FAST SOLVERS FOR THE SYMMETRIC IPDG
DISCRETIZATION OF SECOND ORDER ELLIPTIC PROBLEMS

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Abstract. In this paper, we develop and analyze a preconditioning technique and an iterative solver for the linear systems resulting from the discretization of second order elliptic problems by the symmetric interior penalty discontinuous Galerkin methods. The main ingredient of our approach is a stable decomposition of the piecewise polynomial discontinuous finite element space of arbitrary order into a linear conforming space and a space containing high frequency components. To derive such decomposition, we introduce a novel interpolation operator which projects piecewise polynomials of arbitrary order to continuous piecewise linear functions. We prove that this operator is stable which allows us to derive the required space decomposition easily. Moreover, we prove that both the condition number of the preconditioned system and the convergent rate of the iterative method are independent of the mesh size. Numerical experiments are also shown to confirm these theoretical results.

Key words. Discontinuous Galerkin methods, iterative method, preconditioner.

1. Introduction

Discontinuous Galerkin (DG) methods are widely used numerical methodologies for the numerical solutions of partial differential equations. They have traditionally been used for the numerical solutions of hyperbolic equations [36, 29, 23, 24, 19]. There are many advantages in using DG methods compared with other types of finite element methods. For example, DG methods allow more flexibility in handling equations whose types change within the computational domain and in designing hp-refinement strategies. Besides, they have the ability to provide important conservation properties as well as give block diagonal mass matrices for time-dependent problems [23, 24, 19]. Owing to these unique advantages, DG methods have also been developed for second order elliptic problems [14, 18, 25] and many other problems. In addition, DG methods based on staggered grids are recently developed and analyzed for a large class of problems [21, 22, 26, 27, 28]. On the other hand, one main obstacle in the efficient implementation of DG methods is that the resulting linear systems contain a larger number of unknowns compared with conforming methods. Thus, the construction of fast algorithms is crucial for the efficient implementations of DG methods. In this paper, we will pay our attention to the symmetric interior penalty discontinuous Galerkin (IPDG) for second order elliptic equations.

Fast solvers for DG methods have been widely studied in the literature. For example, additive Schwarz methods [31, 1, 2, 6], multigrid methods [13, 11, 10, 5, 9, 4] and preconditioning techniques [32, 30, 15, 16, 5, 4] are developed for the efficient solution strategies for DG methods. The first work on preconditioning technique, to the best of our knowledge, is given by Gopalakrishnan and Kanschat [32], in which they studied the variable V-cycle multigrid operator as a preconditioner, but assumed that the underlying hierarchy of meshes is quasi-uniform and the solution exhibits a certain (weak) extra regularity. Dobrev et. al. presented a two-level scheme in the framework of the auxiliary space method in [30]. However, they only analyzed the case with discontinuous piecewise linear finite elements and their technique requires the exact solution of a coarse-grid problem. Brix et. al. constructed a multilevel preconditioner and obtained uniformly bounded condition numbers for the preconditioned linear system without the use of the weaker assumptions presented in [15, 16]. Their scheme allows the use of triangulations with hanging nodes or graded meshes. Recently, some iterative and preconditioning techniques are presented and analyzed in [5]. The key idea is a splitting of the discontinuous finite element space into the standard Crouziex-Raviart space and its complementary space with respect to the energy inner product induced by IPDG-0 methods. Such decomposition has also been proposed and used in [17] for obtaining a priori error bounds for some DG methods. Moreover, the results in [5] are extended in [4] to the design of multilevel preconditioners for linear systems resulting from the DG discretization of elliptic problems with discontinuous coefficients. However, the mathematical analysis of these methods is based on the discretization by using discontinuous piecewise linear finite element spaces.

In this paper, we will develop and analyze a preconditioning technique and an iterative method for solving the linear systems resulting from the discretization of elliptic boundary value problems by symmetric IPDG methods. The key to the constructions of these is a stable space decomposition of the discontinuous finite element space $\mathbb{V}_h$ containing piecewise polynomials. More precisely, we will prove the following stable splitting

$$\mathbb{V}_h = \sum_{i=1}^{N} \mathbb{V}_i + \mathbb{V}_h^{Conf},$$

where $\mathbb{V}_i = \text{span}\{\varphi_i\}$, $\{\varphi_i\}_{i=1}^{N}$ is the set of all nodal basis functions in $\mathbb{V}_h$ having dimension N and $\mathbb{V}_h^{Conf}$ denotes the conforming finite element space with homogeneous Dirichlet boundary conditions. The above decomposition can be seen as decomposing the finite element space $\mathbb{V}_h$ as the sum of conforming space, whose fast solution techniques are well-known, and the space $\sum_{i=1}^{N} \mathbb{V}_i$, which can be regarded as a space containing high frequency components. We will prove that these high frequency components can be handled by using Jacobi or Gauss-Seidel smoothers. The use of this type of space decomposition can also be found in [30, 15, 16, 4]. In this paper, we will introduce a new interpolation operator, which gives a simpler approach for establishing the aforementioned stable space decomposition. We will show that the condition number of the preconditioned linear system is uniformly bounded and the iterative method is uniformly convergent with respect to the mesh size. Compared with most of the existing works, our main contributions in this paper are threefold. First, we give a construction of an interpolation operator of Scott-Zhang type, which from $\mathbb{V}_h$ into $\mathbb{V}_h^{Conf}$. Secondly, our ideas can be applied directly to the original discrete variational problems without the need of another equivalent bilinear form. Last and more importantly, our preconditioning technique

and iterative method can be applied to DG methods discretized by higher order discontinuous finite element functions.

In order to avoid repeated use of generic but unspecified constants, we use the notation $\alpha \lesssim \beta$ to mean that the constant α is bounded above by a constant multiple of β uniformly with respect to any parameters on which α and β may depend. Moreover, $\alpha \approx \beta$ means that $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$.

The rest of the paper is organized as follows. In Section 2, we will present a brief review on the symmetric IPDG methods. Then in Section 3, we will construct the required stable space decomposition and the corresponding interpolation operator. Stability bounds for the space decomposition and the interpolation operator are also proved. The design and the analysis of the preconditioner and the iterative method are presented in Section 4 and Section 5 respectively. Finally, in Section 6, we report some numerical results to confirm our theoretical estimates.

2. Discontinuous Galerkin Methods

In this section, we present an overview of the symmetric IPDG methods for which preconditioners and fast iterative solvers are developed. For an extensive review, see [37, 33]. To start, we consider the following model problem of finding $u \in H_0^1(\Omega)$ such that

$$(1) \qquad \int_\Omega \nabla u \cdot \nabla v \mathrm{d}\boldsymbol{x} = \int_\Omega f v \mathrm{d}\boldsymbol{x}, \quad \forall v \in H^1(\Omega),$$

where $\Omega \in \mathbb{R}^d$ is a bounded polygonal (for $d = 2$) or polyhedral (for $d = 3$) domains with Lipschitz boundary $\partial\Omega$.

Let $\mathcal{T}_h$ be a shape-regular family of partitions of Ω into $d-$dimensional simplices T (triangles if $d = 2$ and tetrahedra if $d = 3$). We assume that $\mathcal{T}_h$ is conforming in the sense that it does not contain hanging nodes. A face e in the triangulation $\mathcal{T}_h$ is a $(d - 1)$-dimensional simplex which belongs to the boundary of some $T \in \mathcal{T}_h$. We denote by $\mathcal{E}_h^o$ and $\mathcal{E}_h^\partial$ the collection of all interior faces and boundary faces. The set of all faces (the skeleton of the triangualation) is denoted by $\mathcal{E}_h$. We remark that our methods do not require that the mesh is quasi-uniform. Our numerical experiments, presented in Section 6, show that the proposed schemes still work for graded meshes. Nevertheless, we will prove our results under the assumption of quasi-uniform meshes. The corresponding mathematical analysis for graded meshes can be performed by following the method in [15, 16].

Let $\mathbb{V}_h$ be the space of discontinuous finite element functions defined by

$$\mathbb{V}_h = \{v \in L^2(\Omega) : v_T = v|_T \in P_l(T), \ \forall \ T \in \mathcal{T}_h\},$$

where $P_l(T)$ is the set of polynomials of degree at most l on T and $l \geq 1$ is a fixed integer. In addition, we let $H^s(\Omega; \mathcal{T}_h)$, $s \geq 1$, be the space of piecewise Sobolev functions defined by

$$H^s(\Omega; \mathcal{T}_h) = \left\{v \in L^2(T) : v_T = v|_T \in H^s(T), \quad \forall \ T \in \mathcal{T}_h\right\},$$

and $L^2(\mathcal{E}_h)$ be the set of L^2 functions defined on $\mathcal{E}_h$. Moreover, we define the following inner products

$$
\begin{aligned}
(v, w)_{\mathcal{T}_h} &= \sum_{T \in \mathcal{T}_h} \int_T vw \ \mathrm{d}\boldsymbol{x} \quad \forall \ v, w \in L^2(\Omega), \\
<v, w>_{\mathcal{E}_h} &= \sum_{e \in \mathcal{E}_h} \int_e vw \ \mathrm{d}s \quad \forall \ v, w \in L^2(\mathcal{E}_h).
\end{aligned}
$$

Let $e \in \mathcal{E}_h^o$ be an interior face shared by two elements $T_\pm \in \mathcal{T}_h$, and $\boldsymbol{n}_\pm$ be the unit normals of e pointing towards the outside of $T_\pm$. Then, on e, we define

$$[[v]] = v_+ \boldsymbol{n}_+ + v_- \boldsymbol{n}_-, \qquad \{\!\{\nabla v\}\!\} = \frac{\nabla v_+ + \nabla v_-}{2}, \qquad \forall v \in H^1(\Omega; \mathcal{T}_h),$$

$$\{\!\{\boldsymbol{w}\}\!\} = \frac{\boldsymbol{w}_+ + \boldsymbol{w}_-}{2}, \qquad \forall \boldsymbol{w} \in H^1(\Omega; \mathcal{T}_h) \times H^1(\Omega; \mathcal{T}_h)$$

with $v_\pm = v|_{T_\pm}$ and $\boldsymbol{w}_\pm = \boldsymbol{w}|_{T_\pm}$. For a boundary face $e \in \mathcal{E}_h^\partial$, we define

$$\begin{aligned} [[v]] &= v_T \boldsymbol{n}, \quad \forall v \in H^1(\Omega; \mathcal{T}_h), \\ \{\!\{\boldsymbol{w}\}\!\} &= \boldsymbol{w}_T, \quad \forall \boldsymbol{w} \in H^1(\Omega; \mathcal{T}_h) \times H^1(\Omega; \mathcal{T}_h), \end{aligned}$$

where $\boldsymbol{n}$ is the unit normal of e pointing towards the outside of Ω.

For any face e, we define a lifting operator $r_e : L^2(e) \times L^2(e) \mapsto \mathbb{V}_h \times \mathbb{V}_h$ by

$$\int_\Omega r_e(\boldsymbol{v}) \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} = -\int_e \boldsymbol{v} \cdot \{\!\{\boldsymbol{w}\}\!\} \, \mathrm{d}s, \qquad \boldsymbol{w} \in \mathbb{V}_h \times \mathbb{V}_h.$$

Then the global lifting $r_h : L^2(\mathcal{E}_h) \times L^2(\mathcal{E}_h) \mapsto \mathbb{V}_h \times \mathbb{V}_h$ is defined by

$$r_h(\boldsymbol{v}) = \sum_{e \in \mathcal{E}_h} r_e(\boldsymbol{v}).$$

For any given $f \in L^2(\Omega)$, the symmetric IPDG method [3] for (1) is: find $u_h \in \mathbb{V}_h$ such that

(2) $$a_h(u_h, v) = (f, v)_{\mathcal{T}_h}, \quad \forall v \in \mathbb{V}_h,$$

where

$$\begin{aligned} a_h(w, v) \quad = \quad & (\nabla w, \nabla v)_{\mathcal{T}_h} - <\{\!\{\nabla w\}\!\}, [[v]]>_{\mathcal{E}_h} - <\{\!\{\nabla v\}\!\}, [[w]]>_{\mathcal{E}_h} \\ (3) \qquad & + \delta \left(r_h([[w]]) \cdot r_h([[v]]) \right)_{\mathcal{T}_h} + J_h(w, v), \end{aligned}$$

$\delta = 1$ or 0, and $J_h = J^j$ or J^r. The jump terms J^j and J^r are defined by

$$\begin{aligned} J^j(w, v) &= \eta < h_e^{-1} [[v]], [[w]] >_{\mathcal{E}_h} & \forall v, w \in \mathbb{V}_h, \\ J^r(w, v) &= \eta \left(r_e([[w]]), r_e([[v]]) \right)_{\mathcal{T}_h} & \forall v, w \in \mathbb{V}_h, \end{aligned}$$

where h_e denotes the length of e for $d = 2$ and the diameter of e for $d = 3$, and $\eta > 0$ is a penalty parameter.

Four different IPDG methods can be obtained by using different choices of δ and J_h, and they are all **symmetric**, **consistent** and **stable** under suitable conditions on the penalty parameter η, which can be found in [3, 37] and also Table 1 in [9]. In this paper, we always assume that these conditions on η are satisfied. In the following, we only list some properties of symmetric IPDG methods without proofs. We refer the reader to [3, 9, 37] for the proofs of these relations.

For symmetric IPDG methods, the bilinear form $a_h(\cdot, \cdot)$ is continuous and coercive, namely

(4)   Continuity : $a_h(w, v) \lesssim \|w\|_h \|v\|_h, \quad w, v \in \mathbb{V}_h,$

(5)   Coercivity : $a_h(v, v) \gtrsim \|v\|_h, \quad v \in \mathbb{V}_h,$

where the mesh-dependent energy norm $\|\cdot\|_h$ is defined by

(6) $$\|v\|_h^2 = \sum_{T \in \mathcal{T}_h} \|\nabla v\|_{L^2(T)}^2 + \eta^{-1} \sum_{e \in \mathcal{E}_h} h_e \|\{\!\{\nabla v\}\!\}\|_{L^2(e)}^2 + \eta \sum_{e \in \mathcal{E}_h} h_e^{-1} \|[[v]]\|_{L^2(e)}^2.$$

Then by (4) and (5), we have

(7) $$\|v\|_h^2 \approx a_h(v, v), \quad v \in \mathbb{V}_h$$

and $a_h(\cdot, \cdot)$ is an inner product on $\mathbb{V}_h$.

Furthermore, we introduce another mesh-dependent energy norm $||| \cdot |||_h$ defined by

$$(8) \qquad |||v|||_h^2 = \sum_{T \in \mathcal{T}_h} \|\nabla v\|_{L_2(T)}^2 + \eta \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [[v]] \|_{L_2(e)}^2.$$

It's easy to prove that the two norms $\| \cdot \|_h$ and $||| \cdot |||_h$ are equivalent on $\mathbb{V}_h$, that is,

$$(9) \qquad \|v\|_h \approx |||v|||_h \quad v \in \mathbb{V}_h.$$

Indeed, in view of (6) and (8), $||| \cdot |||_h \leq \| \cdot \|_h$ holds. On the other hand, by using the trace theorem for the average term in $\| \cdot \|_h$, we obtain $\| \cdot \|_h \leq C ||| \cdot |||_h$ (which is (10.5.16) of [12]).

3. A stable space decomposition

Our preconditioner and fast iterative solver are based on a stable space decomposition of the DG space $\mathbb{V}_h$. In this section, we will prove the following overlapping space decomposition of $\mathbb{V}_h$:

$$(10) \qquad \mathbb{V}_h = \mathbb{V}_h + \mathbb{V}_h^{Conf},$$

where $\mathbb{V}_h^{Conf}$ denotes the conforming linear finite element space with homogeneous Dirichlet boundary condition on $\mathcal{T}_h$. To do so, we need to construct a new interpolation operator $\Pi_h : \mathbb{V}_h \mapsto \tilde{\mathbb{V}}_h^{Conf}$, where $\tilde{\mathbb{V}}_h^{Conf}$ denotes the conforming linear finite element space, without zero boundary condition, defined on $\mathcal{T}_h$.

Let $\mathcal{N}_h$ be the collection of all nodes in the triangulation, and $\{\phi_p : p \in \mathcal{N}_h\}$ be the set of nodal basis functions of $\tilde{\mathbb{V}}_h^{Conf}$. We denote by $\mathcal{N}_h^o$ and $\mathcal{N}_h^\partial$ the collection of all interior nodes and boundary nodes. For a node $p \in \mathcal{N}_h^o$, let $e_p \in \mathcal{E}_h^o$ be an interior face with $p \in e_p$. The choice of e_p is not unique. For a boundary node $p \in \mathcal{N}_h^\partial$, we take $e_p \in \mathcal{E}_h^\partial$. Let $\psi_p \in P_1(e_p)$ be the unique function satisfying

$$\int_{e_p} \psi_p \lambda_l \mathrm{d}s = \delta_{l1}, \quad l = 1, \cdots, d,$$

where λ_l is the barycentric coordinates of e_p. It's easy to see that

$$(11) \qquad \left| \int_{e_p} \psi_p \ w \ \mathrm{d}s \right| \lesssim h^{1-d} \int_{e_p} |w| \ \mathrm{d}s,$$

and

$$(12) \qquad \int_{e_p} \psi_p \ w \ \mathrm{d}s = w(p) \quad \text{if} \quad w \in P_1(e_p).$$

The new interpolation operator $\Pi_h : \mathbb{V}_h \mapsto \tilde{\mathbb{V}}_h^{Conf}$ is then defined by

$$(13) \qquad \Pi_h v = \sum_{q \in \mathcal{N}_h^\partial} (\Pi_q v) \phi_q + \sum_{p \in \mathcal{N}_h^o} (\Pi_p v) \phi_p, \qquad v \in \mathbb{V}_h,$$

where

$$(14) \qquad \Pi_q v = \int_{e_q} v \psi_q \ \mathrm{d}s \quad \text{and} \quad \Pi_p v = \int_{e_p} \frac{v^+ + v^-}{2} \psi_p \ \mathrm{d}s,$$

$v^\pm = v|_{T_\pm}$ with $T_\pm \in \mathcal{T}_h$ and $T_+ \cap T_- = e_p$.

Remark 3.1. *If v is continuous on the interelement face e_p, then Π_h is the same as Scott-Zhang quasi-interpolation [38]. In this sense, we can view Π_h as a generalization of the Scott-Zhang interpolation to discontinuous piecewise polynomial spaces. On the other hand, Brix et. al. [15] constructed a similar interpolation by prescribing its nodal values at every interior regular vertex. The proof for the approximation properties (Lemma 3.1) of the operator Π_h considered in this paper is relatively easier than the proof presented in [15]. In this paper, we present an analysis of a preconditioning technique and a fast iterative solver by using the Scott-Zhang interpolation.*

Remark 3.2. *The operator Π_h may also be constructed through the use of the composition of a special interpolation $\mathcal{I}_h : \mathbb{V}_h \mapsto \mathbb{V}_h \cap H_0^1(\Omega)$ (see [7, 16] and the Scott-Zhang quasi-interpolation $\mathcal{I}_{SZ} : \mathbb{V}_h \cap H_0^1 \mapsto \mathbb{V}_h^{Conf}$ (see [38, 15]).*

Remark 3.3. *For linear discontinuous Galerkin finite element space*

$$\mathbb{V}_h^1 := \{v \in L^2(\Omega) : v_T = v|_T \in P_1(T), \ \forall \ T \in \mathcal{T}_h\},$$

Houston et. al. [35] (Section 5.2) introduces an operator $\tilde{\mathcal{A}} : \mathbb{V}_h^1 \mapsto \mathbb{V}_h^1 \cap H_0^1(\Omega)$, which is defined by prescribing its nodal values.

In the following lemma, we will prove some approximation properties for the operator Π_h defined in (13).

Lemma 3.1. *For the interpolation operator defined in (13), we have*

$$(15) \quad \|(I - \Pi_h)v\|_{L^2(T)} \lesssim h \left(\sum_{T' \in \Omega_T} \|\nabla v\|_{L^2(T')}^2 + h^{-1} \sum_{e \in \mathcal{E}(\Omega_T)} \| [\![v]\!] \|_{L^2(e)}^2 \right)^{1/2},$$

$$(16) \quad \|\nabla(\Pi_h v)\|_{L^2(T)} \lesssim \left(\sum_{T' \in \Omega_T} \|\nabla v\|_{L^2(T')}^2 + h^{-1} \sum_{e \in \mathcal{E}(\Omega_T)} \| [\![v]\!] \|_{L^2(e)}^2 \right)^{1/2}.$$

for all $v \in \mathbb{V}_h$ and $T \in \mathcal{T}_h$, where $\Omega_T = \cup_{p \in T} \Omega_p$ and Ω_p is the union of all cells $T' \in \mathcal{T}_h$ with $p \in T'$.

Proof. The proof of this lemma consists of four steps.

    Step 1:: We will prove that, for any $v \in \mathbb{V}_h$ and $p \in \mathcal{N}_h$,

$$(17) \quad \|\Pi_p v\|_{L^2(T)} \lesssim \sum_{T' \in \Omega_p} \left(\|v\|_{L^2(T')} + h_{T'} \|\nabla v\|_{L^2(T')} \right), \quad T \in \Omega_p.$$

To do so, we let $p \in \mathcal{N}_h^o$ be an arbitrary interior node. By the definition of $\Pi_p v$ in (14) and the triangle inequality, we have

$$|\Pi_p v| = \left| \int_{e_p} \frac{v^+ + v^-}{2} \psi_p \, ds \right| \leq \frac{1}{2} \left(\left| \int_{e_p} v^+ \psi_p \, ds \right| + \left| \int_{e_p} v^- \psi_p \, ds \right| \right).$$

Using the estimate (11),

$$|\Pi_p v| \lesssim h^{1-d} \left(\left| \int_{e_p} v^+ \, ds \right| + \left| \int_{e_p} v^- \, ds \right| \right).$$

Then by using Cauchy-Schwarz inequality, the trace inequality together with the scaling argument, we have

$$|\Pi_p v| \lesssim h^{1-d} \sum_{e_p \subset T'} \left(h_{T'}^{-1} |T'|^{1/2} \|v\|_{L^2(T')} + |T'|^{1/2} \|\nabla v\|_{L^2(T')} \right).$$

Since $|T'| \approx h^d$, we obtain

$$|\Pi_p v| \lesssim \sum_{e_p \subset T'} \left(h^{-d/2} \|v\|_{L^2(T')} + h^{1-d/2} \|\nabla v\|_{L^2(T')} \right).$$

The desired estimate (17) is then obtained by combining the above inequality and the following result

$$\|\Pi_p v\|_{L^2(T)} \leq |\Pi_p v| \cdot |T|^{1/2} \lesssim h_T^{d/2} |\Pi_p v|.$$

The proof of (17) for a boundary node $p \in \mathcal{N}_h^\partial$ is similar, and hence omitted.

Step 2:: We will prove that, for any $v \in \mathbb{V}_h$,

$$(18) \qquad \|(I - \Pi_p)v\|_{L^2(\Omega_p)} \lesssim h \left(\sum_{T \in \widetilde{\Omega_p}} \|\nabla v\|_{L^2(T)}^2 + h^{-1} \sum_{e \in \mathcal{E}(\widetilde{\Omega_p})} \| [\![v]\!] \|_{L^2(e)}^2 \right)^{1/2}.$$

where $\widetilde{\Omega_p} = \cup_{T' \in \Omega_p} \Omega_{T'}$ and $\mathcal{E}(\widetilde{\Omega_p})$ is the union of all faces in $\widetilde{\Omega_p}$.

To do so, for any node p, we let $v_p = |\Omega_p|^{-1} \int_{\Omega_p} v d\boldsymbol{x}$ be the mean value of v over Ω_p. Then by using the triangle inequality, we have

$$(19) \qquad \|(I - \Pi_p)v\|_{L^2(\Omega_p)} \leq \|v - v_p\|_{L^2(\Omega_p)} + \|v_p - \Pi_p v\|_{L^2(\Omega_p)}.$$

We can estimate the first term on the right hand side of (19) by using the Friedrichs-Poincare inequality (see Lemma 3.1 of [2]), namely,

$$\|v - v_p\|_{L^2(\Omega_p)}^2 \lesssim h^2 \left(\sum_{T \in \Omega_p} \|\nabla v\|_{L^2(T)}^2 + h^{-1} \sum_{e \in \mathcal{E}(\Omega_p)} \| [\![v]\!] \|_{L^2(e)}^2 \right).$$

For the second term on the right hand side of (19), we use the fact that $\Pi_p v_p = v_p$, the above inequality and (17) to obtain

$$
\begin{aligned}
\|v_p - \Pi_p v\|_{L^2(T)}^2 &= \|\Pi_p (v_p - v)\|_{L^2(T)}^2 \\
&\lesssim \sum_{T' \in \Omega_p} \left(\|v_p - v\|_{L^2(T')}^2 + h_{T'}^2 \|\nabla (v_p - v)\|_{L^2(T')}^2 \right) \\
&\lesssim h^2 \left(\sum_{T \in \Omega_p} \|\nabla v\|_{L^2(T)}^2 + h^{-1} \sum_{e \in \mathcal{E}(\Omega_p)} \| [\![v]\!] \|_{L^2(e)}^2 \right).
\end{aligned}
$$

Step 3:: We will prove (15).

By the definition of Π_p in (13) and the property that $\sum_{p \in T} \phi_p(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in T$, we have

$$\|(I - \Pi_h)v\|_{L^2(T)} = \|v - \sum_{p \in \mathcal{N}_h} (\Pi_p v) \phi_p\|_{L^2(T)} = \|\sum_{p \in T} (v - \Pi_p v) \phi_p\|_{L^2(T)}.$$

Then, using the fact that $|\phi_p| \leq 1$ and the inequality (18), we get

$$
\begin{aligned}
\|(I - \Pi_h)v\|_{L^2(T)} &\leq \sum_{p \in T} \|v - \Pi_p v\|_{L^2(\Omega_p)} \\
&\lesssim h \left(\sum_{T' \in \Omega_T} \|\nabla v\|_{L^2(T')}^2 + h^{-1} \sum_{e \in \mathcal{E}(\Omega_T)} \| [\![v]\!] \|_{L^2(e)}^2 \right)^{1/2}.
\end{aligned}
$$

Step 4:: We will prove (16).

To do so, we let $\bar{v}_T = |T|^{-1} \int_T v \mathrm{d}\boldsymbol{x}$ be the average of v on T. Then by the inverse estimate, we have

$$|\Pi_h v|_{H^1(T)} = \|\nabla(\Pi_h v)\|_{L^2(T)} = \|\nabla(\Pi_h v - \bar{v}_T)\|_{L^2(T)} \lesssim h^{-1}\|\Pi_h v - \bar{v}_T\|_{L^2(T)}.$$

Using (15) and the Poincare inequality, we obtain

$$|\Pi_h v|_{H^1(T)} \lesssim h^{-1} \left(\|\Pi_h v - v\|_{L^2(T)} + \|v - \bar{v}_T\|_{L^2(T)}\right)$$

$$\lesssim \left(\sum_{T' \in \Omega_T} \|\nabla v\|^2_{L^2(T')} + h^{-1} \sum_{e \in \mathcal{E}(\Omega_T)} \| \, [\![v]\!] \, \|^2_{L^2(e)}\right)^{1/2}.$$

$\square$

By using the above estimates for the interpolation operator Π_h, we will prove that the space $\mathbb{V}_h$ can be decomposed in the way described in (10) and that it is stable.

Lemma 3.2. *For any $v_h \in \mathbb{V}_h$, there exist $\tilde{v}_h \in \mathbb{V}_h$ and $p_h \in \mathbb{V}_h^{Conf}$ such that*

$$(20) \qquad\qquad v_h = \tilde{v}_h + p_h$$

and

$$(21) \qquad \sum_{T \in \mathcal{T}_h} \|h^{-1}\tilde{v}_h\|^2_{L^2(T)} + \sum_{T \in \mathcal{T}_h} \|\nabla p_h\|^2_{L^2(T)} \lesssim a_h(v_h, v_h),$$

which implies that the decomposition (20) is stable.

Proof. For any $v_h \in \mathbb{V}_h$, by using (13), we have

$$\begin{aligned} v_h &= (I - \Pi_h)v_h + \Pi_h v_h \\ &= (I - \Pi_h)v_h + \sum_{q \in \mathcal{N}_h^\partial} (\Pi_q v_h)\, \phi_q + \sum_{p \in \mathcal{N}_h^o} (\Pi_p v_h)\, \phi_p. \end{aligned}$$

Then the decomposition (20) is obtained by defining

$$p_h = \sum_{p \in \mathcal{N}_h^o} (\Pi_p v_h)\, \phi_p \quad \text{and} \quad \tilde{v}_h = (I - \Pi_h)v_h + \sum_{q \in \mathcal{N}_h^\partial} (\Pi_q v_h)\, \phi_q.$$

Next we will prove the stability bound (21). First, by the definitions of p_h, $\tilde{v}_h$ and the triangle inequality, we have

$$\sum_{T \in \mathcal{T}_h} \left(\|h^{-1}\tilde{v}_h\|^2_{L^2(T)} + \|\nabla p_h\|^2_{L^2(T)}\right)$$

$$\leq 2 \sum_{T \in \mathcal{T}_h} \left(\|h^{-1}(I - \Pi_h)v_h\|^2_{L^2(T)}\right.$$

$$\left. + \|h^{-1} \sum_{q \in \partial T \cap \mathcal{N}_h^\partial} (\Pi_q v_h)\, \phi_q\|^2_{L^2(T)} + |\sum_{p \in \mathcal{N}_h^o} (\Pi_p v_h)\, \phi_p|^2_{H^1(T)}\right).$$

Next, we notice that

$$\left(\sum_{p \in \mathcal{N}_h^o} (\Pi_p v_h)\, \phi_p\right)\bigg|_T = \sum_{p \in \partial T} (\Pi_p v_h)\, \phi_p = \Pi_h v_h|_T \quad \text{if} \quad \partial T \cap \partial\Omega = \emptyset,$$

$$\left(\sum_{p \in \mathcal{N}_h^o} (\Pi_p v_h)\, \phi_p\right)\bigg|_T = \left(\Pi_h v_h - \sum_{p \in \partial T \cap \partial\Omega} (\Pi_p v_h)\, \phi_p\right)\bigg|_T \quad \text{if} \quad \partial T \cap \partial\Omega \neq \emptyset.$$

Thus, we have

$$\sum_{T \in \mathcal{T}_h} \left(\|h^{-1} \tilde{v}_h\|_{L^2(T)}^2 + \|\nabla p_h\|_{L^2(T)}^2 \right)$$

$$\leq \sum_{T \in \mathcal{T}_h} \Bigg(\|h^{-1}(I - \Pi_h) v_h\|_{L^2(T)}^2 + \|h^{-1} \sum_{q \in \partial T \cap \mathcal{N}_h^\partial} (\Pi_q v_h) \phi_q\|_{L^2(T)}^2 + |(\Pi_h v_h)|_{H^1(T)}^2$$

$$+ | \sum_{q \in \partial T \cap \mathcal{N}_h^\partial} (\Pi_q v_h) \phi_q |_{H^1(T)}^2 \Bigg).$$

Using the inverse inequality, we get

(22)
$$\sum_{T \in \mathcal{T}_h} \left(\|h^{-1} \tilde{v}_h\|_{L^2(T)}^2 + \|\nabla p_h\|_{L^2(T)}^2 \right)$$

$$\lesssim \sum_{T \in \mathcal{T}_h} \Bigg(\|h^{-1}(I - \Pi_h) v_h\|_{L^2(T)}^2 + \|h^{-1} \sum_{q \in \partial T \cap \mathcal{N}_h^\partial} (\Pi_q v_h) \phi_q\|_{L^2(T)}^2 + |(\Pi_h v_h)|_{H^1(T)}^2 \Bigg).$$

For the first and the third terms on the right-hand side of (22), we use Lemma 3.1 to obtain

$$\sum_{T \in \mathcal{T}_h} \left(\|h^{-1}(I - \Pi_h) v_h\|_{L^2(T)}^2 + |(\Pi_h v_h)|_{H^1(T)}^2 \right)$$

$$\lesssim \sum_{T \in \mathcal{T}_h} \left(\sum_{T' \in \Omega_T} \|\nabla v\|_{L^2(T')}^2 + h^{-1} \sum_{e \in \mathcal{E}(\Omega_T)} \| [\![v]\!] \|_{L^2(e)}^2 \right).$$

Using (6) and (5), we have

$$\sum_{T \in \mathcal{T}_h} \left(\|h^{-1}(I - \Pi_h) v_h\|_{L^2(T)}^2 + |(\Pi_h v_h)|_{H^1(T)}^2 \right) \leq \|v_h\|_h^2 \lesssim a_h(v_h, v_h).$$

For the second term in the right-hand side of (22), since $|\phi_q| \leq 1$, we have

$$\|h^{-1} \sum_{q \in \partial T \cap \mathcal{N}_h^\partial} (\Pi_q v_h) \phi_q\|_{L^2(T)}^2 \leq \sum_{q \in \partial T \cap \mathcal{N}_h^\partial} \|h^{-1} (\Pi_q v_h) \phi_q\|_{L^2(T)}^2$$

$$\leq \sum_{q \in \partial T \cap \mathcal{N}_h^\partial} h^{-2} \|\Pi_q v_h\|_{L^2(T)}^2$$

$$\leq \sum_{q \in \partial T \cap \mathcal{N}_h^\partial} h^{-2} |T| \cdot \left| \int_{e_q} v_h \psi_q \, \mathrm{d}s \right|^2.$$

Then, applying the results in (14) and (11), we have

$$\|h^{-1} \sum_{q \in \partial T \cap \mathcal{N}_h^\partial} (\Pi_q v_h) \phi_q\|_{L^2(T)}^2 \leq \sum_{q \in \partial T \cap \mathcal{N}_h^\partial} h^{-2} |T| \cdot \left| \int_{e_q} v_h \psi_q \, \mathrm{d}s \right|^2$$

$$\lesssim \sum_{q \in \partial T \cap \mathcal{N}_h^\partial} h^{-2} |T| \cdot h^{2-2d} \left| \int_{e_q} v_h \, \mathrm{d}s \right|^2.$$

Using the fact that $|T| \lesssim h^d$ and the Cauchy-Schwarz inequality,

$$\|h^{-1} \sum_{q \in \partial T \cap \mathcal{N}_h^\partial} (\Pi_q v_h) \phi_q\|_{L^2(T)}^2 \lesssim \sum_{q \in \partial T \cap \mathcal{N}_h^\partial} h^{-d} \left| \int_{e_q} v_h \, ds \right|^2$$

$$\leq \sum_{q \in \partial T \cap \mathcal{N}_h^\partial} h^{-d} |e_q| \cdot \|v_h\|_{L^2(e_q)}^2.$$

Since the face e_q lies on the domain boundary, we obtain

$$\|h^{-1} \sum_{q \in \partial T \cap \mathcal{N}_h^\partial} (\Pi_q v_h) \phi_q\|_{L^2(T)}^2 \lesssim \sum_{q \in \partial T \cap \mathcal{N}_h^\partial} h^{-d} |e_q| h_{e_q} \cdot h_{e_q}^{-1} \| [\![v_h]\!] \|_{L^2(e_q)}^2$$

$$\lesssim \|v_h\|_h^2 \leq a_h(v_h, v_h)$$

where we have used the fact that $h^{-d} |e_p| h_{e_p} \lesssim 1$. Combining the above results, we complete the proof. $\qquad\square$

Let N be the dimension of $\mathbb{V}_h$, $\{\varphi_i\}_{i=1}^N$ be the set of all nodal basis functions spanning $\mathbb{V}_h$ and $\mathbb{V}_i = \mathrm{span}\{\varphi_i\}$. Then Lemma 3.2 implies that the space decomposition (10) can be rewritten as follows:

$$(23) \qquad \mathbb{V}_h = \sum_{i=1}^N \mathbb{V}_i + \mathbb{V}_h^{Conf}.$$

Next lemma shows that the decomposition (23) is also stable.

Theorem 3.1. *For any $v_h \in \mathbb{V}_h$, there exist $v_i \in \mathbb{V}_i$ and $p_h \in \mathbb{V}_h^{Conf}$ such that*

$$(24) \qquad v_h = \sum_{i=1}^N v_i + p_h$$

and

$$(25) \qquad \sum_{i=1}^N a_h(v_i, v_i) + \sum_{T \in \mathcal{T}_h} \|\nabla p_h\|_{L^2(T)}^2 \lesssim a_h(v_h, v_h),$$

which implies that the decomposition (24) is stable.

Proof. For any $v_h \in \mathbb{V}_h$, in view of Lemma 3.2, there exist $\tilde{v}_h \in \mathbb{V}_h$ and $p_h \in \mathbb{V}_h^{Conf}$ such that $v_h = \tilde{v}_h + p_h$ and

$$\sum_{T \in \mathcal{T}_h} \|h^{-1} \tilde{v}_h\|_{L^2(T)}^2 + \sum_{T \in \mathcal{T}_h} \|\nabla p_h\|_{L^2(T)}^2 \lesssim a_h(v_h, v_h).$$

Recall that $\{\varphi_i\}_{i=1}^N$ are the nodal basis functions spanning $\mathbb{V}_h$ and $\mathbb{V}_i = \mathrm{span}\{\varphi_i\}$, thus there are $v_i = \alpha_i \varphi_i \in \mathbb{V}_i$ such that $\tilde{v}_h = \sum_{i=1}^N v_i$. Therefore, it remains only to verify that

$$(26) \qquad \sum_{i=1}^N a_h(v_i, v_i) \lesssim \sum_{T \in \mathcal{T}_h} \|h^{-1} \tilde{v}_h\|_{L^2(T)}^2.$$

Using the definition of v_i and the inverse inequality, we have

$$
\begin{aligned}
\sum_{i=1}^{N} \sum_{T \in \mathcal{T}_h} \|\nabla v_i\|_{L_2(T)}^2 &= \sum_{i=1}^{N} \sum_{T \in \mathcal{T}_h} \alpha_i^2 \|\nabla \varphi_i\|_{L_2(T)}^2 = \sum_{i=1}^{N} \sum_{T \cap \mathrm{supp}\varphi_i \neq \emptyset} \alpha_i^2 \|\nabla \varphi_i\|_{L_2(T)}^2 \\
&= \sum_{T \in \mathcal{T}_h} \sum_{\mathrm{supp}\varphi_i \cap T \neq \emptyset} \alpha_i^2 \|\nabla \varphi_i\|_{L_2(T)}^2 \\
&\lesssim \sum_{T \in \mathcal{T}_h} \sum_{\mathrm{supp}\varphi_i \cap T \neq \emptyset} \alpha_i^2 h_T^{-2} \|\varphi_i\|_{L_2(T)}^2.
\end{aligned}
\tag{27}
$$

For any $v \in \mathbb{V}_h$ and any interior face $e \in \mathcal{E}_h^0$ shared by two triangles $T_\pm \in \mathcal{T}_h$, combining the triangle inequality, the trace inequality and the inverse inequality, we have

$$
\begin{aligned}
h_e^{-1} \| [\![v]\!] \|_{L_2(e)}^2 &\leq h_e^{-1} \left(\|v_+\|_{L_2(e)}^2 + \|v_-\|_{L_2(e)}^2 \right) \\
&\lesssim \sum_{T=T_\pm} \left(h_T^{-2} \|v\|_{L_2(T)}^2 + \|\nabla v\|_{L_2(T)}^2 \right) \\
&\lesssim \sum_{T=T_\pm} h_T^{-2} \|v\|_{L_2(T)}^2.
\end{aligned}
$$

Then using the above inequality, we have

$$
\begin{aligned}
\sum_{i=1}^{N} \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [\![v_i]\!] \|_{L_2(e)}^2 &\lesssim \sum_{i=1}^{N} \sum_{T \in \mathcal{T}_h} h_T^{-2} \|v_i\|_{L_2(T)}^2 = \sum_{i=1}^{N} \sum_{T \in \mathcal{T}_h} h_T^{-2} \alpha_i^2 \|\varphi_i\|_{L_2(T)}^2 \\
&= \sum_{i=1}^{N} \sum_{T \cap \mathrm{supp}\varphi_i \neq \emptyset} h_T^{-2} \alpha_i^2 \|\varphi_i\|_{L_2(T)}^2 \\
&= \sum_{T \in \mathcal{T}_h} \sum_{\mathrm{supp}\varphi_i \cap T \neq \emptyset} h_T^{-2} \alpha_i^2 \|\varphi_i\|_{L_2(T)}^2.
\end{aligned}
\tag{28}
$$

We remark that a similar result holds if e is a boundary face.

Next, by the assumption that the mesh is quasi-uniform, for any $T \in \mathcal{T}_h$, we have

$$
\sum_{\mathrm{supp}\varphi_i \cap T \neq \emptyset} \alpha_i^2 \|\varphi_i\|_{L_2(T)}^2 \lesssim \|\tilde{v}_h\|_{L_2(T)}^2.
\tag{29}
$$

Then from (7), (9) and (8), we have

$$
\sum_{i=1}^{N} a_h(v_i, v_i) \lesssim \sum_{i=1}^{N} \left(\sum_{T \in \mathcal{T}_h} \|\nabla v_i\|_{L_2(T)}^2 + \eta \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [\![v_i]\!] \|_{L_2(e)}^2 \right).
$$

Consequently, by the conditions (27), (28) and (29), we get

$$
\sum_{i=1}^{N} a_h(v_i, v_i) \lesssim \sum_{T \in \mathcal{T}_h} \sum_{\mathrm{supp}\varphi_i \cap T \neq \emptyset} h_T^{-2} \alpha_i^2 \|\varphi_i\|_{L_2(T)}^2 \lesssim \sum_{T \in \mathcal{T}_h} \|h_T^{-1} \tilde{v}_h\|_{L_2(T)}^2
$$

which proves (26).

To complete the proof of the theorem, it remains to show (29). For a given element T, we let $\widehat{T}$ be a reference element. We note that $\tilde{v}_h|_T = \sum_{\mathrm{supp}\varphi_i \cap T \neq \emptyset} \alpha_i^2 \varphi_i$. Then, by standard scaling arguments, we have

$$
\sum_{\mathrm{supp}\varphi_i \cap T \neq \emptyset} \alpha_i^2 \|\varphi_i\|_{L_2(T)}^2 \lesssim \sum_{\mathrm{supp}\varphi_i \cap T \neq \emptyset} \alpha_i^2 h^{d/2} \|\widehat{\varphi}_i\|_{L_2(\widehat{T})}^2.
$$

Since norms in finite dimensional spaces are equivalent, we have

$$\sum_{\text{supp}_{\varphi_i} \cap T \neq \emptyset} \alpha_i^2 h^{d/2} \|\widehat{\varphi}_i\|_{L_2(\hat{T})}^2 \lesssim h^{d/2} \| \sum_{\text{supp}_{\varphi_i} \cap T \neq \emptyset} \alpha_i^2 \widehat{\varphi}_i\|_{L_2(\hat{T})}^2.$$

Finally, using the above inequality and standard scaling argument again, we obtain

$$\sum_{\text{supp}_{\varphi_i} \cap T \neq \emptyset} \alpha_i^2 \|\varphi_i\|_{L_2(T)}^2 \lesssim h^{d/2} \|\widehat{\tilde{v}_h}\|_{L_2(\hat{T})}^2 \lesssim h^{d/2} h^{-d/2} \|\tilde{v}_h\|_{L_2(T)}^2 \lesssim \|\tilde{v}_h\|_{L_2(T)}^2$$

which gives (29). $\square$

4. An additive two-level preconditioner

In this section, we will develop an additive two-level preconditioner for the algebraic system resulting from (2). For this purpose, we first summarize the abstract theoretical framework in Hiptmair and Xu [34]. Let $\mathbb{V}$ be a real Hilbert space with inner product $a(\cdot, \cdot)$ and (energy) norm $\|\cdot\|_A$. Consider the following variational problem: for any given $f \in \mathbb{V}'$, find $u \in V$ such that

$$a(u, v) = f(v), \quad v \in \mathbb{V}.$$

Let the operator $\mathcal{A} : \mathbb{V} \mapsto \mathbb{V}'$ be the isomorphism associated with $a(\cdot, \cdot)$, namely $< \mathcal{A}v, w >= a(v, w), \quad \forall\, v, w \in \mathbb{V}$, where we denote dual spaces by $'$ and use angle brackets for duality pairings.

Let $\bar{\mathbb{V}}_1, \cdots, \bar{\mathbb{V}}_J, J \in \mathbb{N}$, be Hilbert spaces endowed with inner products $\bar{a}_j(\cdot, \cdot)$ and (energy) norms $\|\cdot\|_{\bar{A}_j}, j = 1, \cdots, J$. The operators $\bar{\mathcal{A}}_j : \bar{\mathbb{V}}_j \mapsto \bar{\mathbb{V}}_j$ are isomorphisms induced by $\bar{a}_j(\cdot, \cdot)$, namely $< \bar{\mathcal{A}}_j \bar{u}_j, \bar{v}_j >= \bar{a}_j(\bar{u}_j, \bar{v}_j), \quad \forall \bar{u}_j, \bar{v}_j \in \bar{\mathbb{V}}_j$. We assume that, for each $\bar{\mathbb{V}}_j$, there exist a continuous transfer operator $\Pi_j : \bar{\mathbb{V}}_j \mapsto \mathbb{V}$. Then we can construct a preconditioner for the operator $\mathcal{A}$ as follows:

$$(30) \qquad \mathcal{B} = \sum_{j=1}^{J} \Pi_j \bar{\mathcal{B}}_j \Pi_j^*,$$

where $\bar{\mathcal{B}}_j : \bar{\mathbb{V}}_j' \mapsto \bar{\mathbb{V}}_j$ are given preconditioners for $\bar{\mathcal{A}}_j$, and Π_j^* are adjoint operators of Π_j.

The following theorem gives an estimate of the spectral condition number of the preconditioner given by (30). This result is proved in Hiptmair and Xu [34].

Theorem 4.1. *Assume that there exist constants c_j such that*

$$(31) \qquad \|\Pi_j \bar{u}_j\|_{\mathcal{A}} \leq c_j \|\bar{u}_j\|_{\bar{A}_j} \qquad \forall\, \bar{u}_j \in \bar{\mathbb{V}}_j,\ 1 \leq j \leq J,$$

and for all $u \in \mathbb{V}$, there exist $\bar{u}_j \in \bar{\mathbb{V}}_j$ such that $u = \sum_{j=1}^{J} \Pi_j \bar{u}_j$ and

$$(32) \qquad \left(\sum_{j=1}^{J} \|\bar{u}_j\|_{\bar{A}_j}^2 \right)^{1/2} \leq c_0 \|u\|_A.$$

Then for the preconditioner $\mathcal{B}$ defined in (30), we have the following estimate for the spectral condition number

$$\kappa(\mathcal{B}\mathcal{A}) \leq \max_{1 \leq j \leq J} \kappa(\bar{\mathcal{B}}_j \bar{\mathcal{A}}_j) c_0^2 \sum_{j=1}^{J} c_j^2.$$

We are now in a position to design a preconditioner for the discrete variational problem (2). We will apply the above theory by letting $\mathbb{V} = \mathbb{V}_h$ and $a(\cdot, \cdot) = a_h(\cdot, \cdot)$ given by (3). The auxiliary spaces and the corresponding transfer operators are defined as follows:

(1): $\bar{V}_1 = \mathbb{V}_h$ with inner product $\bar{a}_1(\cdot, \cdot)$ defined by

$$\bar{a}_1(v, v) = \sum_{i=1}^{N} a_h(v_i, v_i), \quad \forall\, v = \sum_{i=1}^{N} v_i,\ v_i \in \mathbb{V}_i.$$

The corresponding transfer operator is $\Pi_1 = Id$.

(2): $\bar{V}_2 = \mathbb{V}_h^{Conf}$ with inner product $\bar{a}_2(\cdot, \cdot)$ defined by

$$\bar{a}_2(w, w) = (\nabla w, \nabla w)_{\mathcal{T}_h}, \quad \forall w \in \bar{\mathbb{V}}_2.$$

The transfer operator is $\Pi_2 = Id$.

Let

$$\mathbb{A} = \mathbb{D} - \mathbb{L} - \mathbb{L}^t$$

be the stiffness matrix whose (i,j)-th entry is $a_h(\varphi_j, \varphi_i)$, where $\mathbb{D}$ and $\mathbb{L}$ are the diagonal and the strict lower triangular part of $\mathbb{A}$. Let Δ be the discrete Laplacian matrix for the linear Lagrangian finite element space $\mathbb{V}_h^{Conf}$. Then the operator of the resulting auxiliary space preconditioner for the discrete variational problem (2) reads

$$(33) \qquad\qquad\qquad \mathcal{B} = \bar{\mathcal{B}}_1 + \bar{\mathcal{B}}_2,$$

where $\bar{\mathcal{B}}_j (j = 1, 2)$ denote the preconditioners for the diagonal matrix $\mathbb{D}$ and the discrete Laplacian matrix Δ.

In terms of implementations, we will take $\bar{\mathcal{B}}_1$ as the Jacobi (or Gauss-Seidel) smoothing operator for $\mathbb{A}$, For the discrete Laplacian matrix Δ, we will take the preconditioner $\bar{\mathcal{B}}_2$ as the BPX preconditioner for structured grid and algebraic multigrid method for unstructured grid.

In the following, we will give an estimate for the condition number of the precoditioner $\mathcal{B}$ given by (33). First, we prove that the above transfer operators satisfy the condition (31) of Theorem 4.1. Using the definitions of the inner product and the transfer operator in space $\bar{\mathbb{V}}_1$, for any given $v = \sum_{i=1}^{N} \alpha_i \varphi_i$ with $\alpha_i \in \mathbb{R}$, we have

$$\|\Pi_1 v\|_A^2 \;=\; \|v\|_A^2 = \left\| \sum_{i=1}^{N} \alpha_i \varphi_i \right\|_A^2 = \sum_{T \in \mathcal{T}_h} \left\| \sum_{\mathrm{supp}\varphi_i \cap T \neq \emptyset} \alpha_i \varphi_i \right\|_{A,T}^2$$

$$(34) \qquad\qquad \leq\; M \sum_{T \in \mathcal{T}_h} \sum_{i=1}^{N} \alpha_i^2 \, \|\varphi_i\|_{A,T}^2 = M\|v\|_{\bar{A}_1}^2,$$

where the constant M bounds the number of basis functions whose support overlaps with a single element T.

For any given $w \in \bar{\mathbb{V}}_2$, it's easy to obain

$$(35) \qquad\qquad\qquad \|\Pi_2 w\|_A = \|w\|_A = \|w\|_{\bar{A}_2}.$$

Combining (34) and (35), we conclude that (31) holds with the constants $c_1 = M$ and $c_2 = 1$.

Secondly, we prove that the above transfer operators satisfy the condition (32) in Theorem 4.1. Indeed, this is true since Theorem 3.1 implies that: for any $v \in \mathbb{V}$, there exist $\tilde{v} = \sum_{i=1}^{N} v_i \in \bar{\mathbb{V}}_1, v_i \in \mathbb{V}_i$ and $p \in \bar{\mathbb{V}}_2$ such that $v = \tilde{v} + p$ and

$$(36) \qquad\qquad\qquad \|\tilde{v}\|_{\bar{A}_1}^2 + \|p\|_{\bar{A}_2}^2 \lesssim \|v\|_A^2.$$

Finally, as a direct consequence of (34), (35) and (36), we have

Theorem 4.2. *For the preconditioner $\mathcal{B}$ given by* (33), *we have*

$$\kappa(\mathcal{B}\mathcal{A}) \lesssim 1.$$

5. A fast iterative method

In this section, we propose a two-level iterative method for solving (2). We will then analyze the rate of convergence of this method. In the following, we first give a detail description of the method.

Algorithm 5.1 (A two-level algorithm). *Let $u_h^0 = 0$ be an initial guess. Assume that the l-th iterate $u_h^l \in \mathbb{V}_h$, $l \geq 0$, has been obtained. Then the next iterate $u_h^{l+1} \in \mathbb{V}_h$ is obtained as follows:*

Step 1: *Let $s_0^l = u_h^l$. Then for each $j = 1, \cdots, N$, we find $s_j^l \in \mathbb{V}_j$ such that*

$$a_h(s_j^l, v_j) = (f, v_j) - a_h(s_{j-1}^l, v_j), \quad \forall v_j \in \mathbb{V}_j.$$

Step 2: *Find $e^l \in \mathbb{V}_h^{Conf}$ such that*

$$a_h(e^l, v_h^{Conf}) = (f, v_h^{Conf}) - a_h(u_h^k + s_N^l, v_h^{Conf}), \quad \forall v_h^{Conf} \in \mathbb{V}_h^{Conf}.$$

Step 3: *Let*

$$(37) \qquad\qquad u_h^{l+1} = u_h^l + e^l.$$

Remark 5.1. *Step 1 in Algorithm 5.1 corresponds to applying the Gauss-Seidel smoother once for the discrete variational problem* (2). *It can also be replaced by the Jacobi method and related results can be found in Proposition 6.12 of* [39] *or Lemma 3.3 of* [41]. *One can also apply this smoother a few times before doing the next step. Step 2 in Algorithm 5.1 corresponds to solving the Laplacian problem by the piecewise linear conforming finite element method. In practice, we use the CG method.*

We are now in a position to analyze the above algorithm. Let $\mathbb{V}_{N+1} = \mathbb{V}_h^{Conf}$. We then define the energy projections $P_j : \mathbb{V}_h \mapsto \mathbb{V}_j (j = 1, \cdots, N+1)$ by

$$(38) \qquad a_h(P_j v, w_j) = a_h(v, w_j) \quad \text{for all } v \in \mathbb{V}_h, w_j \in \mathbb{V}_j.$$

Let E be the error propagation operator for each iteration in Algorithm 5.1. We have the following well-known identity [40, 20]

$$u - u_h^{k+1} = E\left(u - u_h^k\right)$$

where u is the solution of (1) and

$$E = (I - P_{N+1})(I - P_N) \cdots (I - P_1).$$

Furthermore, by using the Xu-Zikatanov identity (see Corollary 4.3 in [40] or Corollary 2.1 in [20]), we get

$$\|E\|_A^2 = 1 - \frac{1}{1 + \tilde{c}_0},$$

where

$$(39) \qquad \tilde{c}_0 \quad = \quad \sup_{\|v\|_A=1} \inf_{\sum_{k=1}^{N+1} v_k = v} \sum_{k=1}^{N+1} \|P_k \sum_{j=k+1}^{N+1} v_j\|_A^2.$$

The rest of the section is devoted to the proof for the result that the constant $\tilde{c}_0$ is independent of mesh size.

First Theorem 3.1 implies that: for any $v \in \mathbb{V}_h$, there exist $v_k \in \mathbb{V}_k, k = 1, \cdots, N+1$ such that $v = \sum_{k=0}^{N+1} v_k$ and

$$(40) \qquad \sum_{k=0}^{N+1} a_h(v_k, v_k) \leq K_0 \|v\|_A^2.$$

Let $u_k = P_k \left(\sum_{j=k+1}^{N+1} v_j \right) \in \mathbb{V}_k$. We then apply (38) to obtain

$$\sum_{k=1}^{N+1} \|u_k\|_A^2 = \sum_{k=1}^{N+1} a_h(u_k, P_k \sum_{j=k+1}^{N+1} v_j) = \sum_{k=1}^{N+1} a_h(u_k, \sum_{j=k+1}^{N+1} v_j)$$
$$= \sum_{k=1}^{N+1} \sum_{\text{supp}_{v_j} \cap \text{supp}_{u_k} \neq \emptyset} a_h(u_k, v_j).$$

Using the Cauchy-Schwarz inequality, we have

$$\sum_{k=1}^{N+1} \|u_k\|_A^2 \leq \sum_{k=1}^{N+1} \sum_{\text{supp}_{v_j} \cap \text{supp}_{u_k} \neq \emptyset} a_h(u_k, u_k)^{1/2} a_h(v_j, v_j)^{1/2}$$
$$\leq \sum_{k=1}^{N+1} a_h(u_k, u_k)^{1/2} \sum_{\text{supp}_{v_j} \cap \text{supp}_{u_k} \neq \emptyset} a_h(v_j, v_j)^{1/2}$$
$$\leq \left(\sum_{k=1}^{N+1} a_h(u_k, u_k) \right)^{1/2} \left(\sum_{k=1}^{N+1} \sum_{\text{supp}_{v_j} \cap \text{supp}_{u_k} \neq \emptyset} a_h(v_j, v_j) \right)^{1/2}$$
$$\leq \left(\sum_{k=1}^{N+1} a_h(u_k, u_k) \right)^{1/2} \left(\bar{M} \sum_{k=1}^{N+1} a_h(v_k, v_k) \right)^{1/2}$$

which implies that

$$(41) \qquad \sum_{k=1}^{N+1} \|P_k \sum_{j=k+1}^{N+1} v_j\|_A^2 = \sum_{k=1}^{N+1} \|u_k\|_A \leq \bar{M} \sum_{k=1}^{N+1} a_h(v_k, v_k).$$

Here, $\bar{M}$ bounds the number of basis functions whose support overlaps with the support of the functions $u_k \in \mathbb{V}_k$ for all $k = 1, 2, \cdots, N+1$.

As a direct consequence of (39), (40) and (41), we have

Theorem 5.1. *Let u be the solution of (1), and let u_h^l and u_h^{l+1} be two consecutive iterates obtained by Algorithm 5.1. Then exists a positive number $\rho < 1$, which is independent of l and h, such that*

$$\|u - u_h^{l+1}\|_A \leq \rho \|u - u_h^l\|_A.$$

Thus, given a tolerance level, the algorithm 5.1 will terminate in finite steps.

6. Numerical Results

In this section, we will present numerical experiments showing the performance of the proposed preconditioner and the convergence of our iterative method for

solving the problem (1). We will use the symmetric IPDG method defined by

$$a_h(w, v) = \sum_{T \in \mathcal{T}_h} \int_T \nabla w \cdot \nabla v \mathrm{d}\boldsymbol{x} - \sum_{e \in \mathcal{E}_h} \int_e (\{\!\!\{\nabla w\}\!\!\} [\![v]\!] + \{\!\!\{\nabla v\}\!\!\} [\![w]\!]) \, \mathrm{d}s$$

$$+ \eta \sum_{e \in \mathcal{E}_h} h_e^{-1} \int_e [\![w]\!] \cdot [\![v]\!] \, \mathrm{d}s.$$

The space $\mathbb{V}_h$ is taken as the space of discontinuous piecewise linear functions.

In our first example, we choose domain $\Omega = (-1, 1)^2$, and test the performance of our methods with the use of a structured grid as well as an unstructured grid. For the convenience of computing errors, we solve a problem whose exact solution is defined by

$$u = \sin(\pi x) \sin(\pi y).$$

To test convergence, we use a sequence of structured grids and unstructured grids. For the structured grids, Figure 1 shows an initial triangulation $\mathcal{T}_h$ with $h = 1/4$. We can then obtain the refined meshes by dividing each element uniformly into 4 sub-elements by connecting the midpoints of the edges of the triangles. For the unstructured grids, a mesh generator is employed to get a sequence of increasingly finer meshes, whose triangles all have about the same size with little distortion, for example, Figure 2 shows a triangulation $\mathcal{T}_l$ with level $l = 2$, which is a mesh obtained by refining an initial mesh once.
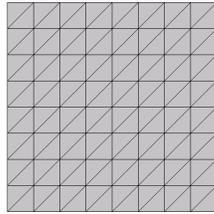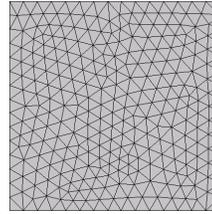

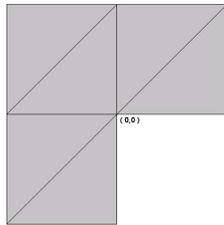FIGURE 1. A structured grid.

FIGURE 2. An unstructured grid.
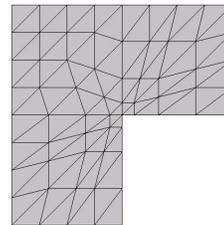

FIGURE 3. $\mathcal{T}_0$: The initial mesh.

FIGURE 4. $\mathcal{T}_2$: Graded mesh after 2 refinement.

In our second example, we consider a L-shaped domain $\Omega = (-1, 1)^2 \setminus [0, 1] \times [-1, 0]$ with the use of graded meshes. We take the exact solution to be

$$u = r^{\frac{2}{3}} \sin(\frac{2}{3}\theta).$$

TABLE 1. Number of PCG iterations with preconditioner (33) for structured grids, where $tol = 1.0e - 8$.

h	1/4	1/8	1/16	1/32	1/64	1/128
$\eta = 10$	35	43	46	49	49	51
$\eta = 20$	35	43	46	48	49	50
$\eta = 50$	34	42	46	48	49	49
$\eta = 100$	33	42	46	47	49	48

TABLE 2. Number of PCG iterations with preconditioner (33) for unstructured grids, where $tol = 1.0e - 8$.

Level	# Cells	$\eta = 10$	$\eta = 20$	$\eta = 50$	$\eta = 100$
1	116	14	13	12	11
2	528	15	14	12	12
3	2118	15	14	13	12
4	8142	15	14	13	12
5	32670	15	14	13	12

In this case, we will use the graded refinement procedure. The construction of such graded meshes can be found in Section 4 of [9]. Figure 3 shows the initial mesh $\mathcal{T}_0$ and Figure 4 shows the graded mesh $\mathcal{T}_2$ after 2 refinements.

6.1. Results on using the preconditioner. In this subsection, we will consider the solution of the discrete variational problem (2) by the Preconditioned Conjugate Gradient (PCG) method. We will illustrate the effect on using our preconditioner $\mathcal{B}$ given by (33), where for $\bar{\mathcal{B}}_1$, we use the Jacobi smoothing, while for $\bar{\mathcal{B}}_2$, we choose the BPX preconditioner for Laplacian problems. We stop the iteration when the relative residual is less than 10^{-8}.

TABLE 3. Number of PCG iterations with standard BPX preconditioner (33) for graded meshes, where $tol = 1.0e - 8$.

Mesh	$\eta = 10$	$\eta = 20$	$\eta = 50$	$\eta = 100$
$\mathcal{T}_1$	33	28	20	21
$\mathcal{T}_2$	33	35	39	40
$\mathcal{T}_3$	53	57	60	64
$\mathcal{T}_4$	79	87	94	96
$\mathcal{T}_5$	120	130	141	143
$\mathcal{T}_6$	180	196	212	216

For the first example with a rectangular domain, the iteration counts required for the method with various penalty parameters η and mesh sizes h are reported in Tables 1 and 2, for the use of structured and unstructured grids respectively. From these results, we see that, when the penalty parameter η is fixed, the number of iterations required is essentially independent of the mesh sizes, even though we see a very mild increase in the number of iterations. Furthermore, for fixed mesh sizes, we see that the number of iterations is independent of the penalty parameter η. We also observe that there is actually a small reduction in the number of iterations

when the penalty parameter is increased. Hence, we see that $\mathcal{B}$ given by (33) provides a very effective precondioner for the discrete variational problem (2).

TABLE 4. Number of PCG iterations with modified BPX precon-ditioner (33) for graded meshes, where $tol = 1.0e - 8$.

Mesh	$\eta = 10$	$\eta = 20$	$\eta = 50$	$\eta = 100$
$\mathcal{T}_1$	25	28	24	24
$\mathcal{T}_2$	40	37	36	36
$\mathcal{T}_3$	46	44	43	43
$\mathcal{T}_4$	52	51	50	50
$\mathcal{T}_5$	55	55	54	54
$\mathcal{T}_6$	60	59	58	58

For the second example with L-shaped domain, we remark that the largest edge may be much longer than the shortest edge in the same graded mesh. Therefore, the standard BPX preconditioner with the same mesh size given by the largest edge will not work, as we can see in the results reported in Tables 3.

To improve the iteration counts, we construct a modified BPX preconditioner for graded mesh as follows:

$$\bar{\mathcal{B}}_2 := \sum_{k=0}^{N^C} \sum_{l=1}^{\mathcal{N}_k} h_k^2(p_l) Q_k^l,$$

where N^C is the dimension of $\mathbb{V}_h^{Conf}$, $h_k(p_l)$ is the length of the shortest interior edge in Ω_{p_l}, Q_k^l denotes the L^2 projection from $\mathbb{V}_h^{Conf}$ to $\mathbb{V}_i^C = \mathrm{span}\{\varphi_i^C\}(i = 1, \cdots, N^C)$, $\{\varphi_i^C\}_{i=1}^{N^C}$ is the set of all nodal basis functions spanning $\mathbb{V}_h^{Conf}$. The corresponding results with the use of this new preconditioner $\bar{\mathcal{B}}_2$ are reported in Table 4. We observe improved results and the behavior is similar to that of the first example.

6.2. Results on using Algorithm 5.1. In this section, we will present some results on using Algorithm 5.1 for solving the discrete variational problem (2). We will show the number of iterations required for various mesh sizes h and penalty parameters η. Moreover, we will test the behavior of our Algorithm 5.1 by using the smoothing step (i.e. Step 1 of Algorithm 5.1) more than once before doing Step 2. In the following, we use m, $(m \geq 1)$, to denote the number of smoothing steps done before doing Step 2.

For the first example, we present the number of iterations required with fixed penalty parameter $\eta = 10$ for various h and m in Table 5 and Table 6 respectively.

TABLE 5. Number of iteration for Algorithm 5.1 for structured grids, where $\eta = 10$.

h	1/4	1/8	1/16	1/32	1/64	1/128
$m = 1$	20	19	19	19	20	23
$m = 2$	12	11	11	12	12	12
$m = 3$	10	10	10	10	10	10
$m = 4$	9	9	9	9	9	9
$m = 5$	9	9	8	8	8	9

TABLE 6. Number of iteration for Algorithm 5.1 for unstructured grids, where $\eta = 10$.

Level	# Cells	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
1	116	22	13	10	9	8
2	528	23	13	10	9	8
3	2118	25	14	10	9	8
4	8142	26	14	11	10	8
5	32670	27	15	11	10	8

TABLE 7. Number of iteration for Algorithm 5.1 for structured grids, where $\eta = 20$.

h	1/4	1/8	1/16	1/32	1/64	1/128
$m = 1$	20	20	21	23	27	27
$m = 2$	12	11	12	12	12	15
$m = 3$	9	9	10	10	10	13
$m = 4$	9	9	9	10	10	10
$m = 5$	9	8	8	8	8	8

TABLE 8. Number of iteration for Algorithm 5.1 for unstructured grids, where $\eta = 20$.

Level	# Cells	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
1	116	23	13	10	9	8
2	528	25	14	10	9	8
3	2118	26	14	11	9	8
4	8142	28	15	11	10	8
5	32670	31	15	12	10	8

TABLE 9. Number of iteration for Algorithm 5.1 for graded grids with $\eta = 10$ (left) and $\eta = 20$ (right).

Mesh	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$	$\mathcal{T}_6$
$m = 1$	18	20	20	20	20	20
$m = 2$	13	12	12	12	11	11
$m = 3$	11	11	10	10	10	10
$m = 4$	10	10	9	9	9	9
$m = 5$	10	9	9	8	8	8

Mesh	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$	$\mathcal{T}_6$
$m = 1$	17	20	20	20	20	20
$m = 2$	12	11	11	11	11	11
$m = 3$	10	10	9	9	9	9
$m = 4$	10	9	9	9	8	8
$m = 5$	9	9	8	8	8	8

Numerical results with penalty parameters $\eta = 20$ are reported in Table 7 and Table 8. From these results, we see that the number of iterations required is essentially independent of the mesh size h. Furthermore, we see that the number of iterations can be reduced by increasing the number of smoothing steps used in Step 1 of Algorithm 5.1. We also observe that the number of smoothing steps can be taken as $m = 4$ as this is the smallest m giving iteration counts that are rather stable with respect to mesh sizes.

For the second example, the corresponding results are shown in Table 9. We see that similar conclusions hold in this case.

Acknowledgments

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