A PRIORI ERROR ESTIMATES FOR FINITE VOLUME ELEMENT APPROXIMATIONS TO SECOND ORDER LINEAR HYPERBOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, both semidiscrete and completely discrete finite volume element methods (FVEMs) are analyzed for approximating solutions of a class of linear hyperbolic integro-differential equations in a two-dimensional convex polygonal domain. The effect of numerical quadrature is also examined. In the semidiscrete case, optimal error estimates in $L^\infty(L^2)$ and $L^\infty(H^1)$ norms are shown to hold with minimal regularity assumptions on the initial data, whereas quasi-optimal estimate is derived in $L^\infty(L^\infty)$ norm under higher regularity on the data. Based on a second order explicit method in time, a completely discrete scheme is examined and optimal error estimates are established with a mild condition on the space and time discretizing parameters. Finally, some numerical experiments are conducted which confirm the theoretical order of convergence.

Key words. Finite volume element, hyperbolic integro-differential equation, semidiscrete method, numerical quadrature, Ritz-Volterra projection, completely discrete scheme, optimal error estimates.

1. Introduction

In this paper, we discuss and analyze a finite volume element method for approximating solutions to the following class of second order linear hyperbolic integro-differential equations:

$$
\begin{aligned}
u_{tt} - \nabla \cdot \left(\mathcal{A}(x)\nabla u + \int_0^t \mathcal{B}(x,t,s)\nabla u(s)\,ds \right) &= f(x,t) &&\text{in } \Omega \times J, \\
u(x,t) &= 0 &&\text{on } \partial\Omega \times J, \\
u(x,0) &= u_0(x) &&\text{in } \Omega, \\
u_t(x,0) &= u_1(x) &&\text{in } \Omega,
\end{aligned}
$$
(1)

with given functions u_0 and u_1, where $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain, $J = (0,T]$, $T < \infty$, $u_{tt} = \partial^2 u / \partial t^2$ and f is a given function defined on the space-time domain $\Omega \times J$. Here, $\mathcal{A} = [a_{ij}(x)]$ and $\mathcal{B} = [b_{ij}(x,t,s)]$ are 2×2 matrices with smooth coefficients. Further, assume that $\mathcal{A}$ is symmetric and uniformly positive definite in $\bar{\Omega}$. Problems of this kind arise in linear viscoelastic models, specially in the modelling of viscoelastic materials with memory (cf. Renardy *et al.* [23]).

Earlier, the finite volume difference methods which are based on cell centered grids and approximating the derivatives by difference quotients have been proposed and analyzed, see [15] for a survey. Another approach, which we shall follow in this article was formulated in the framework of Petrov-Galerkin finite element method

using two different grids to define the trial space and test space. This is popularly
known finite volume element methods (FVEMs). Here and also in literatures, the
trial space consists of C^0-piecewise linear polynomials on the finite element partition
$\mathcal{T}_h$ of $\overline{\Omega}$ and the test space is piecewise constants over the control volume $\mathcal{T}_h^*$ to be
defined in Section 2. Earlier, the FVEM has been examined by Bank and Rose [3],
Cai [4], Chatzipantelidis [8], Li *et al.* [17], Ewing *et al.* [12], etc. for elliptic problems,
for parabolic and parabolic type problems by Chou *et al.* [7], Chatzipantelidis *et
al.* [9], Ewing *et al.* [13], Sinha *et al.* [25] and for second order wave equations by
Kumar *et al.* [16]. For a recent survey on FVEM, see, a review article by Lin *et
al.* [19].

For linear elliptic problems, Li *et al.* [17] have established optimal error estimates
in H^1 and L^2 norms. More precisely, for L^2 norm the following estimate is derived:

$$\|u - u_h\|_0 \leq Ch^2 \|u\|_{W^{3,p}(\Omega)}, \ p > 1,$$

where u is the exact solution and u_h is the finite volume element approximation of
u. Compared to the error analysis of finite element methods, it is observed that
this method is optimal in approximation property, but is not optimal with respect
to the regularity of the exact solution as for $O(h^2)$ order convergence, the exact
solution $u \in H^3$. For convex polygonal domain Ω, it may be difficult to prove
H^3 regularity for the solution u. Therefore, an attempt has been made in [12] to
establish optimal L^2 error estimate under the assumption that the exact solution
$u \in H^2$ and the source term $f \in H^1$. A counter example has also been provided
in [12] to show that if $f \in L^2$, then FVE solution may not have optimal error
estimates in L^2 norm. The analysis has been extended to parabolic problems in
convex polygonal domain in [9] and optimal error estimates have been derived under
some compatibility conditions on the initial data. Further, the effect of quadrature,
that is, when the L^2 inner product is replaced by numerical quadrature has been
analyzed. Subsequently, Ewing *et al.* [13] have employed FVEM for approximat-
ing solutions of parabolic integro-differential equations and derived optimal error
estimates under $L^\infty(H^3)$ regularity for the exact solution and $L^2(H^3)$ regularity
for its time derivative. Then on convex polygonal domain, Sinha *et al.* [25] have
examined semidiscrete FVEM and proved optimal error estimates for smooth and
non smooth data. The analysis is further generalized to a second order linear wave
equation defined on a convex polygonal domain and *a priori* error estimates have
been established only for semidiscrete case, see, Kumar *et al.* [16]. Further, the
effect of quadrature and maximum norm estimates are proved under some addi-
tional conditions on the initial data and the forcing function. In the present article,
an attempt has been made to extend the analysis of FVEM to a class of second
order linear hyperbolic integro-differential equations in convex polygonal domains
with minimal regularity assumptions on the initial data. Moreover, a completely
discrete scheme based on a second order explicit method has been analyzed.

In order to put the present investigation into a proper perspective visa-vis earlier
results, we discuss, below, the literature for the second order hyperbolic equations.
Li et al. [17] have proved an optimal order of convergence in H^1 norm without
quadrature using elliptic projection, but the regularity of the exact solution assumed
to be higher than the regularity assumed in our results even when $B = 0$ for the
problem (1). On a related finite element analysis for the second order hyperbolic
equations without quadrature, we refer to Baker [1] and with quadrature, see,
Baker and Dougalis [2] and Dupont [11]. Baker and Dougalis [2] have proved
optimal order of convergence in $L^\infty(L^2)$ for the semidiscrete finite element scheme,
provided the initial displacement $u_0 \in H^5 \cap H_0^1$ and the initial velocity $u_1 \in H^4 \cap H_0^1$.

Subsequently, Rauch [22] has derived the convergence analysis for the Galerkin finite element methods when applied to a second order wave equation by using piecewise linear polynomials and established optimal $L^\infty(L^2)$ estimate with $u_0 \in H^3 \cap H_0^1$ and $u_1 = 0$ which are turned out to be the minimal regularity conditions for the second order wave equation. Subsequently, Pani *et al.* [26] have examined the effect of numerical quadrature on finite element method for hyperbolic integro-differential equations with minimal regularity assumptions on the initial data, that is, $u_0 \in H^3 \cap H_0^1$ and $u_1 \in H^2 \cap H_0^1$. On a related article on a linear second order wave equation, we refer to Sinha [24] and on hyperbolic PIDE, see, [6]. When FVEM is combined with quadrature for approximating solution of (1), we have, in this article, proved optimal $L^\infty(L^2)$ estimate with minimal regularity assumptions on the initial data.

The organization of the present paper is as follows: Section 2 deals with some notations, weak formulation and the regularity results for the exact solution. Section 3 is devoted to the primary and dual meshes for finite volume element method and semidiscrete FVE approximation to the problem (1). Section 4 focuses on *a priori* error estimates for the semidiscrete FVE approximations and optimal order of convergence in L^2 and H^1 norms are established under minimal regularity assumptions on the initial data. Further, quasi-optimal order of convergence in maximum norm has also been derived. Section 5 is on completely discrete scheme which is based on a second order explicit scheme in time and *a priori* error estimates are established. Section 6 deals with the effect of numerical quadrature and the related error estimates are derived again with minimal regularity assumption on the initial data. Finally in Section 7, some numerical experiments are conducted which confirm our theoretical order of convergence.

Through out this paper, C is a generic positive constant independent of discretising parameters h and k.

2. Notation and Preliminaries.

This section is devoted to some notations and preliminary results related to the weak solution of (1).

Let $W^{m,p}(\Omega)$ denote the standard Sobolev space with the norm

$$\|u\|_{m,p,\Omega} = \left(\sum_{|\alpha| \le m} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p} \quad \text{for } 1 \le p < \infty,$$

and for $p = \infty$,

$$\|u\|_{m,\infty,\Omega} = \sup_{|\alpha| \le m} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

When there is no confusion, we denote $\|u\|_{m,p,\Omega}$ by $\|u\|_{m,p}$. For $p = 2$, we simply write $W^{m,2}(\Omega)$ as $H^m(\Omega)$ and denote its norm by $\| \cdot \|_m$. For a Banach space X with norm $\| \cdot \|_X$ and $1 \le p \le \infty$, let $W^{m,p}(0,T;X)$ be defined by

$$W^{m,p}(0,T;X) := \{v : (0,T) \longrightarrow X | \|D_t^j v\|_X \in L^p(0,T), \ 0 \le j \le m\}.$$

with its norm

$$\|v\|_{Wm,p(0,T;X)} = \|u\|_{W^{m,p}(X)} := \sum_{j=0}^m \left(\int_0^T \|D_t^j v\|_X^p \, dt \right)^{1/p},$$

with the standard modification for $p = \infty$, see [14]. For $m = 0$, $W^{m,p}(0, T; X)$ is simply the space $L^p(X)$. Finally, let $(\cdot, \cdot)$ and $\| \cdot \|_0$ denote, respectively, the L^2 inner product and its induced norm on $L^2(\Omega)$.

With $H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$, define the bilinear forms $A(\cdot, \cdot)$ and $B(\cdot, \cdot) = B(t, s; \cdot, \cdot)$ on $H_0^1(\Omega) \times H_0^1(\Omega)$ by

$$A(u, v) = \int_\Omega \mathcal{A}(x)\nabla u \cdot \nabla v \, dx,$$

and

$$B(t, s; u(s), v) = \int_\Omega \mathcal{B}(x, t, s)\nabla u(s) \cdot \nabla v \, dx.$$

Then, the weak formulation for (1) is to seek $u : (0, T] \longrightarrow H_0^1(\Omega)$ such that

$$(2) \qquad (u_{tt}, v) + A(u, v) + \int_0^t B(t, s; u(s), v) \, ds = (f, v) \quad \forall v \in H_0^1(\Omega)$$

with $u(0) = u_0$ and $u_t(0) = u_1$.

Since $\mathcal{A}$ is symmetric and uniformly positive definite in Ω, the bilinear form $A(\cdot, \cdot)$ satisfies the following condition: there exist positive constants α and Λ with $\Lambda \geq \alpha$ such that

$$(3) \qquad \Lambda\|v\|_1^2 \geq A(v, v) \geq \alpha\|v\|_1^2 \quad \forall v \in H_0^1(\Omega).$$

For our subsequent use, we state without proof *a priori* estimates of the solution u of the problem (1) under appropriate regularity conditions and compatibility conditions on u_0, u_1 and f. Its proof can be easily obtained by appropriately modified arguments in the proof of Theorem 3.1 of [26]. For similar estimates for second order linear hyperbolic equations, see Lemma 2.1 of [16].

Lemma 2.1. *Let u be a weak solution of* (1). *Then, there is a positive constant $C = C(T)$ such that the following estimates*

$$\|D_t^{j+2}u(t)\|_0 + \|D_t^{j+1}u(t)\|_1 + \|D_t^j u(t)\|_2 \leq C\Big(\|u_0\|_{j+2} + \|u_1\|_{j+1}$$

$$+ \sum_{k=0}^j \|D_t^k f\|_{L^1(H^{j-k})} + \|D_t^{j+1}f\|_{L^1(L^2)}\Big),$$

hold for $j = 0, 1, 2$, where $D_t^j = (\partial^j/\partial t^j)$.

We shall have occasion to use the following identity for $\phi \in C^1([0, T]; X)$, where X is a Banach space

$$(4) \qquad \phi(t) = \phi(0) + \int_0^t \phi_t(s) \, ds.$$

3. Finite Volume Element Method

This section deals with primary and dual meshes on the domain Ω, construction of finite dimensional spaces, finite volume element formulation and some preliminary results.

Let $\mathcal{T}_h$ be a family of regular triangulations of the closed, convex polygonal domain $\overline{\Omega}$ into closed triangles K, and let $h = \max_{K \in \mathcal{T}_h}(\text{diam}K)$, where h_K denotes the diameter of K. Let N_h be set of nodes or vertices, that is, $N_h := \{P_i : P_i \text{ is a vertex of the element } K \in \mathcal{T}_h \text{ and } P_i \in \overline{\Omega}\}$ and let N_h^0 be the set of interior nodes in $\mathcal{T}_h$ with cardinality N. Further, let $\mathcal{T}_h^*$ be the dual mesh associated with the primary mesh $\mathcal{T}_h$, which is defined as follows. With P_0 as an interior node of the triangulation $\mathcal{T}_h$, let P_i $(i = 1, 2 \cdots m)$ be its adjacent nodes
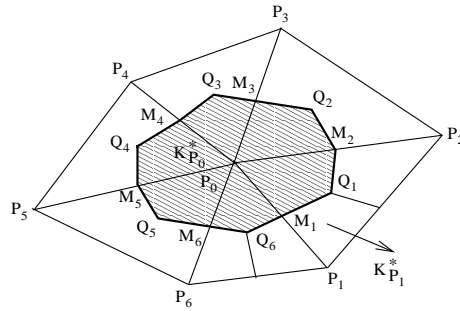
FIGURE 1

(see, FIGURE 1 with $m = 6$). Let M_i, $i = 1, 2 \cdots m$ denote the midpoints of $\overline{P_0 P_i}$ and let Q_i, $i = 1, 2 \cdots m$, be the barycenters of the triangle $\triangle P_0 P_i P_{i+1}$ with $P_{m+1} = P_1$. The *control volume* $K_{P_0}^*$ is constructed by joining successively M_1, $Q_1, \cdots$, M_m, Q_m, M_1. With Q_i ($i = 1, 2 \cdots m$) as the nodes of *control volume* $K_{p_i}^*$, let N_h^* be the set of all dual nodes Q_i. For a boundary node P_1, the control volume $K_{P_1}^*$ is shown in the FIGURE 1. Note that the union of the control volumes forms a partition $\mathcal{T}_h^*$ of $\overline{\Omega}$.

Assume that the partitions $\mathcal{T}_h$ and $\mathcal{T}_h^*$ are quasi-uniform in the sense that there exist positive constants C_1 and C_2 independent of h such that

$$(5) \qquad\qquad C_1\, h^2 \leq |K_{Q_i}| \leq C_2\, h^2 \ \forall Q_i \in N_h^*,$$

$$(6) \qquad\qquad C_1\, h^2 \leq |K_{P_i}^*| \leq C_2\, h^2 \ \forall P_i \in N_h,$$

where $|K| = \text{ meas } (K)$.

We consider a finite volume element discretization of (1) in the standard C^0-conforming piecewise linear finite element space U_h on the primary mesh $\mathcal{T}_h$, which is defined by

$$U_h = \{v_h \in C^0(\overline{\Omega}) \ : \ v_h|_K \text{ is linear for all } K \in \mathcal{T}_h \text{ and } v_h|_{\partial\Omega} = 0\},$$

and the dual volume element space U_h^* on the dual mesh $\mathcal{T}_h^*$ given by

$$U_h^* = \{v_h \in L^2(\Omega) \ : \ v_h|_{K_{P_0}^*} \text{ is constant for all } K_{P_0}^* \in \mathcal{T}_h^* \text{ and } v_h|_{\partial\Omega} = 0\}.$$

Now, $U_h = \text{span}\{\phi_i \ : \ P_i \in N_h^0\}$ and $U_h^* = \text{span}\{\chi_i \ : \ P_i \in N_h^0\}$, where ϕ_i's are the standard nodal basis functions associated with nodes P_i and χ_i's are the characteristic basis functions corresponding to the control volume $K_{P_i}^*$ given by

$$\chi_i(x) = \begin{cases} 1, & \text{if } x \in K_{P_i}^* \\ 0, & \text{elsewhere.} \end{cases}$$

The semidiscrete finite volume element formulation for (1) is to seek $u_h : (0, T] \longrightarrow U_h$ such that

$$(7) \quad (u_{h,tt}, v_h) + A_h(u_h, v_h) + \int_0^t B_h(t, s; u_h(s), v_h)\, ds = (f, v_h) \ \ \forall v_h \in U_h^*,$$

with given initial data $u_h(0)$ and $u_{h,t}(0)$ in U_h to be defined later. Here, the bilinear forms $A_h(\cdot, \cdot)$ and $B_h(t, s; \cdot, \cdot)$ are defined, respectively, by

$$A_h(u_h, v_h) = - \sum_{P_i \in N_h^0} v_h(P_i) \int_{\partial K_{P_i}^*} \mathcal{A}(x)\nabla u_h \cdot \mathbf{n}\, ds,$$

and

$$B_h(t, s; u_h, v_h) = - \sum_{P_i \in N_h^0} v_h(P_i) \int_{\partial K_{P_i}^*} \mathcal{B}(x, t, s) \nabla u_h \cdot \mathbf{n} \, ds$$

for all $(u_h, v_h) \in U_h \times U_h^*$, with $\mathbf{n}$ denoting the outward unit normal to the boundary of the control volume $K_{P_i}^*$. Notice that by taking the L^2 inner product of (1) with $v_h \in U_h^*$ and then integrating, we obtain a similar equation for u as

$$(8) \qquad (u_{tt}, v_h) + A_h(u, v_h) + \int_0^t B_h(t, s; u(s), v_h) \, ds = (f, v_h) \quad \forall v_h \in U_h^*.$$

For the error analysis, we first introduce two interpolation operators. Let $\Pi_h : C(\Omega) \longrightarrow U_h$ be the piecewise linear interpolation operator and $\Pi_h^* : C(\Omega) \longrightarrow U_h^*$ be the piecewise constant interpolation operator. These interpolation operators are defined, respectively, by

$$(9) \qquad \Pi_h u = \sum_{P_i \in N_h^0} u(P_i) \phi_i(x) \text{ and } \Pi_h^* u = \sum_{P_i \in N_h^0} u(P_i) \chi_i(x).$$

Now for $\psi \in H^2$, Π_h has the following approximation property, (see, Ciarlet [10]):

$$(10) \qquad \|\psi - \Pi_h \psi\|_0 \le C h^2 \|\psi\|_2.$$

Further, we introduce the following discrete norms

$$\|v_h\|_{0,h} = \left(\sum_{K \in T_h} |v_h|_{0,h,K}^2 \right)^{1/2} \text{ and } \|v_h\|_{1,h} = \left(\|v_h\|_{0,h}^2 + |v_h|_{1,h}^2 \right)^{1/2},$$

where the seminorm $|v_h|_{1,h} = \left(\sum_{K \in T_h} |v_h|_{1,h,K}^2 \right)^{1/2}$, and for $K = K_Q = \triangle P_1 P_2 P_3$,

$$|v_h|_{0,h,K} = \left\{ \frac{1}{3} \left(v_h^2(P_1) + v_h^2(P_2) + v_h^2(P_3) \right) |K| \right\}^{1/2}$$

$$|v_h|_{1,h,K} = \left\{ \left(\left| \frac{\partial v_h}{\partial x} \right|^2 + \left| \frac{\partial v_h}{\partial y} \right|^2 \right) |K| \right\}^{1/2}.$$

In the following Lemma, a relation between discrete norms and standard Sobolev norms is stated without proof. For a proof, see, [17, pp. 124] and [4].

Lemma 3.1. *For $v_h \in U_h$, $|\cdot|_{1,h}$ and $|\cdot|_1$ are identical; $\|\cdot\|_{0,h}$ and $\|\cdot\|_{1,h}$ are equivalent to $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively, that is, there exist positive constants C_3 and $C_4 > 0$, independent of h, such that*

$$(11) \qquad C_3 \|v_h\|_{0,h} \le \|v_h\|_0 \le C_4 \|v_h\|_{0,h} \quad \forall v_h \in U_h$$

and

$$(12) \qquad C_3 \|v_h\|_{1,h} \le \|v_h\|_1 \le C_4 \|v_h\|_{1,h} \quad \forall v_h \in U_h.$$

Note that $\|v_h\|_{0,h} = \|\Pi_h^* v_h\|_0$. Below, we state without proof the properties of the interpolation operator Π_h^*. For a proof, we refer to [17, pp. 192].

Lemma 3.2. *The following statements hold true.*
(i) For $\Pi_h^* : U_h \longrightarrow U_h^*$ defined in (9),

$$(13) \qquad (\phi_h, \Pi_h^* v_h) = (v_h, \Pi_h^* \phi_h) \quad \forall \phi_h, v_h \in U_h.$$

(ii) With $\|\|\phi_h\|\| := (\phi_h, \Pi_h^* \phi_h)^{1/2}$, the norms $\|\|\cdot\|\|$ and $\|\cdot\|_0$ are equivalent on U_h, that is, there exist positive constants c_{eq} and C_{eq}, independent of h, such that

$$(14) \qquad c_{eq} \|\phi_h\|_0 \le \|\|\phi_h\|\| \le C_{eq} \|\phi_h\|_0 \quad \forall \phi_h \in U_h.$$

4. A Priori Error Estimates

This section is devoted to *a priori* error estimates of the approximation u_h to the spatial semidiscrete scheme (7).

For the derivation of optimal error estimates, we split $e := u - u_h$ as

$$e = (u - V_h u) + (V_h u - u_h) =: \rho + \theta,$$

where $V_h : L^\infty(H_0^1 \cap H^2) \to L^\infty(U_h)$ is the Ritz-Volterra projection defined by

$$(15) \qquad A(u - V_h u, \chi_h) + \int_0^t B(t, s; u - V_h u, \chi_h) \, ds = 0 \quad \forall \chi_h \in U_h.$$

With some abuse of notations, we will denote by $V_h u_0$ the Ritz projection of u_0 onto U_h defined by

$$A(u_0 - V_h u_0, \chi_h) = 0 \quad \forall \chi_h \in U_h.$$

For our subsequent analysis, we state without proof following error estimates for the Ritz-Volterra projection. For a proof, see, [26], [5], [18], [20] and [21].

Lemma 4.1. *There exist positive constants C, independent of h, such that for $j = 0, 1, 2$, and $r = 1, 2$ the following estimates hold:*

$$(16) \qquad \|D_t^j \rho(t)\|_0 + h\|D_t^j \rho(t)\|_1 \le Ch^r \left[\sum_{l=0}^j \|D_t^l u(t)\|_r + \int_0^t \|u(s)\|_r \, ds \right],$$

and

$$(17) \qquad \|\rho(t)\|_{0,\infty} \le Ch^2 \left(\log \frac{1}{h} \right) \left(\|u(t)\|_{2,\infty} + \int_0^t \|u(s)\|_{2,\infty} ds \right).$$

Now, define

$$\epsilon_h(f, \chi) = (f, \chi) - (f, \Pi_h^* \chi) \quad \forall \chi \in U_h,$$
$$\epsilon_A(\psi, \chi) = A(\psi, \chi) - A_h(\psi, \Pi_h^* \chi) \quad \forall \psi, \chi \in U_h,$$

and

$$\epsilon_B(t, s; \psi, \chi) = B(t, s; \psi, \chi) - B_h(t, s; \psi, \Pi_h^* \chi) \quad \forall \psi, \chi \in U_h.$$

Then, the following lemma will be of frequent use in our analysis and the proof of which can be found in [8].

Lemma 4.2. *Assume that the coefficient matrices $\mathcal{A}, \mathcal{B}(t, s) \in W^{1+i,\infty}(\Omega; \mathbb{R}^{2 \times 2})$ for $i = 0, 1$. Then, there exist positive constant C, independent of h, such that the following estimates hold for $\chi \in U_h$ and for $i, j = 0, 1$*

$$(18) \qquad |\epsilon_h(f, \chi)| \le Ch^{i+j} \|f\|_{H^i} \|\chi\|_{H^j} \quad \forall f \in H^i,$$

and for $u \in H^{1+i} \cap H_0^1$

$$(19) \qquad |\epsilon_A(V_h u, \chi)| \le Ch^{i+j} \left(\|u\|_{H^{1+i}} + \int_0^t \|u(s)\|_{H^{1+i}} \, ds \right) \|\chi\|_{H^j}.$$

Moreover,

$$(20) \qquad |\epsilon_A(w_h, \chi)| \le Ch \|w_h\|_{H^1} \|\chi\|_{H^1} \, \forall w_h \in U_h.$$

The estimates (19) and (20) are also valid if ϵ_A is replaced by ϵ_B.

Now, for $\psi \in H_0^1$ and for each $t \in (0, T]$, introduce a linear functional $G(\psi) = G(t, \psi)$ defined on U_h by

$$G(\psi)(\chi) = \epsilon_A(\psi, \chi) + \int_0^t \epsilon_B(t, s; \psi(s), \chi) \, ds, \quad \chi \in U_h.$$

Notice that, by using the definition of G, (2) and (8), there follows that

$$
\begin{aligned}
G(\rho)(\chi) \;=\;& A(u,\chi) + \int_0^t B(t,s;u(s),\chi)\,ds \\
& - A_h(u,\Pi_h^*\chi) - \int_0^t B_h(t,s;u(s),\Pi_h^*\chi)\,ds - G(V_h u)(\chi) \\
=\;& (f - u_{tt},\chi) - (f - u_{tt},\Pi_h^*\chi) - G(V_h u)(\chi)
\end{aligned}
$$

$$
(21) \qquad\qquad =\; \epsilon_h(f - u_{tt},\chi) - G(V_h u)(\chi).
$$

From (7) and (8), we obtain the equation in θ for $v_h \in U_h^*$ as

$$
(\theta_{tt},v_h) + A_h(\theta,v_h) + \int_0^t B_h(t,s;\theta(s),v_h)\,ds
$$

$$
= -A_h(\rho,v_h) - \int_0^t B_h(t,s;\rho,\chi)\,ds - (\rho_{tt},v_h).
$$

Choosing $v_h = \Pi_h^*\chi$ and using the definition of G and (15), we find that

$$
(\theta_{tt},\Pi_h^*\chi) + A(\theta,\chi)\,ds \;\;+\;\; \int_0^t B(t,s;\theta(s),\chi)\,ds = G(\rho)(\chi)
$$

$$
(22) \qquad\qquad\qquad\qquad +\;\; G(\theta)(\chi) - (\rho_{tt},\Pi_h^*\chi) \;\; \forall \chi \in U_h.
$$

For any continuous function ϕ in $[0,t]$, define $\hat\phi$ by

$$
\hat\phi(t) = \int_0^t \phi(s)\,ds.
$$

Notice that $\hat\phi(0) = 0$ and $(d\hat\phi/dt)(t) = \phi(t)$. Then, integrate (22) from 0 to t to obtain

$$
\begin{aligned}
(\theta_t,\Pi_h^*\chi) + A(\hat\theta,\chi) \;=\;& \hat{G}(\rho)(\chi) + \hat{G}(\theta)(\chi) + (-\rho_t,\Pi_h^*\chi) + (e_t(0),\Pi_h^*\chi) \\
(23) \qquad & - \int_0^t B(s,s;\hat\theta(s),\chi)\,ds + \int_0^t\int_0^s B_\tau(s,\tau;\hat\theta(\tau),\chi)\,d\tau\,ds,
\end{aligned}
$$

where

$$
\hat{G}(\phi)(\chi) = \epsilon_A(\hat\phi,\chi) + \int_0^t \epsilon_B(s,s;\hat\phi(s),\chi)\,ds - \int_0^t\int_0^s \epsilon_{B_\tau}(s,\tau;\hat\phi(\tau),\chi)\,d\tau\,ds.
$$

For a linear functional F defined on U_h, set

$$
\|F\|_{-1,h} = \sup_{0 \neq \chi \in U_h} \frac{|F(\chi)|}{\|\chi\|_1}.
$$

We shall need the following lemmas in our subsequent analysis.

Lemma 4.3. *With G and $\hat{G}$ as above, there exists a positive constant $C = C(T)$ such that the following estimates*

$$
(24) \qquad \|D_t^j G(V_h u)\|_{-1,h} \leq Ch^2 \left(\sum_{\ell=0}^j \|D_t^\ell u(t)\|_2 + \int_0^t \|u(s)\|_2\,ds \right),
$$

and

$$
(25) \qquad \|D_t^j \hat{G}(V_h u)\|_{-1,h} \leq Ch^2 \left(\sum_{\ell=0}^j \|D_t^j \hat{u}(t)\|_2 + \int_0^t \|\hat{u}(s)\|_2\,ds \right),
$$

hold for $j = 0,1$.

Proof. Using (19) and the estimates in Lemma 2.1, we obtain

$$
\begin{aligned}
|G(V_h u)(\chi)| &\le |\epsilon_A(V_h u, \chi)| + \int_0^t |\epsilon_B(t, s; V_h u(s), \chi)| \, ds \\
&\le Ch^2 \left(\|u\|_2 + \int_0^t \|u(s)\|_2 \, ds \right) \|\chi\|_1,
\end{aligned}
$$

and

$$
\begin{aligned}
|G_t(V_h u)(\chi)| &\le Ch^2 \left(\|u_t\|_2 + \|u\|_2 + \int_0^t \|u(s)\|_2 \, ds \right) \|\chi\|_1 \\
&\le Ch^2 \left(\|u_t\|_2 + \|u\|_2 + \int_0^t \|u(s)\|_2 \, ds \right) \|\chi\|_1
\end{aligned}
$$

In a similar manner, we derive the second estimate (25) and this completes the rest of the proof. ∎

In the error analysis, we shall frequently use the following inverse assumption:

$$
(26) \qquad \|\chi\|_1 \le C_{inv} h^{-1} \|\chi\|_0, \quad \chi \in U_h.
$$

4.1. H^1 error estimate.

Theorem 4.1. *Let u and u_h be the solutions of (1) and (7), respectively, and assume that $f \in L^1(H^1)$, f_t, $f_{tt} \in L^1(L^2)$, $u_0 \in H^3 \cap H_0^1$ and $u_1 \in H^2 \cap H_0^1$. Further, assume that $u_h(0) = \Pi_h u_0$ and $u_{h,t}(0) = \Pi_h u_1$, where Π_h is the interpolation operator defined in (9). Then, there exists a positive constant $C = C(T)$, independent of h, such that for $t \in (0, T]$ the following estimate*

$$
\|u(t) - u_h(t)\|_1 \le C h \left(\|u_0\|_3 + \|u_1\|_2 + \int_0^t \left(\|f\|_1 + \|f_t\|_0 + \|f_{tt}\|_0 \right) ds \right)
$$

holds.

Proof. Since $u - u_h = \rho + \theta$ and estimates of ρ are known from the Lemma 4.1, it is sufficient to estimate θ. Choose $\chi = \theta_t$ in (22) and use (21) to obtain

$$
\begin{aligned}
(\theta_{tt}, \Pi_h^* \theta_t) + A(\theta, \theta_t) + \int_0^t B(t, s; \theta(s), \theta_t) \, ds &= \epsilon_h(f - u_{tt}, \theta_t) - G(V_h u)(\theta_t) \\
&\quad + G(\theta)(\theta_t) - (\rho_{tt}, \Pi_h^* \theta_t).
\end{aligned}
$$

Now use (13) and the symmetry of the bilinear form $A(\cdot, \cdot)$ to arrive at

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left[(\theta_t, \Pi_h^* \theta_t) + A(\theta, \theta) \right] &= \epsilon_h(f - u_{tt}, \theta_t) - G(V_h u)(\theta_t) + G(\theta)(\theta_t) - (\rho_{tt}, \Pi_h^* \theta_t) \\
&\quad - \int_0^t B(t, s; \theta(s), \theta_t(t)) \, ds.
\end{aligned}
$$

Integration from 0 to t yields

$$
\begin{aligned}
\frac{1}{2}\left(\||\theta_t\||^2 + A(\theta, \theta) \right) &= \frac{1}{2}\left(\||\theta_t(0)\||^2 + A(\theta(0), \theta(0)) \right) + \int_0^t \epsilon_h(f - u_{tt}, \theta_t) ds \\
&\quad - \int_0^t G(V_h u)(\theta_t) ds + \int_0^t G(\theta)(\theta_t) ds + \int_0^t (-\rho_{tt}, \Pi_h^* \theta_t) ds \\
&\quad - \int_0^t \int_0^s B(s, \tau; \theta(\tau), \theta_t(s)) \, d\tau ds \\
(27) \qquad &= J_1 + J_2 + J_3 + J_4 + J_5 + J_6.
\end{aligned}
$$

For the first term on the right hand side of (27), a use of the boundedness of $A(\cdot, \cdot)$ with (10) and (14) shows

(28) $$|J_1| \leq C \left(\|\theta_t(0)\|_0^2 + \|\theta(0)\|_1^2 \right) \leq Ch^2 (\|u_1\|_1^2 + \|u_0\|_2^2).$$

For estimating J_2, an application of (18) with $j = 0$ implies

(29) $$|J_2| \leq Ch \int_0^t (\|f\|_1 + \|u_{tt}\|_1) \|\theta_t\|_0 \, ds.$$

To estimate J_3, a use of the inverse inequality (26) shows that

(30) $$|J_3| \leq C \int_0^t \|G(V_h u)\|_{-1,h} \|\theta_t\|_1 \, ds \leq Ch^{-1} \int_0^t \|G(V_h u)\|_{-1,h} \|\theta_t\|_0 \, ds.$$

Using the definition of G, (20) and the inverse inequality, it follows that

$$
\begin{aligned}
|J_4| &\leq \int_0^t |G(\theta)(\theta_t)| \, ds \\
&\leq Ch \left[\int_0^t \|\theta\|_1 \|\theta_t\|_1 ds + \int_0^t \int_0^s \|\theta(\tau)\|_1 \|\theta_t(s)\|_1 d\tau ds \right] \\
&\leq C \left[\int_0^t \|\theta\|_1 \|\theta_t\|_0 ds + \left(\int_0^t \|\theta\|_1 ds \right) \left(\int_0^t \|\theta_t\|_0 ds \right) \right].
\end{aligned}
$$
(31)

For J_5, apply the Cauchy-Schwarz inequality, L^2 stability of Π_h^* and (16) with $r = 1$ to obtain

(32) $$|J_5| \leq \int_0^t \|\rho_{tt}\|_0 \|\theta_t\|_0 \, ds \leq C(T) \, h \int_0^t \left(\|u_{tt}\|_1 + \|u_t\|_1 + \|u\|_1 \right) \|\theta_t\|_0 \, ds.$$

For the term J_6, we note that an integration by parts yields

$$
\begin{aligned}
\int_0^t \int_0^s B(s, \tau; \theta(\tau), \theta_t(s)) \, d\tau \, ds &= \int_0^t B(t, s; \theta(s), \theta(t)) \, ds - \int_0^t B(s, s; \theta(s), \theta(s)) \\
&\quad - \int_0^t \int_0^s B_s(s, \tau; \theta(\tau), \theta(s)) \, d\tau \, ds,
\end{aligned}
$$

and hence, deduce that

(33) $$|J_6| \leq C \left(\|\theta(t))\|_1 \int_0^t \|\theta(s)\|_1 \, ds + \int_0^t \|\theta(s)\|_1^2 \, ds \right).$$

Now, set $\mathcal{E}_1^2(t) = \|\theta_t\|_0^2 + \|\theta\|_1^2$ and

$$\mathcal{E}_1(t^*) = \max_{0 \leq \tau \leq t} \mathcal{E}_1(t),$$

for some $t^* \in [0, t]$. Then, substituting the estimates (28)-(33) in (27), using the coercivity of $A(\cdot, \cdot)$, equivalence of norms $\|\|\cdot\|\|$ and $\|\cdot\|_0$, apply standard kick back arguments to find that

$$
\begin{aligned}
\mathcal{E}_1(t^*) &\leq Ch \left(\|u_0\|_2 + \|u_1\|_1 + \int_0^T (\|u_{tt}(s)\|_1 + \|u_t(s)\|_1 + \|u(s)\|_1) \, ds \right) \\
&\quad + Ch \int_0^T \left(\|f(s)\|_1 + h^{-2} \|G(V_h u)(s)\|_{-1,h} \right) \, ds + \int_0^{t^*} \mathcal{E}_1(s) \, ds.
\end{aligned}
$$

Now replace t^* by t and apply Gronwall's lemma with the estimate (24) to conclude that

$$\mathcal{E}_1(t) \leq Ch \left(\|u_0\|_2 + \|u_1\|_1 + \int_0^T (\|u\|_2 + \|u_t\|_1 + \|u_{tt}\|_1 + \|f\|_1) \, ds \right).$$

A use of triangle inequality with (16) and the estimates from Lemma 2.1 completes the rest of the proof. ∎

4.2. Optimal L^2 error estimates. In this subsection, we shall discuss optimal $L^\infty(L^2)$ estimates

Theorem 4.2. *Under the assumptions of Theorem 4.1, there exists a positive constant $C = C(T)$, independent of h, such that*

$$\|u(t) - u_h(t)\|_0 \le Ch^2 \left(\|u_0\|_3 + \|u_1\|_2 + \int_0^t \left(\|f\|_1 + \|f_t\|_0 + \|f_{tt}\|_0 \right) ds \right).$$

Proof. By setting $\chi = \theta$ in (23), and the using (13) with symmetry of the bilinear form $A(\cdot, \cdot)$, we find that

$$\frac{1}{2} \frac{d}{dt} \left[(\theta, \Pi_h^* \theta) + A(\hat{\theta}, \hat{\theta}) \right]$$
$$= \hat{G}(\rho)(\theta) + \hat{G}(\theta)(\theta) + (-\rho_t, \Pi_h^* \theta) + (u_1 - \Pi_h u_1, \Pi_h^* \theta)$$
$$- \int_0^t B(s, s; \hat{\theta}(s), \theta(t)) \, ds + \int_0^t \int_0^s B_\tau(s, \tau; \hat{\theta}(\tau), \theta(t)) \, d\tau \, ds.$$

Integrate from 0 to t to obtain

$$\frac{1}{2} \left[\|\|\theta\|\|^2 + A(\hat{\theta}, \hat{\theta}) \right]$$
$$= \frac{1}{2} \|\|\theta(0)\|\|^2 + \int_0^t \hat{G}(\rho)(\theta) \, ds + \int_0^t \hat{G}(\theta)(\theta) \, ds + \int_0^t (-\rho_t, \Pi_h^* \theta) \, ds$$
$$+ (u_1 - \Pi_h u_1, \Pi_h^* \hat{\theta}) - \int_0^t \int_0^s B(\tau, \tau; \hat{\theta}(\tau), \theta(s)) \, d\tau \, ds$$
$$+ \int_0^t \int_0^s \int_0^{\tau'} B_{\tau'}(\tau, \tau'; \hat{\theta}(\tau'), \theta(s)) \, d\tau' \, d\tau \, ds$$
$$\text{(34)} \qquad = \frac{1}{2} \|\|\theta(0)\|\|^2 + I_1 + I_2 + I_3 + I_4 + I_5 + I_6.$$

To estimate I_1, we note from (21) that

$$\hat{G}(\rho)(\theta) = \epsilon_h(\hat{f} - \hat{u}_{tt}, \theta) - \hat{G}(V_h u)(\theta)$$
$$\text{(35)} \qquad = \frac{d}{dt} \left(\epsilon_h(\hat{f} - \hat{u}_{tt}, \hat{\theta}) - \hat{G}(V_h u)(\hat{\theta}) \right) - \left(\epsilon_h(f - u_{tt}, \hat{\theta}) - \hat{G}_t(V_h u)(\hat{\theta}) \right),$$

and hence,

$$I_1 = \left(\epsilon_h(\hat{f} - \hat{u}_{tt}, \hat{\theta}) - \hat{G}(V_h u)(\hat{\theta}) \right) - \int_0^t \left(\epsilon_h(f - u_{tt}, \hat{\theta}) - \hat{G}_s(V_h u)(\hat{\theta}) \right) ds.$$

A use of (18) for $j = 1$ shows

$$|I_1| \le |\epsilon_h(\hat{f} - (u_t - u_1), \hat{\theta})| + |\hat{G}(V_h u)(\hat{\theta})|$$
$$+ \int_0^t \left(|\epsilon_h(f - u_{tt}, \hat{\theta})| + |\hat{G}_s(V_h u)(\hat{\theta})| \right) ds$$
$$\le C \left[h^2 \left(\|\hat{f}\|_1 + \|u_t\|_1 + \|u_1\|_1 \right) + \|\hat{G}(V_h u)\|_{-1,h} \right] \|\hat{\theta}\|_1$$
$$\text{(36)} \qquad + C \int_0^t \left(h^2 (\|f\|_1 + \|u_{tt}\|_1) + \|\hat{G}_s(V_h u)\|_{-1,h} \right) \|\hat{\theta}\|_1 \, ds.$$

Notice that I_2 can be written as

$$(37) \qquad I_2 \;=\; \int_0^t \epsilon_A(\hat{\theta}, \theta)ds + \int_0^t \int_0^s \epsilon_B(\tau, \tau; \hat{\theta}(\tau), \theta(s)) \; d\tau \; ds$$

$$(38) \qquad\qquad - \int_0^t \int_0^s \int_0^\tau \epsilon_{B_{\tau'}}(\tau, \tau'; \hat{\theta}(\tau'), \theta(s)) \; d\tau' \; d\tau \; ds$$

$$(39) \qquad\qquad = \; I_{21} + I_{22} + I_{23}.$$

For I_{21}, we apply (20) and the inverse inequality (26) to find that

$$(40) \qquad |I_{21}| = \int_0^t |\epsilon_A(\hat{\theta}, \theta)|ds \le Ch \int_0^t \|\theta\|_1 \|\hat{\theta}\|_1 \le C C_{inv} \int_0^t \|\theta\|_0 \, \|\hat{\theta}\|_1.$$

In order to estimate I_{22}, we integrate by parts in time so that

$$\begin{aligned} |I_{22}| \;&=\; \left| \int_0^t \epsilon_B(s, s; \hat{\theta}(s), \hat{\theta}(t)) \; ds - \int_0^t \epsilon_B(s, s; \hat{\theta}(s), \hat{\theta}(s)) \; ds \right| \\ &\le\; Ch \left\{ \|\hat{\theta}(t)\|_1 \int_0^t \|\hat{\theta}(s)\|_1 \; ds + \int_0^t \|\hat{\theta}(s)\|_1^2 \; ds \right\}. \end{aligned}$$

Similarly for I_{23}, we note that

$$\begin{aligned} |I_{23}| \;&=\; \left| \int_0^t \int_0^s \epsilon_{B_\tau}(s, \tau; \hat{\theta}(\tau), \hat{\theta}(t)) \; d\tau \; ds - \int_0^t \int_0^s \epsilon_{B_\tau}(s, \tau; \hat{\theta}(\tau), \hat{\theta}(s)) \; d\tau \; ds \right| \\ &\le\; C(T)h \left\{ \|\hat{\theta}(t)\|_1 \int_0^t \|\hat{\theta}(s)\|_1 \; ds + \int_0^t \|\hat{\theta}(s)\|_1^2 \; ds \right\}. \end{aligned}$$

Using the stability of Π_h^* (i.e., $\|\Pi_h^*\theta\|_0 \le C\|\theta\|_0$) and the Cauchy-Schwarz inequality, it follows that

$$(41) \qquad |I_3| \le \int_0^t |(\rho_t, \Pi_h^*\theta)|ds \le C \int_0^t \|\rho_t(s)\|_0 \|\theta(s)\|_0 ds.$$

For I_4, we apply (10) and $\|\Pi_h^*\hat{\theta}\|_0 \le C\|\hat{\theta}\|_1$ to obtain

$$(42) \qquad |I_4| \le \|u_1 - \Pi_h u_1\|_0 \, \|\Pi_h^*\hat{\theta}\|_0 \le Ch^2\|u_1\|_2 \, \|\hat{\theta}\|_1.$$

Finally, similarly for I_{22} and I_{23}, an integration by parts leads to

$$(43) \qquad |I_5| + |I_6| \le C(T) \left\{ \|\hat{\theta}(t)\|_1 \int_0^t \|\hat{\theta}(s)\|_1 \; ds + \int_0^t \|\hat{\theta}(s)\|_1^2 \; ds \right\}.$$

Now, define $\mathcal{E}_0^2(t) = \|\theta(t)\|_0^2 + \|\hat{\theta}(t)\|_1^2$ and let $t^* \in [0, t]$ be such that

$$\mathcal{E}_0(t^*) = \max_{0 \le s \le t} \mathcal{E}_0(t).$$

At $t = t^*$, substitute the estimates (35)-(43) in (34) and use the equivalence of the norms $\|\|\cdot\|\|$ and $\|\cdot\|_0$ from (14) along with the coercivity property (3) of $A(\cdot, \cdot)$. Then a standard use of kick back arguments yields

$$\mathcal{E}_0(t^*) \le C\|\theta(0)\|_0 + Ch^2 \left[\|u_0\|_2 + \|u_1\|_2 + \int_0^{t^*} (\|f\|_1 + \|u_{tt}\|_1) \; ds \right]$$

$$+ C \left[\|\hat{G}(V_h u)\|_{-1,h} + \int_0^{t^*} (\|\rho_t\|_0 + \|\hat{G}_s(V_h u)\|_{-1,h}) \; ds \right] + C \int_0^{t^*} \mathcal{E}_0(s) \; ds.$$

Note that with the choice of $u_h(0) = \Pi_h u_0$,

$$\|\theta(0)\|_0 \le \|\rho(0)\|_0 + \|u_0 - \Pi_h u_0\|_0 \le Ch^2\|u_0\|_2.$$

Now apply Lemmas 4.3, 4.1 along with the estimates in Lemma 2.1 to obtain

$$\mathcal{E}_0(t^*) \leq Ch^2 \left(\|u_0\|_3 + \|u_1\|_2 + \int_0^{t^*} (\|f\|_1 + \|f_t\|_0 + \|f_{tt}\|_0) \, ds \right)$$
$$+ C \int_0^{t^*} \mathcal{E}_0(s) \, ds.$$

Then replace t^* by t and use Gronwall's lemma for $t \leq T$ to conclude that

$$\|\theta(t)\|_0 \leq C(T)h^2 \left(\|u_0\|_3 + \|u_1\|_2 + + \int_0^T (\|f\|_1 + \|f_t\|_0 + \|f_{tt}\|_0) \, ds \right).$$

Finally, a use of the triangle inequality completes the proof. ∎

Remark 4.1. *Note that it is possible to choose $u_{h,t}(0)$ as the L^2 projection of u_1 onto U_h^* and in that case, the term $(u_t(0) - u_{h,t}(0), \Pi_h^* \theta_t)$ becomes zero.*

4.3. Maximum norm estimates. In this subsection, a superconvergent result for $\|\theta\|_1$ is first derived and it is then used to analyze quasi-optimal maximum error estimates.

Lemma 4.4. *Assume that $f \in L^1(H^2)$, $f_t \in L^1(H^1)$, f_{tt}, $f_{ttt} \in L^1(L^2)$, $u_0 \in H^4 \cap H_0^1$ and $u_1 \in H^3 \cap H_0^1$. With $u_h(0) = V_h u_0$ and $u_{h,t}(0) = \Pi_h u_1$, there exists a positive constant $C = C(T)$, independent of h, such that the following holds for $t \in (0, T]$*

$$\|\theta_t(t)\|_0 + \|\theta(t)\|_1 \leq C \, h^2 \left(\|u_0\|_4 + \|u_1\|_3 + \|D_t^3 f\|_{L^1(L^2)} + \sum_{j=0}^2 \|D_t^j f\|_{L^1(H^{2-j})} \right).$$

Proof. We now modify the estimates of J_1, J_2, J_3 and J_5 in (27) of Theorem 4.1 to obtain a superconvergence result for $\|\theta(t)\|_1$ norm. As $u_h(0) = V_h u_0$, it follows that $A(\theta(0), \theta(0)) = 0$. Now with $u_{h,t}(0) = \Pi_h u_1$, we obtain

(44) $$|J_1| \leq Ch^4 \|u_1\|_2^2.$$

To estimate J_2, observe that

$$\epsilon_h(f - u_{tt}, \theta_t) = \frac{d}{dt} \epsilon_h(f - u_{tt}, \theta) - \epsilon_h(f_t - u_{ttt}, \theta),$$

and thus, rewrite J_2 as

$$J_2 = \epsilon_h(f - u_{tt}, \theta) - \int_0^t \epsilon_h(f_t - u_{ttt}, \theta) ds.$$

Then, a use of (18) yields

(45) $$|J_2| \leq Ch^2 \left[(\|f\|_1 + \|u_{tt}\|_1) \|\theta\|_1 + \int_0^t (\|f_t\|_1 + \|u_{ttt}\|_1) \|\theta\|_1 ds \right].$$

For J_3, rewrite G term as

$$G(V_h u)(\theta_t) = \frac{d}{dt} \{G(V_h u)(\theta)\} - G_t(V_h u)(\theta),$$

and hence, a use of (19) shows that

$$|J_3| \leq |G(V_h u)(\theta)| + \int_0^t |G_s(V_h u)(\theta)| \, ds$$

(46) $$\leq 2 \left[\|G(V_h u)\|_{-1,h} + \int_0^t \|G_s(V_h u)\|_{-1,h} \, ds \right] \|\theta\|_1.$$

For J_5, apply (16) to obtain

$$(47) \quad |J_5| \leq 2 \int_0^t \|\rho_{tt}\|_0 \|\theta_t\|_0 ds \leq C(T)h^2 \int_0^t (\|u_{tt}\|_2 + |u_t|_2 + \|u\|_2) \|\theta_t\|_0 \, ds.$$

Substituting the estimates (44)-(47) in (27), and apply standard kick back arguments to arrive at

$$\begin{aligned}
\mathcal{E}_1(t) &\leq Ch^2 \Big[\|u_1\|_2 + \|f\|_1 + \|u_{tt}\|_1 + |u_t|_2 + \|u\|_2 \\
&\quad + \int_0^t (\|f_t\|_1 + \|u\|_2 + \|u_t\|_2 + \|u_{tt}\|_2 + \|u_{ttt}\|_1) \, ds \Big] \\
&\quad + C \int_0^t \mathcal{E}_1(s) \, ds.
\end{aligned}$$

An application of the integral identity (4) shows

$$\|f\|_1 \leq \|f(0)\|_1 + \int_0^t \|f_t\|_1 ds.$$

Then using the estimates in Lemma 2.1 we arrive at

$$\begin{aligned}
\mathcal{E}_1(t) &\leq Ch^2 \Big(\|u_1\|_3 + \|u_0\|_4 + \|f(0)\|_1 \\
&\quad + \int_0^T (\|f\|_2 + \|f_t\|_1 + \|f_{tt}\|_0 + \|f_{ttt}\|_0) \, ds \Big) \\
&\quad + C \int_0^t \mathcal{E}_1(s) \, ds.
\end{aligned}$$

Since $W^{1,1}([0,T]; H^1)$ is continuously imbedded in $C^0([0,T]; H^1)$, that is $\|f(0)\|_1 \leq C \|f\|_{W^{1,1}(H^1)}$, a use of Gronwall's lemma completes the rest of the proof. ∎

Remark 4.2. *As a result of Lemma 4.4, we obtain a super-convergence estimate for θ in H^1 norm.*

For $\|\theta\|_\infty$, a use of Sobolev inequality

$$(48) \qquad \|\chi\|_\infty \leq C \left(\log \frac{1}{h} \right)^{1/2} \|\nabla \chi\|_0 \quad \forall \chi \in U_h$$

with Lemma 4.4 yields

$$\begin{aligned}
\|\theta\|_\infty &\leq C(T)h^2 \left(\log \frac{1}{h} \right)^{1/2} \Big[\|u_0\|_4 + \|u_1\|_3 \\
(49) &\quad + \int_0^T (\|f\|_2 + \|f_t\|_1 + \|f_{tt}\|_0 + \|f_{ttt}\|_0) \, ds \Big].
\end{aligned}$$

Below, we discuss the maximum norm estimate in form of a theorem.

Theorem 4.3. *Let u and u_h be the solutions of (1) and (7) respectively. Further, let the assumptions of Lemma 4.4 hold. Then,*

$$\begin{aligned}
\|u(t) - u_h(t)\|_\infty &\leq C h^2 \left(\log \frac{1}{h} \right) \Big(\|u_0\|_4 + \|u_1\|_3 + \|D_t^3 f\|_{L^1(L^2)} \\
&\quad + \sum_{j=0}^2 \|D_t^j f\|_{L^1(H^{2-j})} \Big),
\end{aligned}$$

where $C = C(T)$ is a positive constant independent of h.

Proof. By the triangle inequality

$$\|u(t) - u_h(t)\|_\infty \le \|\theta\|_\infty + \|\rho\|_\infty.$$

Now, combine the estimates obtained in (49) and in (17) with Lemma 2.1 to obtain the required result. ∎

5. Error Estimates for a Completely Discrete Scheme

In this section, we introduce further notations and formulate a completely discrete scheme by applying an explicit finite difference method to discretize the time variable of the semidiscrete system (7). Then, we discuss optimal error estimates.

Let k $(0 < k < 1)$ be the time step, $k = T/N$ for some positive integer N, and $t_n = nk$. For any function ϕ of time, let ϕ^n denote $\phi(t_n)$. We shall use this notation for functions defined for continuous in time as well as those defined for discrete in time. Set $\phi^{n+1/2} = (\phi^{n+1} + \phi^n)/2$, and define the following notations for the difference quotients:

$$\delta_t \phi^n = \frac{\phi^{n+1} - \phi^{n-1}}{2k}, \quad \partial_t \phi^{n+1/2} = \frac{\phi^{n+1} - \phi^n}{k}, \quad \partial_t^2 \phi^n = \frac{\phi^{n+1} - 2\phi^n + \phi^{n-1}}{k^2}.$$

Note that

$$\delta_t \phi^n = \frac{\partial_t \phi^{n+1/2} + \partial_t \phi^{n-1/2}}{2}, \qquad \partial_t^2 \phi^n = \frac{\partial_t \phi^{n+1/2} - \partial_t \phi^{n-1/2}}{k}.$$

Then, the discrete-in-time scheme of (7) is to seek $U^n \in U_h$ such that for $\chi \in U_h$

$$(50) \quad \frac{2}{k}(\partial_t U^{1/2}, \Pi_h^* \chi) + A_h(U^0, \Pi_h^* \chi) = (f^0 + \frac{2}{k}u_1, \Pi_h^* \chi), \quad \forall \chi \in U_h,$$

$$(51) \quad (\partial_t^2 U^n, \Pi_h^* \chi) + A_h(U^n, \Pi_h^* \chi) + k \sum_{j=0}^{n-1} B_h(t_n, t_{j+1/2}; U^{j+1/2}, \Pi_h^* \chi) = (f^n, \Pi_h^* \chi),$$

$n \ge 1$, with a given initial data U^0 in U_h. This choice of time discretization leads to a second order accuracy in k. The integral term in (7) is computed by using the second order quadrature formula

$$\sigma^n(g) = k \sum_{j=0}^{n-1} g(t_{j+1/2}) \approx \int_0^{t_n} g(s)\, ds, \quad \text{with} \quad t_{j+1/2} = (j+1/2)k.$$

We shall use a shorthand notation $\sigma^n(B_h^n(U, \Pi_h^* \chi))$ for $k \sum_{j=0}^{n-1} B_h(t_n, t_{j+1/2};$ $U^{j+1/2}, \Pi_h^* \chi)$. The quadrature error $q^n(g)$ is defined by

$$q^n(g) = \sigma^n(g) - \int_0^{t_n} g(s)\, ds = \sum_{j=0}^{n-1} \left(k g^{j+1/2} - \int_{t_j}^{t_{j+1}} g(s)\, ds \right).$$

Similarly, for $\phi \in U_h$, we define a linear functional $q_B^n(\phi)$ representing the error in the quadrature formula by

$$q_B^n(\phi)(\chi) = \sigma^n \left(B^n(\phi, \chi) \right) - \int_0^{t_n} B(t_n, s; \phi(s), \chi)\, ds.$$

Notice that $q_B^0(\phi) = 0$.

For our future use, we state without proof the following lemma. For a proof, see, [21].

Lemma 5.1. *There exists a positive constant C, independent of k and h, such that the following estimate holds:*

$$k \sum_{n=0}^{m} ||\partial_t q_B^{n+1/2}(\phi)||_{-1,h} \le Ck^2 \int_0^{t_{m+1}} (||\phi||_1 + ||\phi_t||_1 + ||\phi_{tt}||_1) \, ds.$$

Now, define $e^n := u^n - U^n$. We split $e^n = \rho^n + \xi^n$ with $\rho^n = u^n - V_h u^n$ and $\xi^n = V_h u^n - U^n$. From (50)-(51) and (7), we derive equations in e^n as follows

$$(52) \qquad \frac{2}{k}(\partial_t e^{1/2}, \Pi_h^* \chi) + A_h(e^0, \Pi_h^* \chi) = (2r^0, \Pi_h^* \chi),$$

$$(53) \qquad (\partial_t^2 e^n, \Pi_h^* \chi) + A_h(e^n, \Pi_h^* \chi) + \sigma^n (B_h^n(e, \Pi_h^* \chi) = (r^n, \Pi_h^* \chi) + q_{B_h}^n(u)(\Pi_h^* \chi),$$

$n \ge 1$, for all $\chi \in U_h$, where $r^0 = \frac{1}{k}\left(\partial_t u^{1/2} - u_1\right) - \frac{1}{2}u_{tt}^0 = \frac{1}{2k^2} \int_0^k (t-k)^2 \frac{\partial^3 u}{\partial t^3}(t) \, dt$, and

$$(54) \qquad r^n = \partial_t^2 u^n - u_{tt}^0 = -\frac{1}{6k^2} \int_{-k}^k (|t| - k)^3 \frac{\partial^4 u}{\partial t^4}(t^n + t) \, dt, \quad n \ge 1.$$

Since estimates for ρ are known from Lemma 4.1, it is sufficient to estimate ξ. From (52)-(53), we obtain the following equations in ξ^n:

$$\frac{2}{k}(\partial_t \xi^{1/2}, \Pi_h^* \chi) \quad + \quad A(\xi^0, \chi) = -\frac{2}{k}(\partial_t \rho^{1/2}, \Pi_h^* \chi) + (2r^0, \Pi_h^* \chi)$$

$$(55) \qquad\qquad\qquad + \quad \epsilon_h(f^0 - u_{tt}^0, \chi) - \epsilon_A(V_h u(0), \chi),$$

$$(\partial_t^2 \xi^n, \Pi_h^* \chi) \quad + \quad A(\xi^n, \chi) = (r^n, \Pi_h^* \chi) - (\partial_t^2 \rho^n, \Pi_h^* \chi) + H^n(\xi)(\chi)$$

$$(56) \qquad\qquad\qquad - \quad \sigma^n (B^n(\xi, \chi)) - H^n(V_h u)(\chi) + \epsilon_h(f^n - u_{tt}^n, \chi) + q_B^n(V_h u)(\chi),$$

where

$$H^n(\xi)(\chi) = \epsilon_A(\xi^n, \chi) + k \sum_{j=0}^{n-1} \epsilon_B(t_n, t_{j+1/2}; \xi^{j+1/2}, \chi).$$

Below, we shall obtain $l^\infty(H^1)$ estimate for $\xi^{n+1/2}$.

Lemma 5.2. *Assume that $f \in L^1(H^2)$, $f_t \in L^1(H^1)$, f_{tt}, $f_{ttt} \in L^1(L^2)$, $u_0 \in H^4 \cap H_0^1$ and $u_1 \in H^3 \cap H_0^1$. Further, assume that the CFL condition*

$$(57) \qquad\qquad\qquad\qquad \frac{k^2}{h^2} \le \frac{4c_{eq}}{\Lambda C_{inv}}$$

is satisfied, where $\Lambda > 0$ is the constant given in (3), C_{inv} appears in the inverse inequality (26) and c_{eq} is stated in the equivalence of norms as in (14). Then, with $u_h(0) = V_h u_0$ and $u_{h,t}(0) = \Pi_h u_1$, there exists a positive constant $C = C(T)$, independent of h and k, such that the following estimate

$$\|\partial_t \xi^{m+1/2}\|_0 + \|\xi^{m+1/2}\|_1 \quad \le \quad C(T)(k^2 + h^2)\Big(\|u_0\|_4 + \|u_1\|_3$$

$$(58) \qquad\qquad\qquad\qquad + \|D_t^3 f\|_{L^1(L^2)} + \sum_{j=0}^{2} \|D_t^j f\|_{L^1(H^{2-j})}\Big),$$

holds for $m = 0, 1, \cdots, N - 1$.

Proof. Choose $\chi = \delta_t \xi^n$ in (56) and obtain

$$
\begin{aligned}
\frac{1}{2}\bar{\partial}_t\Big(|||\partial_t\xi^{n+1/2}|||^2 \quad + \quad A(\xi^{n+1},\xi^n)\Big) &= (r^n - \partial_t^2\rho^n, \delta_t\xi^n) + H^n(\xi)(\delta_t\xi^n) \\
&\quad -\sigma^n\left(B^n(\xi,\delta_t\xi^n)\right) - H^n(V_h u)(\delta_t\xi^n) \\
&\quad +\epsilon_h(f^n - u_{tt}^n, \delta_t\xi^n) + q_B^n(V_h u)(\delta_t\xi^n) \\
&= I_1^n + I_2^n + I_3^n + I_4^n + I_5^n + I_6^n,
\end{aligned}
$$

(59)

where $\bar{\partial}_t$ denotes backward differencing. Next multiply (59) by $2k$ and sum the resulting one from $n = 2$ to m to arrive at

$$
\begin{aligned}
\frac{1}{2}\Big(|||\partial_t\xi^{m+1/2}|||^2 + A(\xi^{m+1},\xi^m)\Big) &\leq \frac{1}{2}\Big(|||\partial_t\xi^{3/2}|||_0^2 + A(\xi^2,\xi^1)\Big) \\
&\quad + k\left|\sum_{n=2}^m (I_1^n + I_2^n + I_3^n + I_4^n + I_5^n + I_6^n)\right|.
\end{aligned}
$$

(60)

Now define

$$
|||\xi^{n+1/2}|||_1^2 = ||\partial_t\xi^{n+1/2}||_0^2 + ||\xi^{n+1/2}||_1^2,
$$

and let for some $m^\star$ with $0 \leq m^\star \leq m$,

$$
|||\xi^{m^\star+1/2}|||_1 = \max_{0 \leq n \leq m} |||\xi^{n+1/2}|||_1.
$$

To estimate the sum in I_1^n, an application of the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
k\left|\sum_{n=2}^m I_1^n\right| &\leq Ck\sum_{n=2}^m \left(||\partial_t^2\rho^n||_0 + ||r^n||_0\right)\left(||\partial_t\xi^{n+1/2}||_0 + ||\partial_t\xi^{n-1/2}||_0\right) \\
&\leq 2Ck\sum_{n=2}^m \left(||\partial_t^2\rho^n||_0 + ||r^n||_0\right)|||\xi^{m^\star+1/2}|||_1.
\end{aligned}
$$

For the second sum on the right hand side of (60), we use the fact that

(61) $$ \psi^n\delta_t\xi^n = \bar{\partial}_t(\psi^n\xi^{n+1/2}) - \partial_t\psi^{n+1/2}\xi^{n-1/2} $$

and conclude

$$
k\sum_{n=2}^m \epsilon_A(\xi^n, \delta_t\xi^n) = \epsilon_A(\xi^m, \xi^{m+1/2}) - \epsilon_A(\xi^1, \xi^{1+1/2}) - k\sum_{n=2}^m \epsilon_A(\partial_t\xi^{n+1/2}, \xi^{n-1/2}).
$$

Using (20) and the inverse inequality (26), we obtain

$$
\begin{aligned}
\left|k\sum_{n=2}^m \epsilon_A(\xi^n, \delta_t\xi^n)\right| &\leq Ch\left\{||\xi^m||_1||\xi^{m+1/2}||_1 + ||\xi^1||_1||\xi^{1+1/2}||_1\right\} \\
&\quad + Chk\sum_{n=2}^m ||\partial_t\xi^{n+1/2}||_1||\xi^{n-1/2}||_1 \\
&\leq C\left\{||\xi^m||_0 + ||\xi^1||_0 + k\sum_{n=2}^m ||\partial_t\xi^{n+1/2}||_0\right\}|||\xi^{m^\star+1/2}|||_1.
\end{aligned}
$$

Since $\xi^0 = 0$, $\xi^m = k\sum_{n=0}^{m-1}\partial_t\xi^{n+1/2}$, and it follows that

$$
\left|k\sum_{n=2}^m \epsilon_A(\xi^n, \delta_t\xi^n)\right| \leq Ck\left(\sum_{n=0}^{m-1}||\partial_t\xi^{n+1/2}||_0\right)|||\xi^{m^\star+1/2}|||_1.
$$

Similarly, we obtain

$$\left| k^2 \sum_{n=2}^{m} \sum_{j=0}^{n-1} \epsilon_B(t_n, t_{j+1/2}; \xi^{j+1/2}, \delta_t \xi^n) \right| \leq k^2 \sum_{n=2}^{m} \left(C \sum_{j=0}^{n-1} \|\xi^{j+1/2}\|_1 \right) \||\xi^{m^\star+1/2}\||_1$$

$$\leq CTk \left(\sum_{j=0}^{m-1} \|\xi^{j+1/2}\|_1 \right) \||\xi^{m^\star+1/2}\||_1,$$

and hence,

$$k \left| \sum_{n=2}^{m} I_2^n \right| \leq C(T)k \left(\sum_{n=0}^{m-1} \||\xi^{n+1/2}\||_1 \right) \||\xi^{m^\star+1/2}\||_1.$$

To estimate the sum in I_3^n, we again use (61) and rewrite the sum as:

$$k \sum_{n=2}^{m} I_3^n = \sigma^m \left(B^m(\xi, \xi^{m+1/2}) \right) - \sigma^1 \left(B^1(\xi, \xi^{1+1/2}) \right)$$

$$- k^2 \sum_{n=2}^{m} \sum_{j=0}^{n-1} (\bar{\partial}_{t,1} B)(t_n, t_{j+1/2}; \xi^{j+1/2}, \xi^{n-1/2})$$

$$+ k \sum_{n=2}^{m} B(t_{n-1}, t_{n-1/2}; \xi^{n-1/2}, \xi^{n-1/2}),$$

where $\bar{\partial}_{t,1} B$ denotes the difference quotient of B with respect to its first argument. Since, $|\bar{\partial}_{t,1} B| \leq C \|B_t\|_\infty < \infty$, it follows that

$$\left| k \sum_{n=2}^{m} I_3^n \right| \leq C(T)k \left(\sum_{j=0}^{m-1} \|\xi^{j+1/2}\|_1 \right) \||\xi^{m^\star+1/2}\||_1.$$

For the sum involving I_4^n, we note that

$$\left| k \sum_{n=2}^{m} \epsilon_A(V_h u^n, \delta_t \xi^n) \right|$$

$$= \left| \epsilon_A(V_h u^m, \xi^{m+1/2}) - \epsilon_A(V_h u^1, \xi^{1+1/2}) - k \sum_{n=2}^{m} \epsilon_A(\partial_t V_h u^{n+1/2}, \xi^{n-1/2}) \right|$$

$$\leq Ch^2 \left\{ \|u^m\|_2 + \|u^1\|_2 + k \sum_{n=0}^{m} \|\partial_t V_h u^{n+1/2}\|_2 \right\} \||\xi^{m^\star+1/2}\||_1$$

$$\leq Ch^2 \left\{ \|u_0\|_2 + \|u_t\|_{L^1(H^2)} \right\} \||\xi^{m^\star+1/2}\||_1.$$

Similarly, we have

$$\left| k^2 \sum_{n=2}^{m} \sum_{j=0}^{n-1} \epsilon_B(t_n, t_{j+1/2}; V_h u^{j+1/2}, \delta_t \xi^n) \right| \leq CTh^2 \left\{ \|u_0\|_2 + \|u_t\|_{L^1(H^2)} \right\} \||\xi^{m^\star+1/2}\||_1.$$

In order to estimate the sum in I_5^n, we repeat the previous arguments and use (18) to arrive at

$$
\left| k \sum_{n=2}^{m} \epsilon_h(f^n - u_{tt}^n, \delta_t \xi^n) \right| = \left| \epsilon_h(f^m - u_{tt}^m, \xi^{m+1/2}) - \epsilon_h(f^1 - u_{tt}^1, \xi^{1+1/2}) \right.
$$

$$
\left. - k \sum_{n=2}^{m} \epsilon_h\left(\partial_t(f^{n+1/2} - u_{tt}^{n+1/2}), \xi^{n-1/2} \right) \right|
$$

$$
\leq Ch^2 \left\{ \|f^0 - u_{tt}^0\|_1 + \|f_t - u_{ttt}\|_{L^1(H^1)} \right\} |||\xi^{m^\star + 1/2}|||_1.
$$

For the last sum, we rewrite it as

$$
k \sum_{n=2}^{m} I_6^n = q_B^m(V_h u)(\xi^{m+1/2}) - q_B^1(V_h u)(\xi^{1+1/2}) - k \sum_{n=2}^{m} (\partial_t q_B^{n+1/2}(V_h u))(\xi^{n-1/2}).
$$

Since $q_B^0(V_h u) = 0$, $q_B^m(V_h u) = k \sum_{n=0}^{m} \partial_t q_B^{n+1/2}(V_h u)$, we obtain

$$
k \left| \sum_{n=2}^{m} I_6^n \right| \leq Ck \left\{ \sum_{n=0}^{m} \|\partial_t q_B^{n+1/2}(V_h u)\|_{-1,h} \right\} |||\xi^{m^\star + 1/2}|||_1.
$$

Combining all the previous estimates, we conclude that

$$
|||\partial_t \xi^{m+1/2}|||^2 + A(\xi^{m+1}, \xi^m)
$$

$$
\leq |||\partial_t \xi^{3/2}|||^2 + A(\xi^2, \xi^1) + Ck \left\{ \sum_{n=2}^{m} (\|\partial_t^2 \rho^n\|_0 + \|r^n\|_0) \right.
$$

$$
\left. + \sum_{n=0}^{m} \|\partial_t q_B^{n+1/2}(V_h u)\|_{-1,h} + \sum_{j=0}^{m-1} |||\xi^{j+1/2}|||_1 \right\} |||\xi^{m^\star + 1/2}|||_1
$$

$$
(62) \qquad + h^2 C(T, f, u) |||\xi^{m^\star + 1/2}|||_1,
$$

where

$$
C(T, f, u) = \|u_0\|_2 + \|u_t\|_{L^1(H^2)} + \|u_{tt}(0)\|_1 + \|u_{ttt}\|_{L^1(H^1)} + \|f^0\|_1 + \|f_t\|_{L^1(H^1)}.
$$

In order to estimate the first two terms on the right hand side of (62), we choose $\chi = \partial_t \xi^{3/2}$ in (56) for $n = 1$ and obtain

$$
|||\partial_t \xi^{3/2}|||^2 + A(\xi^2, \xi^1) \leq |||\partial_t \xi^{1/2}|||^2 + h^2 \left(\|u^1\|_2 + \|u_0\|_2 + k\|\partial_t u^{1/2}\|_2 \right)
$$

$$
+ h^2 \left(\|f^0 - u_{tt}^0\|_2 + \|f_t - u_{ttt}\|_{L^1(0,k;H^1)} \right) + \|\partial_t q_B^{1/2}\|_{-1,h}.
$$

Next, we choose $\chi = \partial_t \xi^{1/2}$ in (55) to find that

$$
|||\partial_t \xi^{1/2}||| \leq C \left\{ \|\partial_t \rho^{1/2}\|_0 + k\|r^0\|_0 + h^2\|f^0 - u_{tt}^0\|_2 + h^2\|u_0\|_2 \right\}.
$$

A use of these estimates in (62) results in

$$
|||\partial_t \xi^{m+1/2}|||^2 + A(\xi^{m+1}, \xi^m) \leq C \left\{ \|\partial_t \rho^{1/2}\|_0 + k \sum_{n=1}^{m} \|\partial_t^2 \rho^n\|_0 + k \sum_{n=0}^{m} \|r^n\|_0 \right.
$$

$$
\left. + k \sum_{n=0}^{m} \|\partial_t q_B^{n+1/2}(V_h u)\|_{-1,h} + k \sum_{j=0}^{m-1} \|\xi^{j+1/2}\|_1 \right\} |||\xi^{m^\star + 1/2}|||_1
$$

$$
(63) \qquad + h^2 C(T, f, u) |||\xi^{m^\star + 1/2}|||_1.
$$

Note that

$$A(\xi^{m+1}, \xi^m) = A(\xi^{m+1/2}, \xi^{m+1/2}) - \frac{k^2}{4}A(\partial_t\xi^{m+1/2}, \partial_t\xi^{m+1/2}).$$

Hence,

$$|||\partial_t\xi^{m+1/2}|||^2 + A(\xi^{m+1}, \xi^m)$$

$$\geq c_{eq} \|\partial_t\xi^{m+1/2}\|_0^2 + \alpha\|\xi^{m+1/2}\|_1^2 - \frac{k^2}{4}A(\partial_t\xi^{m+1/2}, \partial_t\xi^{m+1/2}).$$

Since the CFL condition (57) holds, choose k so that $C_* = \left(c_{eq} - \Lambda C_{inv}\frac{k^2}{4h^2}\right) > 0$, where the constants Λ, c_{eq} and C_{inv} appear in (3), (14) and (26), respectively. Then

$$|||\partial_t\xi^{m+1/2}|||^2 + A(\xi^{m+1}, \xi^m) \geq \min\{C_*, \alpha\}|||\xi^{m+1/2}|||_1.$$

Altogether, it now results in

$$|||\xi^{m+1/2}|||_1 \leq |||\xi^{m^\star+1/2}|||_1 \leq C\left\{\|\partial_t\rho^{1/2}\|_0 + k\sum_{n=1}^{m}\|\partial_t^2\rho^n\|_0 + k\sum_{n=0}^{m}\|r^n\|_0\right.$$

$$\left. + k\sum_{n=0}^{m}\|\partial_tq_B^{n+1/2}(V_hu)\|_{-1,h} + k\sum_{j=0}^{m-1}|||\xi^{j+1/2}|||_1\right\}$$

$$(64) \qquad\qquad + h^2C(T, f, u).$$

To estimate the first two terms on the right hand side of (64), it is observed that

$$(65) \qquad\qquad \|\partial_t\rho^{1/2}\|_0 \leq \frac{1}{k}\int_0^k \|\rho_t(s)\|_0\, ds,$$

and a use of Taylor series expansion yields

$$k\sum_{n=1}^{m}\|\partial_t^2\rho^n\|_0 \leq \frac{1}{k}\sum_{n=1}^{m}\left\{\int_{t_n}^{t_{n+1}}(t_{n+1}-s)\|\rho_{tt}(s)\|_0\, ds + \int_{t_{n-1}}^{t_n}(s-t_{n-1})\|\rho_{tt}(s)\|_0\, ds\right\}$$

$$(66) \qquad \leq 2\int_0^{t_{m+1}}\|\rho_{tt}(s)\|_0\, ds.$$

Further, from (54) it follows that

$$\|r^n\|_0 \leq Ck\int_{t_{n-1}}^{t_{n+1}}\|D_t^4u(s)\|_0\, ds, \quad n \geq 1,$$

and

$$\|r^0\|_0 \leq Ck\|u_{ttt}\|_{L^\infty(0,k/2;L^2(\Omega))} \leq Ck\int_0^{t_{m+1}}\|D_t^3u(s)\|_0\, ds.$$

Thus, we arrive at

$$(67) \qquad k\sum_{n=0}^{m}\|r^n\|_0 \leq Ck^2\int_0^{t_{m+1}}\left(\|D_t^3u(s)\|_0 + \|D_t^4u(s)\|_0\right)\, ds.$$

Finally, a use of Lemma 5.1 and the triangle inequality yields

$$k\sum_{n=1}^{m}\|\partial_tq_B^{n+1/2}(V_hu)\|_{-1,h} \leq Ck^2\sum_{j=0}^{2}\int_0^{t_{m+1}}\left(\|D_t^ju(s)\|_1 + \|D_t^j\rho(s)\|_1\right)\, ds.$$

Substitute now (65)-(67) in (64) and use the estimates in Lemmas 4.1 and 2.1. Then, an application of the discrete Gronwall's lemma completes the rest of the proof. ∎

By Sobolev inequality, it follows that

$$(68) \qquad \|\xi^{n+1/2}\|_\infty \le C \left(\log \frac{1}{h} \right)^{1/2} \|\xi^{n+1/2}\|_1.$$

Using Lemma 5.2, the triangle inequality and the estimates (68) and (17), we obtain the result of the following theorem.

Theorem 5.1. *Let the assumptions of Lemma* 5.2 *hold. Then,*

$$\|u(t_{m+1/2}) - U^{m+1/2}\|_\infty \quad \le \quad C(T) \left(\log \frac{1}{h} \right) (k^2 + h^2) \Big(\|u_0\|_4 + \|u_1\|_3$$

$$(69) \qquad\qquad\qquad +\|D_t^3 f\|_{L^1(L^2)} + \sum_{j=0}^{2} \|D_t^j f\|_{L^1(H^{2-j})} \Big)$$

for $m = 0, 1, \cdots, N-1$.

6. FVEM with Quadrature

In this section, we discuss the effect of numerical quadrature on FVEM, when the L^2 inner product $(\cdot, \cdot)$ and the bilinear forms $A_h(\cdot, \cdot)$ and $B_h(t, s; \cdot, \cdot)$ appearing in (7) are approximated by simple quadrature formulae.

For a continuous function ϕ on a triangle K, consider the quadrature formula

$$(70) \qquad \mathcal{Q}_{K,h}(\phi) = \frac{1}{3}|K| \sum_{l=1}^{3} \phi(P_l) \approx \int_K \phi(x)dx \quad \forall K \in \mathcal{T}_h,$$

where P_l, $1 \le l \le 3$ denote the vertices of the triangle K and $|K|$ denotes the area of the triangle K. Now the quadrature formula given by (70) is exact for $\phi \in P_1(K) \; \forall K \in \mathcal{T}_h$. Using (70), we replace the L^2 inner product by the following discrete L^2 inner product:

$$(\chi, \Pi_h^* \psi)_h \quad = \quad \sum_{K \in \mathcal{T}_h} \mathcal{Q}_{K,h}(\chi \Pi_h^* \psi)$$

$$(71) \qquad\qquad = \quad \sum_{P_i \in N_h^0} \chi(P_i)\psi(P_i)|S_{K_{P_i}}^*| \; \forall \chi, \; \psi \in U_h.$$

This is known as lumping of mass in the literature. Observe that $\|\chi\|_h^2 = (\chi, \chi)_h$ is a norm on U_h, which is equivalent to the L^2 norm, i.e., there exist positive constants C_5 and C_6, independent of h, such that

$$(72) \qquad C_5\|\chi\|_0 \le \|\chi\|_h \le C_6\|\chi\|_0.$$

Define the quadrature error by

$$\bar{\epsilon}_h(\chi, \psi) = (\chi, \Pi_h^* \psi) - (\chi, \Pi_h^* \psi)_h.$$

Since the quadrature formula involves only the values of the functions at the interior nodes and $\Pi_h^* \psi(P_i) = \psi(P_i) \; \forall P_i \in N_h^0$ and $\psi \in U_h$, it follows that

$$(73) \qquad (\chi, \psi)_h = (\chi, \Pi_h^* \psi)_h \quad \forall \chi, \; \psi \in U_h.$$

Below, we state the estimates related to quadrature error, whose proof can be found in [16].

Lemma 6.1. *For* $\chi, \; \psi \in U_h$, *there is a positive constant* C, *independent of* h, *such that the following estimate holds:*

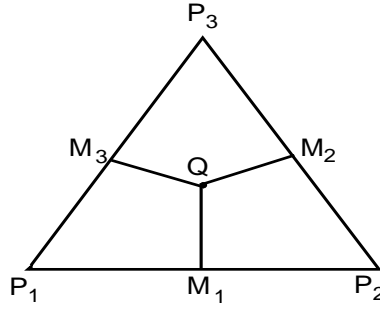$$(74) \qquad |\bar{\epsilon}_h(\chi, \psi)| \le Ch^2 \|\chi\|_1 \|\psi\|_1.$$

FIGURE 2

Further, for $\chi \in H^2$ and $\psi \in U_h$, there holds:

$$(75) \qquad\qquad |\bar{\epsilon}_h(\chi, \psi)| \leq Ch^2 \|\chi\|_2 \|\psi\|_1.$$

Now define the following quadrature approximation over each element K by

$$(76) \qquad \int_{\overline{M_lQ} \cap K} v(z) \, ds \approx \frac{\overline{M_lQ}}{2} \left(v(M_l) + v(Q) \right) = \tilde{\mathcal{Q}}_{h,l}(v),$$

where M_l is the midpoint of P_lP_{l+1} and Q is the barycenter of the triangle $\triangle P_lP_{l+1}P_{l+2}$, (see FIGURE 2 for $l = 1$). Associated with (76), we now introduce the quadrature error as

$$\mathcal{E}_{\overline{M_lQ} \cap K}(v) = \int_{\overline{M_lQ} \cap K} v(s) ds - \tilde{\mathcal{Q}}_{h,l}(v).$$

Then, we have the following estimate related to the above quadrature error. For a proof, see, Cai [4, pp 732].

Lemma 6.2. *Let $v \in W^{2,\infty}(\overline{M_lQ} \cap K)$. Then, there is a positive constant C, independent of h_K, such that*

$$(77) \qquad\qquad |\mathcal{E}_{\overline{M_lQ} \cap K}(v)| \leq Ch_K^3 \|v\|_{2,\infty,\overline{M_lQ} \cap K},$$

where h_K is the diam(K).

Now to replace the integral in the definition of $A_h(\cdot, \cdot)$, we observe that

$$
\begin{aligned}
A_h(u_h, \Pi_h^* v_h) &= - \sum_{P_l \in N_h} v_i \int_{\partial K_{P_l}^*} A\nabla u_h \cdot \mathbf{n} \, ds \quad \left(v_i = v_h(P_i) \right) \\
&= \sum_K I_K(u_h, \Pi_h^* v_h),
\end{aligned}
$$

where

$$
\begin{aligned}
I_K(u_h, \Pi_h^* v_h) &= - \sum_{P_l (1 \leq l \leq 3)} v_l \int_{\partial K_{P_l}^* \cap K} A\nabla u_h \cdot \mathbf{n_l} ds \\
&= \sum_{P_l (1 \leq l \leq 3)} (v_{l+1} - v_l) \int_{\overline{M_lQ} \cap K} A\nabla u_h \cdot \mathbf{n_l} ds,
\end{aligned}
$$

$v_4 = v_1$ and $\mathbf{n_l}$ is the outward unit normal vector to $\overline{M_lQ}$. Since $\nabla u_h \cdot \mathbf{n_l}$ is constant on each element K, we define the quadrature rule as

$$(78) \qquad \tilde{I}_K(u_h, \Pi_h^* v_h) = \sum_{P_l (1 \leq l \leq 3)} \mathcal{E}_{\overline{M_lQ} \cap K}(A) \nabla u_h \cdot \mathbf{n_l}(v_{l+2} - v_{l+1}).$$

and set

$$\tilde{A}_h(\chi, \Pi_h^* \psi) = \sum_{K \in T_h} \tilde{I}_K(\chi, \Pi_h^* \psi).$$

Note that the bilinear form $A_h(\cdot, \cdot)$ in (7) is approximated by $\tilde{A}_h(\cdot, \cdot)$. Define $\tilde{B}_h(\cdot, \cdot)$ as an approximation of $B_h(\cdot, \cdot)$.

With the definitions as above, define quadrature error functional for the bilinear form $A_h(\cdot, \cdot)$ as

$$(79) \qquad \bar{\epsilon}_A(\chi, \psi) = A_h(\chi, \Pi_h^* \psi) - \tilde{A}_h(\chi, \Pi_h^* \psi) \quad \forall \chi, \ \psi \in U_h.$$

Below, we state without proof the estimate of (79) whose proof can be found in [16].

Lemma 6.3. *Assume that* $\mathcal{A} \in W^{2,\infty}(\Omega; \mathbb{R}^{2\times 2})$. *Then, there exists a positive constant* C, *independent of* h, *such that*

$$\bar{\epsilon}_A(\chi, \psi) \le Ch^2 \|\chi\|_1 \|\psi\|_1 \quad \forall \chi, \ \psi \in U_h.$$

Similar results hold for $\bar{\epsilon}_B(t, s; \cdot, \cdot)$ which is defined as in (79). For the rest of our analysis, we introduce the functionals $S(t) = S$ and $\hat{S}(t) = \hat{S}$ defined on U_h for a given ψ and $t \in (0, T]$ as

$$S(\psi)(\chi) = \bar{\epsilon}_A(\psi, \chi) + \int_0^t \bar{\epsilon}_B(t, s; \psi(s), \chi) ds,$$

and

$$\hat{S}(\psi)(\chi) = \bar{\epsilon}_A(\hat{\psi}, \chi) + \int_0^t \bar{\epsilon}_B(s, s; \hat{\psi}(s), \chi) \, ds - \int_0^t \int_0^s \bar{\epsilon}_{B_\tau}(s, \tau; \hat{\psi}(\tau), \chi) \, d\tau ds.$$

Then using Lemma 6.3, we derive the following estimate for S in a similar manner to those obtained in Lemma 4.3

$$\|S(\psi)\|_{-1,h} \le Ch^2 \left(\|\psi(t)\|_2 + \int_0^t \|\psi(s)\|_2 \, ds \right).$$

Similar result can be obtain for the estimate of $\hat{S}$ again following proof of Lemma 4.3.

Now the semidiscrete finite volume element method combined with quadrature is to seek $u_h : (0, T] \longrightarrow U_h$ such that

$$(80) \qquad (u_{h,tt}, v_h)_h + \tilde{A}_h(u_h, v_h) + \int_0^t \tilde{B}_h(t, s; u_h(s), v_h) \, ds = (f, v_h)_h \ \forall v_h \in U_h^*,$$

with appropriate initial data $u_h(0)$ and $u_{h,t}(0)$ in U_h.

6.1. Optimal error estimates. In this subsection, we discuss optimal estimates in $L^\infty(L^2)$ as well as in $L^\infty(H^1)$ norms and quasi-optimal estimates in $L^\infty(L^\infty)$ norm.

Now replace v_h by $\Pi_h^* \chi$ in (80) and subtract the resulting equation from (8) to obtain

$$
\begin{aligned}
(u_{tt}, \Pi_h^* \chi) - (u_{h,tt}, \Pi_h^* \chi)_h \quad & + \quad A_h(u, \Pi_h^* \chi) - \tilde{A}_h(u_h, \Pi_h^* \chi) \\
& + \quad \int_0^t B_h(t, s; u, \Pi_h^* \chi) \, ds - \int_0^t \tilde{B}_h(t, s; u_h, \Pi_h^* \chi) \, ds \\
(81) \qquad & = (f, \Pi_h^* \chi) - (f, \Pi_h^* \chi)_h \quad \forall \chi \in U_h.
\end{aligned}
$$

Using the definitions of Ritz-Volterra projection $V_h u$ and S, we arrive at an equation in θ as

$$
\begin{aligned}
(\theta_{tt}, \Pi_h^* \chi)_h + A(\theta, \chi) = {}& -(\rho_{tt}, \Pi_h^* \chi) + G(\rho)(\chi) + G(\theta)(\chi) \\
& + S(\theta)(\chi) - S(V_h u)(\chi) + \bar{\epsilon}_h(f, \chi) - \bar{\epsilon}_h((V_h u)_{tt}, \chi) \\
& - \int_0^t B(t, s; \theta, \chi)\, ds.
\end{aligned}
\tag{82}
$$

Below, we establish $L^\infty(H^1)$ estimate.

Theorem 6.1. *Let u and u_h be the solutions of (1) and (80), respectively, and assume that $f \in L^1(H^1)$, f_t, $f_{tt} \in L^1(L^2)$, $u_0 \in H^3 \cap H_0^1$ and $u_1 \in H^2 \cap H_0^1$. With $u_h(0) = \Pi_h u_0$ and $u_{h,t}(0) = \Pi_h u_1$, there exists a positive constant $C = C(T)$, independent of h, such that*

$$
\|u(t) - u_h(t)\|_1 \le Ch \left(\|u_0\|_3 + \|u_1\|_2 + \|f\|_{L^1(H^1)} + \sum_{j=1}^{2} \|D_t^j f\|_{L^1(L^2)} \right)
$$

holds for $t \in (0, T]$.

Proof. Choose $\chi = \theta_t$ in (82) so that

$$
\begin{aligned}
(\theta_{tt}, \Pi_h^* \theta_t)_h + A(\theta, \theta_t) = {}& G(\rho)(\theta_t) + G(\theta)(\theta_t) - (\rho_{tt}, \Pi_h^* \theta_t) + S(\theta)(\theta_t) \\
& - S(V_h u)(\theta_t) + \bar{\epsilon}_h(f, \theta_t) - \bar{\epsilon}_h((V_h u)_{tt}, \theta_t) \\
& - \int_0^t B(t, s; \theta, \theta_t)\, ds.
\end{aligned}
\tag{83}
$$

Then, use (73) and the symmetric property of $A(\cdot, \cdot)$ to obtain

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left[(\theta_t, \theta_t)_h + A(\theta, \theta) \right] = {}& G(\rho)(\theta_t) + G(\theta)(\theta_t) - (\rho_{tt}, \Pi_h^* \theta_t) + S(\theta)(\theta_t) \\
& - S(V_h u)(\theta_t) + \bar{\epsilon}_h(f, \theta_t) - \bar{\epsilon}_h((V_h u)_{tt}, \theta_t) \\
& - \int_0^t B(t, s; \theta, \theta_t)\, ds.
\end{aligned}
$$

Integrate from 0 to t and use the equivalence of the norms in (72) to find that

$$
\begin{aligned}
\frac{1}{2} \left[\|\theta_t\|_h^2 + A(\theta, \theta) \right] = {}& \left\{ \frac{1}{2} \|\theta_t(0)\|_h^2 + \frac{1}{2} A(\theta(0), \theta(0)) + \int_0^t \left[G(\rho)(\theta_t) + G(\theta)(\theta_t) \right. \right. \\
& \left. - (\rho_{tt}, \Pi_h^* \theta_t) - \int_0^s B(s, \tau; \theta(\tau), \theta_t)\, d\tau \right] ds \Bigg\} + \int_0^t S(\theta)(\theta_t)\, ds \\
& - \int_0^t S(V_h u)(\theta_t)\, ds + \int_0^t \bar{\epsilon}_h(f, \theta_t)\, ds - \int_0^t \bar{\epsilon}_h((V_h u)_{tt}, \theta_t)\, ds \\
= {}& I + J_1 + J_2 + J_3 + J_4.
\end{aligned}
\tag{84}
$$

Estimates for the first term I have already been derived in Theorem 4.1. In order to estimate J_1, use Lemma 6.3 and the inverse inequality (26) to obtain

$$
\begin{aligned}
|J_1| &\le \int_0^t |S(\theta)(\theta_t)|\, ds \le \int_0^t \|S(\theta)\|_{-1,h} \|(\theta_t)\|_1\, ds \\
&\le Ch^2 \left[\int_0^t \|\theta\|_1 \|\theta_t\|_1 ds + \int_0^t \int_0^s \|\theta_t(\tau)\|_1 \|\theta(s)\|_1\, d\tau\, ds \right] \\
&\le Ch \left[\int_0^t \|\theta\|_1 \|\theta_t\|_0 ds + \left(\int_0^t \|\theta\|_1\, ds \right) \left(\int_0^t \|\theta_t\|_0\, ds \right) \right].
\end{aligned}
\tag{85}
$$

For J_2, we find that

$$(86) \qquad |J_2| \leq \int_0^t |S(V_h u)(\theta_t)|\, ds \leq Ch^{-1} \int_0^t \|S(V_h u)\|_{-1,h} \|\theta_t\|_0\, ds.$$

In view of Lemma 6.1, the terms J_3 and J_4 are bounded as

$$
\begin{aligned}
|J_3| + |J_4| \quad &\leq \quad \int_0^t |\bar{\epsilon}_h(f,\theta_t)| ds + 2\int_0^t |\bar{\epsilon}_h((V_h u)_{tt}, \theta_t)| ds \\
&\leq \quad Ch^2 \int_0^t (\|f\|_2 + \|\rho_{tt}\|_1 + \|u_{tt}\|_1)\|\theta_t\|_1 ds \\
(87) \qquad &\leq \quad Ch \int_0^t (\|f\|_2 + \|\rho_{tt}\|_1 + \|u_{tt}\|_1)\|\theta_t\|_0 ds.
\end{aligned}
$$

Now, substitute (85)-(87) in (84). Use the coercivity property of the bilinear form $A(\cdot, \cdot)$ and equivalence of norms (72). Then, proceed as in Theorem 4.1 to complete the rest of the proof. ∎

In the following theorem, we prove optimal $L^\infty(L^2)$ estimate.

Theorem 6.2. *Under the assumptions of Theorem 6.1, there exists a positive constant $C = C(T)$, independent of h, such that*

$$\|u(t) - u_h(t)\|_0 \leq Ch^2 \left(\|u_0\|_3 + \|u_1\|_2 + \|f\|_{L^1(H^1)} + \sum_{j=1}^{2} \|D_t^j f\|_{L^1(L^2)} \right)$$

holds for all $t \in (0, T]$.

Proof. Integrate (82) from 0 to t to arrive at

$$
\begin{aligned}
(\theta_t, \Pi_h^* \chi)_h \quad &+ \quad A(\hat{\theta}, \chi) = -(\rho_t, \Pi_h^* \chi) + \hat{G}(\rho)(\chi) + \hat{G}(\theta)(\chi) \\
&+ \quad \hat{S}(\theta)(\chi) - \hat{S}(V_h u)(\chi) + \bar{\epsilon}_h(\hat{f}, \chi) - \bar{\epsilon}_h((V_h u)_t, \chi) \\
(88) \qquad &+ \quad (u_t(0), \Pi_h^* \chi) - (u_{h,t}(0), \Pi_h^* \chi)_h - \int_0^t \int_0^s B(s, \tau; \theta(\tau), \chi)\, d\tau\, ds.
\end{aligned}
$$

Choose $\chi = \theta$ in (88) and use (73) with the symmetry of the bilinear form $A(\cdot, \cdot)$ to obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left[(\theta, \theta)_h + A(\hat{\theta}, \hat{\theta})\right] \quad &= \quad I(t) + \hat{S}(\theta)(\theta) - \hat{S}(V_h u)(\theta) + \bar{\epsilon}_h(\hat{f}, \theta) \\
(89) \qquad &\quad -\bar{\epsilon}_h((V_h u)_t, \theta) + (u_t(0), \Pi_h^* \theta) - (u_{h,t}(0), \Pi_h^* \theta)_h,
\end{aligned}
$$

where

$$I(t) = -(\rho_t, \Pi_h^* \theta) + \hat{G}(\rho)(\theta) + \hat{G}(\theta)(\theta) - \int_0^t \int_0^s B(s, \tau; \theta(\tau), \theta)\, d\tau ds.$$

Integrate (89) from 0 to t to find that

$$
\begin{aligned}
\frac{1}{2}\left(\|\theta(t)\|_h^2 + A(\hat{\theta}, \hat{\theta})\right) \quad &= \quad \frac{1}{2}\|\theta(0)\|_h^2 + \int_0^t I(s)\, ds + \int_0^t \hat{S}(\theta)(\theta)\, ds \\
&\quad - \int_0^t \hat{S}(V_h u)(\theta)\, ds - \int_0^t \bar{\epsilon}_h((V_h u)_t, \theta)\, ds \\
&\quad + \int_0^t \bar{\epsilon}_h(\hat{f}, \theta)\, ds + \left[(u_t(0), \Pi_h^* \hat{\theta}) - (u_{h,t}(0), \Pi_h^* \hat{\theta})_h\right] \\
(90) \qquad &= \quad \frac{1}{2}\|\theta(0)\|_h^2 + \int_0^t I(s)\, ds + J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}
$$

Note that estimates for the first two terms on the right hand sides of (90) have already been derived in Theorem 4.2. For J_1, use the definition of $\hat{S}$ and integrate by parts to arrive at

$$(91) \qquad J_1 \;=\; \int_0^t \bar{\epsilon}_A(\hat{\theta}, \theta)\, ds + \int_0^t \int_0^s \bar{\epsilon}_B(\tau, \tau; \hat{\theta}(\tau), \theta(s))\, d\tau\, ds$$

$$(92) \qquad\qquad - \int_0^t \int_0^s \int_0^\tau \bar{\epsilon}_{B_{\tau'}}(\tau, \tau'; \hat{\theta}(\tau'), \theta)\, d\tau'\, d\tau\, ds$$

$$(93) \qquad\qquad = \;\; J_{11} + J_{12} + J_{13}.$$

For J_{11}, a use of Lemma 6.3 with the inverse inequality (26) yields

$$(94) \qquad |J_{11}| \le \int_0^t |\bar{\epsilon}_A(\hat{\theta}, \theta)| ds \le Ch^2 \int_0^t \|\theta\|_1 \|\hat{\theta}\|_1 ds \le Ch \int_0^t \|\theta\|_0 \|\hat{\theta}\|_1 \, ds.$$

For J_{12}, an integration by parts shows

$$\begin{aligned} |J_{12}| \;&=\; \left| \int_0^t \bar{\epsilon}_B(s, s; \hat{\theta}(s), \hat{\theta}(t)) ds - \int_0^t \bar{\epsilon}_B(s, s; \hat{\theta}(s), \hat{\theta}(s)) ds \right| \\ &\le\; Ch^2 \left\{ \|\hat{\theta}(t)\|_1 \int_0^t \|\hat{\theta}(s)\|_1 ds + \int_0^t \|\hat{\theta}(s)\|_1^2 ds \right\}. \end{aligned}$$

Similarly for J_{13}, we have

$$\begin{aligned} |J_{13}| \;&=\; \left| \int_0^t \int_0^s \bar{\epsilon}_{B_\tau}(s, \tau; \hat{\theta}(\tau), \hat{\theta}(t)) d\tau ds - \int_0^t \int_0^s \bar{\epsilon}_{B_\tau}(s, \tau; \hat{\theta}(\tau), \hat{\theta}(s)) d\tau ds \right| \\ &\le\; C(T) h^2 \left\{ \|\hat{\theta}(t)\|_1 \int_0^t \|\hat{\theta}(s)\|_1 ds + \int_0^t \|\hat{\theta}(s)\|_1^2 ds \right\}. \end{aligned}$$

For J_2, we obtain

$$(95) \qquad |J_2| \le \|\hat{S}(V_h u)\|_{-1,h} \|\hat{\theta}\|_1 + \int_0^t \|\hat{S}_s(V_h u)\|_{-1,h} \|\hat{\theta}\|_1 \, ds.$$

To bound J_3 and J_4, we integrate by parts and apply Lemma 6.1 to arrive at

$$\begin{aligned} |J_3| \;&\le\; |\bar{\epsilon}_h((V_h u)_t, \hat{\theta})| + \int_0^t |\bar{\epsilon}_h((V_h u)_{tt}, \hat{\theta})| ds \\ (96) \qquad &\le\; Ch^2 \left((\|\rho_t\|_1 + \|u_t\|_1) \|\hat{\theta}\|_1 + \int_0^t (\|\rho_{tt}\|_1 + \|u_{tt}\|_1) \|\hat{\theta}\|_1 \, ds \right) \end{aligned}$$

and

$$\begin{aligned} |J_4| \;&\le\; |\bar{\epsilon}_h(\hat{f}, \hat{\theta})| + \int_0^t |\bar{\epsilon}_h(f, \hat{\theta})| ds \\ (97) \qquad &\le\; Ch^2 \left(\|\hat{f}\|_2 \|\hat{\theta}\|_1 + \int_0^t \|f\|_2 \|\hat{\theta}\|_1 ds \right). \end{aligned}$$

Finally, since $u_{h,t}(0) = \Pi_h u_t(0)$, we have $J_5 = (u_t(0) - \Pi_h u_t(0), \Pi_h^* \hat{\theta}) + \bar{\epsilon}_h(\Pi_h u_t(0), \hat{\theta})$. Hence,

$$\begin{aligned} |J_5| \;&\le\; Ch^2 \left(\|u_t(0)\|_2 + \|\Pi_h u_t(0)\|_1 \right) \|\hat{\theta}\|_1 \\ (98) \qquad &\le\; Ch^2 \|u_t(0)\|_2 \|\hat{\theta}\|_1 \le Ch^2 \|u_t(0)\|_2 \|\hat{\theta}\|_1. \end{aligned}$$

Substitute (94)-(98) in (90). We use the coercivity property of the bilinear form $A(\cdot, \cdot)$ and the equivalence of the norms, and proceed as in Theorem 4.2 to complete the rest of the proof. ∎

Finally, we prove quasi-optimal maximum norm estimate.

Theorem 6.3. *Let u and u_h be the solutions of (1) and (7), respectively. Further, let the assumptions of Lemma 4.4 hold. Then,*

$$\|u(t)-u_h(t)\|_\infty \leq C(T)h^2 \left(\log\frac{1}{h}\right)\left(\|u_0\|_4+\|u_1\|_3+\|D_t^3 f\|_{L^1(L^2)}+\sum_{j=0}^{2}\|D_t^j f\|_{L^1(H^{2-j})}\right),$$

where $C(T)$ is a positive constant, independent of h.

Proof: Since $u_h(0) = V_h u_0$, it follows that $\theta(0) = 0$. Then, we modify our estimates for J_2 to J_4 in (84) to arrive at a superconvergence result for θ in H^1 norm

$$\|\theta_t\|_0 + \|\theta\|_1 \quad \leq \quad C(T)h^2\Big(\|u_0\|_4 + \|u_1\|_3 + \|D_t^3 f\|_{L^1(L^2)}$$

(99)
$$+ \sum_{j=0}^{2} \|D_t^j f\|_{L^1(H^{2-j})}\Big).$$

Now, a use of (48) and (99) completes the rest of the proof. ∎

7. Numerical Experiment

In this section, we present numerical results to illustrate the performance of the finite volume element method applied to (1). Assume that $\mathcal{T}_h$ is an admissible regular, uniform triangulation of $\overline{\Omega}$ into closed triangles and $0 = t_0 < t_1 < \cdots t_M = T$ is a given partition of the time interval $(0,T]$ with step length $k = \frac{T}{M}$ for some positive integer M. With U^n denoting the approximation of u_h at $t = t_n$, consider the discrete-in-time scheme derived in Section 5, with discrete L^2 inner product $(\cdot,\cdot)_h$ and the bilinear forms $A_h(\cdot,\cdot)$ and $B_h(t,s;\cdot,\cdot)$ evaluated using numerical quadrature formulae.

Thus, the time discretization scheme is to seek $U^n \in U_h$ for given U^0, such that

(100)
$$\frac{2}{k}(\partial_t U^{1/2}, \Pi_h^* \chi)_h \quad + \quad \tilde{A}_h(U^0, \Pi_h^* \chi) = (f^0 + \frac{2}{k}u_1, \Pi_h^* \chi)_h,$$

$$(\partial_t^2 U^n, \Pi_h^* \chi)_h + \tilde{A}_h(U^n, \Pi_h^* \chi) \quad + \quad k\sum_{j=0}^{n-1} \tilde{B}_h(t_n, t_{j+1/2}; U^{j+1/2}, \Pi_h^* \chi)$$

(101)
$$= \quad (f^n, \Pi_h^* \chi)_h,$$

$n \geq 1$, for all $\chi \in U_h$. The method is explicit in time in the sense that the calculation of U^n involves only the inversion of a mass-type matrix associated with the space U_h and the corresponding dual volume element space U_h^*.

Let $\{\phi_j\}_{j=1,2,\cdots,N}$ be the standard nodal basis functions for the trial space U_h and $\{\chi_j\}_{j=1,2,\cdots,N}$ be the characteristic basis functions corresponding to the control volumes for the test space U_h^*. Then, express U^n as

$$U^n = \sum_{j=1}^{N} \alpha_j^n \phi_j(x), \text{ where } \alpha_j^n = U^n(x_j).$$

Define now the following matrices

$$\mathbb{M} = [(\phi_i, \chi_j)_h]_{N\times N}, \quad \mathbb{A} = [\tilde{A}_h(\phi_i, \chi_j)]_{N\times N}, \quad \mathbb{B}(t,s) = [\tilde{B}_h(t,s;\phi_i, \chi_j)]_{N\times N},$$

and the vector $\mathbb{F}(t) = [(f(t), \chi_j)_h]_{1\times N}$. Then, for instance, (101) can be written as the following system of linear equations which can be solved for $\bar{\alpha}^{n+1}$:

$$\mathbb{M}\bar{\alpha}^{n+1} = (2\mathbb{M} - k^2\mathbb{A})\bar{\alpha}^{n-1} - \mathbb{M}\bar{\alpha}^{n-1} - k^3\sum_{l=0}^{n-1}\mathbb{B}(t_n, t_{l+1/2})\bar{\alpha}^{l+1/2} + k^2\mathbb{F}^n,$$
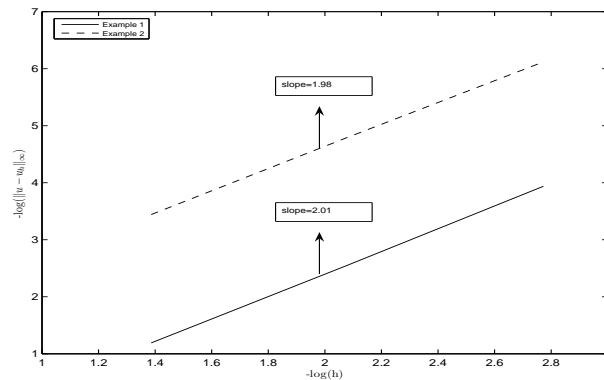
FIGURE 3. Convergence order estimate in L^∞-norm.

where $\bar{\alpha}^n = (\alpha_1^n,\ \alpha_2^n, \cdots, \alpha_N^n)^T$. Since we have used mass lumping for $(\cdot,\cdot)_h$, the mass matrix $\mathbb{M}$ is a diagonal matrix.

In order to illustrate the performance of the finite volume element method for solving (1), we consider the following test problems where the computational domain $\Omega = (0,1) \times (0,1)$ and the final time $T = 1$.

Example 1: We choose $u_0(x,y) = \sin(\pi x)\sin(\pi y)$, $u_1(x,y) = \sin(\pi x)\sin(\pi y)$, $A = I$ and $B(t,s) = e^{(t-s)}I$. The function f is chosen so that the exact solution is

$$u = e^t \sin(\pi x)\sin(\pi y).$$

Example 2: Set $u_0(x,y) = xy(x-1)(y-1)$, $u_1(x,y) = xy(x-1)(y-1)$, $A = \begin{pmatrix} 1+x^2 & 0 \\ 0 & 1+x^2 \end{pmatrix}$ and $B(t,s) = e^{(t-s)}A$. The function f is chosen in such a way that the exact solution is

$$u = e^t xy(x-1)(y-1).$$

The order of convergence is computed in L^∞ norm. In both examples, Fig 3 shows that the computed order of convergence for $\|u - u_h\|_\infty$ in the log-log scale matches with the theoretical order of convergence that we have derived.

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