \rangle + \langle q_{\alpha_1} - q_{\alpha_2}\, , \nabla \cdot (\eta_{2\rho}\mathbf{u}_d) \rangle \big) \Big| \\
=& \Big| \frac{1}{\beta_m - \beta_p} \big(- \langle \mathbf{w}_{\alpha_1} - \mathbf{w}_{\alpha_2}\, , -\mu \triangle \mathbf{x} + \nabla k \rangle + \langle q_{\alpha_1} - q_{\alpha_2}\, , \nabla \cdot \mathbf{x} \rangle \Big) \Big| \\
=& \Big| \frac{1}{\beta_m - \beta_p} \big(- \mu \left\langle \nabla \left(\mathbf{w}_{\alpha_1} - \mathbf{w}_{\alpha_2} \right), \nabla \mathbf{x} \right\rangle \\
&+ \left\langle \nabla \cdot \left(\mathbf{w}_{\alpha_1} - \mathbf{w}_{\alpha_2} \right), k \right\rangle - \left\langle \nabla \left(q_{\alpha_1} - q_{\alpha_2} \right), \mathbf{x} \right\rangle \Big) \Big|.
\end{aligned}$$

Invoking (25) with $\mathbf{v} = \mathbf{x}$ and $\phi = k$, we have

$$\begin{aligned}
&\Big| T_\alpha \circ T_{\mathbf{f}}(\alpha_1) - T_\alpha \circ T_{\mathbf{f}}(\alpha_2) \Big| \\
=& \Big| \frac{\alpha_1 - \alpha_2}{\beta_m - \beta_p} \big(\langle -\mu \triangle (\eta_\rho \mathbf{u}_s) + \nabla (\eta_\rho p_s)\, , \mathbf{x} \rangle - \langle \nabla \cdot (\eta_\rho \mathbf{u}_s)\, , k \rangle \Big) \Big|.
\end{aligned}$$

Integration by parts derives

$$\Big| T_\alpha \circ T_{\mathbf{f}}(\alpha_1) - T_\alpha \circ T_{\mathbf{f}}(\alpha_2) \Big| = \Big| \frac{\alpha_1 - \alpha_2}{\beta_m - \beta_p} \big(\langle \eta_\rho \mathbf{u}_s\, , -\mu \triangle \mathbf{x} + \nabla k \rangle - \langle \eta_\rho p_s\, , \nabla \cdot \mathbf{x} \rangle \Big) \Big|.$$

We now test $\eta_\rho \mathbf{u}_s$ and $\eta_\rho p_s$ with the first and the second equations in (26) to obtain

$$\begin{aligned}
\langle -\mu \triangle \mathbf{x} + \nabla k\, , \eta_\rho \mathbf{u}_s \rangle &= \langle -\mu \triangle (\eta_{2\rho}\mathbf{u}_d) + \nabla (\eta_{2\rho} p_d)\, , \eta_\rho \mathbf{u}_s \rangle\, , \\
\langle \nabla \cdot \mathbf{x}\, , \eta_\rho p_s \rangle &= \langle \nabla \cdot (\eta_{2\rho}\mathbf{u}_d)\, , \eta_\rho p_s \rangle
\end{aligned}$$

We can check easily that the right hand side terms of above equations are identically zero by Lemma 2.1, because of the distinct supports of η_ρ and $\eta_{2\rho}$. Therefore, we complete well-posedness of the problem (19).

4. Error Analysis for the finite element dual singular function method

In this section, we will prove Theorem 1 which are errors of FE-DSFM (20)∼(21) by comparing them with (18)∼(19). To do this, we first introduce an assumption.

Assumption 1 (Discrete inf-sup). *For given $p_h \in \mathbb{P}_h$, there exists a constant $\gamma > 0$ such that*

$$\gamma \|p_h\|_0 \le \sup_{v_h \in \mathbb{V}_h} \frac{\langle \nabla \cdot \mathbf{v}_h\, , p_h \rangle}{\|\mathbf{v}_h\|_1}.$$

In order to introduce a useful lemma in [5], we consider the Stokes equations:

$$\begin{aligned} -\mu\triangle\mathbf{x} + \nabla k &= \mathbf{f}, \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{x} &= \chi, \quad \text{in } \Omega, \\ \mathbf{x} &= \mathbf{0}, \quad \text{on } \partial\Omega. \end{aligned} \tag{28}$$

Lemma 4.1 (Upper bound of regular solution)**.** *Let Ω be a polygonal domain with non-convex vertices. If $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ and $\chi \in L_0^2(\Omega)$, then there exist a unique solution $(\mathbf{x}, k) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ of (28), with*

$$\|\mathbf{x}\|_1 + \|k\|_0 \le C\left(\|\mathbf{f}\|_{-1} + \|\chi\|_0\right).$$

Moreover, if $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\chi \in H^1(\Omega)$, then the solution $(\mathbf{x}, k) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ can be rewritten in the form of $\mathbf{x} = \mathbf{x}_R + \alpha_1\eta_1\mathbf{u}_{s1} + \alpha_2\eta_2\mathbf{u}_{s2}$ and $k = k_R + \alpha_1\eta_1 p_{s1} + \alpha_2\eta_2 p_{s2}$, with $(\mathbf{x}_R, k_R) \in \left(\mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)\right) \times \left(L_0^2(\Omega) \cap H^1(\Omega)\right)$ satisfying

$$\|\mathbf{x}_R\|_2 + \|k_R\|_1 + |\alpha_1| + |\alpha_2| \le C\left(\|\mathbf{f}\|_0 + \|\chi\|_1\right),$$

where $(\eta_1\mathbf{u}_{s1}, \eta_1 p_{s1})$ and $(\eta_1\mathbf{u}_{s2}, \eta_1 p_{s2})$ are singular functions.

We evaluate errors under the notations:

$$\begin{aligned} \mathbf{E} &:= \mathbf{w} - \mathbf{w}_h, & \mathbf{E}_h &:= \mathcal{I}_h\mathbf{w} - \mathbf{w}_h, & \mathcal{I}_h\mathbf{E} &:= \mathbf{w} - \mathcal{I}_h\mathbf{w}, \\ e &:= q - q_h, & e_h &:= \mathcal{I}_h q - q_h, & \mathcal{I}_h e &:= q - \mathcal{I}_h q, \end{aligned}$$

and

$$\varepsilon := \alpha - \alpha_h,$$

where $\mathcal{I}_h$ the Clement interpolant. Because $(\mathbf{w}, q) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$, we can use the well known results

$$\|\mathcal{I}_h\mathbf{E}\|_0 + h\|\mathcal{I}_h\mathbf{E}\|_1 \le Ch^2\|\mathbf{w}\|_2 \quad \text{and } \|\mathcal{I}_h e\|_0 \le Ch\|q\|_1. \tag{29}$$

In proof of the main theorem, We will use the solution $(\mathbf{z}, r)$ of, for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ and for all $\phi \in L^2(\Omega)$,

$$\begin{aligned} \mu\left\langle\nabla\mathbf{z}\,,\,\nabla\mathbf{v}\right\rangle + \left\langle\nabla r\,,\,\mathbf{v}\right\rangle + (1-\kappa)\left(c(\mathbf{z}) - d(r)\right)a(\mathbf{v}) &= \left\langle\mathbf{E}\,,\,\mathbf{v}\right\rangle, \\ \left\langle\nabla\cdot\mathbf{z}\,,\,\phi\right\rangle - (1-\kappa)\left(c(\mathbf{z}) - d(r)\right)b(\phi) &= 0, \end{aligned} \tag{30}$$

where κ is an arbitrary small positive constant. We will establish the well-posedness of equations (30) via the same manner within §3. To do this, we rewrite (30) with $\bar{\alpha} := (1-\kappa)(c(\mathbf{z}) - d(r))$ and

$$\begin{aligned} \mu\left\langle\nabla\mathbf{z}\,,\,\nabla\mathbf{v}\right\rangle + \left\langle\nabla r\,,\,\mathbf{v}\right\rangle + \bar{\alpha}a(\mathbf{v}) &= \left\langle\mathbf{E}\,,\,\mathbf{v}\right\rangle, \\ \left\langle\nabla\cdot\mathbf{z}\,,\,\phi\right\rangle - \bar{\alpha}b(\phi) &= 0. \end{aligned} \tag{31}$$

We will prove well-posedness of (30) in next lemma, by verifying existence and uniqueness of the solution $(\mathbf{z}, r, \bar{\alpha})$ of (31).

Lemma 4.2 (Well-posedness and regularity of (30))**.** *If κ is any number in $(0,1)$, then equations (30) have a unique solution $(\mathbf{z}, r)$ in $\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ and there exists a positive constant C_1 satisfying*

$$\|\mathbf{z}\|_1 + \|r\|_0 \le C_1\|\mathbf{E}\|_0. \tag{32}$$

Proof. We first show the compatibility condition for (30). since we have

$$\int_\Omega \nabla\cdot\mathbf{z}dx = \int_{\partial\Omega}\mathbf{z}\cdot\boldsymbol{\nu}ds = 0,$$

where $\boldsymbol{\nu}$ is the outward unit normal vector, we conclude $\int_\Omega \nabla\cdot\mathbf{z}dx = \int_\Omega \nabla\cdot(\eta_{2\rho}\mathbf{u}_d)dx$=0, because of $b(1) = 0$ in (27).

We now start to prove existence and uniqueness of solution of (30). It is well known that (31) has unique solution $(\mathbf{z}_{\bar\alpha}, r_{\bar\alpha})$ for any $\bar\alpha \in \mathbb{R}$, if $\mathbf{E} \in \mathbf{L}^2(\Omega)$. So we can define a mapping $F_{\mathbf{E}}$ from $\mathbb{R}$ to $\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ by $F_{\mathbf{E}}(\bar\alpha) := (\mathbf{z}_{\bar\alpha}, r_{\bar\alpha})$ to be the solution of (31), for any $\bar\alpha \in \mathbb{R}$. And we define $F_{\bar\alpha}$ from $\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ to $\mathbb{R}$ by

$$\begin{aligned} F_{\bar\alpha}(z, r) :&= (1-\kappa)\left(c(\mathbf{z}) - d(r)\right) \\ &= (1-\kappa)\left(\langle -\mu\triangle(\eta_\rho \mathbf{u}_s) + \nabla(\eta_\rho p_s)\,,\, \mathbf{z}\rangle - \langle \nabla\cdot(\eta_\rho \mathbf{u}_s)\,,\, r\rangle\right). \end{aligned} \tag{33}$$

Then $F_{\bar\alpha} \circ F_{\mathbf{E}}$ becomes a mapping from $\mathbb{R}$ to $\mathbb{R}$. So we need to show, by the contraction mapping theorem [11], that $F_{\bar\alpha} \circ F_{\mathbf{E}}$ has a unique fixed point to finish this proof. And it is equivalent to proving $\|F_{\bar\alpha} \circ F_{\mathbf{E}}\| < 1$. To show this, let $\bar\alpha_1$ and $\bar\alpha_2$ be arbitrary real numbers and let

$$F_{\mathbf{E}}(\bar\alpha_1) = (\mathbf{z}_{\bar\alpha_1}, r_{\bar\alpha_1}) \quad \text{and} \quad F_{\mathbf{E}}(\bar\alpha_2) = (\mathbf{w}_{\bar\alpha_2}, r_{\bar\alpha_2}). \tag{34}$$

From (31), we obtain

$$\begin{aligned} \mu\langle \nabla(\mathbf{z}_{\bar\alpha_1} - \mathbf{z}_{\bar\alpha_2})\,,\, \nabla\mathbf{v}\rangle + \langle \nabla(r_{\bar\alpha_1} - r_{\bar\alpha_2})\,,\, \mathbf{v}\rangle \\ = \frac{\bar\alpha_1 - \bar\alpha_2}{\beta_m - \beta_p}\langle -\mu\triangle(\eta_{2\rho}\mathbf{u}_d) + \nabla(\eta_{2\rho}p_d)\,,\, \mathbf{v}\rangle, \\ \langle \nabla\cdot(\mathbf{z}_{\bar\alpha_1} - \mathbf{z}_{\bar\alpha_2})\,,\, \phi\rangle = \frac{\bar\alpha_1 - \bar\alpha_2}{\beta_m - \beta_p}\langle \nabla\cdot(\eta_{2\rho}\mathbf{u}_d)\,,\, \phi\rangle. \end{aligned} \tag{35}$$

And we define the Stokes equations

$$\begin{aligned} -\mu\triangle\mathbf{x} + \nabla k &= -\mu\triangle(\eta_\rho\mathbf{u}_s) + \nabla(\eta_\rho p_s), && \text{in } \Omega, \\ \nabla\cdot\mathbf{x} &= \nabla\cdot(\eta_\rho\mathbf{u}_s), && \text{in } \Omega, \\ \mathbf{x} &= \mathbf{0}, && \text{on } \partial\Omega. \end{aligned} \tag{36}$$

We note here that the right hand side terms are smooth functions which come from Lemma 2.1 and the compatibility condition hold because of $\int_\Omega \nabla\cdot(\eta_\rho\mathbf{u}_s)dx = \int_{\partial\Omega}(\eta_\rho\mathbf{u}_s)\cdot\boldsymbol{\nu}ds = 0$. So (36) has a unique solution $(\mathbf{x}, k) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$. We now apply (34), (33) and (36) as a sequence and then we use integration by parts to get

$$\begin{aligned} &\left|F_{\bar\alpha}\circ F_{\mathbf{E}}(\bar\alpha_1) - F_{\bar\alpha}\circ F_{\mathbf{E}}(\bar\alpha_2)\right| = \left|F_{\bar\alpha}(\mathbf{z}_{\bar\alpha_1}, r_{\bar\alpha_1}) - F_{\bar\alpha}(\mathbf{w}_{\bar\alpha_2}, r_{\bar\alpha_2})\right| \\ =&\left|(1-\kappa)\left(\langle -\mu\triangle(\eta_\rho\mathbf{u}_s) + \nabla(\eta_\rho p_s)\,,\, \mathbf{z}_{\bar\alpha_1} - \mathbf{z}_{\bar\alpha_2}\rangle - \langle \nabla\cdot(\eta_\rho\mathbf{u}_s)\,,\, r_{\bar\alpha_1} - r_{\bar\alpha_2}\rangle\right)\right| \\ =&\left|(1-\kappa)\left(\langle -\mu\triangle\mathbf{x} + \nabla k\,,\, \mathbf{z}_{\bar\alpha_1} - \mathbf{z}_{\bar\alpha_2}\rangle - \langle \nabla\cdot\mathbf{x}\,,\, r_{\bar\alpha_1} - r_{\bar\alpha_2}\rangle\right)\right| \\ =&\left|(1-\kappa)\left(\mu\langle \nabla\mathbf{x}\,,\, \nabla(\mathbf{z}_{\bar\alpha_1} - \mathbf{z}_{\bar\alpha_2})\rangle - \langle k\,,\, \nabla\cdot(\mathbf{z}_{\bar\alpha_1} - \mathbf{z}_{\bar\alpha_2})\rangle + \langle \mathbf{x}\,,\, \nabla(r_{\bar\alpha_1} - r_{\bar\alpha_2})\rangle\right)\right|. \end{aligned}$$

We now consider $\mathbf{v} = \mathbf{x}$ and $\phi = k$ in (35) to derive

$$\begin{aligned} &\left|F_{\bar\alpha}\circ F_{\mathbf{E}}(\bar\alpha_1) - F_{\bar\alpha}\circ F_{\mathbf{E}}(\bar\alpha_2)\right| \\ &\quad = \left|(1-\kappa)\frac{\bar\alpha_1 - \bar\alpha_2}{\beta_m - \beta_p}\left(\langle -\mu\triangle(\eta_{2\rho}\mathbf{u}_d) + \nabla(\eta_{2\rho}p_d)\,,\, \mathbf{x}\rangle - \langle \nabla\cdot(\eta_{2\rho}\mathbf{u}_d)\,,\, k\rangle\right)\right|, \end{aligned}$$

and then we apply Lemma 2.2 again to obtain

$$\begin{aligned} &\left|F_{\bar\alpha}\circ F_{\mathbf{E}}(\bar\alpha_1) - F_{\bar\alpha}\circ F_{\mathbf{E}}(\bar\alpha_2)\right| \\ =&\left|(1-\kappa)\frac{\bar\alpha_1 - \bar\alpha_2}{\beta_m - \beta_p}\left(\langle \eta_{2\rho}\mathbf{u}_d\,,\, -\mu\triangle\mathbf{x} + \nabla k\rangle - \langle \eta_{2\rho}p_d\,,\, \nabla\cdot\mathbf{x}\rangle\right)\right|, \end{aligned}$$

In light of (17), testing $\eta_{2\rho}\mathbf{u}_d$ and $\eta_{2\rho}p_d$ with the first and the second in (36), respectively, leads

$$\langle -\mu\triangle\mathbf{x}+\nabla k\,,\,\eta_{2\rho}\mathbf{u}_d\rangle = \langle -\mu\triangle(\eta_\rho\mathbf{u}_s)+\nabla(\eta_\rho p_s)\,,\,\eta_{2\rho}\mathbf{u}_d\rangle = \beta_m,$$
$$\langle \nabla\cdot\mathbf{x}\,,\,\eta_{2\rho}p_d\rangle = \langle \nabla\cdot(\eta_\rho\mathbf{u}_s)\,,\,\eta_{2\rho}p_d\rangle = \beta_p.$$

Finally, we arrive

$$\left|F_{\bar\alpha}\circ F_{\mathbf{E}}(\bar\alpha_1) - F_{\bar\alpha}\circ F_{\mathbf{E}}(\bar\alpha_2)\right| = |1-\kappa||\bar\alpha_1-\bar\alpha_2| < |\bar\alpha_1-\bar\alpha_2|,$$

provided $\kappa\in(0,1)$. Therefore, we prove that (30) has a unique solution $(\mathbf{z_f}, r_\mathbf{f})\in \mathbf{H}^1(\Omega)\times L^2_0(\Omega)$ for any $\mathbf{f}\in\mathbf{H}^{-1}(\Omega)$. From now, we will derive (32) to finish this proof. Let $F:\mathbf{H}^{-1}(\Omega)\to\mathbf{H}^1(\Omega)\times L^2_0(\Omega)$ be the operator corresponding to the solution of (30), for all $\mathbf{f}\in\mathbf{H}^{-1}(\Omega)$, namely $F(\mathbf{f})=(\mathbf{z_f}, r_\mathbf{f})$. According to (30), the inverse operator F^{-1} is well-defined and

$$\begin{aligned}\|\mathbf{f}\|_{-1} &= \left\|F^{-1}(\mathbf{z_f}, r_\mathbf{f})\right\|_{-1}\\ &= \sup_{\mathbf{v}\in\mathbf{H}^1_0(\Omega)} \frac{\mu\langle\nabla\mathbf{z_f}\,,\,\nabla\mathbf{v}\rangle + \langle\nabla r_\mathbf{f}\,,\,\mathbf{v}\rangle + (1-\kappa)\left(c(\mathbf{z_f})-d(r_\mathbf{f})\right)a(\mathbf{v})}{\|\mathbf{v}\|_1}\\ &\le (1+C)\left(\|\mathbf{z_f}\|_1+\|r_\mathbf{f}\|_0\right),\end{aligned}$$

where C depends on only given datum of domain Ω, and given smooth functions $\|\nabla\cdot(\eta_{2\rho}\mathbf{u}_d)\|_0$, $\|-\mu\triangle(\eta_{2\rho}\mathbf{u}_d)+\nabla(\eta_{2\rho}p_d)\|_0$, $\|-\mu\triangle(\eta_\rho\mathbf{u}_s)+\nabla(\eta_\rho p_s)\|_0$ and $\|\nabla\cdot(\eta_\rho\mathbf{u}_s)\|_0$. So $\mathbf{F}^{-1}$ is bounded bilinear transform. If we choose $C_1 = 1/\left\|\mathbf{F}^{-1}\right\|$, then we can have

$$\|\mathbf{z_f}\|_1+\|r_\mathbf{f}\|_0 \le \|F\|\|\mathbf{f}\|_{-1}\le C_1\|\mathbf{f}\|_0.$$

If we choose $\mathbf{f}=\mathbf{E}$, we arrive at (32) and it is the proof. ■

Lemma 4.3 (Properties of the solution $(\mathbf{z}, r)$ of (30))**.** *Let $(\mathbf{z}, r)$ be the solutions of (30). Then there is a singular function representation*

$$\mathbf{z} = \mathbf{w_z}+\alpha_\mathbf{z}\eta_\rho\mathbf{u}_s \quad \textit{and} \quad r = q_\mathbf{z}+\alpha_\mathbf{z}\eta_\rho p_s, \tag{37}$$

where $\mathbf{w_z}\in\mathbf{H}^2(\Omega)\cap\mathbf{H}^1_0(\Omega)$, $q_\mathbf{z}\in H^1(\Omega)\cap L^2_0(\Omega)$ and $\alpha_\mathbf{z}$ satisfy the regularity estimate

$$\|\mathbf{w_z}\|_2+\|q_\mathbf{z}\|_1+|\alpha_\mathbf{z}|\le C\|\mathbf{E}\|_0. \tag{38}$$

Proof. It is easy to check the solution (31) satisfies

$$\begin{aligned}-\mu\triangle\mathbf{z}+\nabla r &= \mathbf{E}+\frac{\bar\alpha}{\beta_m-\beta_p}\left(-\mu\triangle(\eta_{2\rho}\mathbf{u}_d)+\nabla(\eta_{2\rho}p_d)\right),\\ \nabla\cdot\mathbf{z} &= \frac{\bar\alpha}{\beta_m-\beta_p}\nabla\cdot(\eta_{2\rho}\mathbf{u}_d).\end{aligned} \tag{39}$$

The right hand side terms are in $\mathbf{L}^2(\Omega)$ and $H^1(\Omega)\cap L^2_0(\Omega)$ for the fist and the second equations, respectively, which come from (11) and (27). So we can directly get the representation of (37) from Lemma 4.1 and

$$\|\mathbf{w_z}\|_2+\|q_\mathbf{z}\|_1+|\alpha_\mathbf{z}|\le C\|-\mu\triangle\mathbf{z}+\nabla r\|_0+\|\nabla\cdot\mathbf{z}\|_1.$$

According to (39), we arrive at

$$\begin{aligned}\|\mathbf{w_z}\|_2+\|q_\mathbf{z}\|_1+|\alpha_\mathbf{z}| &\le C\left\|\mathbf{E}+\frac{\bar\alpha}{\beta_m-\beta_p}\left(-\mu\triangle\eta_{2\rho}\mathbf{u}_d+\nabla\eta_{2\rho}p_d\right)\right\|_0\\ &\quad + C\left\|\frac{\bar\alpha}{\beta_m-\beta_p}\nabla\cdot\eta_{2\rho}\mathbf{u}_d\right\|_1 \le C\left(\|\mathbf{E}\|_0+|\bar\alpha|\right).\end{aligned}$$

Because $|\bar{\alpha}| = |(1-\kappa)(c(\mathbf{z}) - d(r))| \leq C\left(\|\mathbf{z}\|_0 + \|r\|_0\right)$, (32) leads $|\bar{\alpha}| \leq C\|\mathbf{E}\|_0$ and (38). It is the proof. ∎

If we denote

$$\mathbf{G} := \mathbf{z} - \mathcal{I}_h\mathbf{z} \quad \text{and} \quad g := r - \mathcal{I}_h r,$$

then thanks to Lemma 30, we have

$$\|\mathbf{G}\|_1 + \|g\|_0 \leq Ch^{\lambda}\|\mathbf{E}\|_0, \tag{40}$$

because of $\|\eta_\rho \mathbf{u}_s - \mathcal{I}_h(\eta_\rho \mathbf{u}_s)\|_1 + \|\eta_\rho p_s - \mathcal{I}_h(\eta_\rho p_s)\|_0 \leq Ch^{\lambda}$.

Remark 4.4 (The reason of sub-optimality)**.** The inequality (40) is the main restriction to get optimal accuracy in the next lemma and the reason of sub-optimality $|\alpha - \alpha_h| + \|\mathbf{w} - \mathbf{w}_h\|_0 \leq Ch^{1+\lambda}$ in Theorem 1. Also the sub-optimality (3) is due to (40).

From now, we impose $\kappa = 0$ in (30) to simple explanation, because κ is an arbitrary small positive constant. We start to estimate errors in $\mathbf{L}^2(\Omega)$.

Lemma 4.5 (Estimate $\|\mathbf{E}\|_0$)**.** *Let Assumption 1 hold. Then we have*

$$\|\mathbf{E}\|_0 \leq Ch^{\lambda}\left(\|\mathbf{E}\|_1 + \|e\|_0\right), \tag{41}$$

$$|\varepsilon| \leq C\|\mathbf{E}\|_0 \quad \textit{and} \quad \|e\|_0 \leq C\left(\|\mathbf{E}\|_1 + h\|q\|_1\right). \tag{42}$$

Proof. We start this proof with constructing error equations by subtracting (20) from (19) to get

$$\begin{aligned} \mu\left\langle \nabla\mathbf{E}\,,\, \nabla\mathbf{v}_h\right\rangle + \left\langle \nabla e\,,\, \mathbf{v}_h\right\rangle + \left(a(\mathbf{E}) + b(e)\right)c(\mathbf{v}_h) &= 0,\\ \left\langle \nabla\cdot\mathbf{E}\,,\, \phi_h\right\rangle + \left(a(\mathbf{E}) + b(e)\right)d(\phi_h) &= 0. \end{aligned} \tag{43}$$

We first prove (42). From the second equation in (43), we have

$$|b(e)d(\phi_h)| = |\left\langle \mathbf{E}\,,\, \nabla\phi_h\right\rangle - a(\mathbf{E})d(\phi_h)|.$$

We fix $\phi_h = C_2 x$ with C_2 satisfying $d(C_2 x) = 1$, then $\|\nabla\phi_h\|_0 = |C_2||\Omega|^{1/2}$ is a bounded number, because the space $\mathbb{P}_h$ is composed of continuous functions. So we can readily obtain

$$|b(e)| \leq C\|\mathbf{E}\|_0. \tag{44}$$

Therefore $\varepsilon = a(\mathbf{E}) + b(e)$ which comes from subtracting (21) from (18) yields

$$|\varepsilon| \leq C\left(\|\mathbf{E}\|_0 + |b(e)|\right) \leq C\|\mathbf{E}\|_0,$$

and, in light of (43), Assumption 1 leads

$$\begin{aligned} \gamma\|e_h\|_0 &\leq \sup_{v_h\in\mathbb{V}_h} \frac{\mu\left\langle \nabla\mathbf{E}\,,\, \nabla\mathbf{v}_h\right\rangle + \left\langle \mathcal{I}_h e\,,\, \nabla\cdot\mathbf{v}_h\right\rangle + \left(a(\mathbf{E}) + b(e)\right)c(\mathbf{v}_h)}{\|\mathbf{v}_h\|_1}\\ &\leq C\left(\|\mathbf{E}\|_1 + h\|q\|_1 + \|\mathbf{E}\|_0 + |b(e)|\right). \end{aligned}$$

Thus, in conjunction with (44), we arrive at (42). We now prove (41) with choosing $\mathbf{v}_h = \mathcal{I}_h\mathbf{z} = \mathbf{z} - \mathbf{G}$ and $\phi = \mathcal{I}_h r = r - g$ in (43):

$$\begin{aligned} \mu\left\langle \nabla\mathbf{E}\,,\, \nabla\left(\mathbf{z} - \mathbf{G}\right)\right\rangle + \left\langle \nabla e\,,\, \mathbf{z} - \mathbf{G}\right\rangle + \left(a(\mathbf{E}) + b(e)\right)c(\mathbf{z} - \mathbf{G}) &= 0,\\ \left\langle \nabla\cdot\mathbf{E}\,,\, r - g\right\rangle + \left(a(\mathbf{E}) + b(e)\right)d(r - g) &= 0, \end{aligned} \tag{45}$$

And then we choose $\mathbf{v} = \mathbf{E}$ and $\phi = e$ in (30) to get

$$\begin{aligned} \mu\left\langle \nabla\mathbf{z}\,,\, \nabla\mathbf{E}\right\rangle + \left\langle \nabla r\,,\, \mathbf{E}\right\rangle + \left(c(\mathbf{z}) - d(r)\right)a(\mathbf{E}) &= \left\langle \mathbf{E}\,,\, \mathbf{E}\right\rangle,\\ \left\langle \nabla\cdot\mathbf{z}\,,\, e\right\rangle - \left(c(\mathbf{z}) - d(r)\right)b(e) &= 0. \end{aligned} \tag{46}$$

We now replace $\langle \nabla e\,,\,\mathbf{z}\rangle$ at the first equation in (45) with the second equation in (46) to obtain

(47)
$$\mu\,\langle \nabla \mathbf{E}\,,\,\nabla\,(\mathbf{z}-\mathbf{G})\rangle - \langle \nabla e\,,\,\mathbf{G}\rangle + (a(\mathbf{E})+b(e))\,c(\mathbf{z}-\mathbf{G}) - (c(\mathbf{z})-d(r))\,b(e) = 0.$$

By the same manner, we replace $\langle \nabla r\,,\,\mathbf{E}\rangle$ at the first equation in (46) with the second equation in (45)

(48)
$$\mu\,\langle \nabla \mathbf{z}\,,\,\nabla \mathbf{E}\rangle - \langle \nabla\cdot\mathbf{E}\,,\,g\rangle + (c(\mathbf{z})-d(r))\,a(\mathbf{E}) + (a(\mathbf{E})+b(e))\,d(r-g) = \langle \mathbf{E}\,,\,\mathbf{E}\rangle\,.$$

In light of (40), subtracting (47) from (48) yields

$$\begin{aligned}\|\mathbf{E}\|_0^2 &= \mu\,\langle \nabla \mathbf{E}\,,\,\nabla \mathbf{G}\rangle - \langle \nabla\cdot\mathbf{E}\,,\,g\rangle - \langle e\,,\,\nabla\cdot\mathbf{G}\rangle + (a(\mathbf{E})+b(e))\,(c(\mathbf{G})-d(g))\\ &\le Ch^{\lambda}\,(\|\nabla\mathbf{E}\|_0 + \|\nabla\cdot\mathbf{E}\|_0 + \|e\|_0)\,\|\mathbf{E}\|_0.\end{aligned}$$

Therefore we arrive at (41) and finish the proof of the theorem. ■

We now estimate error in $\mathbf{H}_0^1(\Omega)$ space.

Lemma 4.6 (Estimate $\|\mathbf{E}\|_1 + \|e\|_0$). *Let Assumption 1 hold. If the mesh size h be small enough, then we have*

$$\|\mathbf{E}\|_1 + \|e\|_0 \le Ch. \tag{49}$$

Proof. We choose $\mathbf{v}_h = \mathbf{E}_h = \mathcal{I}_h\mathbf{w} - \mathbf{w}_h = \mathbf{E} - \mathcal{I}_h\mathbf{E} \in \mathbb{V}_h$ and $\phi_h = e_h = \mathcal{I}_h q - q_h = e - \mathcal{I}_h e \in \mathbb{P}_h$ in (43), then we have

$$\begin{aligned}\mu\,\langle \nabla\mathbf{E}\,,\,\nabla(\mathbf{E}-\mathcal{I}_h\mathbf{E})\rangle + \langle \nabla e\,,\,\mathbf{E}-\mathcal{I}_h\mathbf{E}\rangle + (a(\mathbf{E})+b(e))\,c(\mathbf{E}_h) &= 0,\\ \langle \nabla\cdot\mathbf{E}\,,\,e-\mathcal{I}_h e\rangle + (a(\mathbf{E})+b(e))\,d(e_h) &= 0.\end{aligned} \tag{50}$$

And then we replace $\langle \nabla e\,,\,\mathbf{E}\rangle$ in the first equation with the second equation in (50) to have

$$\begin{aligned}\mu\,\langle \nabla\mathbf{E}\,,\,\nabla(\mathbf{E}-\mathcal{I}_h\mathbf{E})\rangle - \langle \nabla e\,,\,\mathcal{I}_h\mathbf{E}\rangle - \langle \nabla\cdot\mathbf{E}\,,\,\mathcal{I}_h e\rangle&\\ + (a(\mathbf{E})+b(e))\,(c(\mathbf{E}_h)+d(e_h)) &= 0.\end{aligned}$$

In conjunction with Lemma 4.5 and $|b(e)| \le C\|\mathbf{E}\|_0$ in (44), (29) yields

$$\begin{aligned}\mu\|\nabla\mathbf{E}\|_0^2 &\le C\Big(\|\nabla\mathbf{E}\|_0\|\nabla\mathcal{I}_h\mathbf{E}\|_0 + \|e\|_0\|\nabla\cdot\mathcal{I}_h\mathbf{E}\|_0\\ &\qquad + \|\mathcal{I}_h e\|_0\|\nabla\cdot\mathbf{E}\|_0 + \|\mathbf{E}\|_0\,(\|\mathbf{E}_h\|_0 + \|e_h\|_0)\Big)\\ &\le Ch\Big(\,(\|\nabla\mathbf{E}\|_0 + \|e\|_0)\,\|\mathbf{w}\|_2 + \|\nabla\cdot\mathbf{E}\|_0\|q\|_1\Big)\\ &\qquad + Ch^{\lambda}\,(\|\mathbf{E}\|_1 + \|e\|_0)\,(\|\mathbf{E}_h\|_0 + \|e_h\|_0)\\ &\le Ch\Big(\,(\|\nabla\mathbf{E}\|_0 + h\|q\|_0)\,\|\mathbf{w}\|_2 + \|\nabla\cdot\mathbf{E}\|_0\|q\|_1\Big) + Ch^{\lambda}\,(\|\mathbf{E}\|_1 + h\|q\|_1)^2\,.\end{aligned}$$

Assumption of a small enough h yields

$$\|\mathbf{E}\|_1 \le Ch\,(\|\mathbf{w}\|_2 + \|q\|_1)\,.$$

Finally, we arrive at (49) by combining with (42) and complete this proof. ■

5. Numerical test

In this section, we document the computational performance of the FE-DSFM within a polygonal Γ shape domain $([-1,1]\times[-1,1])\setminus([0,1]\times[-1,0])$. So, in this test, $\omega = 1.5\pi$ and the solution λ of (4) becomes $\lambda = 0.544483736782463925$. Let the solution be given by

$$\begin{aligned} u &= -\sin^2(\pi x)\sin(2\pi y) + u_s, \\ v &= \sin(2\pi x)\sin^2(\pi y) + v_s, \\ p &= (2+\cos(\pi x))(2+\cos(\pi y)) - 4.0 + p_s. \end{aligned}$$

We note that the solution for velocity has not vanished on Γ_{out}. The forcing term $\mathbf{f}$ is determined accordingly for any μ; here $\mu = 1$. In order to impose FE-DSFM, we choose the cut-off function $\eta_\rho \in H^3(\Omega)$ as

$$\eta_\rho = \begin{cases} 1, & \text{in } B(\frac{1}{2}\rho R), \\ \frac{1}{32}\left(16 - 35\psi + 35\psi^3 - 21\psi^5 + 5\psi^7\right), & \text{in } B(\frac{1}{2}\rho R;\rho R), \\ 0, & \text{in } \Omega\setminus\bar{B}(\rho R), \end{cases}$$

with $\psi = \frac{4r}{\rho R} - 3$ with $R = 1$. Then the solution $(\mathbf{u}, p)$ can be rewritten by

$$\begin{aligned} \mathbf{u} &= \mathbf{w} + \eta_\rho \mathbf{u}_s, \\ p &= q + \eta_\rho p_s, \end{aligned}$$

where $(\mathbf{w}, q)$ is the regular part of the solution. We note that the regularities of $\mathbf{w} = \mathbf{u} - \eta_\rho \mathbf{u}_s$ and $q = p - \eta_\rho p_s$ are equal to that of η_ρ, and so $(\mathbf{w}, q) \in \mathbf{H}^3(\Omega)\times H^3(\Omega)$ in this example. Computations are carried out with the Taylor-Hood $(\mathcal{P}^2, \mathcal{P}^1)$ finite element pair on the union jack shape uniform meshes of size h.

TABLE 1. Error table for the standard mixed method.

h	1/8	1/16	1/32	1/64	1/128
$\Vert\mathbf{E}\Vert_0$	0.00525989	0.00203043	0.000863199	0.000381872	0.000173168
	Order	1.373247	1.234020	1.176604	1.140917
$\Vert\mathbf{E}\Vert_{\mathbf{L}^\infty}$	0.027998	0.0195703	0.0135074	0.00928508	0.00637329
	Order	0.516658	0.534916	0.540764	0.542876
$\Vert\mathbf{E}\Vert_{\mathbf{H}^1}$	0.217839	0.121565	0.0805253	0.0549791	0.0376842
	Order	0.841534	0.594214	0.550559	0.544924
$\Vert e\Vert_{L^2}$	0.209838	0.142047	0.0968696	0.0662486	0.0453665
	Order	0.562908	0.552252	0.548154	0.546263
$\Vert e\Vert_{\mathbf{L}^\infty}$	0.713876	0.956344	1.31637	1.81283	2.49197
	Order	-0.421856	-0.460963	-0.461679	-0.459043

Table 1 is the error decay for the standard mixed method. We can check that the errors consist with (3) and we see that the error $\Vert e\Vert_{\mathbf{L}^\infty}$ does not converge to 0. It is natural behavior, because s in (3) is less than 1. More precisely, $s = \lambda = 0.544483736782463925$ in this test.

Tables 2, 3 and 4 are the results of mesh analysis of FE-DSFM with $\rho = 0.125$, $\rho = 0.3$ and $\rho = 0453$, respectively. The convergence orders in these experiments are the same as the optimal accuracy in (2) with $s = 2$, because of $(\mathbf{w}, q) \in \mathbf{H}^3(\Omega)\times H^3(\Omega)$. So we can conclude that FE-DSFM has optimal accuracy in numerical tests, even though we get only suboptimal accuracy $\Vert E\Vert_0 \le Ch^{1+\lambda}$

TABLE 2. Error decay for the FE-DSFM with $\rho = 0.125$.

h	1/8	1/16	1/32	1/64	1/128
$\|\mathbf{E}\|_0$	0.654576	0.00606273	0.00103749	4.86052e-05	5.53585e-06
	Order	6.754449	2.546870	4.415843	3.134234
$\|\mathbf{E}\|_{\mathbf{L}^\infty}$	4.86035	0.0539083	0.00987302	0.000588627	7.25307e-05
	Order	6.494409	2.448944	4.068066	3.020690
$\|\mathbf{E}\|_{\mathbf{H}^1}$	18.6427	0.32717	0.0799223	0.0249258	0.00637899
	Order	5.832426	2.033370	1.680958	1.966240
$\|e\|_{L^2}$	4.70916	0.0830669	0.0193997	0.00418722	0.00119974
	Order	5.825052	2.098239	2.211970	1.803271
$\|e\|_{\mathbf{L}^\infty}$	55.3433	1.55404	0.254753	0.1594	0.0539996
	Order	5.154313	2.608853	0.676447	1.561631
$\|\varepsilon\|$	0.0385324	0.224605	0.0435706	0.00112617	7.46192e-05
	Order	2.543246	2.365963	5.273858	3.915734

.

TABLE 3. Error decay for the FE-DSFM with $\rho = 0.3$.

h	1/8	1/16	1/32	1/64	1/128
$\|\mathbf{E}\|_0$	0.00930983	0.000962465	0.000102994	-05	1.63735e-06
	Order	3.273949	3.224174	2.941771	3.033283
$\|\mathbf{E}\|_{\mathbf{L}^\infty}$	0.0477212	0.00387738	0.000521333	7.35078e-05	8.82426e-06
	Order	3.621476	2.894805	2.826236	3.058350
$\|\mathbf{E}\|_{\mathbf{H}^1}$	0.383561	0.116552	0.0285883	0.00747874	0.00188788
	Order	1.718482	2.027477	1.934558	1.986028
$\|e\|_{L^2}$	0.0615555	0.016293	0.00492944	0.00139146	0.000361693
	Order	1.917635	1.724757	1.824824	1.943762
$\|e\|_{\mathbf{L}^\infty}$	0.422723	0.184322	0.0851973	0.0267362	0.00702548
	Order	1.197484	1.113349	1.672013	1.928126
$\|\varepsilon\|$	0.091617	0.00650135	0.000434171	3.40152e-05	2.34765e-06
	Order	3.816804	3.904404	3.674012	3.856890

.

TABLE 4. Error decay for the FE-DSFM with $\rho = 0453$.

h	1/8	1/16	1/32	1/64	1/128
$\|\mathbf{E}\|_0$	0.0111371	0.000491094	7.48313e-05	8.11691e-06	9.89508e-07
	Order	4.503231	2.714285	3.204639	3.036147
$\|\mathbf{E}\|_{\mathbf{L}^\infty}$	0.0351972	0.00146173	0.000238326	2.64804e-05	3.11194e-06
	Order	4.589712	2.616669	3.169940	3.089039
$\|\mathbf{E}\|_{\mathbf{H}^1}$	0.273671	0.0690917	0.0179002	0.0045319	0.00113814
	Order	1.985858	1.948537	1.981788	1.993438
$\|e\|_{L^2}$	0.051732	0.0107246	0.00309985	0.000820025	0.000208841
	Order	2.270133	1.790654	1.918459	1.973263
$\|e\|_{\mathbf{L}^\infty}$	0.262159	0.0874078	0.0365356	0.0103794	0.00267379
	Order	1.584608	1.258459	1.815580	1.956765
$\|\varepsilon\|$	0.0697584	0.00162356	0.000205428	1.66353e-05	2.40529e-06
	Order	5.425134	2.982456	3.626313	2.789965

.

theoretically in Theorem 1. In addition, we can check that the errors are smaller for the bigger ρ, so the errors are reduced in Tables 2, 3 and 4 successively. In Table 2, we can see also that the errors of the case $h = 1/8$ are relatively bigger than others casees and it is due to $h = \rho = 0.125$.

6. Singular functions and dual Singular functions

In this section, we derive corner singular functions and dual Singular functions for Stoke equations (1). To do this, we use notations U_r and U_θ as polar components of the vector function $\mathbf{u} = (u, v)^T$ which means

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} U_r \\ U_\theta \end{pmatrix}. \tag{51}$$

If we define $\mathbf{e}_r = (\cos(\theta), \sin(\theta))^T$ and $\mathbf{e}_\theta = (-\sin(\theta), \cos(\theta))^T$, then (51) becomes

$$\mathbf{u} = U_r \mathbf{e}_r + U_\theta \mathbf{e}_\theta,$$

and (1) can be written as the polar coordinate Stokes system

$$\begin{aligned} -\left(\partial_{rr}U_r + \frac{1}{r}\partial_r U_r + \frac{1}{r^2}\left(-U_r + \partial_{\theta\theta}U_r - 2\partial_\theta U_\theta\right)\right) + \partial_r P = 0, \\ -\left(\partial_{rr}U_\theta + \frac{1}{r}\partial_r U_\theta + \frac{1}{r^2}\left(-U_\theta + \partial_{\theta\theta}U_\theta + 2\partial_\theta U_r\right)\right) + \frac{1}{r}\partial_\theta P = 0, \\ \partial_r U_r + \frac{1}{r}\left(U_r + \partial_\theta U_\theta\right) = 0. \end{aligned} \tag{52}$$

We will find the singular solutions in §6.1 and the dual singular solutions §6.2 via solving (52).

6.1. Singular functions. In this section, we will find solutions of (52) by using the separation of variables. We first assume that the solution (U_r, U_θ, P) of (52) has the singular form, with $\lambda > 0$,

$$\begin{pmatrix} U_r(r,\theta) \\ U_\theta(r,\theta) \end{pmatrix} = r^\lambda \begin{pmatrix} u_r(\theta) \\ u_\theta(\theta) \end{pmatrix} \quad \text{and} \quad P(r,\theta) = r^{\lambda-1}p(\theta).$$

Then, as computed in [10], we obtain 4 solutions of (52)

$$\begin{pmatrix} u_r^1 \\ u_\theta^1 \\ p^1 \end{pmatrix} = \begin{pmatrix} \dfrac{1-\lambda}{\mu}\cos((1-\lambda)\theta) \\ -\dfrac{1+\lambda}{\mu}\sin((1-\lambda)\theta) \\ -4\lambda\cos((1-\lambda)\theta) \end{pmatrix}, \quad \begin{pmatrix} u_r^3 \\ u_\theta^3 \\ p^3 \end{pmatrix} = \begin{pmatrix} \cos((1+\lambda)\theta) \\ -\sin((1+\lambda)\theta) \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} u_r^2 \\ u_\theta^2 \\ p^2 \end{pmatrix} = \begin{pmatrix} \dfrac{1-\lambda}{\mu}\sin((1-\lambda)\theta) \\ \dfrac{1+\lambda}{\mu}\cos((1-\lambda)\theta) \\ -4\lambda\sin((1-\lambda)\theta) \end{pmatrix}, \quad \begin{pmatrix} u_r^4 \\ u_\theta^4 \\ p^4 \end{pmatrix} = \begin{pmatrix} \sin((1+\lambda)\theta) \\ \cos((1+\lambda)\theta) \\ 0 \end{pmatrix},$$

and so the general solution becomes

$$\begin{pmatrix} U_r(r,\theta) \\ U_\theta(r,\theta) \\ P(r,\theta) \end{pmatrix} = C_1 \begin{pmatrix} r^\lambda u_r^1 \\ r^\lambda u_\theta^1 \\ r^{\lambda-1} p^1 \end{pmatrix} + C_2 \begin{pmatrix} r^\lambda u_r^2 \\ r^\lambda u_\theta^2 \\ r^{\lambda-1} p^2 \end{pmatrix} + C_3 \begin{pmatrix} r^\lambda u_r^3 \\ r^\lambda u_\theta^3 \\ r^{\lambda-1} p^3 \end{pmatrix} + C_4 \begin{pmatrix} r^\lambda u_r^4 \\ r^\lambda u_\theta^4 \\ r^{\lambda-1} p^4 \end{pmatrix}. \tag{53}$$

To make a homogeneous condition at Γ_{in}, we impose

$$U_r(r,0) = U_\theta(r,0) = 0 \quad \text{and} \quad U_r(r,\omega) = U_\theta(r,\omega) = 0. \tag{54}$$

Then the first equations in (54) lead to

$$C_1 \frac{1-\lambda}{\mu} + C_3 = 0 \quad \text{and} \quad C_2 \frac{1+\lambda}{\mu} + C_4 = 0$$

and so, in conjunction with (51), (53) becomes

$$\begin{pmatrix} u \\ v \end{pmatrix} = 2C_1 \frac{r^\lambda}{\mu} \begin{pmatrix} \lambda \sin(\theta)\sin((1-\lambda)\theta) \\ \sin(\lambda\theta) - \lambda\sin(\theta)\cos((1-\lambda)\theta) \end{pmatrix} - 2C_2 \frac{r^\lambda}{\mu} \begin{pmatrix} \sin(\lambda\theta) + \lambda\sin(\theta)\cos((1-\lambda)\theta) \\ \lambda\sin(\theta)\sin((1-\lambda)\theta) \end{pmatrix}.$$

From $U_r(r,\omega) = 0$ in (54), C_1 and C_2 have to be

$$C_1 = \sin(\lambda\omega) + \lambda\sin(\omega)\cos((1-\lambda)\omega) \quad \text{and} \quad C_2 = \lambda\sin(\omega)\sin((1-\lambda)\omega), \tag{55}$$

And the last equation in (54), $U_\theta(r,\omega) = 0$, yields

$$\lambda^2 \sin^2(\omega) = \sin^2(\lambda\omega). \tag{56}$$

Finally, we arrive at the singular function

$$\begin{pmatrix} u \\ v \\ p \end{pmatrix} = C_1 \begin{pmatrix} \frac{r^\lambda}{\mu}\lambda\sin(\theta)\sin((1-\lambda)\theta) \\ \frac{r^\lambda}{\mu}(\sin(\lambda\theta) - \lambda\sin(\theta)\cos((1-\lambda)\theta)) \\ -2r^{\lambda-1}\lambda\cos((1-\lambda)\theta) \end{pmatrix} - C_2 \begin{pmatrix} \frac{r^\lambda}{\mu}(\sin(\lambda\theta) + \lambda\sin(\theta)\cos((1-\lambda)\theta)) \\ \frac{r^\lambda}{\mu}\lambda\sin(\theta)\sin((1-\lambda)\theta) \\ 2r^{\lambda-1}\lambda\sin((1-\lambda)\theta) \end{pmatrix},$$

where λ is the solution of (56), and C_1 and C_2 are the same as in (55).

In conjunction with Figure 1, we can readily get the following properties for (56):

Lemma 6.1 (Variability of λ on ω). *If we denote $\beta :\approx 1.430296653124203$, then we have that*

(1) $\lambda = 0$ and $\lambda = 1$ are solutions for all ω and we call these trivial solutions,
(2) there are only trivial solutions for $0 \le \omega \le \pi$,
(3) there is a unique solution for $\pi \le \omega \le \beta\pi$ except trivial solutions,
(4) there are 2 solutions for $\beta\pi \le \omega < 2\pi$ except trivial solutions,
(5) $\lambda = 0.5$ is the unique solution except trivial solutions, if $\omega = 2\pi$.

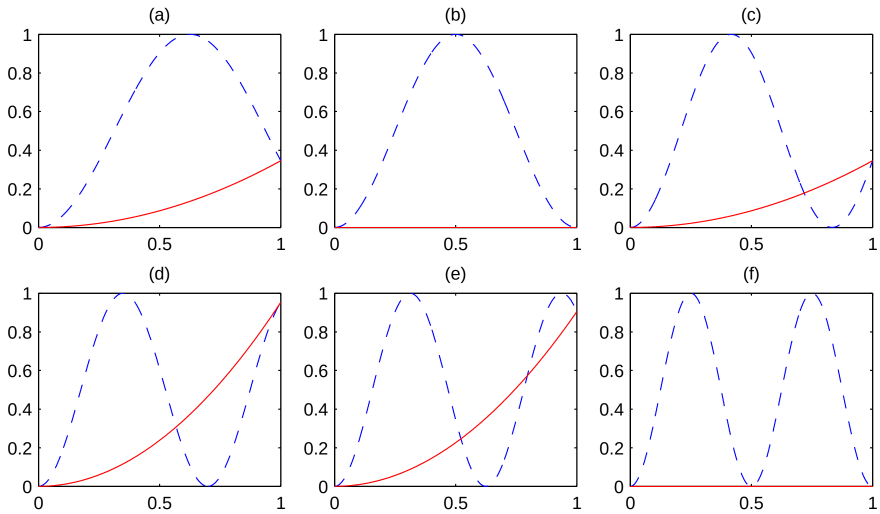

FIGURE 1. Graphs of $\sin^2(\lambda\omega)$ (dashed line) and $\lambda^2\sin^2(\omega)$ (solid line) with variable λ and fixed ω's as (a) $\omega = 0.8\pi$, (b) $\omega = \pi$, (c) $\omega = 1.2\pi$, (d) $\omega = 1.430296653124203\pi$, (e) $\omega = 1.6\pi$, and (f) $\omega = 2\pi$.

6.2. dual Singular functions. By the same manner within §6.1, we assume the solution (U_r, U_θ, P) of (52) has the form, for $\lambda > 0$,

$$\begin{pmatrix} U_r(r,\theta) \\ U_\theta(r,\theta) \end{pmatrix} = r^{-\lambda} \begin{pmatrix} u_r(\theta) \\ u_\theta(\theta) \end{pmatrix} \quad \text{and} \quad P(r,\theta) = r^{-\lambda-1}p(\theta).$$

Then we can readily get 4 solutions of (52)

$$\begin{pmatrix} u_r^1 \\ u_\theta^1 \\ p^1 \end{pmatrix} = \begin{pmatrix} \dfrac{\lambda+1}{\mu}\cos((\lambda+1)\theta) \\ \dfrac{\lambda-1}{\mu}\sin((\lambda+1)\theta) \\ 4\lambda\cos((\lambda+1)\theta) \end{pmatrix}, \quad \begin{pmatrix} u_r^3 \\ u_\theta^3 \\ p^3 \end{pmatrix} = \begin{pmatrix} \cos((\lambda-1)\theta) \\ \sin((\lambda-1)\theta) \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} u_r^2 \\ u_\theta^2 \\ p^2 \end{pmatrix} = \begin{pmatrix} \dfrac{\lambda+1}{\mu}\sin((\lambda+1)\theta) \\ -\dfrac{\lambda-1}{\mu}\cos((\lambda+1)\theta) \\ 4\lambda\sin((\lambda+1)\theta) \end{pmatrix}, \quad \begin{pmatrix} u_r^4 \\ u_\theta^4 \\ p^4 \end{pmatrix} = \begin{pmatrix} \sin((\lambda-1)\theta) \\ -\cos((\lambda-1)\theta) \\ 0 \end{pmatrix},$$

and so the general solution becomes

$$\begin{pmatrix} U_r(r,\theta) \\ U_\theta(r,\theta) \\ P(r,\theta) \end{pmatrix} = D_1 \begin{pmatrix} r^{-\lambda}u_r^1 \\ r^{-\lambda}u_\theta^1 \\ r^{-\lambda-1}p^1 \end{pmatrix} + D_2 \begin{pmatrix} r^{-\lambda}u_r^2 \\ r^{-\lambda}u_\theta^2 \\ r^{-\lambda-1}p^2 \end{pmatrix} + D_3 \begin{pmatrix} r^{-\lambda}u_r^3 \\ r^{-\lambda}u_\theta^3 \\ r^{-\lambda-1}p^3 \end{pmatrix} + D_4 \begin{pmatrix} r^{-\lambda}u_r^4 \\ r^{-\lambda}u_\theta^4 \\ r^{-\lambda-1}p^4 \end{pmatrix}. \tag{57}$$

To make a homogeneous condition on Γ_{in}, we impose

$$U_r(r,0) = U_\theta(r,0) = U_r(r,\omega) = U_\theta(r,\omega) = 0. \tag{58}$$

Then the first 2 conditions in (58), $U_r(r,0) = U_\theta(r,0) = 0$, leads us

$$D_1 \frac{\lambda+1}{\mu} + D_3 = 0 \quad \text{and} \quad D_2 \frac{\lambda-1}{\mu} + D_4 = 0$$

and so, in conjunction with (51), (57) becomes

$$\begin{pmatrix} u \\ v \end{pmatrix} = -\,2D_1 \frac{r^{-\lambda}}{\mu} \begin{pmatrix} \lambda \sin(\theta)\sin((\lambda+1)\theta) \\ \sin(\lambda\theta) - \lambda\sin(\theta)\cos((\lambda+1)\theta) \end{pmatrix} + 2D_2 \frac{r^{-\lambda}}{\mu} \begin{pmatrix} \sin(\lambda\theta) + \lambda\sin(\theta)\cos((\lambda+1)\theta) \\ \lambda\sin(\theta)\sin((\lambda+1)\theta) \end{pmatrix}.$$

In order to make hold the third equation in (58), $U_r(r,\omega) = 0$, D_1 and D_2 have to be

$$D_1 = \sin(\lambda\omega) + \lambda\sin(\omega)\cos((\lambda+1)\omega) \quad \text{and} \quad D_2 = \lambda\sin(\omega)\sin((\lambda+1)\omega). \tag{59}$$

Then the last equation in (58), $U_\theta(r,\omega) = 0$, yields the same equation as (56)

$$\lambda^2 \sin^2(\omega) = \sin^2(\lambda\omega). \tag{60}$$

Finally, we arrive at the dual singular function

$$\begin{pmatrix} u \\ v \\ p \end{pmatrix} = D_1 \begin{pmatrix} -\frac{r^{-\lambda}}{\mu}\lambda\sin(\theta)\sin((\lambda+1)\theta) \\ -\frac{r^{-\lambda}}{\mu}\left(\sin(\lambda\theta) - \lambda\sin(\theta)\cos((\lambda+1)\theta)\right) \\ 2r^{-\lambda-1}\lambda\cos((\lambda+1)\theta) \end{pmatrix} + D_2 \begin{pmatrix} \frac{r^{-\lambda}}{\mu}\left(\sin(\lambda\theta) + \lambda\sin(\theta)\cos((\lambda+1)\theta)\right) \\ \frac{r^{-\lambda}}{\mu}\lambda\sin(\theta)\sin((\lambda+1)\theta) \\ 2r^{-\lambda-1}\lambda\sin((\lambda+1)\theta) \end{pmatrix},$$

where λ is the solution of (60) and D_1 and D_2 are the same as in (59).

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