A FINITE ELEMENT DUAL SINGULAR FUNCTION METHOD TO SOLVE THE STOKES EQUATIONS INCLUDING CORNER SINGULARITIES

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Abstract. The finite element dual singular function method [FE-DSFM] has been constructed and analyzed accuracy by Z. Cai and S. Kim to solve the Laplace equation on a polygonal domain with one reentrant corner. In this paper, we impose FE-DSFM to solve the Stokes equations via the mixed finite element method. To do this, we compute the singular and the dual singular functions analytically at a non-convex corner. We prove well-posedness by using the contraction mapping theorem and then estimate errors of the algorithm. We obtain optimal accuracy $O(h)$ for velocity in $H^1(\Omega)$ and pressure in $L^2(\Omega)$, but we are able to prove only $O(h^{1+\lambda})$ error bounds for velocity in $L^2(\Omega)$ and stress intensity factor, where $\lambda$ is the eigenvalue (solution of (4)). However, we get optimal accuracy results in numerical experiments.

Key words. Stokes equations, dual singular function method, corner singularity, incompressible fluids.

1. Introduction

Solutions of elliptic boundary value problems on a domain with corners have singular behaviors near the corners. This occurs even when the given data of the governed equations is very smooth. Such singular behavior affects the accuracy of the finite element method throughout the whole domain. In order to overcome the singularity problem, the finite element dual singular function method [FE-DSFM] has been constructed in [3] to solve the Laplace equation and performed numerical tests in [4]. And then it is extended to solve the Helmholtz equation in [9] and the interface problem in [8]. The goal of this paper is to reconstruct FE-DSFM to solve Stokes equations:

\begin{align*}
-\mu \Delta u + \nabla p &= f, \quad \text{in } \Omega, \\
\nabla \cdot u &= 0, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{align*}

with $f$ is a given function in $H^{-1}(\Omega)$, $\Omega$ is a computational domain, and $\mu = Re^{-1}$ is the reciprocal of the Reynolds number. Here the unknowns are the (vector) velocity field $u \in H^1_0(\Omega)$ and the (scalar) pressure $p \in L^2(\Omega)$.

If the solution of (1) is smooth enough, namely $(u, p) \in H^{s+1}(\Omega) \times H^s(\Omega)$ with $s \geq 1$, and if a suitable finite element pair is imposed for velocity and pressure, then the finite element solution $(u_h, p_h)$ using the standard mixed method has optimal error bounds as shown in [1, 6]:

\begin{align*}
|u - u_h|_0 + h|u - u_h|_1 + h|p - p_h|_0 &\leq Ch^{s+1}(|u|_{s+1} + |p|_s),
\end{align*}

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where \( h \) is the biggest mesh size. However, if \( s < 1 \), then the error bounds of the method become only

\[
\|u - u_h\|_0 + h^s\|u - u_h\|_1 + h^s\|p - p_h\|_0 \leq C h^{2s} \left(\|u\|_{s+1} + \|p\|_s\right).
\]

So we call \((u, p)\) a singular solution for the case \( s < 1 \), otherwise a regular solution. Because the singularity is due to reentrant corners of computational domain \( \Omega \), we assume that \( \Omega \) is an open and bounded polygonal domain in \( \mathbb{R}^2 \) with one reentrant corner. Extension to the domain with a finite number of reentrant corners is straightforward.

Let \( \omega \) be the internal angle. Without the loss of generality, we assume that the corresponding vertex is at the origin and that the internal angle \( \omega \) is spanned by the two half-lines \( \theta = 0 \) and \( \theta = \omega \). We denote \( \Gamma_{in} \) for 2 edges on the boundary including the reentrant corner and \( \Gamma_{out} \) for other parts of the boundary. Even though the singular functions are already computed in [10], we will derive those again in §6 to get more advanced properties of the singular functions and newly find out the dual singular functions in (8) below.

The singular function \((u_s, p_s)\), where \( u_s = (u_s, v_s) \), can be summarized with the eigenvalue \( \lambda(>0) \) which is the solution of

\[
\sin^2(\lambda \omega) = \lambda^2 \sin^2(\omega),
\]

by

\[
\begin{pmatrix}
u_d \\
p_d
\end{pmatrix} = d_1 \begin{pmatrix}
- r^{-\lambda} \frac{\lambda}{\mu} \sin(\theta) \sin((1 + \lambda)\theta) \\
- r^{-\lambda} \frac{1}{\mu} (\sin(\lambda \theta) - \lambda \sin(\theta) \cos((1 + \lambda)\theta)) \\
2r^{-\lambda-1} \lambda \cos((1 + \lambda)\theta)
\end{pmatrix} + d_2 \begin{pmatrix}
r^{-\lambda} \frac{1}{\mu} (\sin(\lambda \theta) + \lambda \sin(\theta) \cos((1 + \lambda)\theta)) \\
r^{-\lambda} \frac{\lambda}{\mu} \sin(\theta) \sin((1 + \lambda)\theta) \\
2r^{-\lambda-1} \lambda \sin((1 + \lambda)\theta)
\end{pmatrix},
\]

where

\[
C_1 = \sin(\lambda \omega) + \lambda \sin(\omega) \cos((1 - \lambda)\omega) \quad \text{and} \quad C_2 = \lambda \sin(\omega) \sin((1 - \lambda)\omega).
\]

We note that the singular function \((u_s, p_s)\) is the solution of homogeneous Stokes equations with vanishing Dirichlet boundary condition at \( \Gamma_{in} \). And \( \lambda \) has to be a positive real number and \((u_s, p_s) \in H^{1+s}(\Omega) \times H^s(\Omega) \). As the conclusion in Lemma 6.1 below, \( \lambda = 1 \) for any \( \omega \leq \pi \), so \((u_s, p_s) \in H^2(\Omega) \times H^1(\Omega) \) is a regular solution and it becomes a singular solution for the case \( \lambda < 1 \), namely \( \omega > \pi \). Moreover (4) has a unique non-trivial solution \( \lambda \in \mathbb{R} \) for the case \( \pi < \omega \leq \beta \pi \). where \( \beta \approx 1.4302966553124203 \). And (4) has 2 non-trivial real solutions \( 0.5 < \lambda_1 < \lambda_2 < 1 \) for the case \( \omega \in (\beta \pi, 2\pi) \). In addition, \( \lambda = 0.5 \) is the unique non-trivial solution for \( \omega = 2\pi \).

Let \( \eta \) be a smooth cut-off function which is equal one identically in neighborhood of origin, and the support of \( \eta \) is small enough so that the functions \( \eta u_s \) vanishes identically on \( \partial \Omega \). Then, in general, the solution \((u, p)\) including singular parts of
(1) can be rewritten with 2 singular functions as

\[
\begin{pmatrix}
u \\
p
\end{pmatrix} = \begin{pmatrix} w \\ q \end{pmatrix} + \alpha_1 \begin{pmatrix} \eta_1 u_{s,1} \\ \eta_1 p_{s,1} \end{pmatrix} + \alpha_2 \begin{pmatrix} \eta_2 u_{s,2} \\ \eta_2 p_{s,2} \end{pmatrix},
\]

where \( \alpha_1 \) and \( \alpha_2 \) are the stress intensity factors and \( (w, q) \in H^2(\Omega) \times H^1(\Omega) \).

However, for a simpler explanation, we consider \((u, p)\) including only one singular function, namely,

\[
\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} w \\ q \end{pmatrix} + \alpha \begin{pmatrix} \eta u_s \\ \eta p_s \end{pmatrix},
\]

because FE-DSFM can be extended to problems with 2 singular solutions of the form (6) by applying the adjusted Sherman-Morrison-Woodbury formula without any additional skill, as mentioned in Remark 2.4 below.

The goal of this paper is to construct FE-DSFM to have optimal error decay for the Stokes equations (1) in a non-convex domain with \( \omega > \pi \). The main strategy of FE-DSFM is to compute the regular solution \((w, q)\) and the stress intensity factors \( \alpha \) by applying the finite element method. To built FE-DSFM, we need to use the following dual singular functions \((u_d, p_d)\), where \( u_d = (u_d, v_d) \), which is derived in \( \S 6.2 \):

\[
\begin{pmatrix} u_d \\ v_d \\ p_d \end{pmatrix} = d_1 \begin{pmatrix} -r^{-\lambda} \frac{\lambda}{\mu} \sin(\theta) \sin((1+\lambda)\theta) \\ -r^{-\lambda} \frac{1}{\mu} (\sin(\lambda\theta) - \lambda \sin(\theta) \cos((1+\lambda)\theta)) \\ 2r^{-\lambda-1} \cos((1+\lambda)\theta) \end{pmatrix} + d_2 \begin{pmatrix} r^{-\lambda} \frac{1}{\mu} (\sin(\lambda\theta) + \lambda \sin(\theta) \cos((1+\lambda)\theta)) \\ r^{-\lambda} \frac{\lambda}{\mu} \sin(\theta) \sin((1+\lambda)\theta) \\ 2r^{-\lambda-1} \sin((1+\lambda)\theta) \end{pmatrix},
\]

where

\[
d_1 = \sin(\lambda\omega) + \lambda \sin(\omega) \cos((1+\lambda)\omega) \quad \text{and} \quad d_2 = \lambda \sin(\omega) \sin((1+\lambda)\omega).
\]

The paper is organized as follows. FE-DSFM to find the smooth part \((w_h, q_h)\) of the solution and stress intensity factor \( \alpha_h \) is constructed in (20) and (21) in \( \S 2 \). And we establish the well-posedness of the variational form of FE-DSFM in \( \S 3 \). We carry out the following error estimates in \( \S 4 \) by proving several lemmas.

**Theorem 1 (Main theorem).** Let Assumption 1 below hold. If \( h \) is small enough, then we have

\[
|\alpha - \alpha_h| + \|w - w_h\|_0 \leq C h^{1+\lambda} \quad \text{and} \quad \|w - w_h\|_1 + \|q - q_h\|_0 \leq C h.
\]

The sub-optimality for \( |\alpha - \alpha_h| + \|w - w_h\|_0 \) is due to the weak regularity of the functions in duality argument, as discussed in Remark 4.4. In \( \S 5 \), we perform numerical tests with known solution to compare to theoretical results in the Theorem 1. In these tests, we obtain the optimal accuracy (2) which is better than (9). Finally, we present the singular and the dual singular functions in \( \S 6 \).
2. The finite element dual singular function method

In this section, we build a new variational formulation to find the regular part of solution \((w, q)\) and the stress intensity factor \(\alpha\). We start this section with introducing the following lemma for the properties of the singular and the dual singular functions.

**Lemma 2.1 (Properties of singular and dual singular functions).** The singular function \((u_s, p_s) \in H^{1+\lambda}(\Omega) \times H^{\lambda}(\Omega)\) and the dual singular function \((\mathbf{u}_d, p_d) \notin H^1(\Omega) \times L^2(\Omega)\) are solutions of

\[
-\mu \Delta u_s + \nabla p_s = 0, \quad \text{in } \Omega,
\]

and

\[
-\mu \Delta u_d + \nabla p_d = 0, \quad \text{in } \Omega,
\]

respectively. The boundary conditions of \(u_s\) and \(u_d\) vanish on \(\Gamma_{in}\), but the boundary value of \(u_d\) is not defined at the origin. Both of \(u_s\) and \(u_d\) are not \(0\) on \(\Gamma_{out}\).

In order to derive an explicit form of the singular functions, we set

\[
B(r_1; r_2) = \{(r, \theta) : r_1 < r < r_2 \text{ and } 0 < \theta < \omega\} \cap \Omega
\]

and

\[
B(r_1) = B(0; r_1),
\]

and define a smooth enough cut-off function of \(r, \eta_r\), as follows:

\[
\eta_r = \begin{cases} 
1, & \text{in } B(\frac{1}{2} \rho R), \\
\text{very smooth function}, & \text{in } B(\frac{1}{2} \rho R; \rho R), \\
0, & \text{in } \Omega \setminus B(\rho R),
\end{cases}
\]

where \(\rho\) is a parameter in \((0, 2]\) and \(R\) is a fixed real number which will be determined later so that the singular part of \(\eta_r u_s\) has \(0\) on whole \(\partial \Omega\). Here and thereafter, we choose that \(\eta = \eta_r\) in (7) and assume that \(0 < \rho < 1\). That is, the singular function representation of the solution of problem (1) has the form

\[
\left( \begin{array}{c} u \\
p \end{array} \right) = \left( \begin{array}{c} w \\
q \end{array} \right) + \alpha \left( \begin{array}{c} \eta_r u_s \\
\eta_r p_s \end{array} \right),
\]

where \(w \in H^1_0(\Omega) \cap H^2(\Omega)\) and \(q \in L^2(\Omega) \cap H^1(\Omega)\) satisfy

\[
-\mu \Delta w + \nabla q + \alpha (-\mu \Delta (\eta_r u_s) + \nabla (\eta_r p_s)) = f, \quad \text{in } \Omega,
\]

\[
\nabla \cdot (w + \alpha \eta_r u_s) = 0, \quad \text{in } \Omega.
\]

For the sake of a clear explanation, we note that the inner product of vectors \(\mathbf{a} = (a_1, a_2)\) and \(\mathbf{b} = (b_1, b_2)\) is

\[
\langle \mathbf{a}, \mathbf{b} \rangle = \langle a_1, b_1 \rangle + \langle a_2, b_2 \rangle
\]

and

\[
\langle \nabla \mathbf{a}, \nabla \mathbf{b} \rangle = \langle \partial_x a_1, \partial_y b_1 \rangle + \langle \partial_x a_2, \partial_y b_2 \rangle + \langle \partial_y a_1, \partial_x b_1 \rangle + \langle \partial_y a_2, \partial_x b_2 \rangle.
\]

Then we can obtain the weak form for (12) by the standard finite element technique: find \((w, q) \in H^1_0(\Omega) \times L^2(\Omega)\) satisfying, for all \(\mathbf{v} \in H^1_0(\Omega)\) and \(\phi \in L^2(\Omega),

\[
\mu \langle \nabla w, \nabla \mathbf{v} \rangle + \langle \nabla q, \mathbf{v} \rangle + \alpha \langle -\mu \Delta (\eta_r u_s) + \nabla (\eta_r p_s), \mathbf{v} \rangle = \langle f, \mathbf{v} \rangle,
\]

\[
\langle \nabla \cdot w, \phi \rangle + \alpha \langle \nabla \cdot (\eta_r u_s), \phi \rangle = 0.
\]
By the following extraction formula:
\[ \text{Lemma 2.3} \]

\[ \langle -\mu \Delta \mathbf{w} + \nabla \mathbf{q}, \eta_2 \mathbf{u}_d \rangle + \alpha \langle -\mu \Delta (\eta_\rho \mathbf{u}_s) + \nabla (\eta_\rho \mathbf{p}_s), \eta_2 \mathbf{u}_d \rangle = (\mathbf{f}, \eta_2 \mathbf{u}_d), \]
\[ \langle \nabla \cdot (\mathbf{w} + \alpha \eta_\rho \mathbf{u}_s), \eta_2 \mathbf{p}_d \rangle = 0. \]

Because the dual singular functions are not smooth enough to apply the integration by parts directly in (14), the following lemma is crucial.

**Lemma 2.2 (Integration by parts for dual singular functions).** For \( \rho \in (0, 1] \), we have that
\[ \langle \nabla \cdot \mathbf{w}, \eta_2 \mathbf{u}_d \rangle = \langle \mathbf{w}, -\mu \Delta (\eta_2 \mathbf{u}_d) + \nabla (\eta_2 \mathbf{p}_d) \rangle \]
and
\[ \langle \nabla \cdot \mathbf{q}, \eta_2 \mathbf{u}_d \rangle = -\langle \mathbf{q}, \nabla \cdot (\eta_2 \mathbf{u}_d) \rangle. \]

**Proof.** We can obtain (15) by integration by parts, if all functions inside the inner products are smooth enough. So, in light of density argument of Hilbert space, we need to show boundedness of both sides. Since \( \mathbf{w} \in H^1(\Omega) \cap H^2(\Omega) \) and \( \eta_2 \mathbf{u}_d \in L^2(\Omega) \) and since \( \nabla \cdot \mathbf{w} = 0 \) in \( B(\frac{x}{2}) \), we can get the boundedness of the left hand side in (15). On the other hand, the right hand side in (15) is also bounded, because of (11) and the definition of \( \eta_2 \rho \). So we conclude (15). By the same manner, the properties \( \mathbf{q} \in H^1(\Omega), \eta_2 \mathbf{u}_d \in L^2(\Omega), \) and \( \nabla \cdot (\eta_2 \mathbf{u}_d) = 0 \) in \( B(\rho) \) yield (16).

We subtract the second equation from the first equation in (14) and then apply Lemma 2.2 to obtain
\[ \alpha (\beta_m - \beta_p) = \beta_c - \langle -\mu \Delta \mathbf{w} + \nabla \mathbf{q}, \eta_2 \mathbf{u}_d \rangle + \langle \nabla \cdot \mathbf{w}, \eta_2 \mathbf{p}_d \rangle \]
\[ = \beta_c - \langle \mathbf{w}, -\mu \Delta (\eta_2 \mathbf{u}_d) + \nabla (\eta_2 \mathbf{p}_d) \rangle + \langle \mathbf{q}, \nabla \cdot (\eta_2 \mathbf{u}_d) \rangle, \]
where
\[ \beta_c := (\mathbf{f}, \eta_2 \mathbf{u}_d), \]
\[ \beta_m := -\mu \Delta (\eta_\rho \mathbf{u}_s) + \nabla (\eta_\rho \mathbf{p}_s), \eta_2 \mathbf{u}_d, \]
\[ \beta_p := \langle \nabla \cdot (\eta_\rho \mathbf{u}_s), \eta_2 \mathbf{p}_d \rangle. \]

Finally we arrive at the following lemma to compute the stress intensity factor \( \alpha \).

**Lemma 2.3 (Formula for the stress intensity factor \( \alpha \)).** The values \( |\beta_c|, |\beta_m|, \) and \( |\beta_p| \) are bounded and the stress intensity factor \( \alpha \) can be expressed in terms of \( \langle \mathbf{w}, \mathbf{q} \rangle \) by the following extraction formula:
\[ \alpha = \frac{1}{\beta_m - \beta_p} - \langle \mathbf{w}, -\mu \Delta (\eta_2 \mathbf{u}_d) + \nabla (\eta_2 \mathbf{p}_d) \rangle + \langle \mathbf{q}, \nabla \cdot (\eta_2 \mathbf{u}_d) \rangle. \]

**Proof.** Because (18) comes directly from (17), it is enough to show boundedness of 3 values \( \beta_c, \beta_m, \) and \( \beta_p \). First, \( |\beta_c| < \infty \) is trivial, because of both \( \mathbf{f} \) and \( \eta_2 \mathbf{u}_d \in L^2(\Omega) \). In light of the definition of \( \eta_\rho \), (10) yields \( -\mu \Delta (\eta_\rho \mathbf{u}_s) + \nabla (\eta_\rho \mathbf{p}_s) = 0 \) on \( B(\frac{x}{2}) \cup (\Omega - B(\rho)) \), or \( -\mu \Delta (\eta_\rho \mathbf{u}_s) + \nabla (\eta_\rho \mathbf{p}_s) \) is a smooth enough function. So we readily get \( |\beta_m| < \infty \). By the same manner, we can readily obtain \( |\beta_p| < \infty \) from \( \nabla \cdot (\eta_\rho \mathbf{u}_s) = 0 \) in \( B(\frac{x}{2}) \).

We note that both \( -\mu \Delta (\eta_2 \mathbf{u}_d) + \nabla (\eta_2 \mathbf{p}_d) \) and \( \nabla \cdot (\eta_2 \mathbf{u}_d) \) are smooth enough functions, because the singular parts are removed by the properties \( -\mu \Delta (\eta_2 \mathbf{u}_d) + \nabla (\eta_2 \mathbf{p}_d) = 0 \) and \( \nabla \cdot (\eta_2 \mathbf{u}_d) = 0 \) in \( B(\rho) \). So we can compute (18) without any
special technique to treat singular functions. In order to solve the coupled system (13) and (18) to find 3 unknown variables \((w, q)\) and \(\alpha\), we insert \(\alpha\) in (18) into (13), and then we obtain a mixed formula

\[
\mu \langle \nabla w, \nabla v \rangle + \langle \nabla q, v \rangle + (a(w) + b(q)) \ c(v) = \langle f, v \rangle - \frac{\beta_r}{\beta_m - \beta_p} c(v),
\]

\[
\langle \nabla \cdot w, \phi \rangle + (a(w) + b(q)) \ d(\phi) = -\frac{\beta_r}{\beta_m - \beta_p} d(\phi),
\]

where

\[
a(w) = -\frac{1}{\beta_m - \beta_p} \langle w, -\mu \Delta (\eta_2 p_u) + \nabla (\eta_2 p_d) \rangle, \quad b(q) = \frac{1}{\beta_m - \beta_p} \langle q, \nabla \cdot (\eta_2 p_u) \rangle,
\]

\[
c(v) = -\mu \Delta (\eta_3 u) + \nabla (\eta_3 p_s), \quad d(\phi) = \langle \nabla \cdot (\eta_3 p_s), \phi \rangle.
\]

In order to introduce the finite element discretization, we need further notations. Let \(\mathcal{T} = \{K\}\) be a shape-regular quasi-uniform partition of \(\Omega\) of meshsize \(h\) into closed elements \(K\) [1, 2, 6]. The vector and scalar finite element spaces are:

\[
\mathcal{W}_h := \{w_h \in L^2(\Omega) : w_h|_K \in \mathcal{P}(K) \quad \forall K \in \mathcal{T}\}, \quad \mathcal{V}_h := \mathcal{W}_h \cap \mathcal{H}_0^1(\Omega),
\]

\[
\mathcal{P}_h := \{q_h \in L^2(\Omega) \cap C^0(\Omega) : q_h|_K \in \mathcal{Q}(K) \quad \forall K \in \mathcal{T}\},
\]

where \(\mathcal{P}(K)\) and \(\mathcal{Q}(K)\) are spaces of polynomials with degree bounded uniformly with respect to \(K \in \mathcal{T}\). We stress that the space \(\mathcal{P}_h\) is composed of continuous functions to use integration by parts: for all \(q_h \in \mathcal{P}_h\)

\[
\langle \nabla \cdot v_h, q_h \rangle = -\langle v_h, \nabla q_h \rangle, \quad \forall v_h \in \mathcal{V}_h.
\]

Finally, we arrive at FE-DSFM from (18) and (19): find \((w_h, q_h) \in \mathcal{V}_h \times \mathcal{P}_h\) as the solution of, for all \(v_h \in \mathcal{V}_h\) and for all \(\phi_h \in \mathcal{P}_h\),

\[
\mu \langle \nabla w_h, \nabla v_h \rangle + \langle \nabla q_h, v_h \rangle + (a(w_h) + b(q_h)) \ c(v_h) = \langle f, v_h \rangle - \frac{\beta_r}{\beta_m - \beta_p} c(v_h),
\]

\[
\langle \nabla \cdot w_h, \phi_h \rangle + (a(w_h) + b(q_h)) \ d(\phi_h) = -\frac{\beta_r}{\beta_m - \beta_p} d(\phi_h),
\]

and then find \(\alpha_h \in \mathbb{R}\) by computing

\[
\alpha_h = \frac{\beta_r}{\beta_m - \beta_p} + a(w_h) + b(q_h).
\]

We note that \(a(\cdot), b(\cdot), c(\cdot), \) and \(d(\cdot)\) are the same functionals within (19).

The matrix form of the coupled system (20) becomes

\[
\begin{bmatrix}
A & B T \\
B & 0
\end{bmatrix}
+\begin{bmatrix}
c \\
d
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix}
= \begin{bmatrix}
w \\
q
\end{bmatrix}
= \begin{bmatrix}
L \\
l
\end{bmatrix},
\]

and it is solvable by the Sherman-Morrison-Woodbury formula in [7].

\[
(M + U \cdot V^T)^{-1} = M^{-1} - \frac{M^{-1} U V^T M^{-1}}{1 + V^T M^{-1} U}
\]

and then we find \(\alpha_h\) by computing (21). So we conclude that FE-DSFM (20) and (21) is an applicable algorithm.

Even though we consider the solution including only one singular part in this paper, FE-DSFM can be applied to find a solution including 2 singular parts by the following remark:
Remark 2.4 (The solution with 2 singular parts). If the solution has 2 stress intensity factors, then the solution is expressed by the form (6) and the matrix form (22) becomes

\[
\begin{pmatrix}
A & B^T \\
B & 0
\end{pmatrix} + \sum_{i=1}^{2} \begin{pmatrix} a_i \\ d_i \end{pmatrix} \begin{pmatrix} a_i, b_i \end{pmatrix} \begin{pmatrix} w \\ q \end{pmatrix} = \begin{pmatrix} L \\ l \end{pmatrix}.
\]

It is also solvable by using the generalized Sherman-Morrison-Woodbury formula:

\[
(M + U_1 \cdot V_1^T + U_2 \cdot V_2^T)^{-1} = M^{-1} - M^{-1}[U_1, U_2]Q^{-1}[V_1, V_2]^T M^{-1},
\]

where \( Q \) is a \( 2 \times 2 \) matrix given by

\[
Q = \begin{bmatrix}
1 + V_1^T M^{-1} U_1, & V_1^T M^{-1} U_2 \\
V_2^T M^{-1} U_1, & 1 + V_2^T M^{-1} U_2
\end{bmatrix}.
\]

3. Well-posedness

In this section, we establish the well-posedness of the coupled system (19) by the use of the contraction mapping theorem. Because solving the system (19) is equivalent to solving equations of (13) and (18), we will prove the well-posedness of the system (13) and (18). To do this, we first check the compatibility condition of (13) as

\[
\int_{\Omega} \nabla \cdot (\eta_\alpha u_\alpha) \, dx = \int_{\partial \Omega} \eta_\alpha u_\alpha \cdot \nu \, ds = 0,
\]

where \( \nu \) is the outward unit normal vector. The system (13) is a standard saddle point problem and has a unique solution for all given \( \alpha \in \mathbb{R} \) and for all \( f \in L^2(\Omega) \) in [1, 2, 6]. So we can define a mapping \( T_\alpha \) from \( \mathbb{R} \) to \( H^1(\Omega) \times L^2_0(\Omega) \) by the unique solution of (13) for any given \( f \in L^2(\Omega) \). It means that \( T_\alpha(\alpha) := (w_\alpha, q_\alpha) \) is the solution of (13) with \( \alpha \in \mathbb{R} \). Also we define a mapping \( T_\alpha \) from \( H^1(\Omega) \times L^2_0(\Omega) \) to \( \mathbb{R} \) by using (18) as

\[
T_\alpha(w, q) := \frac{1}{\beta_m - \beta_p} (\beta_f - \langle w, -\mu \nabla (\eta_\alpha u_\alpha) + \nabla (\eta_\alpha p_\alpha) \rangle + \langle q, \nabla \cdot (\eta_\alpha u_\alpha) \rangle).
\]

Then the composition \( T_\alpha \circ T_\gamma \) is a mapping from \( \mathbb{R} \) to \( \mathbb{R} \). To prove the well-posedness of the problem (19) is equivalent to prove existence of the unique fixed point of \( T_\alpha \circ T_\gamma \), and it is equivalent to prove \( \|T_\alpha \circ T_\gamma\| < 1 \). We now start to prove \( \|T_\alpha \circ T_\gamma\| < 1 \) by applying the contraction mapping theorem [11]. To do this, we choose arbitrary real numbers \( \alpha_1 \) and \( \alpha_2 \), and let

\[
T_\alpha(\alpha_1) = (w_{\alpha_1}, q_{\alpha_1}) \quad \text{and} \quad T_\alpha(\alpha_2) = (w_{\alpha_2}, q_{\alpha_2}).
\]

Then we can get from (13),

\[
\langle \nabla w_{\alpha_2}, \nabla v \rangle + \langle \nabla q_{\alpha_2}, v \rangle = -\langle \alpha_1 - \alpha_2 \rangle (-\mu \nabla (\eta_\alpha u_\alpha) + \nabla (\eta_\alpha p_\alpha), v),
\]

\[
\langle \nabla \cdot (w_{\alpha_2} - w_{\alpha_1}), \phi \rangle = -\langle \alpha_1 - \alpha_2 \rangle \langle \nabla \cdot (\eta_\alpha u_\alpha), \phi \rangle.
\]

And we define the following Stokes type problem:

\[
\begin{align*}
-\mu \Delta x + \nabla k &= -\mu \nabla (\eta_\alpha u_\alpha) + \nabla (\eta_\alpha p_\alpha), \quad \text{in } \Omega, \\
\nabla \cdot x &= \nabla \cdot (\eta_\alpha u_\alpha), \quad \text{in } \Omega, \\
x &= 0, \quad \text{on } \partial \Omega.
\end{align*}
\]
We note here that the right hand side terms are smooth functions which come from Lemma 2.1. In order to assert the existence and uniqueness of the solution of (26), we need to check the compatibility condition. Since we have
\[ \int_{\Omega} \nabla \cdot \mathbf{x} \, dx = \int_{\partial \Omega} \mathbf{x} \cdot \mathbf{n} \, ds = 0, \]
it is enough to prove \( \int_{\Omega} \nabla \cdot (\eta_{2\rho} \mathbf{u}_d) = 0 \). Because \( \nabla \cdot (\eta_{2\rho} \mathbf{u}_d) = 0 \) in \( B(\rho) \) and \( \eta_{2\rho} \mathbf{u}_d = 0 \) on \( \partial \Omega \) except the origin, there exists a function \( \mathbf{y} \in H^1_0(\Omega) \) satisfying
\[ \langle \nabla \cdot \mathbf{y}, \phi \rangle = \langle \nabla \cdot (\eta_{2\rho} \mathbf{u}_d), \phi \rangle, \]
for all \( \phi \in L^2(\Omega) \) and \( \mathbf{y} = 0 \) on \( \partial \Omega \). So we can get
\[ \int_{\Omega} \nabla \cdot (\eta_{2\rho} \mathbf{u}_d) \, dx = \int_{\Omega} \nabla \cdot \mathbf{y} \, dx = \int_{\partial \Omega} \mathbf{y} \cdot \mathbf{n} \, ds = 0 \]
and we obtain \( \int_{\Omega} \nabla \cdot \mathbf{x} \, dx = \int_{\Omega} \nabla \cdot (\eta_{2\rho} \mathbf{u}_d) \, dx = 0 \). Thus we conclude that there exist a unique solution \((\mathbf{x}, \mathbf{k}) \in H^1(\Omega) \times L^2(\Omega) \) of (26).

We apply (24), (23) and (26) as a sequence and then use the integration by parts to obtain
\[
| T_\alpha \circ T_\Gamma(\alpha_1) - T_\alpha \circ T_\Gamma(\alpha_2) | = \left| T_\alpha (\mathbf{w}_{\alpha_1}, q_{\alpha_1}) - T_\alpha (\mathbf{w}_{\alpha_2}, q_{\alpha_2}) \right|
\]
\[
= \frac{1}{\beta_m - \beta_p} \left( - \langle \mathbf{w}_{\alpha_1} - \mathbf{w}_{\alpha_2}, -\mu \Delta (\eta_{2\rho} \mathbf{u}_d) + \nabla (\eta_{2\rho} p_d) \rangle + \langle q_{\alpha_1} - q_{\alpha_2}, \nabla \cdot (\eta_{2\rho} \mathbf{u}_d) \rangle \right)
\]
\[
= \frac{1}{\beta_m - \beta_p} \left( - \langle \mathbf{w}_{\alpha_1} - \mathbf{w}_{\alpha_2}, -\mu \Delta x + \nabla k \rangle + \langle q_{\alpha_1} - q_{\alpha_2}, \nabla \cdot x \rangle \right)
\]
\[
+ \langle \nabla \cdot (\mathbf{w}_{\alpha_1} - \mathbf{w}_{\alpha_2}) \rangle \rangle \rangle = \langle \nabla \cdot (q_{\alpha_1} - q_{\alpha_2}) \rangle \rangle = \langle \nabla \cdot k \rangle \rangle.
\]
Invoking (25) with \( \mathbf{v} = \mathbf{x} \) and \( \phi = k \), we have
\[
| T_\alpha \circ T_\Gamma(\alpha_1) - T_\alpha \circ T_\Gamma(\alpha_2) | = \left| \frac{\alpha_1 - \alpha_2}{\beta_m - \beta_p} \langle -\mu \Delta (\eta_{\rho} \mathbf{u}_s) + \nabla (\eta_{\rho} p_s) \rangle \rangle \rangle - \langle \nabla \cdot (\eta_{\rho} \mathbf{u}_s) \rangle \rangle \rangle \langle k \rangle \rangle \rangle.
\]
Integration by parts derives
\[
| T_\alpha \circ T_\Gamma(\alpha_1) - T_\alpha \circ T_\Gamma(\alpha_2) | = \left| \frac{\alpha_1 - \alpha_2}{\beta_m - \beta_p} \langle -\mu \Delta x + \nabla k \rangle \rangle \rangle - \langle \eta_{\rho} p_s \rangle \rangle \rangle \langle \nabla \cdot \mathbf{x} \rangle \rangle \rangle \langle \eta_{\rho} \mathbf{u}_s \rangle \rangle \rangle.
\]
We now test \( \eta_{\rho} \mathbf{u}_s \) and \( \eta_{\rho} p_s \) with the first and the second equations in (26) to obtain
\[ \langle -\mu \Delta x + \nabla k \rangle \rangle \rangle \langle \eta_{\rho} \mathbf{u}_s \rangle \rangle \rangle = \langle -\mu \Delta (\eta_{2\rho} \mathbf{u}_d) + \nabla (\eta_{2\rho} p_d) \rangle \rangle \rangle \langle \eta_{\rho} \mathbf{u}_s \rangle \rangle \rangle,
\]
\[ \langle \nabla \cdot \mathbf{x} \rangle \rangle \rangle \langle \eta_{\rho} p_s \rangle \rangle \rangle = \langle \nabla \cdot (\eta_{2\rho} \mathbf{u}_d) \rangle \rangle \rangle \langle \eta_{\rho} \mathbf{p}_s \rangle \rangle \rangle.
\]
We can check easily that the right hand side terms of above equations are identically zero by Lemma 2.1, because of the distinct supports of \( \eta_{\rho} \) and \( \eta_{2\rho} \). Therefore, we complete well-posedness of the problem (19).

4. Error Analysis for the finite element dual singular function method

In this section, we will prove Theorem 1 which are errors of FE-DSFM (20)~(21) by comparing them with (18)~(19). To do this, we first introduce an assumption.

**Assumption 1** (Discrete inf-sup). For given \( p_h \in \Pi_h \), there exists a constant \( \gamma > 0 \) such that
\[ \| p_h \|_0 \leq \sup_{\mathbf{v}_h \in V_h} \frac{\langle \nabla \cdot \mathbf{v}_h, p_h \rangle}{\| \mathbf{v}_h \|_1}. \]
In order to introduce a useful lemma in \([5]\), we consider the Stokes equations:

\[-\mu\Delta \mathbf{x} + \nabla k = \mathbf{f}, \quad \text{in } \Omega,\]
\[\nabla \cdot \mathbf{x} = \chi, \quad \text{in } \Omega,\]
\[\mathbf{x} = \mathbf{0}, \quad \text{on } \partial \Omega.\]

**Lemma 4.1** (Upper bound of regular solution). Let \(\Omega\) be a polygonal domain with non-convex vertices. If \(\mathbf{f} \in H^{-1}(\Omega)\) and \(\chi \in L^2(\Omega)\), then there exist a unique solution \((\mathbf{x}, k) \in H_0^1(\Omega) \times L^2_0(\Omega)\) of (28), with

\[\|\mathbf{x}\|_1 + \|k\|_0 \leq C (\|\mathbf{f}\|_{-1} + \|\chi\|_0).\]

Moreover, if \(\mathbf{f} \in L^2(\Omega)\) and \(\chi \in H^1(\Omega)\), then the solution \((\mathbf{x}, k) \in H_0^1(\Omega) \times L^2_0(\Omega)\) can be rewritten in the form of \(\mathbf{x} = \mathbf{x}_R + \alpha_1 \eta_1 u_{a1} + \alpha_2 \eta_2 u_{a2}\) and \(k = k_R + \alpha_1 \eta_1 p_{a1} + \alpha_2 \eta_2 p_{a2}\), with \((\mathbf{x}_R, k_R) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times (L^2_0(\Omega) \cap H^2(\Omega))\) satisfying

\[\|\mathbf{x}_R\|_2 + \|k_R\|_1 + \|\alpha_1\| + \|\alpha_2\| \leq C (\|\mathbf{f}\|_0 + \|\chi\|_1),\]

where \((\eta_1 u_{a1}, \eta p_{a1})\) and \((\eta u_{a2}, \eta_2 p_{a2})\) are singular functions.

We evaluate errors under the notations:

\[E := \mathbf{w} - w_h, \quad E_h := \mathcal{I}_h \mathbf{w} - w_h, \quad \mathcal{I}_h \mathbf{E} := \mathbf{w} - \mathcal{I}_h \mathbf{w}, \quad E_h := q - q_h, \quad I_h e := q - \mathcal{I}_h q,\]

and

\[\varepsilon := \alpha - \alpha_h,\]

where \(\mathcal{I}_h\) the Clement interpolant. Because \((\mathbf{w}, q) \in H^2(\Omega) \times H^1(\Omega)\), we can use the well known results

(29) \[\|I_h \mathbf{E}\|_0 + h\|E_h\|_1 \leq C h^2 \|\mathbf{w}\|_2 \quad \text{and} \quad \|I_h e\|_0 \leq C h \|q\|_1.\]

In proof of the main theorem, We will use the solution \((\mathbf{z}, r)\) of, for all \(\mathbf{v} \in H^1_0(\Omega)\) and for all \(\phi \in L^2(\Omega)\),

\[\mu \langle \nabla \mathbf{z}, \nabla \mathbf{v} \rangle + \langle \nabla r, \mathbf{v} \rangle + (1 - \kappa) (c(\mathbf{z}) - d(\mathbf{r})) a(\mathbf{v}) = \langle \mathbf{E}, \mathbf{v} \rangle,\]
\[\langle \nabla \cdot \mathbf{z}, \phi \rangle - (1 - \kappa) (c(\mathbf{z}) - d(\mathbf{r})) b(\phi) = 0,\]

where \(\kappa\) is an arbitrary small positive constant. We will establish the well-posedness of equations (30) via the same manner within §3. To do this, we rewrite (30) with \(\alpha := (1 - \kappa)(c(\mathbf{z}) - d(\mathbf{r}))\) and

(30) \[\mu \langle \nabla \mathbf{z}, \nabla \mathbf{v} \rangle + \langle \nabla r, \mathbf{v} \rangle + \bar{\alpha} a(\mathbf{v}) = \langle \mathbf{E}, \mathbf{v} \rangle,\]
\[\langle \nabla \cdot \mathbf{z}, \phi \rangle - \bar{\alpha} b(\phi) = 0.\]

We will prove well-posedness of (30) in next lemma, by verifying existence and uniqueness of the solution \((\mathbf{z}, r, \bar{\alpha})\) of (31).

**Lemma 4.2** (Well-posedness and regularity of (30)). If \(\kappa\) is any number in \((0, 1)\), then equations (30) have a unique solution \((\mathbf{z}, r)\) in \(H^1_0(\Omega) \times L^2_0(\Omega)\) and there exists a positive constant \(C_1\) satisfying

(32) \[\|\mathbf{z}\|_1 + \|r\|_0 \leq C_1 \|\mathbf{E}\|_0.\]

**Proof.** We first show the compatibility condition for (30). since we have

\[\int_\Omega \nabla \cdot \mathbf{z} \, dx = \int_{\partial \Omega} \mathbf{z} \cdot \mathbf{n} \, ds = 0,\]

where \(\mathbf{n}\) is the outward unit normal vector, we conclude \(\int_\Omega \nabla \cdot \mathbf{z} \, dx = \int_\Omega \nabla \cdot (\eta_2 u_{a2}) \, dx = 0\), because of \(b(1) = 0\) in (27).
We now start to prove existence and uniqueness of solution of (30). It is well known that (31) has unique solution \((z_{\alpha}, r_{\alpha})\) for any \(\alpha \in \mathbb{R}\), if \(E \in L^2(\Omega)\). So we can define a mapping \(F_E\) from \(\mathbb{R}\) to \(H^1_0(\Omega) \times L^2_0(\Omega)\) by \(F_E(\alpha) := (z_{\alpha}, r_{\alpha})\) to be the solution of (31), for any \(\alpha \in \mathbb{R}\). And we define \(F_\alpha\) from \(H^1_0(\Omega) \times L^2_0(\Omega)\) to \(\mathbb{R}\) by
\[
F_\alpha(z, r) := (1 - \kappa)(c(z) - d(r)) = (1 - \kappa)((-\mu \Delta(\eta_{p}u_s) + \nabla(\eta_{p}p_s), r) - \langle \nabla \cdot (\eta_{p}u_s), r \rangle).
\]
Then \(F_\alpha \circ F_E\) becomes a mapping from \(\mathbb{R}\) to \(\mathbb{R}\). So we need to show, by the contraction mapping theorem [11], that \(F_\alpha \circ F_E\) has a unique fixed point to finish this proof. And it is equivalent to proving \(\|F_\alpha \circ F_E\| < 1\). To show this, let \(\bar{\alpha}_1\) and \(\bar{\alpha}_2\) be arbitrary real numbers and let
\[
F_E(\bar{\alpha}_1) = (z_{\bar{\alpha}_1}, r_{\bar{\alpha}_1}) \quad \text{and} \quad F_E(\bar{\alpha}_2) = (w_{\bar{\alpha}_2}, r_{\bar{\alpha}_2}).
\]
From (31), we obtain
\[
\mu \langle \nabla (z_{\alpha_1} - z_{\alpha_2}), \nabla v \rangle + \langle \nabla (r_{\alpha_1} - r_{\alpha_2}), v \rangle = \frac{\bar{\alpha}_1 - \bar{\alpha}_2}{\beta_m - \beta_p} \langle -\mu \Delta(\eta_{p}u_s) + \nabla(\eta_{p}p_s), v \rangle,
\]
\[
\langle \nabla \cdot (z_{\alpha_1} - z_{\alpha_2}), \phi \rangle = \frac{\bar{\alpha}_1 - \bar{\alpha}_2}{\beta_m - \beta_p} \langle \nabla \cdot (\eta_{p}u_s), \phi \rangle.
\]
And we define the Stokes equations
\[
\begin{align*}
-\mu \Delta x + \nabla k &= -\mu \Delta(\eta_{p}u_s) + \nabla(\eta_{p}p_s), \quad \text{in} \ \Omega, \\
\nabla \cdot x &= \nabla \cdot (\eta_{p}u_s), \quad \text{in} \ \Omega, \\
\nabla \cdot x &= 0, \quad \text{on} \ \partial \Omega.
\end{align*}
\]
We note here that the right hand side terms are smooth functions which come from Lemma 2.1 and the compatibility condition hold because of \(\int terminated
\]
\[
F_\alpha \circ F_E(\bar{\alpha}_1) - F_\alpha \circ F_E(\bar{\alpha}_2) = F_\alpha(z_{\bar{\alpha}_1}, r_{\bar{\alpha}_1}) - F_\alpha(w_{\bar{\alpha}_2}, r_{\bar{\alpha}_2})
\]
\[
= (1 - \kappa)((-\mu \Delta(\eta_{p}u_s) + \nabla(\eta_{p}p_s), z_{\bar{\alpha}_1} - z_{\bar{\alpha}_2}) - \langle \nabla \cdot (\eta_{p}u_s), r_{\bar{\alpha}_1} - r_{\bar{\alpha}_2} \rangle)
\]
\[
= (1 - \kappa)((-\mu \Delta x + \nabla k, z_{\bar{\alpha}_1} - z_{\bar{\alpha}_2}) - \langle \nabla \cdot (z_{\bar{\alpha}_1} - z_{\bar{\alpha}_2}), \phi \rangle)
\]
\[
= (1 - \kappa)((-\mu \Delta x, \nabla(z_{\bar{\alpha}_1} - z_{\bar{\alpha}_2})) - \langle k, \nabla \cdot (z_{\bar{\alpha}_1} - z_{\bar{\alpha}_2}) \rangle + \langle x, \nabla (r_{\bar{\alpha}_1} - r_{\bar{\alpha}_2}) \rangle).
\]
We now consider \(v = x\) and \(\phi = k\) in (35) to derive
\[
F_\alpha \circ F_E(\bar{\alpha}_1) - F_\alpha \circ F_E(\bar{\alpha}_2) = \left| \frac{\bar{\alpha}_1 - \bar{\alpha}_2}{\beta_m - \beta_p} \langle -\mu \Delta(\eta_{p}u_s) + \nabla(\eta_{p}p_s), x \rangle - \langle \nabla \cdot (\eta_{p}u_s), k \rangle \right|,
\]
and then we apply Lemma 2.2 again to obtain
\[
F_\alpha \circ F_E(\bar{\alpha}_1) - F_\alpha \circ F_E(\bar{\alpha}_2) = \left| \frac{\bar{\alpha}_1 - \bar{\alpha}_2}{\beta_m - \beta_p} ((\eta_{p}u_s, -\mu \Delta x + \nabla k) - \langle \eta_{p}p_s, \nabla \cdot x \rangle) \right|,
\]
In light of (17), testing $\eta_{2p}u_d$ and $\eta_{2p}p_d$ with the first and the second in (36), respectively, leads
\[
-\mu \Delta x + \nabla k \cdot \eta_{2p}u_d = -\mu \nabla (\eta_p p_s) + \eta_{2p}u_d = \beta_m, \\
\nabla \cdot x \cdot \eta_{2p}p_d = \nabla \cdot (\eta_p u_s) + \eta_{2p}p_d = \beta_p.
\]
Finally, we arrive
\[
|F_2 \circ F_1 (\tilde{x}) - F_2 \circ F_1 (\tilde{x})| = |1 - \kappa| < |\tilde{x} - \tilde{x}|
\]
provided $\kappa \in (0, 1)$. Therefore, we prove that (30) has a unique solution $(z_f, r_f) \in H^1(\Omega) \times L^2_0(\Omega)$ for any $f \in H^{-1}(\Omega)$. From now on, we will derive (32) to finish this proof. Let $F : H^{-1}(\Omega) \to H^1(\Omega) \times L^2_0(\Omega)$ be the operator corresponding to the solution of (30), for all $f \in H^{-1}(\Omega)$, namely $F(f) = (z_f, r_f)$. According to (30), the inverse operator $F^{-1}$ is well-defined and
\[
\|f\|_1 = \|F^{-1}(z_f, r_f)\|_{-1} = \sup_{\|v\|_1} \mu (\nabla z_f, \nabla v) + (\nabla r_f, v) + (1 - \kappa)(c(z_f) - d(r_f))a(v)
\]
where $C$ depends only on given datum of domain $\Omega$, and given smooth functions $\nabla \cdot (\eta_{2p}u_d)$, $-\mu \Delta (\eta_{2p}u_d) + \nabla (\eta_{2p}p_d)$, and $\nabla \cdot (\eta_{2p}u_s)$, $-\mu \Delta (\eta_{2p}u_s) + \nabla (\eta_{2p}p_s)$, and $\nabla \cdot (\eta_{2p}u_d)$. So $F^{-1}$ is bounded bilinear transform. If we choose $C_1 = 1/\|F^{-1}\|$, then we can have
\[
\|z_f\|_1 + \|r_f\|_0 \leq \|f\||f||f|_{-1} \leq C_1\|f\|_0.
\]
If we choose $f = E$, we arrive at (32) and it is the proof.

**Lemma 4.3 (Properties of the solution $(z, r)$ of (30)).** Let $(z, r)$ be the solutions of (30). Then there is a singular function representation
\[
z = w_z + \alpha_z \eta_{2p}u_s \\
r = q_z + \alpha_z \eta_{2p}p_s,
\]
where $w_z \in H^2(\Omega) \cap H^1_0(\Omega)$, $q_z \in H^1(\Omega) \cap L^2_0(\Omega)$ and $\alpha_z$ satisfy the regularity estimate
\[
\|w_z\|_1 + \|q_z\|_1 + |\alpha_z| \leq C\|E\|_0.
\]

**Proof.** It is easy to check the solution (31) satisfies
\[
-\mu \Delta z + \nabla r = E + \frac{\alpha}{\beta_m - \beta_p} (\mu \nabla (\eta_{2p}u_d) + \nabla (\eta_{2p}p_d)),
\]
\[
\nabla \cdot z = \frac{\alpha}{\beta_m - \beta_p} \nabla \cdot (\eta_{2p}u_d).
\]
The right hand side terms are in $L^2(\Omega)$ and $H^1(\Omega) \cap L^2_0(\Omega)$ for the first and the second equations, respectively, which come from (11) and (27). So we can directly get the representation of (37) from Lemma 4.1 and
\[
\|w_z\|_1 + \|q_z\|_1 + \|\alpha_z\| \leq C\|E\|_1 + \|\alpha\|_1.
\]
According to (39), we arrive at
\[
\|w_z\|_1 + \|q_z\|_1 + \|\alpha_z\| \leq C \left[ E + \frac{\alpha}{\beta_m - \beta_p} (-\mu \nabla \eta_{2p}u_d + \nabla \eta_{2p}p_d) \right]_0 + C \left[ \frac{\alpha}{\beta_m - \beta_p} \nabla \cdot \eta_{2p}u_d \right]_1 \leq C (\|E\|_0 + \|\alpha\|).
\]
Because $|\alpha| = |(1 - \kappa)(c(z) - d(r))| \leq C(|z|_0 + |r|_0)$, (32) leads $|\alpha| \leq C\|E\|_0$ and (38). It is the proof.

If we denote
\[ G := z - I_h z \quad \text{and} \quad g := r - I_h r, \]
then thanks to Lemma 30, we have
\[ \|G\|_1 + \|g\|_0 \leq Ch^{1/2} ||E||_0, \]
because of $\|\eta_p u_s - I_h (\eta_p u_s)\|_1 + \|\eta_p p_s - I_h (\eta_p p_s)\|_0 \leq Ch^{1/2}$.

**Remark 4.4 (The reason of sub-optimality).** The inequality (40) is the main restriction to get optimal accuracy in the next lemma and the reason of sub-optimality $|\alpha - \alpha_h| + \|w - w_h\|_0 \leq Ch^{1+\lambda}$ in Theorem 1. Also the sub-optimality (3) is due to (40).

From now, we impose $\kappa = 0$ in (30) to simple explanation, because $\kappa$ is an arbitrary small positive constant. We start to estimate errors in $L^2(\Omega)$.

**Lemma 4.5 (Estimate $||E||_0$).** Let Assumption 1 hold. Then we have
\[ ||E||_0 \leq Ch^{1/2} (||E||_1 + ||e||_0), \]
\[ ||e||_0 \leq C ||E||_0 \quad \text{and} \quad ||e||_0 \leq C (||E||_1 + h||q||_1). \]

**Proof.** We start this proof with constructing error equations by subtracting (20) from (19) to get
\[ \mu \langle \nabla E, \nabla v_h \rangle + \langle \nabla e, v_h \rangle + (a(E) + b(e)) c(v_h) = 0, \]
\[ \langle \nabla \cdot E, \phi_h \rangle + (a(E) + b(e)) d(\phi_h) = 0. \]

We first prove (42). From the second equation in (43), we have
\[ |b(e)|d(\phi_h) = |\langle E, \nabla \phi_h - a(E)d(\phi_h) \rangle|. \]

We fix $\phi_h = C_2 x$ with $C_2$ satisfying $d(C_2 x) = 1$, then $||\nabla \phi_h||_0 = |C_2||\Omega|^{1/2}$ is a bounded number, because the space $P_h$ is composed of continuous functions. So we can readily obtain
\[ |b(e)| \leq C ||E||_0. \]

Therefore $e = a(E) + b(e)$ which comes from subtracting (21) from (18) yields
\[ |e| \leq C (||E||_0 + |b(e)|) \leq C ||E||_0, \]
and, in light of (43), Assumption 1 leads
\[ \gamma ||e_h||_0 \leq \sup_{v_h \in V_h} \mu \frac{\langle \nabla E, \nabla v_h \rangle + (I_h e, \nabla \cdot v_h) + (a(E) + b(e)) c(v_h)}{\|v_h\|_1} \]
\[ \leq C \left( ||E||_1 + h||q||_1 + ||E||_0 + |b(e)| \right). \]

Thus, in conjunction with (44), we arrive at (42). We now prove (41) with choosing $v_h = I_h z = z - G$ and $\phi = I_h r = r - g$ in (43):
\[ \mu \langle \nabla E, \nabla (z - G) \rangle + \langle \nabla e, z - G \rangle + (a(E) + b(e)) c(z - G) = 0, \]
\[ \langle \nabla \cdot E, r - g \rangle + (a(E) + b(e)) d(r - g) = 0. \]

And then we choose $v = E$ and $\phi = e$ in (30) to get
\[ \mu \langle \nabla z, \nabla E \rangle + \langle \nabla r, E \rangle + (c(z) - d(r)) a(E) = \langle E, E \rangle, \]
\[ \langle \nabla \cdot z, e \rangle - (c(z) - d(r)) b(e) = 0. \]
Therefore we arrive at (41) and finish the proof of the theorem.

In light of (40), subtracting (47) from (48) yields
\[ \mu \langle \nabla E, \nabla (z - G) \rangle - \langle \nabla e, G \rangle + (a(E) + b(e)) c(z - G) - (c(z) - d(r)) b(e) = 0. \]

By the same manner, we replace \( \langle \nabla r, E \rangle \) at the first equation in (46) with the second equation in (45)
\[ \mu \langle \nabla z, \nabla E \rangle - \langle \nabla \cdot E, g \rangle + (c(z) - d(r)) a(E) + (a(E) + b(e)) d(r - g) = \langle E, E \rangle. \]

In light of (40), subtracting (47) from (48) yields
\[
\| E \|_0^2 = \mu \langle \nabla E, \nabla G \rangle - \langle \nabla \cdot E, g \rangle - (c(E) + b(e)) c(G) - d(g) \leq Ch^2 (\| \nabla E \|_0 + \| \nabla \cdot E \|_0 + \| e \|_0) \| E \|_0.
\]

Therefore we arrive at (41) and finish the proof of the theorem.

We now estimate error in \( H^1_0(\Omega) \) space.

**Lemma 4.6 (Estimate \( \| E \|_1 + \| e \|_0 \))**. Let Assumption 1 hold. If the mesh size \( h \) be small enough, then we have
\[ \| E \|_1 + \| e \|_0 \leq Ch. \]

**Proof**. We choose \( v_h = E_h = I_h w - w_h = E - I_h E \in V_h \) and \( \phi_h = e_h = I_h q - q_h = e - I_h e \in P_h \) in (43), then we have
\[ \mu \langle \nabla E, \nabla (E - I_h E) \rangle + \langle \nabla e, E - I_h E \rangle + (a(E) + b(e)) c(E_h) = 0, \]
\[ \langle \nabla \cdot E, e - I_h e \rangle + (a(E) + b(e)) d(e_h) = 0. \]

And then we replace \( \langle \nabla e, E \rangle \) in the first equation with the second equation in (50) to have
\[ \mu \langle \nabla E, \nabla (E - I_h E) \rangle - \langle \nabla e, I_h E \rangle - \langle \nabla \cdot E, I_h e \rangle + (a(E) + b(e)) c(E_h) + d(e_h) = 0. \]

In conjunction with Lemma 4.5 and \( |b(e)| \leq C\| E \|_0 \) in (44), (29) yields
\[
\mu \| \nabla E \|_0^2 \leq C \left( \| \nabla E \|_0 \| \nabla I_h E \|_0 + \| e \|_0 \| \nabla \cdot I_h E \|_0 \\
+ \| I_h e \|_0 \| \nabla \cdot E \|_0 + \| E \|_0 (\| E_h \|_0 + \| e_h \|_0) \right) \]
\[
\leq Ch \left( \| \nabla E \|_0 + \| e \|_0 \right) \| \nabla \cdot E \|_0 \| q \|_1 \\
+ Ch^2 (\| E \|_1 + \| e \|_0) \| E_h \|_0 + \| e_h \|_0 \]
\[
\leq Ch \left( \| \nabla E \|_0 + \| q \|_0 \| \nabla \cdot E \|_0 \| q \|_1 \right) + Ch^2 (\| E \|_1 + \| q \|_1)^2. \]

Assumption of a small enough \( h \) yields
\[ \| E \|_1 \leq Ch \left( \| \nabla \|_2 + \| q \|_1 \right). \]

Finally, we arrive at (49) by combining with (42) and complete this proof.
5. Numerical test

In this section, we document the computational performance of the FE-DSFM within a polygonal $\Gamma$ shape domain $([-1, 1] \times [-1, 1]) \setminus ([0, 1] \times [-1, 0])$. So, in this test, $\omega = 1.5\pi$ and the solution $\lambda$ of (4) becomes $\lambda = 0.544483736782463925$. Let the solution be given by

\begin{align*}
u &= -\sin^2(\pi x) \sin(2\pi y) + u_s, \\
v &= \sin(2\pi x) \sin^2(\pi y) + v_s, \\
p &= (2 + \cos(\pi x))(2 + \cos(\pi y)) - 4.0 + p_s.
\end{align*}

We note that the solution for velocity has not vanished on $\Gamma_{out}$. The forcing term $f$ is determined accordingly for any $\mu$; here $\mu = 1$. In order to impose FE-DSFM, we choose the cut-off function $\eta_\rho \in H^3(\Omega)$ as

$$\eta_\rho = \begin{cases} 1, & \text{in } B_{1/2}(\rho R), \\
\frac{1}{\rho^3} (16 - 35\psi + 35\psi^3 - 21\psi^5 + 5\psi^7), & \text{in } B(\rho R) \setminus B(\rho R), \\
0, & \text{in } \Omega \setminus B(\rho R),
\end{cases}$$

with $\psi = \frac{4r}{\rho R} - 3$ with $R = 1$. Then the solution $(u, p)$ can be rewritten by

\begin{align*}u &= w + \eta_\rho u_s, \\
p &= q + \eta_\rho p_s,
\end{align*}

where $(w, q)$ is the regular part of the solution. We note that the regularities of $w = u - \eta_\rho u_s$ and $p = p - \eta_\rho p_s$ are equal to that of $\eta_\rho$, and so $(w, q) \in H^3(\Omega) \times H^3(\Omega)$ in this example. Computations are carried out with the Taylor-Hood $(P^2, P^1)$ finite element pair on the union jack shape uniform meshes of size $h$.

### Table 1. Error table for the standard mixed method.

<table>
<thead>
<tr>
<th>$h$</th>
<th>1/8</th>
<th>1/16</th>
<th>1/32</th>
<th>1/64</th>
<th>1/128</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|E|_0$</td>
<td>0.00525989</td>
<td>0.00203043</td>
<td>0.000863199</td>
<td>0.000381872</td>
<td>0.000173168</td>
</tr>
<tr>
<td>Order</td>
<td>1.373247</td>
<td>1.234020</td>
<td>1.176604</td>
<td>1.140917</td>
<td>-</td>
</tr>
<tr>
<td>$|E|_{L^\infty}$</td>
<td>0.027998</td>
<td>0.0195703</td>
<td>0.0135074</td>
<td>0.00928508</td>
<td>0.00637329</td>
</tr>
<tr>
<td>Order</td>
<td>0.516658</td>
<td>0.534916</td>
<td>0.540764</td>
<td>0.542876</td>
<td>-</td>
</tr>
<tr>
<td>$|E|_{H^1}$</td>
<td>0.217839</td>
<td>0.121565</td>
<td>0.0805253</td>
<td>0.0549791</td>
<td>0.0376842</td>
</tr>
<tr>
<td>Order</td>
<td>0.841534</td>
<td>0.594214</td>
<td>0.550559</td>
<td>0.544924</td>
<td>-</td>
</tr>
<tr>
<td>$|e|_{L^2}$</td>
<td>0.209838</td>
<td>0.142047</td>
<td>0.0968696</td>
<td>0.0662486</td>
<td>0.0453065</td>
</tr>
<tr>
<td>Order</td>
<td>0.562908</td>
<td>0.552252</td>
<td>0.548154</td>
<td>0.546263</td>
<td>-</td>
</tr>
<tr>
<td>$|e|_{L^\infty}$</td>
<td>0.713876</td>
<td>0.956344</td>
<td>1.31637</td>
<td>1.81283</td>
<td>2.49197</td>
</tr>
<tr>
<td>Order</td>
<td>-0.421856</td>
<td>-0.460963</td>
<td>-0.461679</td>
<td>-0.459043</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1 is the error decay for the standard mixed method. We can check that the errors consist with (3) and we see that the error $\|e\|_{L^\infty}$ does not converge to 0. It is natural behavior, because $s$ in (3) is less than 1. More precisely, $s = \lambda = 0.544483736782463925$ in this test.

Tables 2, 3 and 4 are the results of mesh analysis of FE-DSFM with $\rho = 0.125$, $\rho = 0.3$ and $\rho = 0.453$, respectively. The convergence orders in these experiments are the same as the optimal accuracy in (2) with $s = 2$, because of $(w, q) \in H^3(\Omega) \times H^3(\Omega)$. So we can conclude that FE-DSFM has optimal accuracy in numerical tests, even though we get only suboptimal accuracy $\|E\|_0 \leq Ch^{1+\lambda}$.
Table 2. Error decay for the FE-DSFM with $\rho = 0.125$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>1/8</th>
<th>1/16</th>
<th>1/32</th>
<th>1/64</th>
<th>1/128</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|E|_0$</td>
<td>0.654576</td>
<td>0.000606273</td>
<td>0.000103749</td>
<td>4.86052e-05</td>
<td>5.53585e-06</td>
</tr>
<tr>
<td>Order</td>
<td>6.75449</td>
<td>2.548497</td>
<td>4.15843</td>
<td>3.132434</td>
<td></td>
</tr>
<tr>
<td>$|E|_{L^\infty}$</td>
<td>4.86035</td>
<td>0.0539083</td>
<td>0.00897302</td>
<td>0.000588627</td>
<td>7.25307e-05</td>
</tr>
<tr>
<td>Order</td>
<td>6.494409</td>
<td>2.448944</td>
<td>4.068066</td>
<td>3.026990</td>
<td></td>
</tr>
<tr>
<td>$|E|_{H^1}$</td>
<td>18.6427</td>
<td>0.32717</td>
<td>0.0799223</td>
<td>0.0249258</td>
<td>0.00637999</td>
</tr>
<tr>
<td>Order</td>
<td>5.832426</td>
<td>2.033370</td>
<td>1.689588</td>
<td>1.966240</td>
<td></td>
</tr>
<tr>
<td>$|\epsilon|_{L^2}$</td>
<td>4.70916</td>
<td>0.0830669</td>
<td>0.0193997</td>
<td>0.00418722</td>
<td>0.00119974</td>
</tr>
<tr>
<td>Order</td>
<td>5.853524</td>
<td>2.028239</td>
<td>2.211970</td>
<td>1.803271</td>
<td></td>
</tr>
<tr>
<td>$|\epsilon|_{L^\infty}$</td>
<td>55.3433</td>
<td>1.55404</td>
<td>0.254753</td>
<td>0.1594</td>
<td>0.0539996</td>
</tr>
<tr>
<td>Order</td>
<td>5.154313</td>
<td>2.608853</td>
<td>0.676447</td>
<td>1.561631</td>
<td></td>
</tr>
</tbody>
</table>

| $\|\epsilon\|_{L^2}$ | 0.00930983 | 0.000962465 | 0.000102994 | -05 | 1.63735e-06 |
| Order | 3.273949 | 3.221721 | 2.941771 | 3.033283 |
| $\|\epsilon\|_{L^\infty}$ | 0.0477212 | 0.00387738 | 0.000521333 | 7.355078e-05 | 8.82426e-06 |
| Order | 3.621476 | 2.894793 | 2.826236 | 3.058350 |
| $\|\epsilon\|_{H^1}$ | 0.383561 | 0.116552 | 0.0285883 | 0.00747874 | 0.00188788 |
| Order | 1.718482 | 2.027477 | 1.934588 | 1.980289 |
| $\|\epsilon\|_{L^2}$ | 0.0615555 | 0.016293 | 0.00402944 | 0.00139146 | 0.00061693 |
| Order | 1.917635 | 1.724757 | 1.824824 | 1.943762 |
| $\|\epsilon\|_{L^\infty}$ | 0.422723 | 0.08551973 | 0.0267362 | 0.00702548 |
| Order | 1.97484 | 1.113449 | 1.672013 | 1.928126 |
| $\|\epsilon\|_{L^2}$ | 0.091617 | 0.000504317 | 3.40152e-05 | 2.34765e-06 |
| Order | 3.816804 | 3.904404 | 3.674012 | 3.856890 |

Table 3. Error decay for the FE-DSFM with $\rho = 0.3$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>1/8</th>
<th>1/16</th>
<th>1/32</th>
<th>1/64</th>
<th>1/128</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|E|_0$</td>
<td>0.00930983</td>
<td>0.00962465</td>
<td>0.000102994</td>
<td>-05</td>
<td>1.63735e-06</td>
</tr>
<tr>
<td>Order</td>
<td>3.273949</td>
<td>3.221721</td>
<td>2.941771</td>
<td>3.033283</td>
<td></td>
</tr>
<tr>
<td>$|E|_{L^\infty}$</td>
<td>0.0477212</td>
<td>0.00387738</td>
<td>0.000521333</td>
<td>7.355078e-05</td>
<td>8.82426e-06</td>
</tr>
<tr>
<td>Order</td>
<td>3.621476</td>
<td>2.894793</td>
<td>2.826236</td>
<td>3.058350</td>
<td></td>
</tr>
<tr>
<td>$|E|_{H^1}$</td>
<td>0.383561</td>
<td>0.116552</td>
<td>0.0285883</td>
<td>0.00747874</td>
<td>0.00188788</td>
</tr>
<tr>
<td>Order</td>
<td>1.718482</td>
<td>2.027477</td>
<td>1.934588</td>
<td>1.980289</td>
<td></td>
</tr>
<tr>
<td>$|\epsilon|_{L^2}$</td>
<td>0.0615555</td>
<td>0.016293</td>
<td>0.00402944</td>
<td>0.00139146</td>
<td>0.00061693</td>
</tr>
<tr>
<td>Order</td>
<td>1.917635</td>
<td>1.724757</td>
<td>1.824824</td>
<td>1.943762</td>
<td></td>
</tr>
<tr>
<td>$|\epsilon|_{L^\infty}$</td>
<td>0.422723</td>
<td>0.08551973</td>
<td>0.0267362</td>
<td>0.00702548</td>
<td></td>
</tr>
<tr>
<td>Order</td>
<td>1.97484</td>
<td>1.113449</td>
<td>1.672013</td>
<td>1.928126</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Error decay for the FE-DSFM with $\rho = 0.453$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>1/8</th>
<th>1/16</th>
<th>1/32</th>
<th>1/64</th>
<th>1/128</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|E|_0$</td>
<td>0.0113171</td>
<td>0.000491094</td>
<td>7.48313e-05</td>
<td>8.11691e-06</td>
<td>9.85058e-07</td>
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<tr>
<td>Order</td>
<td>4.503321</td>
<td>2.714285</td>
<td>3.204639</td>
<td>3.036147</td>
<td></td>
</tr>
<tr>
<td>$|E|_{L^\infty}$</td>
<td>0.0351972</td>
<td>0.00146173</td>
<td>0.00238326</td>
<td>2.648046e-05</td>
<td>3.11194e-06</td>
</tr>
<tr>
<td>Order</td>
<td>4.587912</td>
<td>2.616669</td>
<td>3.169940</td>
<td>3.089039</td>
<td></td>
</tr>
<tr>
<td>$|E|_{H^1}$</td>
<td>0.273671</td>
<td>0.0690017</td>
<td>0.02849002</td>
<td>0.00435319</td>
<td>0.00113814</td>
</tr>
<tr>
<td>Order</td>
<td>1.985858</td>
<td>1.948537</td>
<td>1.987158</td>
<td>1.934384</td>
<td></td>
</tr>
<tr>
<td>$|\epsilon|_{L^2}$</td>
<td>0.051732</td>
<td>0.0107246</td>
<td>0.00050095</td>
<td>0.000820025</td>
<td>0.00020841</td>
</tr>
<tr>
<td>Order</td>
<td>2.270133</td>
<td>1.790584</td>
<td>1.918459</td>
<td>1.973263</td>
<td></td>
</tr>
<tr>
<td>$|\epsilon|_{L^\infty}$</td>
<td>0.262159</td>
<td>0.0874078</td>
<td>0.00651546</td>
<td>0.00103794</td>
<td>0.000267379</td>
</tr>
<tr>
<td>Order</td>
<td>1.584608</td>
<td>1.258499</td>
<td>1.815808</td>
<td>1.950765</td>
<td></td>
</tr>
<tr>
<td>$|\epsilon|_{L^2}$</td>
<td>0.096784</td>
<td>0.00123556</td>
<td>0.00205428</td>
<td>1.66353e-05</td>
<td>2.40529e-06</td>
</tr>
<tr>
<td>Order</td>
<td>5.425134</td>
<td>2.982456</td>
<td>3.626313</td>
<td>2.789965</td>
<td></td>
</tr>
</tbody>
</table>
then theoretically in Theorem 1. In addition, we can check that the errors are smaller for the bigger $\rho$, so the errors are reduced in Tables 2, 3 and 4 successively. In Table 2, we can see also that the errors of the case $h = 1/8$ are relatively bigger than others cases and it is due to $h = \rho = 0.125$.

6. Singular functions and dual Singular functions

In this section, we derive corner singular functions and dual Singular functions for Stoke equations (1). To do this, we use notations $U_r$ and $U_\theta$ as polar components of the vector function $\mathbf{u} = (u, v)^T$ which means

$$
\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} U_r \\ U_\theta \end{pmatrix}.
$$

If we define $\mathbf{e}_r = (\cos(\theta), \sin(\theta))^T$ and $\mathbf{e}_\theta = (-\sin(\theta), \cos(\theta))^T$, then (51) becomes

$$
\mathbf{u} = U_r \mathbf{e}_r + U_\theta \mathbf{e}_\theta,
$$

and (1) can be written as the polar coordinate Stokes system

$$
\begin{align*}
- \left( \partial_{rr} U_r + \frac{1}{r} \partial_r U_r + \frac{1}{r^2} \left(-U_r + \partial_\theta U_\theta - 2\partial_\theta U_\theta \right) \right) + \partial_r P &= 0, \\
- \left( \partial_{rr} U_\theta + \frac{1}{r} \partial_r U_\theta + \frac{1}{r^2} \left(-U_\theta + \partial_\theta U_r + 2\partial_\theta U_r \right) \right) + \frac{1}{r} \partial_\theta P &= 0,
\end{align*}
$$

$$
\partial_r U_r + \frac{1}{r} (U_r + \partial_\theta U_\theta) = 0.
$$

We will find the singular solutions in §6.1 and the dual singular solutions §6.2 via solving (52).

6.1. Singular functions. In this section, we will find solutions of (52) by using the separation of variables. We first assume that the solution $(U_r, U_\theta, P)$ of (52) has the singular form, with $\lambda > 0$,

$$
\begin{pmatrix} U_r(r, \theta) \\ U_\theta(r, \theta) \end{pmatrix} = r^{\lambda} \begin{pmatrix} u_r(\theta) \\ u_\theta(\theta) \end{pmatrix} \quad \text{and} \quad P(r, \theta) = r^{\lambda - 1} p(\theta).
$$

Then, as computed in [10], we obtain 4 solutions of (52)

$$
\begin{pmatrix} u^1_r \\ u^1_\theta \\ p^1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{\lambda}{\mu} \cos((1 - \lambda)\theta) \\ \frac{1 + \lambda}{\mu} \sin((1 - \lambda)\theta) \\ -4\lambda \cos((1 - \lambda)\theta) \end{pmatrix}, \quad \begin{pmatrix} u^3_r \\ u^3_\theta \\ p^3 \end{pmatrix} = \begin{pmatrix} \cos((1 + \lambda)\theta) \\ -\sin((1 + \lambda)\theta) \\ 0 \end{pmatrix},
$$

$$
\begin{pmatrix} u^2_r \\ u^2_\theta \\ p^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\mu} \sin((1 - \lambda)\theta) \\ \frac{1 + \lambda}{\mu} \cos((1 - \lambda)\theta) \\ -4\lambda \sin((1 - \lambda)\theta) \end{pmatrix}, \quad \begin{pmatrix} u^4_r \\ u^4_\theta \\ p^4 \end{pmatrix} = \begin{pmatrix} \sin((1 + \lambda)\theta) \\ \cos((1 + \lambda)\theta) \\ 0 \end{pmatrix},
$$
Finally, we arrive at the singular function

\[ \begin{pmatrix} U_r(r, \theta) \\ U_\theta(r, \theta) \\ P(r, \theta) \end{pmatrix} = C_1 \begin{pmatrix} r^\lambda u_r^1 \\ r^\lambda u_\theta^1 \\ r^{\lambda-1} p^1 \end{pmatrix} + C_2 \begin{pmatrix} r^\lambda u_r^2 \\ r^\lambda u_\theta^2 \\ r^{\lambda-1} p^2 \end{pmatrix} \\
+ C_3 \begin{pmatrix} r^\lambda u_r^3 \\ r^\lambda u_\theta^3 \\ r^{\lambda-1} p^3 \end{pmatrix} + C_4 \begin{pmatrix} r^\lambda u_r^4 \\ r^\lambda u_\theta^4 \\ r^{\lambda-1} p^4 \end{pmatrix}. \]

(53)

To make a homogeneous condition at \( \Gamma_{in} \), we impose

\[ U_r(r, 0) = U_\theta(r, 0) = 0 \quad \text{and} \quad U_r(r, \omega) = U_\theta(r, \omega) = 0. \]

Then the first equations in (54) lead to

\[ C_1 \frac{1 - \lambda}{\mu} + C_3 = 0 \quad \text{and} \quad C_2 \frac{1 + \lambda}{\mu} + C_4 = 0 \]

and so, in conjunction with (51), (53) becomes

\[ \begin{pmatrix} u \\ v \\ p \end{pmatrix} = 2C_1 \frac{r^\lambda}{\mu} \begin{pmatrix} \lambda \sin(\theta) \sin((1 - \lambda) \theta) \\ \sin(\lambda \theta) - \lambda \sin(\theta) \cos((1 - \lambda) \theta) \\ -2r^{\lambda-1} \lambda \cos((1 - \lambda) \theta) \end{pmatrix} - 2C_2 \frac{r^\lambda}{\mu} \begin{pmatrix} \lambda \sin(\theta) + \lambda \sin(\theta) \cos((1 - \lambda) \theta) \\ \lambda \sin(\theta) \sin((1 - \lambda) \theta) \\ 2r^{\lambda-1} \lambda \sin((1 - \lambda) \theta) \end{pmatrix}. \]

From \( U_r(r, \omega) = 0 \) in (54), \( C_1 \) and \( C_2 \) have to be

\[ C_1 = \sin(\lambda \omega) + \lambda \sin(\omega) \cos((1 - \lambda) \omega) \quad \text{and} \quad C_2 = \lambda \sin(\omega) \sin((1 - \lambda) \omega), \]

And the last equation in (54), \( U_\theta(r, \omega) = 0 \), yields

\[ \lambda^2 \sin^2(\omega) = \sin^2(\lambda \omega). \]

Finally, we arrive at the singular function

\[ \begin{pmatrix} u \\ v \\ p \end{pmatrix} = C_1 \begin{pmatrix} \frac{r^\lambda}{\mu} \lambda \sin(\theta) \sin((1 - \lambda) \theta) \\ \frac{r^\lambda}{\mu} (\sin(\lambda \theta) - \lambda \sin(\theta) \cos((1 - \lambda) \theta)) \\ -2r^{\lambda-1} \lambda \cos((1 - \lambda) \theta) \end{pmatrix} - C_2 \begin{pmatrix} \frac{r^\lambda}{\mu} (\sin(\lambda \theta) + \lambda \sin(\theta) \cos((1 - \lambda) \theta)) \\ \frac{r^\lambda}{\mu} \lambda \sin(\theta) \sin((1 - \lambda) \theta) \\ 2r^{\lambda-1} \lambda \sin((1 - \lambda) \theta) \end{pmatrix}, \]

where \( \lambda \) is the solution of (56), and \( C_1 \) and \( C_2 \) are the same as in (55).

In conjunction with Figure 1, we can readily get the following properties for (56):

**Lemma 6.1 (Variability of \( \lambda \) on \( \omega \)).** *If we denote \( \beta := \approx 1.430296653124203 \), then we have that*

1. \( \lambda = 0 \) and \( \lambda = 1 \) are solutions for all \( \omega \) and we call these trivial solutions,
2. there are only trivial solutions for \( 0 \leq \omega \leq \pi \),
3. there is a unique solution for \( \pi \leq \omega \leq \beta \pi \) except trivial solutions,
4. there are 2 solutions for \( \beta \pi \leq \omega < 2\pi \) except trivial solutions,
5. \( \lambda = 0.5 \) is the unique solution except trivial solutions, if \( \omega = 2\pi \).
6.2. dual Singular functions. By the same manner within §6.1, we assume the solution \((U_r, U_\theta, P)\) of (52) has the form, for \(\lambda > 0\),
\[
\begin{pmatrix} U_r(r, \theta) \\ U_\theta(r, \theta) \end{pmatrix} = r^{-\lambda} \begin{pmatrix} u_r(\theta) \\ u_\theta(\theta) \end{pmatrix} \quad \text{and} \quad P(r, \theta) = r^{-\lambda-1} p(\theta).
\]
Then we can readily get 4 solutions of (52)
\[
\begin{align*}
\begin{pmatrix} u^1_r \\ u^1_\theta \\ p^1 \end{pmatrix} &= \begin{pmatrix} \lambda + 1 \\ \lambda - 1 \\ 4\lambda \end{pmatrix} \begin{pmatrix} \cos((\lambda + 1)\theta) \\ \sin((\lambda + 1)\theta) \\ \cos((\lambda + 1)\theta) \end{pmatrix}, \\
\begin{pmatrix} u^3_r \\ u^3_\theta \\ p^3 \end{pmatrix} &= \begin{pmatrix} \cos((\lambda - 1)\theta) \\ \sin((\lambda - 1)\theta) \\ 0 \end{pmatrix}, \\
\begin{pmatrix} u^2_r \\ u^2_\theta \\ p^2 \end{pmatrix} &= \begin{pmatrix} \lambda + 1 \\ \lambda - 1 \\ 4\lambda \end{pmatrix} \begin{pmatrix} \sin((\lambda + 1)\theta) \\ -\cos((\lambda + 1)\theta) \\ \sin((\lambda + 1)\theta) \end{pmatrix}, \\
\begin{pmatrix} u^4_r \\ u^4_\theta \\ p^4 \end{pmatrix} &= \begin{pmatrix} -\cos((\lambda - 1)\theta) \\ 0 \\ 0 \end{pmatrix},
\end{align*}
\]

and so the general solution becomes
\[
\begin{pmatrix} U_r(r, \theta) \\ U_\theta(r, \theta) \\ P(r, \theta) \end{pmatrix} = D_1 \begin{pmatrix} r^{-\lambda} u^1_r \\ r^{-\lambda} u^1_\theta \\ r^{-\lambda-1} p^1 \end{pmatrix} + D_2 \begin{pmatrix} r^{-\lambda} u^2_r \\ r^{-\lambda} u^2_\theta \\ r^{-\lambda-1} p^2 \end{pmatrix} + D_3 \begin{pmatrix} r^{-\lambda} u^3_r \\ r^{-\lambda} u^3_\theta \\ r^{-\lambda-1} p^3 \end{pmatrix} + D_4 \begin{pmatrix} r^{-\lambda} u^4_r \\ r^{-\lambda} u^4_\theta \\ r^{-\lambda-1} p^4 \end{pmatrix}.
\]

Figure 1. Graphs of \(\sin^2(\lambda \omega)\) (dashed line) and \(\lambda^2 \sin^2(\omega)\) (solid line) with variable \(\lambda\) and fixed \(\omega\)’s as (a) \(\omega = 0.8\pi\), (b) \(\omega = \pi\), (c) \(\omega = 1.2\pi\), (d) \(\omega = 1.430296653124203\pi\), (e) \(\omega = 1.6\pi\), and (f) \(\omega = 2\pi\).
To make a homogeneous condition on $\Gamma_n$, we impose
\begin{equation}
U_r(r,0) = U_\theta(r,0) = U_r(r,\omega) = U_\theta(r,\omega) = 0.
\end{equation}

Then the first 2 conditions in (58), $U_r(r,0) = U_\theta(r,0) = 0$, leads us
\begin{equation}
D_1 \frac{\lambda + 1}{\mu} + D_3 = 0 \quad \text{and} \quad D_2 \frac{\lambda - 1}{\mu} + D_4 = 0
\end{equation}

and so, in conjunction with (51), (57) becomes
\begin{equation}
\begin{pmatrix}
u \\
u \\
p \\
\end{pmatrix} = -2D_1 \frac{r^{-\lambda}}{\mu} \begin{pmatrix}
\lambda \sin(\theta) \sin((\lambda + 1)\theta) \\
\sin(\lambda \theta) - \lambda \sin(\theta) \cos((\lambda + 1)\theta) \\
2r^{-\lambda-1} \lambda \cos((\lambda + 1)\theta) \\
\end{pmatrix}
+ 2D_2 \frac{r^{-\lambda}}{\mu} \begin{pmatrix}
\sin(\lambda \theta) + \lambda \sin(\theta) \cos((\lambda + 1)\theta) \\
\lambda \sin(\theta) \sin((\lambda + 1)\theta) \\
2r^{-\lambda-1} \lambda \sin((\lambda + 1)\theta) \\
\end{pmatrix}.
\end{equation}

In order to make hold the third equation in (58), $U_r(r,\omega) = 0$, $D_1$ and $D_2$ have to be
\begin{equation}
D_1 = \sin(\lambda \omega) + \lambda \sin(\omega) \cos((\lambda + 1)\omega) \quad \text{and} \quad D_2 = \lambda \sin(\omega) \sin((\lambda + 1)\omega).
\end{equation}

Then the last equation in (58), $U_\theta(r,\omega) = 0$, yields the same equation as (56)
\begin{equation}
\lambda^2 \sin^2(\omega) = \sin^2(\lambda \omega).
\end{equation}

Finally, we arrive at the dual singular function
\begin{equation}
\begin{pmatrix}
u \\
u \\
p \\
\end{pmatrix} = D_1 \begin{pmatrix}
-\frac{r^{-\lambda}}{\mu} \lambda \sin(\theta) \sin((\lambda + 1)\theta) \\
-\frac{r^{-\lambda}}{\mu} (\sin(\lambda \theta) - \lambda \sin(\theta) \cos((\lambda + 1)\theta)) \\
2r^{-\lambda-1} \lambda \cos((\lambda + 1)\theta) \\
\end{pmatrix}
+ D_2 \begin{pmatrix}
\frac{r^{-\lambda}}{\mu} (\sin(\lambda \theta) + \lambda \sin(\theta) \cos((\lambda + 1)\theta)) \\
\frac{r^{-\lambda}}{\mu} \lambda \sin(\theta) \sin((\lambda + 1)\theta) \\
2r^{-\lambda-1} \lambda \sin((\lambda + 1)\theta) \\
\end{pmatrix},
\end{equation}

where $\lambda$ is the solution of (60) and $D_1$ and $D_2$ are the same as in (59).

References

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