Abstract. In this paper, an explicit a posteriori error estimator is developed for second and fourth order ODEs solved with the Galerkin method that, remarkably, provides exact pointwise error estimates. The error estimator is derived from the variational multiscale theory, in which the subgrid scales are approximated making use of fine-scale Green’s functions. This methodology can be extended to any element type and order. Second and fourth order differential equations cover a great variety of problems in mechanics. Two examples with application in elasticity have been studied: the axially loaded beam and the Euler-Bernoulli beam. Because the error estimator is explicit, it can be very easily implemented and its computational cost is very small. Apart from pointwise error estimates, we present local and global a posteriori error estimates in the $L^1$-norm, the $L^2$-norm and the $H^1$-seminorm. Finally, convergence rates of the error and the efficiencies of the estimator are analyzed.

Key words. a posteriori error estimation, 1D linear elasticity, Euler-Bernoulli beam, pointwise error, variational multiscale theory.

1. Introduction

Second and 4th-order ODEs and, in general, elliptic differential equations have been thoroughly studied using finite element methods (FEM). Specially, the standard Galerkin method gives rise to satisfactory solutions for these types of equations, helping the FEM achieve a widespread use by scientist and engineers. However, it is well-known that numerical methods have an inherent error that, basically, depends on the discretization and the order of the numerical method. Accordingly, in order to evaluate the quality of the FEM solution, it is convenient to quantify the numerical error which is committed. Furthermore, a posteriori error estimation can be exploited by adaptive methods to reduce the error where it is more beneficial.

There exists a broad literature on a posteriori error estimation for FEM which can be classified in three groups [2]:

1) Residual-based methods. The proposed estimator belongs to this category. They are also called explicit methods since the error estimate is based only on the information provided by the FEM solution. The error is computed via interior residuals or/and inter-element residuals. They were proposed for the first time by Babuška et al. [4, 5].

2) Recovery-based methods. Zienkiewicz and Zhu [38] developed these techniques, which take advantage of superconvergent properties of the solution. Satisfactory results are achieved for a wide variety of problems. The main idea consists of estimating the error by comparing a smoothed gradient with the gradient of the FEM solution. A general background might be found in [1].

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Auxiliary-problem-based methods. Lately, important advances have been made in this category in which the error is estimated by solving new differential equations. For this reason, they also receive the name of implicit methods. They involve the calculation of residuals of the FEM solution. Typically, these differential equations are applied to a subdomain (an element or a patch of elements). Many researchers have worked in this matter [6, 3]. Ladevéze et al. proposed a local error estimator based on the recovery of equilibrated fluxes [28, 29]. Applications to linear elasticity have been made recently by Verfürth et al. [31], Carstensen et al. [8], Parés, Diez, Huerta and co-workers [35, 9] and Masud et al. [32], where nearly incompressible elasticity is studied. Goal-oriented error estimators have been developed, see Oden et al. [34, 37], in which the dual problem is solved and it involves the computation of influence functions in order to relate the residuals to specific quantities of interest.

In this work, the error estimator is based on the variational multiscale theory (VMS) [24, 26], in which the solution is split into resolved and unresolved scales. Precisely, the unresolved scales present a paradigm from which the error of the finite element solution can be calculated or estimated. In the theory, the interior residuals and inter-element jumps emerge naturally as error sources.

Previous works of the group on this technology were devoted to the transport equation solved with stabilized methods [18, 19, 21, 17, 20, 15, 22]. Here, this technology is extended to second and fourth order differential equations which can be solved with the Galerkin method. Furthermore, following [27] the fine-scale Green’s functions have been numerically computed and exploited to obtain expressions for the local and global errors. This procedure has been clarified in the Appendix, so it can be extended to other equations and any element type.

Following the variational multiscale theory, a few relevant articles have studied the fine scales, or unresolved scales, [11, 10, 27, 30], revealing that under most circumstances they are nearly local. That is, for certain class of methods the error is mostly confined inside the element. This is an important property that has been exploited in this work to calculate the error in each element.

Also, beyond existing work on VMS error estimation, pointwise error estimates are studied in this paper. This field has been treated previously for other authors in elliptic problems such as Nochetto [33] using regularized Green’s functions. A prominent work was carried out by Prudhomme et al. [36] where quantities of interest of the error are measured and tested in one-dimensional problems. However, the present theory provides a simple way to attain an exact representation of the pointwise error.

Following the introduction, we present in Sec. 2 the background of VMS error estimation. The split of coarse and fine-scale spaces is discussed and it is shown that the error can be assessed using the fine-scale variational form. At the end of the section, the general expressions for the pointwise error estimator and for a domain are established. In Sec. 3 and 4, we address the error estimation for the 1-D axially loaded beam and the Euler-Bernoulli beam problem, respectively. In both sections, numerical examples illustrate the behavior of the error estimator. In Section 5, we explain how this estimator can be extended to multi-dimensional problems. Finally, we remark the conclusions of this work.

2. The VMS error estimation framework

2.1. FEM formulation.
2.1.1. **Strong form.** Let \( \Omega \) be a bounded domain in \( \mathbb{R} \) with boundary \( \Gamma \), which can be divided into two parts, \( \Gamma_g \) and \( \Gamma_h \), such that \( \Gamma_g \cup \Gamma_h = \Gamma \) and \( \Gamma_g \cap \Gamma_h = \emptyset \). The essential boundary condition, \( g \), is applied on \( \Gamma_g \) and the natural boundary condition, \( h \), on \( \Gamma_h \).

The strong form, which stems from the equilibrium equations, consists of finding \( u \in S : \Omega \rightarrow \mathbb{R} \), such that for the given functions \( f : \Omega \rightarrow \mathbb{R} \), \( g : \Gamma_g \rightarrow \mathbb{R} \) and \( h : \Gamma_h \rightarrow \mathbb{R} \), the following equations are satisfied:

\[
\begin{aligned}
L u &= f \quad \text{in} \quad \Omega \\
u &= g \quad \text{on} \quad \Gamma_g \\
B u &= h \quad \text{on} \quad \Gamma_h
\end{aligned}
\]

where \( L \) is a generic differential operator and \( B \) an operator acting on \( \Gamma_h \).

2.1.2. **Weak form.** The weak form is obtained by multiplying the strong form equation by weighting functions. Both the trial solution and the weighting functions must belong to suitable spaces. Therefore, let \( w \in \{ V \subset (H^r(\Omega)) : w|_\Gamma_g = 0 \} \) be the weighting functions and let \( u \in \{ S \subset (H^r(\Omega)) , u|_\Gamma_g = g \} \) be the trial solution with \( r \) denoting the order of the spaces. In our case, we adopt \( r = 1 \) for the beam problem with an axial force (2\(^{\text{nd}}\)-order ODE) and \( r = 2 \) for the Euler-Bernoulli beam problem (4\(^{\text{th}}\)-order ODE).

Accordingly, we can establish the variational form as: Find \( u \in S \) such that

\[
a(w, u) = (w, f) + (w, h)_{\Gamma_h} \quad \forall w \in V
\]

where \( a(\cdot, \cdot) \) is the bilinear form, \( (\cdot, \cdot) \) the \( L^2(\Omega) \) inner product and \( (\cdot, \cdot)_{\Gamma_h} \) the \( L^2(\Gamma_h) \) inner product on \( \Gamma_h \). For each particular problem, we will expand these operators later.

So as to introduce the FEM method, the domain, \( \Omega \), must be discretized into \( n_E \) non-overlapping elements with domain \( \Omega^e \). We define \( \Omega \) as the union of element interiors, \( \Omega = \bigcup_{e=1}^{n_E} \Omega^e \).

We establish the finite spaces \( S^h \subset S \) and \( V^h \subset V \) for the trial FEM solution and the weighting functions, respectively,

\[
S^h = \{ u_h \in (H^r(\Omega)) \mid u_h|_{\Gamma_g} \in P_k, u_h|_{\Gamma_g} = g, \forall \Omega^e \in \tilde{\Omega} \}
\]

\[
V^h = \{ w_h \in (H^r(\Omega)) \mid w_h|_{\Gamma_g} \in P_k, w_h|_{\Gamma_g} = 0, \forall \Omega^e \in \tilde{\Omega} \}
\]

where \( P_k \) denotes the space of polynomials of degree \( k \). Thus, the standard Galerkin method reads:

Find \( u_h \in S^h \) such that

\[
a(w_h, u_h) = (w_h, f) + (w_h, h)_{\Gamma_h} \quad \forall w_h \in V^h
\]

2.2. **Variational multiscale theory.** The variational multiscale method was rigorously developed in [24, 26]. It consists of splitting the weak solution and the weighting functions into coarse scales (or resolved scales) and fine scales (or unresolved scales),

\[
u = \bar{u} + u' \quad \bar{u} \in \bar{S} , \quad u' \in S'
\]

\[
w = \bar{w} + w' \quad \bar{w} \in \bar{S} , \quad w' \in S'
\]

We can identify the coarse scales with the selected finite element spaces, \( S^h \) and \( V^h \). On the other hand, the fine scales represent the unresolved scales, which are defined in an infinite-dimensional space, such that \( S = \bar{S} \oplus S' \) and \( V = \bar{V} \oplus V' \) for the trial functions and the weighting functions, respectively. Thus, in what follows, \( u' \) can be identified with the finite element error.
If the multiscale concept is introduced in Eq. (4), the variational formulation can be transformed into

\begin{align}
(5a) & \quad a(\bar{w}, \bar{u}) + a(\bar{w}, u') = (\bar{w}, f) + (\bar{w}, h)_{\Gamma_h} \quad \forall \bar{w} \in \bar{S} \\
(5b) & \quad a(w', \bar{u}) + a(w', u') = (w', f) + (w', h)_{\Gamma_h} \quad \forall w' \in S'
\end{align}

It may be observed that Eq. (5a) involves the coarse scales, whereas Eq. (5b) represents the fine scales, which play a relevant role in error estimation.

2.3. Error estimation and fine-scale Green’s functions. In this section, the a posteriori error estimator based on the variational multiscale theory is described. It is shown that the interior residual is the only error source and a post-processing a posteriori error estimator based on the variational multiscale theory is described.

For this purpose, we make use of the Eq. (5b) and the fine-scale Green’s function concept [24, 26, 27]. This technology has been applied to the transport equation in [18, 19, 21, 17, 20] with positive results.

Taking Eq. (5b), for nodally exact methods the error can be calculated from the following expression

\begin{align}
(6) & \quad a(w', u') = (w', f - L\bar{u}) + (w', h - B\bar{u})_{\Gamma_h} \quad \forall w' \in S'
\end{align}

Assuming that the FEM solution satisfies exactly the Neumann boundary condition, it can be seen that the error depends only on the FEM solution residual, \( f - L\bar{u} \).

At this point, the fine-scale Green’s function is introduced and the error may be expressed by \( u' = \mathcal{G}'(f - L\bar{u}) \), where \( \mathcal{G}' \) is the fine-scale Green’s operator. Hughes and Sangalli established a general expression for this operator in [27], namely

\begin{align}
(7) & \quad \mathcal{G}' = \mathcal{G} - \mathcal{G}P^T(P\mathcal{G}P^T)^{-1}P\mathcal{G}
\end{align}

where \( \mathcal{G} \) is the classical Green’s function operator which represents the inverse of the differential operator \( L^{-1} \). \( P \) is an orthogonal projector, \( P : S \rightarrow \bar{S} \) such that \( PV = \bar{v} \) and \( PV' = 0 \).

The operator \( \mathcal{G} \) and the projector \( P \) can be approximated by matrix operators \( \mathbf{G} \) and \( \mathbf{P} \), respectively. In order to compute them, the first stage is to set up a fine mesh and the associated basis functions \( \phi_i \). Operating with this basis we can obtain the matrix \( \mathbf{G} \) that can be expressed as \( \mathbf{G} = L^{-1} \), where \( L_{ij} = (L\phi_j, \phi_i) \). This process is also explained in [27]. Therefore, (7) is transformed into

\begin{align}
(8) & \quad \mathbf{G}' = \mathbf{G} - \mathbf{G}^{P^T(\mathbf{P}\mathbf{G}P^T)^{-1}\mathbf{P}\mathbf{G}}
\end{align}

The second step consists of selecting the appropriate projector to estimate the error of the standard Galerkin method. As it was said, the exact solution \( u \) is decomposed into the FEM solution (or coarse solution), \( \bar{u} \), and the unresolved scales, \( u' \), which represent the numerical error. In our case, the projector that is proposed to estimate the error is the bilinear form \( a(\cdot, \cdot) \), i.e., the Galerkin projector.

Accordingly, the fine-scale Green’s functions are orthogonal with respect to \( a(\cdot, \cdot) \) for all \( \bar{w} \) belonging to \( \bar{S} \). Therefore, it is accomplished that the error, \( u' \), belongs to the kernel of this projector, \( \mathcal{P}u' = 0 \), due to the Galerkin orthogonality property \( a(u', \bar{w}) = 0 \ \forall \bar{w} \in \bar{S} \). In the appendix, the process to obtain the projection operator is explained.

In order to obtain numerically the Green’s function matrices \( \mathbf{G} \) and \( \mathbf{G}' \), we have used a (0,1) domain with a uniform fine mesh, with an element size \( h_{\varepsilon}^f \) and homogeneous Dirichlet boundary condition. The shape functions associated with this fine mesh are considered as an approximation of the space \( \bar{S} \). Then, it is selected a coarse mesh with an element size \( h_{\varepsilon}^c \ (h_{\varepsilon}^c >> h_{\varepsilon}^f) \). Similarly, the shape
functions associated with the coarse mesh are assumed to be the \( \bar{S} \) space. As a result, the projection operator is clearly defined \( \mathbf{P} : S \rightarrow \bar{S} \).

The fine-scale Green’s function \( g'(x, y) \) is obtained by means of \( g'(x, y) = \sum_{i,j} G'_{ij} \phi_j(x) \phi_i(y) \), where \( \phi_j(x) \) and \( \phi_i(y) \) are the basis functions of the fine mesh. It will be shown in the numerical examples that \( g'(x, y) \) is a local function, i.e., it is negligible when \( x \) and \( y \) do not belong to the same element.

Once the fine-scale Green’s function is obtained, we can estimate the error. If we recall Eq. (6), the error \( u' \) can be expressed as

\[
 u'(x) = \int_{\Omega} g'(x, y) (f - L\bar{u})(y) \, dy
\]

It is assumed that the fine-scale Green’s function, \( g'(x, y) \), is local and the support of these functions are defined inside the element. Accordingly, the integral in Eq. (9) can be extended to the element domain, \( \Omega^e \),

\[
 u'(x) = \int_{\Omega^e} g'(x, y) (f - L\bar{u})(y) \, dy \quad \text{in } \Omega^e
\]

2.3.1. Piecewise constant residuals. If the residual is piecewise constant, \( (f - L\bar{u})(y) \in P_0 \), the pointwise error expression is

\[
 u'(x) = (f - L\bar{u}) \int_{\Omega^e} g'(x, y) \, dy \quad \text{in } \Omega^e
\]

Since \( g'(x, y) \) is zero on the element boundary, residual-free bubble functions can be introduced. As we can see, this functions are related to the intrinsic scales concept [14, 19, 24, 15, 12, 13, 7], that is,

\[
 b_0^e(x) = \int_{\Omega^e} g'(x, y) \, dy
\]

Obviously, the bubble function depends on the coarse mesh size, \( h_e \). Taking a general norm at both sides of Eq. (11)

\[
 ||u'(x)||_{Z(\Omega^e)} = ||b_0^e(x)||_{Z(\Omega^e)} |(f - L\bar{u})|
\]

where \( ||| \cdot |||_Z \) is a given norm. \( Z \) may refer to the \( L^1 \)-norm, the \( L^2 \)-norm or the \( H^1 \)-seminorm. The intrinsic scales for error estimation, \( \tau_0^Z \), depend on the local mesh size and the bubble function,

\[
 \tau_0^Z = \frac{||b_0^e(x)||_{Z(\Omega^e)}}{\text{meas}(\Omega^e)^{1/r}}
\]

Therefore, the elemental error estimator is expressed as

\[
 ||u'(x)||_{Z(\Omega^e)} = \text{meas}(\Omega^e)^{1/r} \tau_0^Z |(f - L\bar{u})|
\]

2.3.2. Approximation for higher order residuals. When the residual is not piecewise constant, the residual may be decomposed in Taylor series and Eq. (10) can be expressed as

\[
 u'(x) = b_0(x)(f - L\bar{u})(c_i) + \sum_{k=1}^{\infty} b_k^e(x) \frac{d^k}{dy^k} \bigg|_{y=c_i} (f - L\bar{u})
\]
where \( c_i \) is the central point of the element and \( b_k \) is the \( k \)-th-order residual-free bubble, defined as

\[
\begin{align*}
    b_0^k(x) &= \int_{\Omega_e} g'(x, y) \, d\Omega_y \\
    b_1^k(x) &= \int_{\Omega_e} g'(x, y)(y - c_i) \, d\Omega_y \\
    b_2^k(x) &= \int_{\Omega_e} g'(x, y)(y - c_i)^2 \, d\Omega_y \\
    &\vdots \\
    b_k^k(x) &= \int_{\Omega_e} g'(x, y)(y - c_i)^k \, d\Omega_y
\end{align*}
\]

Eq. (16) is the general expression for the pointwise error and it will be particularised in each problem.

Now, we can define the error intrinsic time scales, \( \tau_k^{Z_e} \), which depend on the local mesh size and the bubble functions,

\[
\tau_k^{Z_e} = \frac{||b_k^k(x)||_{Z_e(\Omega_e)}}{\text{meas}(\Omega_e)^{1/r}}
\]

For the \( L^1 \)-norm, \( r \) is equal to 1, and for the \( L^2 \)-norm and the \( H^1 \)-seminorm, \( r \) is 2. Introducing the error time scales, \( \tau_k^{Z_e} \), and taking a suitable norm we obtain an elemental error expression which can be exact when the residual is piecewise constant or an upper bound when the residual is of higher order,

\[
||u'(x)||_{Z_e(\Omega_e)} \leq \text{meas}(\Omega_e)^{1/r} \tau_k^{Z_e} \sum_{k=1}^{\infty} \text{meas}(\Omega_e)^{1/r} \tau_k^{Z_e} \frac{1}{k!} \frac{d^k(f - L\bar{u})}{dy^k} |y=c_i|
\]

2.4. Effectivity index. A way of evaluating the quality and accuracy of an error estimator consists of calculating the efficiency or effectivity index. The effectivity is defined as the ratio between the predicted error norm and the true error norm,

\[
I_{\text{eff}} = \frac{||\text{Predicted error}||}{||\text{True error}||}
\]

This concept can be applied both to the whole domain (obtaining a global effectivity), and to each element (obtaining a local effectivity). Hence, the local and global effectivities are expressed as

\[
I_{\text{eff}}^e = \frac{||\eta^e||_{\Omega_e}}{||u - \bar{u}||_{\Omega_e}} \quad \text{(local effectivity)}
\]

\[
I_{\text{eff}}^G = \frac{||\eta||_{\Omega}}{||u - \bar{u}||_{\Omega}} \quad \text{(global effectivity)}
\]

with \( \eta^e \) being the error estimation in an arbitrary element \( \Omega^e \) and ||·||\( _{\Omega_e} \) represents a norm.

Next, we evaluate the error estimator with several examples. The first benchmark consists of a beam with a sinusoidal load, \( f \). The second problem is the Euler-Bernoulli beam problem, which is a 4-th-order ODE.
3. 1-D axially loaded beam

In Fig. 1, a beam is loaded with a sinusoidal axial force. The general strong form of the problem is

\[
\begin{align*}
EA \frac{d^2 u}{dx^2} &= -f \quad \text{in } \Omega \\
\frac{du}{dx} &= g \quad \text{on } \Gamma_g \\
EA \frac{du}{dx} &= h \quad \text{on } \Gamma_h
\end{align*}
\]

(22)

where \(E\) is the elastic modulus and \(A\) is the transversal section. The variable \(u\) is the unknown axial displacement. The Dirichlet boundary condition expresses the imposed displacement \(g\) on \(\Gamma_g\) and the Neumann boundary condition means the imposed stress \(h\) on \(\Gamma_h\).

![Figure 1. Beam with an axial force.](image)

Particularly, it is seen in Fig. 1 that at both ends of the beam the displacement is zero. Finally, \(f\) is the axial force, which in our case is sinusoidal,

\[
f = f_0 \sin(2^m \pi x)
\]

(23)

with \(f_0\) being an arbitrary constant and \(m\) a positive integer. For \(L = 1\), the analytical solution can be obtained by integrating two times the strong formulation,

\[
u = \frac{1}{4^m \pi^2 EA} f_0 \sin(2^m \pi x)
\]

(24)

To construct the variational form, we define the weighting and trial function spaces. Let \(w \in \{\mathcal{V} \subset (H^1(\Omega)) : w|_{\Gamma_g} = 0\}\) be the weighting functions and let \(u \in \{\mathcal{S} \subset (H^1(\Omega)), u|_{\Gamma_g} = g\}\) be the trial functions. The weak form can be written as

\[
a(w, u) = (w, f) \quad \forall w \in \mathcal{V}
\]

(25)

where

\[
a(w, u) = -\int_0^l w_x EA u_x \, dx + wh|_{\Gamma_h}
\]

(26)

\[(w, f) = -\int_0^l wf \, dx\]

For the FEM formulation, we employ piecewise linear finite elements, \(u_h \in \mathbb{P}_1\) and \(w_h \in \mathbb{P}_1\), in a uniform partition of \(2^n\) elements \((n\) is an integer) with the characteristic that the node positions are at \(x_k = k/2^n\) for \(k = 0, 1, 2, 3, ..., 2^n\).

It is observed that \(u_h\) will be a linear interpolation of the exact solution. Therefore, if we take \(m \geq n\), \(u_h\) is zero on the whole domain. The present example is inspired by the problem proposed by Ainsworth and Oden in [2], in which it is shown some downsides of recovery-based error estimators. For instance, in this problem with \(m \geq n\), recovery-based error estimators predict a zero error whereas
the error is significant. In fact, the FEM error coincides with the exact solution, 
\[ u' = u - u_h = u. \]
Therefore, since the FEM solution is nodally exact, the fine space \( S' \) is the space of bubble functions
\[ S' = \bigoplus_{e=1,...,n_{el}} H_0^1(\Omega^e) \]
In our case, the estimated error may be described as
\[ (27) \quad u'(x) = -\int_0^l g'(x,y)(\mathcal{L}\bar{u} - f)(y) \, d\Omega_y \]
It is observed that \( \mathcal{L}\bar{u} - f = -f \) since the shape functions are piecewise linear. In addition, the residual term can be decomposed in Taylor series around the center of the element, \( x = c \),
\[ (28) \quad f = f_0 \sin(2^m \pi c) + 2^m \pi f_0 \cos(2^m \pi c)(x - c) - \frac{(2^m \pi)^2}{2} f_0 \sin(2^m \pi c)(x - c)^2 + \cdots \]

3.1. Fine-scale Green’s function. The fine-scale Green’s function, \( g'(x,y) \), that appears in Eq. \((27)\), has been obtained via Eq. \((8)\). For that purpose, it is necessary to define a fine mesh and a coarse mesh. In a \((0,1)\) domain, we have selected a uniform coarse mesh of 10 elements and a fine mesh of 1000 elements. In Fig. 2 it is depicted the fine-scale Green’s function with \( EA = 1 \) at the point \( y = 0.48 \).

![Figure 2](image-url)

**Figure 2.** 1-D axially loaded beam. Fine-scale Green’s function at \( y = 0.48 \) and \( EA = 1 \).

As Fig. 2 shows, the fine-scale Green’s function is local since it is zero out of the element in which it is applied the concentrated unit-area source term, \( \delta(x - y) \). In this case, the fine-scale Green’s function is completely defined in the element that goes from 0.4 to 0.5.
This fine-scale Green’s function can also be obtained analytically solving the differential equation below. Since this function is defined within the element, we can identify \( g'(x,y) = g^e(x,y) \), where \( g^e(x,y) \) is termed the *elemental Green’s function* \([17]\),
\[ (29) \quad \left\{ \begin{array}{ll} EA \frac{d^2 g^e(x,y)}{dx^2} = \delta(x - y) & \text{in } \Omega^e \\ g^e(x,y) = 0 & \text{on } \Gamma^e_g \end{array} \right. \]
The result is
\[
g^e(x, y) = \begin{cases} 
\frac{1}{EAh_e}(h_e - x)y & x > y \\
\frac{1}{EAh_e}x(h_e - y) & x < y 
\end{cases}
\]
\[x, y \in [0, h_e]\]

### 3.2. Pointwise error

As for pointwise error, the points where the error is evaluated are represented in Fig. 3. These points are used both in axially loaded beam problem and in Euler-Bernoulli examples.

\[
u'(x) = b_0^e(x)f(c) + \sum_{k=1}^{\infty} b_k^e(x) \frac{1}{k!} \frac{d^k f}{dy^k} \bigg|_{y=c} 
\]

\[
u'(x) = b_0^e(x)f_0 \sin(2^m \pi c) + b_1^e(x)2^m \pi f_0 \cos(2^m \pi c) 
\]

\[
+ b_2^e(x)\left(\frac{2^m \pi)^2}{2!}\right) f_0 \sin(2^m \pi c) + \cdots
\]

where \(b_k^e(x)\) are the residual-free bubble functions of \(k\)th-moment order. They have been defined in Eq. (17) and they are depicted in Fig. 4 up to the 2nd-moment order.

Finally, applying Eq. (31) we have studied the pointwise error at the three points (P1, P2, P3) of an arbitrary element (see Fig. 3). For instance, if we select a domain \(\Omega = (0, 1)\), \(EA = 1\), \(f_0 = 1\) and \(m = n = 2\), we obtain a uniform mesh of 4 elements and a sinusoidal load, \(f = \sin(4\pi x)\), with two cycles (see Fig. 1). Table 1 shows the pointwise error evolution and effectivity indices selecting up to
the bubble function of $9^\text{th}$ moment order. It is observed that the error estimate tends to the exact error as the number of bubble functions is increased.

### 3.3. Local error estimates

Typically, it is also practical to analyse the error in each element. Thus, recalling Eq. (19) and replacing the generic $L^1$-norm with the $L^2$-norm, the $L^2$-norm and the $H^1$-seminorm,

$$
\begin{align}
\|u'(x)\|_{L^1(\Omega^c)} &\leq \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^k}{dy^k} (f_0 \sin(2\pi y)) \bigg|_{y=c} \\
\|u'(x)\|_{L^2(\Omega^c)} &\leq \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^k}{dy^k} (f_0 \sin(2\pi y)) \bigg|_{y=c} \\
\|u'(x)\|_{H^1(\Omega^c)} &\leq \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^k}{dy^k} (f_0 \sin(2\pi y)) \bigg|_{y=c}
\end{align}
$$

(32)

where $c$ is the element centre.

We have selected the intrinsic time scales, $\tau^k_{Z^c}$, from $0^\text{th}$-order to $9^\text{th}$-order (see Table 2). The higher the order of the residual, the more orders that must be taken into consideration to obtain a suitable error estimation.

The exact error in each element equals

$$
\begin{align}
\|u'(x)\|_{L^1(\Omega^c)}^\text{exact} &= 1.00786 \cdot 10^{-3} \text{ m} \cdot \text{m} \\
\|u'(x)\|_{L^2(\Omega^c)}^\text{exact} &= 2.23890 \cdot 10^{-3} \text{ m} \cdot \sqrt{\text{m}} \\
\|u'(x)\|_{H^1(\Omega^c)}^\text{exact} &= 2.81348 \cdot 10^{-6\frac{3}{2}} \text{ m}
\end{align}
$$

(33)

The elemental error estimate and the global effectivity are represented in Table 3. It can be seen that the error estimation is an upper bound of the error. The error has been calculated using error time-scales, $\tau^k_{Z^c}$, up to ninth order.
Table 2. Error time-scales for the axially loaded beam.

<table>
<thead>
<tr>
<th>Order((k))</th>
<th>(L^1)</th>
<th>(L^2)</th>
<th>(H^1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\frac{h^2}{12E_A})</td>
<td>(\frac{h^2}{12E_A})</td>
<td>(\frac{h^2}{12E_A})</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{h^4}{120E_A})</td>
<td>(\frac{h^4}{120E_A})</td>
<td>(\frac{h^4}{120E_A})</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{24h^4}{51840E_A})</td>
<td>(\frac{24h^4}{51840E_A})</td>
<td>(\frac{24h^4}{51840E_A})</td>
</tr>
<tr>
<td>3</td>
<td>(\frac{152h^6}{2956800E_A})</td>
<td>(\frac{152h^6}{2956800E_A})</td>
<td>(\frac{152h^6}{2956800E_A})</td>
</tr>
<tr>
<td>4</td>
<td>(\frac{2304h^6}{46542240E_A})</td>
<td>(\frac{2304h^6}{46542240E_A})</td>
<td>(\frac{2304h^6}{46542240E_A})</td>
</tr>
<tr>
<td>5</td>
<td>(\frac{14400h^6}{22761858624E_A})</td>
<td>(\frac{14400h^6}{22761858624E_A})</td>
<td>(\frac{14400h^6}{22761858624E_A})</td>
</tr>
<tr>
<td>6</td>
<td>(\frac{96768h^6}{120412240E_A})</td>
<td>(\frac{96768h^6}{120412240E_A})</td>
<td>(\frac{96768h^6}{120412240E_A})</td>
</tr>
<tr>
<td>7</td>
<td>(\frac{96768h^6}{120412240E_A})</td>
<td>(\frac{96768h^6}{120412240E_A})</td>
<td>(\frac{96768h^6}{120412240E_A})</td>
</tr>
<tr>
<td>8</td>
<td>(\frac{96768h^6}{120412240E_A})</td>
<td>(\frac{96768h^6}{120412240E_A})</td>
<td>(\frac{96768h^6}{120412240E_A})</td>
</tr>
<tr>
<td>9</td>
<td>(\frac{96768h^6}{120412240E_A})</td>
<td>(\frac{96768h^6}{120412240E_A})</td>
<td>(\frac{96768h^6}{120412240E_A})</td>
</tr>
</tbody>
</table>

(Note: The units for \(h\), \(E\) and \(A\) must be expressed in SI. \(h\) is the element size)

Table 3. Elemental error estimation of beam loaded with an axial force.

<table>
<thead>
<tr>
<th>Norm</th>
<th>Error units</th>
<th>Exact error</th>
<th>Estim. error</th>
<th>Eff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L^1)</td>
<td>(m \cdot m)</td>
<td>(1.007786 \cdot 10^{-03})</td>
<td>(1.65511 \cdot 10^{-03})</td>
<td>(1.6422)</td>
</tr>
<tr>
<td>(L^2)</td>
<td>(m \cdot \sqrt{m})</td>
<td>(2.23890 \cdot 10^{-03})</td>
<td>(3.47197 \cdot 10^{-03})</td>
<td>(1.5507)</td>
</tr>
<tr>
<td>(H^1)</td>
<td>(\sqrt{m})</td>
<td>(2.81348 \cdot 10^{-02})</td>
<td>(4.05022 \cdot 10^{-02})</td>
<td>(1.4396)</td>
</tr>
</tbody>
</table>

4. Euler-Bernoulli beam problem

This section is devoted to study error estimation for the Euler-Bernoulli beam theory, which is a 4th-order ordinary differential equation. The strong form, which expresses the equilibrium equation and the boundary conditions, reads:

Given the distributed transversal load \(q : \Omega \to \mathbb{R}\), find \(u(x) : \Omega \to \mathbb{R}\) such that

\[
\begin{align*}
EI \frac{d^4 u(x)}{dx^4} &= q \quad \text{in } \Omega \\
u &= g \quad \text{on } \Gamma_g \\
u_x &= h \quad \text{on } \Gamma_g \\
E I u_{xx} &= M \quad \text{on } \Gamma_h \\
E I u_{xxx} &= Q \quad \text{on } \Gamma_h
\end{align*}
\]

The unknown function \(u(x)\) represents the deflection of the beam, \(E\) is the elastic modulus and \(I\) is the second moment of area. In the above formulation, \(E\) and \(I\) are assumed to be constants. As regards the boundary conditions, \(g\) and \(h\) are the fixed degrees of freedom for displacements and for rotation, respectively. Also, \(M\) and \(Q\) are the prescribed moment and shear force, respectively.

The weak form of the problem is obtained integrating by parts,

\[
\begin{align*}
\int_0^l a(w, u) &= (w, q) \\
\int_0^l w_{xx} E I u_{xx} dx &= w_{xx} E I u |_{\Gamma_h} - w Q |_{\Gamma_h}
\end{align*}
\]

where

\[
a(w, u) = \int_0^l w_{xx} E I u_{xx} dx - w_{xx} E I u |_{\Gamma_h} - w Q |_{\Gamma_h}
\]

\[
(w, q) = \int_0^l w q dx
\]
According to the weak form, the spaces \( S^h \subset H^2(\Omega) \) and \( V^h \subset H^2(\Omega) \) must be selected. Likewise, in order to guarantee the continuity of the functions and their first derivatives in \( S^h \) and \( V^h \), \( C^1 \) functions must be employed,

\[
S^h = \{ u_h \in (H^2(\Omega)) \mid u_h|\Omega^e \in P_k, u_h|\Gamma_g = g, \forall \Omega^e \in \tilde{\Omega} \}
\]

\[
V^h = \{ w_h \in (H^2(\Omega)) \mid w_h|\Omega^e \in P_k, w_h|\Gamma_g = 0, w_h|\Gamma_x = 0, \forall \Omega^e \in \Omega \}
\]

where \( P_k \) are the polynomials of degree at most \( k \). Therefore, using the standard Galerkin method, the FEM problem reads:

Find \( u_h \in S^h \) such that

\[
a(v_h, u_h) = (v_h, f) + (v_h, h) \quad \forall v_h \in V^h
\]

Remark 1. The same theory can also be applied to similar problems such as rectangular plates with infinite length, \( 4^d w(x) \in \Lambda D \) where \( w(x) \) is the vertical displacement, \( q \) the distributed load and \( D \) the plate rigidity.

4.1. Elements considered in the analysis. In order to approximate the solution, two elements (Element 1 and Element 2) are employed with \( C^1 \)-continuity, i.e., with continuously differentiable displacements.

\[
\begin{array}{c}
1 \\
\downarrow \ \xi \\
2 \\
\xi = -1 & \xi = 1
\end{array}
\]

\text{Displacement} \quad \text{Rotation}

\text{Figure 5. Element 1: Element with two nodes.}

\[ u_{h_E} = N_1^u u_1 + N_2^u u_2 + N_1^\theta \theta_1 + N_2^\theta \theta_2 \]

where

\[
N_1^u(\xi) = \frac{1}{4}(2 - 3\xi + \xi^3) \quad N_1^\theta(\xi) = \frac{1}{4}(1 - \xi - \xi^2 + \xi^3)
\]

\[
N_2^u(\xi) = \frac{1}{4}(2 + 3\xi - \xi^3) \quad N_2^\theta(\xi) = \frac{1}{4}(-1 - \xi + \xi^2 + \xi^3)
\]

with \( \xi \) being the reference coordinate.

The shape functions, \( N_1^u, N_2^u, N_1^\theta \) and \( N_2^\theta \), are represented in Fig. 6 in the reference element.
4.1.2. **Element 2.** In the second element, we add a central node associated with a displacement function, \( N_u \) (see Fig. 7). The shape functions are represented in Fig. 8. The central node has associated a bubble function as shape function and it will be shown that it has a relevant importance in the error.

**Remark 2.** If we add to the central node the rotation degree of freedom, we recover Element 1.

\[
\begin{align*}
\xi &= -1 & \xi &= 0 & \xi &= 1 \\
N_1^u &= -\frac{\xi^4}{4} + \frac{\xi^3}{4} + \xi^2 - \frac{3\xi}{4} \\
N_2^u &= \frac{\xi^4}{4} + \frac{\xi^3}{4} - \frac{\xi^2}{4} - \frac{\xi}{4} \\
N_3^u &= \frac{\xi^4}{4} + \frac{\xi^3}{4} - \frac{\xi^2}{4} + \frac{3\xi}{4} \\
N_4^u &= \frac{\xi^4}{4} - \frac{\xi^3}{4} + \frac{\xi^2}{4} + \frac{3\xi}{4}
\end{align*}
\]

**Figure 7.** Element 2: Element with three nodes.

**Figure 8.** Element 2: Shape functions associated with the nodal variables.

The numerical solution, \( u_{h,E2}(x) \) in an element, is expressed by the formula:

\[
u_{h,E2} = N_1^u u_1 + N_2^u \theta_1 + N_3^u u_2 + N_4^u u_3 + N_4^u \theta_3 \]

Using the reference element, the shape functions are

\[
\begin{align*}
N_1^u(\xi) &= -\frac{\xi^4}{4} + \frac{\xi^3}{4} + \xi^2 - \frac{3\xi}{4} \\
N_2^u(\xi) &= \frac{\xi^4}{4} + \frac{\xi^3}{4} - \frac{\xi^2}{4} - \frac{\xi}{4} \\
N_3^u(\xi) &= \frac{\xi^4}{4} + \frac{\xi^3}{4} - \frac{\xi^2}{4} + \frac{3\xi}{4} \\
N_4^u(\xi) &= \frac{\xi^4}{4} - \frac{\xi^3}{4} + \frac{\xi^2}{4} + \frac{3\xi}{4}
\end{align*}
\]

4.2. **Example problems.** The proposed problems consist of a beam with various loads, \( q \), and boundary conditions. The first problem, which is termed Problem 1, is a beam with a uniform load and fixed displacements at both ends (see Fig. 9a). The second one, Problem 2, is a cantilever beam with a triangular load distribution (Fig. 9b).

For the numerical analysis, it has been chosen \( L = 10 \) m and \( EI = 40000 \) N/m for both problems. The load for Problem 1 is \( q = 100 \) N/m and for Problem 2...
is \( q = 1000 \frac{1}{L^2} \) N/m. The simulations are carried out on uniform meshes. These problems have the following analytical solutions, which can be taken advantage of to calculate the exact error of the FEM,

\[
\begin{align*}
\text{Prob. 1: } u(x)_{\text{prob1}} &= \frac{qx}{24EI} (x^3 - 2Lx^2 + L^3) \\
\text{Prob. 2: } u(x)_{\text{prob2}} &= \frac{(L-x)^3}{24EI} q(0) \left( -\frac{(L-x)^3}{5L} - 2L(L-x) + 4L^2 \right)
\end{align*}
\]

4.3. Error expressions for the Euler-Bernoulli beam. The pointwise error can be determined by means of bubble functions. Applying Eq. (16) to the Euler-Bernoulli beam problem results in

\[
u'(x) = b_0(x)(EI \frac{d^4 \bar{u}(y)}{dy^4} - q)(c) + \sum_{k=1}^{\infty} b_k(y) \frac{1}{k!} \frac{d^k}{dy^k} (EI \frac{d^4 \bar{u}(x)}{dy^4} - q) \bigg|_{y=c}
\]

Also, for these problems, we can measure the error in the whole domain and in the elements making use of the \( L^1 \)-norm, \( L^2 \)-norm and \( H^1 \)-seminorm. For the Euler-Bernoulli beam problem, the elemental error estimate (19) may be rewritten as

\[
\begin{align*}
||u'(x)||_{L^1(\Omega^e)} &\leq \max(\Omega^e) \tau^0_{L^1} \left| (EI \frac{d^4 \bar{u}(y)}{dy^4} - q)(c) \right| + \sum_{k=1}^{\infty} \max(\Omega^e) \tau^k_{L^1} \left| \frac{1}{k!} \frac{d^k}{dy^k} (EI \frac{d^4 \bar{u}(y)}{dy^4} - q) \bigg|_{y=c} \right| \\
||u'(x)||_{L^2(\Omega^e)} &\leq \max(\Omega^e) \tau^0_{L^2} \left| (EI \frac{d^4 \bar{u}(y)}{dy^4} - q)(c) \right| + \sum_{k=1}^{\infty} \max(\Omega^e) \tau^k_{L^2} \left| \frac{1}{k!} \frac{d^k}{dy^k} (EI \frac{d^4 \bar{u}(y)}{dy^4} - q) \bigg|_{y=c} \right| \\
||u'(x)||_{H^1(\Omega^e)} &\leq \max(\Omega^e) \tau^0_{H^1} \left| (EI \frac{d^4 \bar{u}(y)}{dy^4} - q)(c) \right| + \sum_{k=1}^{\infty} \max(\Omega^e) \tau^k_{H^1} \left| \frac{1}{k!} \frac{d^k}{dy^k} (EI \frac{d^4 \bar{u}(y)}{dy^4} - q) \bigg|_{y=c} \right|
\end{align*}
\]
where \( c \) is the centre of the element. Normally, the residual is not a polynomial of high degree. A typical case is when the residual is piecewise constant. In this case the above equations can be reduced to the simpler expressions,

\[
\begin{align*}
||u'(x)||_{L^1(\Omega^c)} &= \text{meas}(\Omega^c) \tau_{L^1}^0 \left( EI \frac{d^4 \bar{u}(y)}{dy^4} - q(c) \right) \\
||u'(x)||_{L^2(\Omega^c)} &= \text{meas}(\Omega^c)^{1/2} \tau_{L^2}^0 \left( EI \frac{d^4 \bar{u}(y)}{dy^4} - q(c) \right) \\
||u'(x)||_{H^1(\Omega^c)} &= \text{meas}(\Omega^c)^{1/2} \tau_{H^1}^0 \left( EI \frac{d^4 \bar{u}(y)}{dy^4} - q(c) \right)
\end{align*}
\]

In the next sections, the intrinsic error scales \( \tau \) which appear in Eqs. (45) and (46) are defined.

It is important to take into consideration that the error scales depend strongly on the element type. It can be proved that the numerical solution and its first derivative is nodally exact [25]. Therefore, the fine-scale Green’s function \( g'(x, y) \) is zero at the nodes and the error is confined within each element. This property can be confirmed computing the fine-scales as proposed Hughes and Sangalli [27].

**4.4. Analysis for Element 1.** The fine-scale Green’s function for Element 1 is shown in Fig. 10. It can be appreciated the local character of \( g'(x, y) \). To obtain the fine-scale Green’s function, the (0,1)-domain is associated with a uniform coarse mesh of 10 elements and a uniform fine mesh of 1000 elements. The fine-scale Green’s function is represented in Fig. 10 at \( y = 0.48 \) for \( EI = 1 \). It can be seen that \( g'(x, y) \) is confined within the element.

![Figure 10. Euler-Bernoulli problem. Fine-scale Green’s function for Element 1 and \( EI = 1 \).](image)

Therefore, we can establish that \( g'(x, y) = g^e(x, y) \), where \( g^e(x, y) \) is the element Green’s function. The value of \( g^e(x, y) \) is only different from zero when \( x \) and \( y \) belong to the same element. Therefore, the error can be calculated in each element without taking into account the error source of other elements.

Another way of obtaining \( g^e(x, y) \) for Element 1 is solving within each element the corresponding element Green’s function problem, in which there are only homogeneous Dirichlet boundary conditions,

\[
\begin{align*}
\left\{ \begin{array}{ll}
EI \frac{d^4 g^e(x, y)}{dx^4} &= \delta(x - y) & \text{in} \quad \Omega^c \\
g^e(x, y) &= 0 & \text{on} \quad \Gamma^c \\
g^e_{,x}(x, y) &= 0 & \text{on} \quad \Gamma^c
\end{array} \right.
\]
\]
In Eq. (47), $\Omega^e$ is the element domain, $\Gamma^e$ the element boundary and $\delta(x, y)$ the Dirac delta distribution. Fig. 11 depicts $g^e(x, y)$ for several values of the coordinate $y$. Note that the fine-scale Green’s functions are the same for both methods.

The solution of (47) is given by the following Green’s function, $x, y \in [0, h_e]$

\[
g^e(x, y) = \begin{cases} 
  \frac{y^2 (h_e - x)^2}{6EI} \left( 2h_e - y + x - 2(h_e - y) \frac{h_e - x}{h_e} \right) & x > y \\
  \frac{(h_e - y)^2 x^2}{6EI} \frac{h_e}{h_e} \left( 3y - x - 2yx \right) & x < y 
\end{cases}
\]

In this case, the fine-scale space $S'$ is the space of bubbles functions belonging to $H^2_0(\Omega)$

\[S' = \bigoplus_{e=1,\ldots, n_{el}} H^2_0(\Omega^e)\]

The bubble functions for the Euler-Bernoulli beam problem are represented in Fig. 12. In order to quantify the bubble function value for error estimation, we
employ the $L^1$-norm, $L^2$-norm and the $H^1$-seminorm in each element,

$$
\|b_{E1}^1(x)\|_{L^1(\Omega^e)} = \frac{h^5_e}{720EI}
$$

$$
\|b_{E1}^1(x)\|_{L^2(\Omega^e)} = \frac{h^9_e/2}{72\sqrt{70EI}}
$$

$$
\|b_{E1}^1(x)\|_{H^1(\Omega^e)} = \frac{h^{7/2}}{12\sqrt{210EI}}
$$

(49)

The superscript $E1$ refers to Element 1. Finally, the intrinsic scales $\tau_{L^1}^0$, $\tau_{L^2}^0$ and $\tau_{H^1}^0$ in Eq. (46) take the form

$$
\tau_{E1,0}^{L^1} = \frac{\|b_{E1}^0(x)\|_{L^1(\Omega^e)}}{\text{meas}(\Omega^e)} = \frac{h^4}{720EI}
$$

$$
\tau_{E1,0}^{L^2} = \frac{\|b_{E1}^0(x)\|_{L^2(\Omega^e)}}{\text{meas}(\Omega^e)^{1/2}} = \frac{h^4}{72\sqrt{70EI}}
$$

$$
\tau_{E1,0}^{H^1} = \frac{\|b_{E1}^0(x)\|_{H^1(\Omega^e)}}{\text{meas}(\Omega^e)^{1/2}} = \frac{h^3}{12\sqrt{210EI}}
$$

(50)

In case that the residual is not piecewise constant, we can approximate it by Taylor series as above, leading to bubble functions and time scales of higher order.

In Prob. 2, the residual is piecewise linear, so it is enough to define the bubble functions and the time scales of first order,

$$
\|b_{E1}^1(x)\|_{L^1(\Omega^e)} = \frac{h^5}{23040EI}
$$

$$
\|b_{E1}^1(x)\|_{L^2(\Omega^e)} = \frac{h^{11/2}}{720\sqrt{770EI}}
$$

$$
\|b_{E1}^1(x)\|_{H^1(\Omega^e)} = \frac{h^{5/2}}{360\sqrt{70EI}}
$$

(51)

$$
\tau_{E1,1}^{L^1} = \frac{\|b_{E1}^1(x)\|_{L^1(\Omega^e)}}{\text{meas}(\Omega^e)} = \frac{h^5}{23040EI}
$$

$$
\tau_{E1,1}^{L^2} = \frac{\|b_{E1}^1(x)\|_{L^2(\Omega^e)}}{\text{meas}(\Omega^e)^{1/2}} = \frac{h^5}{720\sqrt{770EI}}
$$

$$
\tau_{E1,1}^{H^1} = \frac{\|b_{E1}^1(x)\|_{H^1(\Omega^e)}}{\text{meas}(\Omega^e)^{1/2}} = \frac{h^4}{360\sqrt{70EI}}
$$

(52)

4.4.1. Results for Problem 1. In this problem the residual is piecewise constant, $\frac{d^2a(x)}{dx^2} - q \in P_0$. Furthermore, $\frac{d^2a(x)}{dx^2} = 0$. Therefore, the elemental error can be determined by Eq. (46).

Fig. 13 shows the efficiency of the estimator for different number of elements. In this case both the local and global effectivities are one, i.e., the estimated error is the same as the exact error.

The error convergence rates which are achieved with Element 1 are studied in Fig. 14, where the error in the central element is represented. The convergence rates of the elemental error in each norm are steered by the norms of the bubble functions, see Eq. (49). Furthermore, these convergence rates are in concordance
with the theory, which must be adapted to the elemental error (see, for instance, Chapter 4 of [25]).

Eventually, the pointwise error is studied in Table 4. The error is evaluated at points P1, P2 and P3 (see Fig. 3). It can be seen that the error estimate is exact.

<table>
<thead>
<tr>
<th>Point</th>
<th>Exact error (m)</th>
<th>Estimated error (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>$5.859375 \cdot 10^{-5}$</td>
<td>$5.859375 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>P2</td>
<td>$1.941667 \cdot 10^{-4}$</td>
<td>$1.941667 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>P3</td>
<td>$5.859375 \cdot 10^{-5}$</td>
<td>$5.859375 \cdot 10^{-5}$</td>
</tr>
</tbody>
</table>

4.4.2. Results for Problem 2. In Problem 2 the residual is piecewise linear. Therefore, we may approximate the residual by Taylor series and make use of Eq. (45) where only the first two contributions are non zero. The global effectivities are shown in Fig. 15. It can be seen that the effectivities tend to the desired unit value as the number of element is increased.

In this case, although the effectivities are close to one, they are not exact due to the Taylor series approximation. The elemental error convergence measured in different norms is shown in Fig. 16.

Finally, Table 5 shows the pointwise error. Again, the error estimate is exact.

4.5. Analysis for Element 2. Owing to its central node, Element 2 is more precise than Element 1. This aspect makes the fine-scale Green’s function associated with Element 2 significantly different from that of Element 1. In this case, the fine-scale Green’s function of Fig. 17 has been calculated in a uniform coarse mesh of
Figure 15. Global effectivity for Problem 2 and Element 1.

Figure 16. Elemental error for Problem 2 and Element 1.


<table>
<thead>
<tr>
<th>Point</th>
<th>Exact error (m)</th>
<th>Estimated error (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>2.988281 · 10^{-4}</td>
<td>2.988281 · 10^{-4}</td>
</tr>
<tr>
<td>P2</td>
<td>5.208333 · 10^{-4}</td>
<td>5.208333 · 10^{-4}</td>
</tr>
<tr>
<td>P3</td>
<td>2.871094 · 10^{-4}</td>
<td>2.871094 · 10^{-4}</td>
</tr>
</tbody>
</table>

5 elements and a uniform fine mesh of 1000 elements. Fig. 17 depicts \( g'(x, y) \) for \( EI = 1 \) at different points of the central element. It is evaluated in the central element which is defined in the interval (0.4,0.6). It is appreciated that \( g'(x, y) \) is completely local in the element. Fig. 17 shows that the fine-scale Green’s function is different from zero at the central node. This means that the error can be different from zero at the central node, in spite of being zero at the inter-element nodes.

Figure 17. Euler-Bernoulli problem. Fine-scale Green’s functions for Element 2 and \( EI = 1 \).
In this element, the fine-scale space is a subset of bubble functions belonging to \( H^2_0(\Omega) \), due to the central node introducing a bubble function,

\[
S' \subset \bigoplus_{e=1, \ldots, n_{el}} H^2_0(\Omega^e)
\]

Whereas for Element 1 we are able to obtain the fine-scale Green’s functions via two different methods, for Element 2 we can only obtain \( g'(x, y) \) by means of the Hughes-Sangalli formulae. Again, as explained for Element 1, we can define the bubble functions and the intrinsic scales associated with the fine-scale Green’s function of Fig. 17. The bubble functions are necessarily obtained by numerical integration.

A relevant aspect in this element is that the bubble function of 0th-moment order, \( b^{E2}_0(x) \), is zero. Therefore, the intrinsic error scales of 0th-moment order are also zero. The following expressions show the bubble functions of higher order measured in different norms,

\[
\begin{align*}
\|b^{E2}_1(x)\|_{L^1(\Omega^e)} &= 4.339475 \cdot 10^{-5} \frac{h^6}{EI} \\
\|b^{E2}_1(x)\|_{L^2(\Omega^e)} &= 5.010625 \cdot 10^{-5} \frac{h^{2.5}}{EI} \\
\|b^{E2}_1(x)\|_{H^1(\Omega^e)} &= 3.327346 \cdot 10^{-1} \frac{h^{4.5}}{EI} \\
\|b^{E2}_2(x)\|_{L^1(\Omega^e)} &= 2.951922 \cdot 10^{-6} \frac{h^7}{EI} \\
\|b^{E2}_2(x)\|_{L^2(\Omega^e)} &= 3.449933 \cdot 10^{-6} \frac{h^{5.5}}{EI} \\
\|b^{E2}_2(x)\|_{H^1(\Omega^e)} &= 3.222477 \cdot 10^{-7} \frac{h^{5.5}}{EI}
\end{align*}
\]
In order to estimate the error it is enough to handle the 1st intrinsic error scales. In Fig. 4.5.2 Results for Problem 2.

Thus, the predicted error is the same as the exact one. As mentioned in Remark 3, since for Element 2 the 0th-order bubble function is zero, the error source due to the constant part of the residual, \( b_0^I(x)(EI\bar{u}(x) - \bar{q}(x)) \) vanishes (see Eq. (44) or (45)). It can be proved that \( g'(x,y) \) is \( L^2 \)-orthogonal to residuals belonging to \( P_{k-4} \). This idea was proved by Hughes and Sangalli in [27] for the transport equation (2nd-order ODE) where \( g'(x,y) \) is orthogonal to residuals belonging to \( P_{k-2} \)

Remark 4. Observe that the first order bubble functions for Element 1, \( b_1^E(x) \), and Element 2, \( b_1^E(x) \), are the same (see Figs. 12 and 18). Therefore, the advantage of Element 2 over Element 1 is that it gets rid of the 0th-order source error component.

4.5.1. Results for Problem 1. As mentioned in Remark 3, the intrinsic error scales of 0th-order moment are zero. Therefore, there is no error for constant loads, that is to say, the FEM solution equals the analytical solution.

4.5.2. Results for Problem 2. In this problem, the residual is piecewise linear. In order to estimate the error it is enough to handle the 1st-order moment of the intrinsic error scales. In Fig. 19, the efficiency index is depicted in the three analysed norms. It is observed that it is one irrespective of the number of the elements. Thus, the predicted error is the same as the exact one.

Figure 19. Global effectivity for Problem 2 and Element 2.
Fig. 20 expresses the evolution of the elemental error with the number of elements in the central element of the beam. It can be seen that the convergence rate is greater with Element 2 than with Element 1. As for Element 1, the convergence rates of Fig. 20 are governed by the norms of the bubble functions, see Eq. (53).

![Figure 20. Elemental error for Problem 2 and Element 2.](image)

The pointwise error is represented in Table 6. It can be observed that the error at point 2 is zero as expected and that the predicted error is once again exact.

<table>
<thead>
<tr>
<th>Point</th>
<th>Exact error (m)</th>
<th>Estimated error (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>$5.859376 \cdot 10^{-6}$</td>
<td>$5.859376 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>P2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>P3</td>
<td>$5.859376 \cdot 10^{-6}$</td>
<td>$5.859376 \cdot 10^{-6}$</td>
</tr>
</tbody>
</table>

5. Extension to multi-dimensional problems

Taking advantage of the theoretical formulation of this error estimator, the present methodology can be extended to multi-dimensional problems. In the case of pointwise error estimates, this extension would involve multi-dimensional Taylor series expansions, what gives rise to an increased number of bubble functions and more elaborate calculations. In the case of global and local (element) error estimates, the extension is straightforward [23].

One main difference between one-dimensional and multi-dimensional problems is that, in the former case, the error is usually nodally exact, whereas in the latter case, the error is not exact along element edges. Therefore, the challenge to estimate the error in multi-dimensional cases is that, beyond the local error, which has been explained in this paper, we might need to take into account the global (or pollution) error. This global error can be characterized by means of the jump operator which collects the inter-element residuals, as carried out in other manuscripts by Hauke et al. [16, 18]. Currently, we are working on this field.

6. Conclusions

This work extends previous results on a posteriori error estimation based on the variational multiscale theory to second and fourth order differential equations that can be solved using the Galerkin method. Two examples are considered, namely, the axially loaded beam and the Euler-Bernoulli beam, respectively. The theory yields exact error estimates for the pointwise error, and for local and global error.
norms. For high-order residuals, upper bounds for the local and global error norms have been derived.

The paper describes in detail how to compute numerically the corresponding fine-scale and element Green's functions for any element in 2nd and 4th-order ODE's irrespective of the element order. From them, residual-free bubbles, intrinsic error time-scales and their moments have been calculated for various element types. The procedure can be easily extended to other elements.

The numerical examples confirm the theoretical properties of the method, which under certain conditions provides exact pointwise, local and global error estimates. Furthermore, it has been shown that the present technology succeeds when other residual-based methods fail. Therefore, the proposed a posteriori error estimator is a feasible tool for error estimation. The fact that the error estimation is explicit implies very little computational cost. That is to say, the error estimation is carried out by post-processing the FEM solution without solving other differential equations as it happens with implicit error estimators.

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References


Appendix A. Projector calculation

Let \( \mathcal{P} \) be an linear projector such that

\[ \mathcal{P} : \mathcal{V} \rightarrow \mathcal{V}^H \]
where $V$ is an infinite space endowed with a scalar product $(\cdot, \cdot)$ defined on $V \times V$, and $V^H \subset V$ is a finite dimensional space. The superscript $H$ represents that $V^H$ is defined on a coarse mesh.

In order to compute numerically $P$, we also define a fine mesh with a mesh size $h$ associated with the finite space $V^h \subset V$. It is supposed the element size of $V^H$ is much greater than one related to $V^h$, i.e., $h \ll H$. Therefore, let $P : V^h \rightarrow V^H$ be the projection matrix associated with the projector $P$.

Let $v^H$ be the basis functions belonging to $V^H$ and let $v^h$ be the basis functions belonging to $V^h$. The matricial projector $P$ can be computed by the following process:

i. The Galerkin-projection of a generic vector $w^h \in V^h$ into $V^H$ is given by

\[
\int_\Omega L(w^H)v^Hd\Omega_y = \int_\Omega L(w^h)v^Hd\Omega_y \quad \forall v^H \in V^H
\]

where $L$ is the corresponding differential operator. $w^H$ can be expressed as $w^H = \sum_{i=1}^{N} \phi^N_i c^N_i$, where $\phi^N_i$ are the basis functions of the coarse mesh and $c^N_i$ are the coordinates of $w^H$ for $i = 1, ..., N$, being $N$ the degrees of freedom in the coarse mesh. Similarly, $w^h$ can be stated as $w^h = \sum_{j=1}^{n} \phi^h_j c^h_j$, where $\phi^h_j$ are the basis functions of the coarse mesh and $c^h_j$ are the coordinates of $w^h$ for $j = 1, ..., n$, being $n$ the degrees of freedom in the fine mesh.

ii. Eq. (A.1) might be formulated as

\[
M^N c^N = M^n c^n
\]

The matrix $M^N_{i,j}$ is defined as $M^N_{i,j} = \int_\Omega L \phi^N_i \phi^N_j d\Omega$ for $i, j = 1, ..., N$. Likewise, $M^n_{i,j}$ is established as $M^n_{i,j} = \int_\Omega L \phi^h_i \phi^h_j d\Omega$ for $i = 1, ..., N$ and $j = 1, ..., n$.

iii. Therefore, the coordinates $c^N$ of the vector $w^h$ can be solved as

\[
c^N = (M^N)^{-1} M^n c^n
\]

iv. Finally, the matrix $P$ associated with the projector $P$ is

\[
P = (M^N)^{-1} M^n.
\]