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SCHEMES AND ESTIMATES FOR THE LONG-TIME NUMERICAL SOLUTION OF MAXWELL'S EQUATIONS FOR LORENTZ METAMATERIALS

JICHUN LI AND SIMON SHAW

Abstract. We consider time domain formulations of Maxwell's equations for the Lorentz model for metamaterials. The field equations are considered in two different forms which have either six or four unknown vector fields. In each case we use arguments tuned to the physical laws to derive data-stability estimates which do not require Gronwall's inequality. The resulting estimates are, in this sense, sharp. We also give fully discrete formulations for each case and extend the sharp data-stability to these. Since the physical problem is linear it follows (and we show this with examples) that this stability property is also reflected in the constants appearing in the *a priori* error bounds. By removing the exponential growth in time from these estimates we conclude that these schemes can be used with confidence for the long-time numerical simulation of Lorentz metamaterials.

Key words. Maxwell's equations, Lorentz model, metamaterial, Galerkin and mixed finite element method, long-time integration, time stepping.

1. Introduction

Electromagnetic metamaterials are artificially structured materials which exhibit exotic properties such as negative refractive index and reversed Doppler effects. The successful construction of such metamaterials in 2000 triggered a wave of further study of metamaterials and exploration of their applications in diverse areas such as sub-wavelength imaging and cloaking. More details can be found in monographs such as [9, 28, 34, 7] and references cited therein.

Although the finite element approximation of Maxwell's equations has been extensively documented for 'classical' materials (see, for example, [3, 4, 8, 14, 31, 33, 37] and their references), there is now an opportunity to build on this body of knowledge for the development and analysis of finite element methods (FEM) for Maxwell's equations for metamaterials. In this direction we mention [10, 11, 5, 2, 21] for the time-harmonic form, and [19, 20, 16] for the time-domain form. Our focus here is on the Lorentz model which, as we will see below, introduces additional unknowns for electrical and magnetic polarizations. These are governed by ordinary differential equations (in time) which hold at each point in space and have the effect of making the (meta)material dispersive, or 'frequency dependent'. In this context we recall also the work on the time-domain Maxwell's equations in general dispersive media in [1, 17, 24, 35, 27, 36]. In particular, [1] contains a study of numerical dispersion for Debye and Lorentz media and [35] gives long-time stability and error estimates for a Debye model.

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In recent years there have been several efforts in developing and analyzing some FEMs for the time-domain Maxwell's equations for Lorentz metamaterials (see, for example, [22] and the references therein). However most of these previous results for data-stability and error bounds were derived with the use of Gronwall-type inequalities and, hence, are of limited practical use due to the exponential growth, in time, of the constants. This article improves upon this current 'state of the art' by building upon the 'long-time' results in [35] for two popular numerical schemes.

To be precise, in Section 2 we describe the time domain formulation of Maxwell's equations for Lorentz metamaterials. In Sections 3 and 4, respectively, the field equations are considered in two different forms which have, respectively, six and four unknown vector fields. In each case we use arguments tuned to the physical laws to derive data-stability estimates which do not require Gronwall's inequality. The resulting estimates are sharp, in that they contain stability constants that are time independent, and appear to be novel. We also give fully discrete formulations for each case and extend the sharp data stability to these formulations. Moreover, since the physical problem is linear the error terms obey essentially the same stability estimates but with data replaced by approximation error. With this in mind we can therefore show by examples that the long-time stability properties of these schemes are also reflected in the *a priori* error bounds. The time dependence in these constants then arises from the time dependence in the norms of the data and exact solution and produces, at worst, low-order-polynomial growth in time rather than the exponential growth that arises from Gronwall arguments. Hence, we can conclude that the resulting numerical schemes can be used with confidence for the long time numerical simulation of Lorentz metamaterials. This is the major contribution of the work presented below. In Section 5 we close with a short discussion of the formulations.

Throughout our notation is mostly standard. For example, $C \geqslant 0$ will denote a generic positive constant (independent of the finite element mesh size h and time step size τ) and we let $(H^\sigma(\Omega))^3$ be the standard Sobolev space equipped with the norm $\|\cdot\|_\sigma$ and semi-norm $|\cdot|_\sigma$. Specifically, $\|\cdot\|_0$ will mean the $(L^2(\Omega))^3$-norm. From [31] (for example) we also recall the standard spaces for Maxwell problems,

$$\begin{aligned} H(\mathrm{curl};\Omega) &= \{\boldsymbol{v} \in (L^2(\Omega))^3 :\ \nabla\times\boldsymbol{v} \in (L^2(\Omega))^3\}, \\ H_0(\mathrm{curl};\Omega) &= \{\boldsymbol{v} \in H(\mathrm{curl};\Omega) :\ \mathbf{n}\times\boldsymbol{v} = 0 \text{ on } \partial\Omega\}, \\ H^\sigma(\mathrm{curl};\Omega) &= \{\boldsymbol{v} \in (H^\sigma(\Omega))^3 :\ \nabla\times\boldsymbol{v} \in (H^\sigma(\Omega))^3\}, \end{aligned}$$

where $\sigma \geqslant 0$ is a real number, and Ω is a bounded Lipschitz polyhedral domain in $\mathcal{R}^3$ with connected boundary $\partial\Omega$ and outward directed unit normal $\mathbf{n}$. We equip $H(\mathrm{curl};\Omega)$ with norm $\|\boldsymbol{v}\|_{0,\mathrm{curl}} = (\|\boldsymbol{v}\|_0^2 + \|\mathrm{curl}\,\boldsymbol{v}\|_0^2)^{1/2}$, and $H^\sigma(\mathrm{curl};\Omega)$ with norm $\|\boldsymbol{v}\|_{\sigma,\mathrm{curl}} = (\|\boldsymbol{v}\|_\sigma^2 + \|\mathrm{curl}\,\boldsymbol{v}\|_\sigma^2)^{1/2}$. For clarity, in the rest of the paper we introduce the vector notation $\boldsymbol{L}^2(\Omega) = (L^2(\Omega))^3$ and $\boldsymbol{H}^\sigma(\Omega) = (H^\sigma(\Omega))^3$ and also we often omit the explicit display of the dependence of quantitites on $\boldsymbol{x} \in \Omega$ because we want to focus on the handling of their time dependence. The spatial dependencies are handled in a standard way. Further notation is introduced as and when needed.

2. The governing equations

In general terms, the problem of electromagnetic wave propagation requires the solution of Maxwell's equations,

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}, \quad \text{and} \quad \nabla\times\boldsymbol{H} = \frac{\partial\boldsymbol{D}}{\partial t} \quad \text{in } \Omega\times I \tag{1}$$

where $\boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{H}(\boldsymbol{x},t)$ are the electric and magnetic fields, and where $\boldsymbol{D}(\boldsymbol{x},t)$ and $\boldsymbol{B}(\boldsymbol{x},t)$ are the corresponding electric and magnetic flux densities. We will be more specific about inital and boundary data below but here, to close the problem, we note that in a general (linear) medium $\boldsymbol{D}$ and $\boldsymbol{B}$ are related to the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ through the constitutive relations

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} \equiv \epsilon \boldsymbol{E} \quad \text{and} \quad \boldsymbol{B} = \mu_0 \boldsymbol{H} + \boldsymbol{M} \equiv \mu \boldsymbol{H}. \tag{2}$$

Here ϵ_0 is the vacuum permittivity, μ_0 is the vacuum permeability, and $\boldsymbol{P}$ (respectively $\boldsymbol{M}$) is the induced electric (respectively magnetic) polarization. The introduction of ϵ and μ as the permittivity and permeability of the underlying medium implies that there is a functional relationship between $\boldsymbol{E}$ and $\boldsymbol{P}$, and between $\boldsymbol{H}$ and $\boldsymbol{M}$, and it is the form of these relationships that determines the type of medium and its properties. It is convenient to first discuss these models in the frequency domain although later we will revert to the time domain for the specific formulations that we study.

One of the most general models used for modeling wave propagation in metamaterials (see, for example [22]) is the so-called Lorentz model, whose permittivity and permeability are described by

$$\epsilon(\omega) = \epsilon_0 \left(1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{e0}^2 - j\Gamma_e \omega}\right), \quad \mu(\omega) = \mu_0 \left(1 - \frac{\omega_{pm}^2}{\omega^2 - \omega_{m0}^2 - j\Gamma_m \omega}\right), \tag{3}$$

where ω_{pe} (respectively ω_{pm}) is the electric (respectively magnetic) plasma frequency, Γ_e (respectively Γ_m) is the electric (respectively magnetic) damping frequency, ω_{e0} (respectively ω_{m0}) is the electric (respectively magnetic) resonance frequency, $j = \sqrt{-1}$ is the imaginary unit, and ω is a general frequency. Notice that when $\omega_{e0} = \omega_{m0} = 0$, the Lorentz model reduces to the Drude model (e.g. [22]) which is another popular metamaterial model:

$$\epsilon(\omega) = \epsilon_0 \left(1 - \frac{\omega_{pe}^2}{\omega(\omega - j\Gamma_e)}\right), \quad \mu(\omega) = \mu_0 \left(1 - \frac{\omega_{pm}^2}{\omega(\omega - j\Gamma_m)}\right). \tag{4}$$

Furthermore, if we set $\Gamma_e = \Gamma_m = 0$ then this Drude model reduces to the cold plasma model,

$$\epsilon(\omega) = \epsilon_0 \left(1 - \frac{\omega_{pe}^2}{\omega^2}\right), \quad \mu(\omega) = \mu_0 \left(1 - \frac{\omega_{pm}^2}{\omega^2}\right) \tag{5}$$

and so we see that, as long as we allow for these reductions, the study of the Lorentz model, (3), presented below also includes these other models. Therefore, in the rest of this article, unless specified clearly to the contrary, we will assume that all of the physical parameters are positive (i.e. $\epsilon_0, \mu_0, \omega_{e0}, \omega_{m0}, \ldots$).

Moving away from the frequency domain formulation we recall from [20], or infer from (3) above, the following equations for the time-domain Lorentz model for metamaterials:

$$\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial \boldsymbol{P}}{\partial t} - \nabla \times \boldsymbol{H} = 0, \quad \text{in } \Omega \times (0,T), \tag{6}$$

$$\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} + \frac{\partial \boldsymbol{M}}{\partial t} + \nabla \times \boldsymbol{E} = 0, \quad \text{in } \Omega \times (0,T), \tag{7}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}\frac{\partial^2 \boldsymbol{P}}{\partial t^2}+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\frac{\partial \boldsymbol{P}}{\partial t}+\frac{\omega_{e0}^2}{\epsilon_0\omega_{pe}^2}\boldsymbol{P}-\boldsymbol{E}=0, \quad \text{in } \Omega\times(0,T), \tag{8}$$

$$\frac{1}{\mu_0\omega_{pm}^2}\frac{\partial^2 \boldsymbol{M}}{\partial t^2}+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}\frac{\partial \boldsymbol{M}}{\partial t}+\frac{\omega_{m0}^2}{\mu_0\omega_{pm}^2}\boldsymbol{M}-\boldsymbol{H}=0, \quad \text{in } \Omega\times(0,T). \tag{9}$$

To make the problem well-posed, we assume that (6)-(9) are supplemented by the perfectly-conducting boundary condition

$$\mathbf{n}\times\boldsymbol{E}=0 \quad \text{on } \partial\Omega, \tag{10}$$

and, with $\boldsymbol{x}$ dependence suppressed, the initial conditions

$$\boldsymbol{E}(0)=\boldsymbol{E}_0, \quad \boldsymbol{H}(0)=\boldsymbol{H}_0, \tag{11}$$

$$\boldsymbol{P}(0)=\boldsymbol{P}_0, \quad \boldsymbol{M}(0)=\boldsymbol{M}_0, \quad \frac{\partial \boldsymbol{P}}{\partial t}(0)=\boldsymbol{P}_1, \quad \frac{\partial \boldsymbol{M}}{\partial t}(0)=\boldsymbol{M}_1, \tag{12}$$

where $\boldsymbol{E}_0, \boldsymbol{H}_0, \boldsymbol{P}_0, \boldsymbol{M}_0, \boldsymbol{P}_1$, and $\boldsymbol{M}_1$ are given functions.

Now that the physical model is completely specified we notice that there are several ways in which we could approach it in terms of giving a fully discrete numerical approximation. If we work with the model as described then, in the simplest case, we need to store ten vector fields: the current and previous time steps for $\boldsymbol{E}$ and $\boldsymbol{H}$ and three consecutive time levels for $\boldsymbol{P}$ and $\boldsymbol{M}$ (i.e. thirty scalar fields in $\mathcal{R}^3$). And, furthermore, we would need a time stepper that can handle the second time derivatives. We do not consider the discretization of second order ODE's in this article but instead first, in Section 3, we reduce (8) and (9) to first order ODE's by defining $\boldsymbol{J}=\boldsymbol{P}_t$ and $\boldsymbol{K}=\boldsymbol{M}_t$ where, here and below, the subscript denotes partial differentiation. We then need only handle first time derivatives in the time stepping but we will have to store twelve vector fields. Alternatively, in Section 4, we introduce another formulation which uses ony four vector fields at each of two time levels but requires the time integrals of $\boldsymbol{J}$ and $\boldsymbol{K}$. Since, in the time-discrete setting, these can be updated by recursion this scheme requires the storage of only ten vector fields.

We also note that we could formulate this Lorentz model in first-order form with only two vector fields, $\boldsymbol{E}$ and $\boldsymbol{H}$. The result is a system of convolution-type integrodifferential equations with non-monotone kernels of positive type. In the discrete formulation these 'history integrals' can be updated in a recursive way by introducing complex arithmetic and using Euler's formula, and the scheme would need only eight real vector fields to be stored in memory. We do not study this scheme here because of the difficulties associated with proving that the discrete memory term is also of positive type. In general we can expect such a proof to be non-trivial and, for example, refer to the analysis in [30] for the case of monotone kernels.

It is not the purpose of this study to reach any decision as to which of these schemes is the 'best' since that would require a more practical study organised around a well-chosen set of numerical tests. Rather we show, in each case, that a careful treatment of the dispersive terms results in data-stability estimates for both the continuous and discrete problems with constants that grow much more slowly with time that Gronwall-type estimates would suggest. This analysis carries over to the constants in the *a priori* error bounds.

3. Investigation of the Lorentz model in six variables

Following our previous work in [20] we can rewrite the Lorentz model (6)-(9) as,

$$\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{H} + \boldsymbol{J} = 0, \tag{13}$$

$$\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} + \nabla \times \boldsymbol{E} + \boldsymbol{K} = 0, \tag{14}$$

$$\frac{1}{\epsilon_0 \omega_{pe}^2} \frac{\partial \boldsymbol{J}}{\partial t} + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2} \boldsymbol{J} - \boldsymbol{E} + \frac{\omega_{e0}^2}{\epsilon_0 \omega_{pe}^2} \boldsymbol{P} = 0, \tag{15}$$

$$\frac{\omega_{e0}^2}{\epsilon_0 \omega_{pe}^2} \frac{\partial \boldsymbol{P}}{\partial t} - \frac{\omega_{e0}^2}{\epsilon_0 \omega_{pe}^2} \boldsymbol{J} = 0, \tag{16}$$

$$\frac{1}{\mu_0 \omega_{pm}^2} \frac{\partial \boldsymbol{K}}{\partial t} + \frac{\Gamma_m}{\mu_0 \omega_{pm}^2} \boldsymbol{K} - \boldsymbol{H} + \frac{\omega_{m0}^2}{\mu_0 \omega_{pm}^2} \boldsymbol{M} = 0, \tag{17}$$

$$\frac{\omega_{m0}^2}{\mu_0 \omega_{pm}^2} \frac{\partial \boldsymbol{M}}{\partial t} - \frac{\omega_{m0}^2}{\mu_0 \omega_{pm}^2} \boldsymbol{K} = 0, \tag{18}$$

where we define the induced electric and magnetic currents $\boldsymbol{J} = \boldsymbol{P}_t$ and $\boldsymbol{K} = \boldsymbol{M}_t$. Note that redundant coefficients are included in (16) and (18) in order to make the forthcoming stability and error analysis easier to follow.

Letting $\boldsymbol{V}^*$ denote the topological dual space of $\boldsymbol{V} = H_0(\mathrm{curl}; \Omega)$ and denoting the standard $\boldsymbol{L}^2(\Omega)$ inner product as $(\cdot\,, \cdot)$, it is then easy to see that a weak formulation of (13)-(18) can be written as: Find $\boldsymbol{E} \in H^1(0, T; \boldsymbol{V}^*) \cap \boldsymbol{L}^2(0, T; \boldsymbol{V})$, and $\boldsymbol{J}, \boldsymbol{P}, \boldsymbol{H}, \boldsymbol{K}, \boldsymbol{M} \in H^1(0, T; \boldsymbol{L}^2(\Omega))$ such that

$$\epsilon_0(\boldsymbol{E}_t, \boldsymbol{\phi}) - (\boldsymbol{H}, \nabla \times \boldsymbol{\phi}) + (\boldsymbol{J}, \boldsymbol{\phi}) = 0, \quad \forall\ \boldsymbol{\phi} \in H_0(\mathrm{curl}; \Omega), \tag{19}$$

$$\mu_0(\boldsymbol{H}_t, \boldsymbol{\psi}) + (\nabla \times \boldsymbol{E} + \boldsymbol{K}, \boldsymbol{\psi}) = 0, \quad \forall\ \boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega), \tag{20}$$

$$\left(\frac{1}{\epsilon_0 \omega_{pe}^2} \boldsymbol{J}_t + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2} \boldsymbol{J} - \boldsymbol{E} + \frac{\omega_{e0}^2}{\epsilon_0 \omega_{pe}^2} \boldsymbol{P}, \boldsymbol{\phi}_1\right) = 0, \quad \forall\ \boldsymbol{\phi}_1 \in \boldsymbol{L}^2(\Omega), \tag{21}$$

$$\left(\frac{\omega_{e0}^2}{\epsilon_0 \omega_{pe}^2} \boldsymbol{P}_t - \frac{\omega_{e0}^2}{\epsilon_0 \omega_{pe}^2} \boldsymbol{J}, \boldsymbol{\phi}_2\right) = 0, \quad \forall\ \boldsymbol{\phi}_2 \in \boldsymbol{L}^2(\Omega), \tag{22}$$

$$\left(\frac{1}{\mu_0 \omega_{pm}^2} \boldsymbol{K}_t + \frac{\Gamma_m}{\mu_0 \omega_{pm}^2} \boldsymbol{K} - \boldsymbol{H} + \frac{\omega_{m0}^2}{\mu_0 \omega_{pm}^2} \boldsymbol{M}, \boldsymbol{\psi}_1\right) = 0, \quad \forall\ \boldsymbol{\psi}_1 \in \boldsymbol{L}^2(\Omega), \tag{23}$$

$$\left(\frac{\omega_{m0}^2}{\mu_0 \omega_{pm}^2} \boldsymbol{M}_t - \frac{\omega_{m0}^2}{\mu_0 \omega_{pm}^2} \boldsymbol{K}, \boldsymbol{\psi}_2\right) = 0, \quad \forall\ \boldsymbol{\psi}_2 \in \boldsymbol{L}^2(\Omega), \tag{24}$$

subject to the boundary condition (10) and initial conditions

$$\boldsymbol{E}(\mathbf{x}, 0) = \boldsymbol{E}_0(\mathbf{x}), \quad \boldsymbol{J}(\mathbf{x}, 0) = \boldsymbol{J}_0(\mathbf{x}), \quad \boldsymbol{P}(\mathbf{x}, 0) = \boldsymbol{P}_0(\mathbf{x}), \tag{25}$$

$$\boldsymbol{H}(\mathbf{x}, 0) = \boldsymbol{H}_0(\mathbf{x}), \quad \boldsymbol{K}(\mathbf{x}, 0) = \boldsymbol{K}_0(\mathbf{x}), \quad \boldsymbol{M}(\mathbf{x}, 0) = \boldsymbol{M}_0(\mathbf{x}). \tag{26}$$

Choosing the test functions in (19)-(24) as $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{J}, \boldsymbol{P}, \boldsymbol{K}$ and $\boldsymbol{M}$, respectively, and summing up the results, we can easily obtain (see [20, Lemma 5.1]) the following data-stability for the continuous time system:

$$\epsilon_0 ||\boldsymbol{E}(t)||_0^2 + \mu_0 ||\boldsymbol{H}(t)||_0^2 + \frac{1}{\epsilon_0 \omega_{pe}^2} ||\boldsymbol{J}||_0^2 + \frac{1}{\mu_0 \omega_{pm}^2} ||\boldsymbol{K}||_0^2$$
$$+ \frac{\omega_{e0}^2}{\epsilon_0 \omega_{pe}^2} ||\boldsymbol{P}(t)||_0^2 + \frac{\omega_{m0}^2}{\mu_0 \omega_{pm}^2} ||\boldsymbol{M}(t)||_0^2$$

$$\leqslant \epsilon_0||\boldsymbol{E}(0)||_0^2+\mu_0||\boldsymbol{H}(0)||_0^2+\frac{1}{\epsilon_0\omega_{pe}^2}||\boldsymbol{J}(0)||_0^2$$
$$+\frac{1}{\mu_0\omega_{pm}^2}||\boldsymbol{K}(0)||_0^2+\frac{\omega_{e0}^2}{\epsilon_0\omega_{pe}^2}||\boldsymbol{P}(0)||_0^2+\frac{\omega_{m0}^2}{\mu_0\omega_{pm}^2}||\boldsymbol{M}(0)||_0^2.$$

To design a mixed finite element method, we partition Ω by a family of regular tetrahedral (or hexaderal) meshes T_h with maximum mesh size h and, as long as we bear in mind the effect of solution-regularity, we can in principle use any order of Raviart-Thomas-Nédélec finite element spaces on this mesh. For tetrahedra (e.g., [32] and [31]), for any $l \geqslant 1$, and with $S_l = \{\vec{p} \in (\tilde{p}_l)^3, \boldsymbol{x}\cdot\vec{p}=0\}$, these spaces are,

$$\begin{aligned}\boldsymbol{U}_h &= \{\boldsymbol{u}_h \in H(\mathrm{div};\Omega):\ \boldsymbol{u}_h|_K \in (p_{l-1})^3 \oplus \tilde{p}_{l-1}\boldsymbol{x},\ \forall\, K \in T_h\},\\ \boldsymbol{V}_h &= \{\boldsymbol{v}_h \in H(\mathrm{curl};\Omega):\ \boldsymbol{v}_h|_K \in (p_{l-1})^3 \oplus S_l,\ \forall\, K \in T_h\},\end{aligned}$$

while for Raviart-Thomas-Nédélec cubic elements (e.g., [32] and [31]) we have,

$$\begin{aligned}\boldsymbol{U}_h &= \{\boldsymbol{u}_h \in H(\mathrm{div};\Omega):\ \boldsymbol{u}_h|_K \in Q_{l,l-1,l-1}\times Q_{l-1,l,l-1}\times Q_{l-1,l-1,l},\ \forall\, K \in T_h\},\\ \boldsymbol{V}_h &= \{\boldsymbol{v}_h \in H(\mathrm{curl};\Omega):\ \boldsymbol{v}_h|_K \in Q_{l-1,l,l}\times Q_{l,l-1,l}\times Q_{l,l,l-1},\ \forall\, K \in T_h\}.\end{aligned}$$

Here p_k denotes the space of polynomials of degree k, $\tilde{p}_k$ denotes the space of homogeneous polynomials of degree k, and $Q_{i,j,k}$ denotes the space of polynomials whose degrees are less than or equal to i,j,k in variables x,y,z, respectively. To impose the boundary condition (10), we denote $\boldsymbol{V}_h^0 = \{\boldsymbol{v}\in\boldsymbol{V}_h : \boldsymbol{v}\times\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$. It is easy to see that

$$\nabla\times\boldsymbol{V}_h \subset \boldsymbol{U}_h. \tag{27}$$

For error analysis, we need two more operators. The first one is the standard $\boldsymbol{L}^2(\Omega)$-projection operator Π_2: For any $\boldsymbol{H}\in\boldsymbol{L}^2(\Omega)$, $\Pi_2\boldsymbol{H}\in\boldsymbol{U}_h$ satisfies

$$(\Pi_2\boldsymbol{H}-\boldsymbol{H},\boldsymbol{\psi}_h)=0,\quad \forall\ \boldsymbol{\psi}_h\in\boldsymbol{U}_h \tag{28}$$

and where norms of $\Pi_2\boldsymbol{H}-\boldsymbol{H}$ can be estimated by standard best approximation arguments. The second is the standard Nédélec interpolation operator Π_h defined from $H(\mathrm{curl};\Omega)$ to $\boldsymbol{V}_h$. We refer to the literature (e.g. [31, Thm, 5.41]) for full details but here for the $\ell-$th order first-type curl-conforming Nédélec spaces in [32], we will assume error bounds of the form

$$\|\boldsymbol{E}-\Pi_h\boldsymbol{E}\|_0+\|\nabla\times(\boldsymbol{E}-\Pi_h\boldsymbol{E})\|_0 \leqslant Ch^\ell\|\boldsymbol{E}\|_{\boldsymbol{H}^\ell(\mathrm{curl};\Omega)}. \tag{29}$$

These estimates should be regarded in the context of the usual technical assumptions of the mesh being shape-regular.

To define a fully discrete scheme, we assume that the time interval $(0,T)$ is divided into N uniform subintervals by points $0=t_0<t_1<\cdots<t_N=T$, where $t_k=k\tau$ and $\tau=T/N$ is the time step, and denote the k-th subinterval by $I_k=[t_{k-1},t_k]$. Moreover, we introduce the backward and average operators:

$$\delta_\tau\boldsymbol{u}^k=(\boldsymbol{u}^k-\boldsymbol{u}^{k-1})/\tau,\quad \overline{\boldsymbol{u}}^k=(\boldsymbol{u}^k+\boldsymbol{u}^{k-1})/2,$$

for any function $\boldsymbol{u}^k=\boldsymbol{u}(\cdot,k\tau)$, with $0\leqslant k\leqslant N$.

For the purposes of comparison with what follows, let us first recall the Crank-Nicolson scheme constructed in [20, p.634] for solving the system (13)-(18): For

$k = 1, 2, \cdots, N$, find $\boldsymbol{E}_h^k \in \boldsymbol{V}_h^0, \boldsymbol{J}_h^k, \boldsymbol{P}_h^k \in \boldsymbol{V}_h$ and $\boldsymbol{H}_h^k, \boldsymbol{K}_h^k, \boldsymbol{M}_h^k \in \boldsymbol{U}_h$ such that

$$\epsilon_0(\delta_\tau \boldsymbol{E}_h^k, \boldsymbol{\phi}_h) - (\overline{\boldsymbol{H}}_h^k, \nabla \times \boldsymbol{\phi}_h) + (\overline{\boldsymbol{J}}_h^k, \boldsymbol{\phi}_h) = 0, \tag{30}$$

$$\mu_0(\delta_\tau \boldsymbol{H}_h^k, \boldsymbol{\psi}_h) + (\nabla \times \overline{\boldsymbol{E}}_h^k, \boldsymbol{\psi}_h) + (\overline{\boldsymbol{K}}_h^k, \boldsymbol{\psi}_h) = 0, \tag{31}$$

$$\frac{1}{\epsilon_0 \omega_{pe}^2}(\delta_\tau \boldsymbol{J}_h^k, \boldsymbol{\phi}_{1h}) + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2}(\overline{\boldsymbol{J}}_h^k, \boldsymbol{\phi}_{1h}) - (\overline{\boldsymbol{E}}_h^k, \boldsymbol{\phi}_{1h}) + \frac{\omega_{e0}^2}{\epsilon_0 \omega_{pe}^2}(\overline{\boldsymbol{P}}_h^k, \boldsymbol{\phi}_{1h}) = 0, \tag{32}$$

$$\frac{\omega_{e0}^2}{\epsilon_0 \omega_{pe}^2}(\delta_\tau \boldsymbol{P}_h^k, \boldsymbol{\phi}_{2h}) - \frac{\omega_{e0}^2}{\epsilon_0 \omega_{pe}^2}(\overline{\boldsymbol{J}}_h^k, \boldsymbol{\phi}_{2h}) = 0, \tag{33}$$

$$\frac{1}{\mu_0 \omega_{pm}^2}(\delta_\tau \boldsymbol{K}_h^k + \Gamma_m \overline{\boldsymbol{K}}_h^k + \omega_{m0}^2 \overline{\boldsymbol{M}}_h^k, \boldsymbol{\psi}_{1h}) - (\overline{\boldsymbol{H}}_h^k, \boldsymbol{\psi}_{1h}) = 0, \tag{34}$$

$$\frac{\omega_{m0}^2}{\mu_0 \omega_{pm}^2}(\delta_\tau \boldsymbol{M}_h^k, \boldsymbol{\psi}_{2h}) - \frac{\omega_{m0}^2}{\mu_0 \omega_{pm}^2}(\overline{\boldsymbol{K}}_h^k, \boldsymbol{\psi}_{2h}) = 0, \tag{35}$$

hold true for any $\boldsymbol{\phi}_h \in \boldsymbol{V}_h^0, \boldsymbol{\psi}_h, \boldsymbol{\psi}_{1h}, \boldsymbol{\psi}_{2h} \in \boldsymbol{U}_h$ and $\boldsymbol{\phi}_{1h}, \boldsymbol{\phi}_{2h} \in \boldsymbol{V}_h$, and are subject to the initial approximations

$$\begin{aligned} &\boldsymbol{E}_h^0(\boldsymbol{x}) = \Pi_h \boldsymbol{E}_0(\boldsymbol{x}),\ \boldsymbol{J}_h^0(\boldsymbol{x}) = \Pi_h \boldsymbol{J}_0(\boldsymbol{x}),\ \boldsymbol{P}_h^0(\boldsymbol{x}) = \Pi_h \boldsymbol{P}_0(\boldsymbol{x}), \\ &\boldsymbol{H}_h^0(\boldsymbol{x}) = \Pi_2 \boldsymbol{H}_0(\boldsymbol{x}),\ \boldsymbol{K}_h^0(\boldsymbol{x}) = \Pi_2 \boldsymbol{K}_0(\boldsymbol{x}),\ \boldsymbol{M}_h^0(\boldsymbol{x}) = \Pi_2 \boldsymbol{M}_0(\boldsymbol{x}). \end{aligned}$$

Choosing the test functions in (30)-(35) as $\overline{\boldsymbol{E}}_h^k, \overline{\boldsymbol{H}}_h^k, \overline{\boldsymbol{J}}_h^k, \overline{\boldsymbol{P}}_h^k, \overline{\boldsymbol{K}}_h^k$ and $\overline{\boldsymbol{M}}_h^k$, respectively, and summing up the results, we can obtain (cf. [20, Lemma 5.2]) the following discrete stability estimate, which has the exactly same form as the one obtained for the continuous time system: for any $k \geqslant 1$, we have

$$\begin{aligned} &\epsilon_0||\boldsymbol{E}_h^k||_0^2 + \mu_0||\boldsymbol{H}_h^k||_0^2 + \frac{1}{\epsilon_0 \omega_{pe}^2}||\boldsymbol{J}_h^k||_0^2 + \frac{1}{\mu_0 \omega_{pm}^2}||\boldsymbol{K}_h^k||_0^2 + \frac{\omega_{e0}^2}{\epsilon_0 \omega_{pe}^2}||\boldsymbol{P}_h^k||_0^2 \\ &+ \frac{\omega_{m0}^2}{\mu_0 \omega_{pm}^2}||\boldsymbol{M}_h^k||_0^2 \leqslant \epsilon_0||\boldsymbol{E}_h^0||_0^2 + \mu_0||\boldsymbol{H}_h^0||_0^2 + \frac{1}{\epsilon_0 \omega_{pe}^2}||\boldsymbol{J}_h^0||_0^2 \\ &+ \frac{1}{\mu_0 \omega_{pm}^2}||\boldsymbol{K}_h^0||_0^2 + \frac{\omega_{e0}^2}{\epsilon_0 \omega_{pe}^2}||\boldsymbol{P}_h^0||_0^2 + \frac{\omega_{m0}^2}{\mu_0 \omega_{pm}^2}||\boldsymbol{M}_h^0||_0^2. \end{aligned}$$

Although the computational solution of (30)-(35) appears rather demanding it actually requires only minor modifications to a standard Crank-Nicolson Maxwell solver. In each time step, we first solve a linear system for $\boldsymbol{E}_h^k$ and $\boldsymbol{H}_h^k$, after using (32)-(35) to eliminate all but these from (30) and (31). Then we update $\boldsymbol{J}_h^k$ and $\boldsymbol{K}_h^k$ using two simple recursive formulas, and finally update $\boldsymbol{P}_h^k$ and $\boldsymbol{M}_h^k$ through simple algebra operations. For the details see [20, p.635].

Now let us consider a leap-frog scheme for solving the system (13)-(18): Given initial approximations $\boldsymbol{E}_h^0, \boldsymbol{J}_h^{\frac{1}{2}}, \boldsymbol{P}_h^0, \boldsymbol{H}_h^{\frac{1}{2}}, \boldsymbol{K}_h^0, \boldsymbol{M}_h^{\frac{1}{2}}$, for $k = 1, 2, \cdots, N$, find $\boldsymbol{E}_h^k \in \boldsymbol{V}_h^0, \boldsymbol{J}_h^{k+\frac{1}{2}}, \boldsymbol{P}_h^k \in \boldsymbol{V}_h$ and $\boldsymbol{H}_h^{k+\frac{1}{2}}, \boldsymbol{K}_h^k, \boldsymbol{M}_h^{k+\frac{1}{2}} \in \boldsymbol{U}_h$ such that

$$\epsilon_0(\delta_\tau \boldsymbol{E}_h^k, \boldsymbol{\phi}_h) - (\boldsymbol{H}_h^{k-\frac{1}{2}}, \nabla \times \boldsymbol{\phi}_h) + (\boldsymbol{J}_h^{k-\frac{1}{2}}, \boldsymbol{\phi}_h) = 0, \tag{36}$$

$$\mu_0(\delta_\tau \boldsymbol{H}_h^{k+\frac{1}{2}}, \boldsymbol{\psi}_h) + (\nabla \times \boldsymbol{E}_h^k, \boldsymbol{\psi}_h) + (\boldsymbol{K}_h^k, \boldsymbol{\psi}_h) = 0, \tag{37}$$

$$\begin{aligned} &\frac{1}{\epsilon_0 \omega_{pe}^2}(\delta_\tau \boldsymbol{J}_h^{k+\frac{1}{2}}, \boldsymbol{\phi}_{1h}) + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2}(\frac{1}{2}(\boldsymbol{J}_h^{k+\frac{1}{2}} + \boldsymbol{J}_h^{k-\frac{1}{2}}), \boldsymbol{\phi}_{1h}) \\ &\quad - (\boldsymbol{E}_h^k, \boldsymbol{\phi}_{1h}) + \frac{\omega_{e0}^2}{\epsilon_0 \omega_{pe}^2}(\boldsymbol{P}_h^k, \boldsymbol{\phi}_{1h}) = 0, \end{aligned} \tag{38}$$

$$\frac{\omega_{e0}^2}{\epsilon_0\omega_{pe}^2}(\delta_\tau \boldsymbol{P}_h^k, \boldsymbol{\phi}_{2h}) - \frac{\omega_{e0}^2}{\epsilon_0\omega_{pe}^2}(\boldsymbol{J}_h^{k-\frac{1}{2}}, \boldsymbol{\phi}_{2h}) = 0, \tag{39}$$

$$\frac{1}{\mu_0\omega_{pm}^2}(\delta_\tau \boldsymbol{K}_h^k, \boldsymbol{\psi}_{1h}) + \frac{\Gamma_m}{\mu_0\omega_{pm}^2}(\frac{1}{2}(\boldsymbol{K}_h^k + \boldsymbol{K}_h^{k-1}), \boldsymbol{\psi}_{1h})$$

$$- (\boldsymbol{H}_h^{k-\frac{1}{2}}, \boldsymbol{\psi}_{1h}) + \frac{\omega_{m0}^2}{\mu_0\omega_{pm}^2}(\boldsymbol{M}_h^{k-\frac{1}{2}}, \boldsymbol{\psi}_{1h}) = 0, \tag{40}$$

$$\frac{\omega_{m0}^2}{\mu_0\omega_{pm}^2}(\delta_\tau \boldsymbol{M}_h^{k+\frac{1}{2}}, \boldsymbol{\psi}_{2h}) - \frac{\omega_{m0}^2}{\mu_0\omega_{pm}^2}(\boldsymbol{K}_h^k, \boldsymbol{\psi}_{2h}) = 0, \tag{41}$$

hold true for any $\boldsymbol{\phi}_h \in \boldsymbol{V}_h^0, \boldsymbol{\psi}_h, \boldsymbol{\psi}_{1h}, \boldsymbol{\psi}_{2h} \in \boldsymbol{U}_h$ and $\boldsymbol{\phi}_{1h}, \boldsymbol{\phi}_{2h} \in \boldsymbol{V}_h$. For the practical purpose of obtaining the initial half-step values it is usual to start this leap-frog scheme from zero or constant initial data. We then note that (36)-(41) can be implemented as follows: at each time step, first solve (36), (39) and (40) and then solve (37), (38) and (41). Each of these solves can be done in parallel if required.

The following discrete data-stability estimate shows that the solution remains bounded independently of the time interval, $(0,T)$, of integration provided that the time step is small enough.

Theorem 3.1. *Let* $c_v = \frac{1}{\sqrt{\mu_0\epsilon_0}}$ *denote the speed of light in a vacuum, and let* $c_{inv} > 0$ *be the constant in the standard inverse estimate,*

$$||\nabla \times \boldsymbol{u}_h||_0 \leqslant c_{inv}h^{-1}||\boldsymbol{u}_h||_0, \quad \forall\ \boldsymbol{u}_h \in \boldsymbol{V}_h. \tag{42}$$

Then, under the time step constraint

$$\tau \leqslant \min\left\{\frac{1}{2\omega_{pe}}, \frac{1}{2\omega_{pm}}, \frac{1}{2\omega_{e0}}, \frac{1}{2\omega_{m0}}, \frac{h}{2c_vc_{inv}}\right\}, \tag{43}$$

for any $k \geqslant 1$, *we have*

$$\epsilon_0||\boldsymbol{E}_h^k||_0^2 + \mu_0||\boldsymbol{H}_h^{k+\frac{1}{2}}||_0^2 + \frac{1}{\epsilon_0\omega_{pe}^2}||\boldsymbol{J}_h^{k+\frac{1}{2}}||_0^2 + \frac{1}{\mu_0\omega_{pm}^2}||\boldsymbol{K}_h^k||_0^2$$

$$+\frac{\omega_{e0}^2}{\epsilon_0\omega_{pe}^2}||\boldsymbol{P}_h^k||_0^2 + \frac{\omega_{m0}^2}{\mu_0\omega_{pm}^2}||\boldsymbol{M}_h^{k+\frac{1}{2}}||_0^2 \leqslant 3\Bigg(\epsilon_0||\boldsymbol{E}_h^0||_0^2 + \mu_0||\boldsymbol{H}_h^{\frac{1}{2}}||_0^2$$

$$+\frac{1}{\epsilon_0\omega_{pe}^2}||\boldsymbol{J}_h^{\frac{1}{2}}||_0^2 + \frac{1}{\mu_0\omega_{pm}^2}||\boldsymbol{K}_h^0||_0^2 + \frac{\omega_{e0}^2}{\epsilon_0\omega_{pe}^2}||\boldsymbol{P}_h^0||_0^2 + \frac{\omega_{m0}^2}{\mu_0\omega_{pm}^2}||\boldsymbol{M}_h^{\frac{1}{2}}||_0^2\Bigg). \tag{44}$$

Remark 3.2. *We have implicitly assumed that none of the* ω*'s are zero in (43). These cases can be dealt with by simple adaptations of this argument.*

Proof. Choosing $\boldsymbol{\phi}_h = \frac{\tau}{2}(\boldsymbol{E}_h^k + \boldsymbol{E}_h^{k-1}), \boldsymbol{\psi}_h = \frac{\tau}{2}(\boldsymbol{H}_h^{k+\frac{1}{2}} + \boldsymbol{H}_h^{k-\frac{1}{2}}), \boldsymbol{\phi}_{1h} = \frac{\tau}{2}(\boldsymbol{J}_h^{k+\frac{1}{2}} + \boldsymbol{J}_h^{k-\frac{1}{2}}), \boldsymbol{\phi}_{2h} = \frac{\tau}{2}(\boldsymbol{P}_h^k + \boldsymbol{P}_h^{k-1}), \boldsymbol{\psi}_{1h} = \frac{\tau}{2}(\boldsymbol{K}_h^k + \boldsymbol{K}_h^{k-1}), \boldsymbol{\psi}_{2h} = \frac{\tau}{2}(\boldsymbol{M}_h^{k+\frac{1}{2}} + \boldsymbol{M}_h^{k-\frac{1}{2}})$ in

(36)-(41), respectively, summing up the results, and using the following identities:

$$\begin{aligned}
&(\nabla\times\boldsymbol{E}_h^k,\boldsymbol{H}_h^{k+\frac12}+\boldsymbol{H}_h^{k-\frac12})-(\boldsymbol{H}_h^{k-\frac12},\nabla\times(\boldsymbol{E}_h^k+\boldsymbol{E}_h^{k-1}))\\
&\qquad=(\nabla\times\boldsymbol{E}_h^k,\boldsymbol{H}_h^{k+\frac12})-(\nabla\times\boldsymbol{E}_h^{k-1},\boldsymbol{H}_h^{k-\frac12}),\\
&(\boldsymbol{J}_h^{k-\frac12},\boldsymbol{E}_h^k+\boldsymbol{E}_h^{k-1})-(\boldsymbol{E}_h^k,\boldsymbol{J}_h^{k+\frac12}+\boldsymbol{J}_h^{k-\frac12})=(\boldsymbol{J}_h^{k-\frac12},\boldsymbol{E}_h^{k-1})-(\boldsymbol{J}_h^{k+\frac12},\boldsymbol{E}_h^k),\\
&(\boldsymbol{K}_h^k,\boldsymbol{H}_h^{k+\frac12}+\boldsymbol{H}_h^{k-\frac12})-(\boldsymbol{H}_h^{k-\frac12},\boldsymbol{K}_h^k+\boldsymbol{K}_h^{k-1})\\
&\qquad=(\boldsymbol{K}_h^k,\boldsymbol{H}_h^{k+\frac12})-(\boldsymbol{K}_h^{k-1},\boldsymbol{H}_h^{k-\frac12}),\\
&(\boldsymbol{P}_h^k,\boldsymbol{J}_h^{k+\frac12}+\boldsymbol{J}_h^{k-\frac12})-(\boldsymbol{J}_h^{k-\frac12},\boldsymbol{P}_h^k+\boldsymbol{P}_h^{k-1})=(\boldsymbol{P}_h^k,\boldsymbol{J}_h^{k+\frac12})-(\boldsymbol{P}_h^{k-1},\boldsymbol{J}_h^{k-\frac12}),\\
&(\boldsymbol{M}_h^{k-\frac12},\boldsymbol{K}_h^k+\boldsymbol{K}_h^{k-1})-(\boldsymbol{K}_h^k,\boldsymbol{M}_h^{k+\frac12}+\boldsymbol{M}_h^{k-\frac12})\\
&\qquad=(\boldsymbol{M}_h^{k-\frac12},\boldsymbol{K}_h^{k-1})-(\boldsymbol{M}_h^{k+\frac12},\boldsymbol{K}_h^k),
\end{aligned}$$

we have

$$\begin{aligned}
0=&\frac{\epsilon_0}{2}(||\boldsymbol{E}_h^k||_0^2-||\boldsymbol{E}_h^{k-1}||_0^2)+\frac{\mu_0}{2}(||\boldsymbol{H}_h^{k+\frac12}||_0^2-||\boldsymbol{H}_h^{k-\frac12}||_0^2)\\
&+\frac{||\boldsymbol{J}_h^{k+\frac12}||_0^2-||\boldsymbol{J}_h^{k-\frac12}||_0^2}{2\epsilon_0\omega_{pe}^2}+\frac{\omega_{e0}^2}{2\epsilon_0\omega_{pe}^2}(||\boldsymbol{P}_h^k||_0^2-||\boldsymbol{P}_h^{k-1}||_0^2)+\frac{||\boldsymbol{K}_h^k||_0^2-||\boldsymbol{K}_h^{k-1}||_0^2}{2\mu_0\omega_{pm}^2}\\
&+\frac{\omega_{m0}^2}{2\mu_0\omega_{pm}^2}(||\boldsymbol{M}_h^{k+\frac12}||_0^2-||\boldsymbol{M}_h^{k-\frac12}||_0^2)+\frac{\Gamma_e}{\tau\epsilon_0\omega_{pe}^2}||\frac{\tau}{2}(\boldsymbol{J}_h^{k+\frac12}+\boldsymbol{J}_h^{k-\frac12})||_0^2\\
&+\frac{\Gamma_m}{\tau\mu_0\omega_{pm}^2}||\frac{\tau}{2}(\boldsymbol{K}_h^k+\boldsymbol{K}_h^{k-1})||_0^2+\frac{\tau}{2}[(\nabla\times\boldsymbol{E}_h^k,\boldsymbol{H}_h^{k+\frac12})-(\nabla\times\boldsymbol{E}_h^{k-1},\boldsymbol{H}_h^{k-\frac12})]\\
&+\frac{\tau}{2}[(\boldsymbol{J}_h^{k-\frac12},\boldsymbol{E}_h^{k-1})-(\boldsymbol{J}_h^{k+\frac12},\boldsymbol{E}_h^k)]+\frac{\tau}{2}[(\boldsymbol{K}_h^k,\boldsymbol{H}_h^{k+\frac12})-(\boldsymbol{K}_h^{k-1},\boldsymbol{H}_h^{k-\frac12})]\\
&+\frac{\tau\omega_{e0}^2}{2\epsilon_0\omega_{pe}^2}[(\boldsymbol{P}_h^k,\boldsymbol{J}_h^{k+\frac12})-(\boldsymbol{P}_h^{k-1},\boldsymbol{J}_h^{k-\frac12})]\\
&+\frac{\tau\omega_{m0}^2}{2\mu_0\omega_{pm}^2}[(\boldsymbol{M}_h^{k-\frac12},\boldsymbol{K}_h^{k-1})-(\boldsymbol{M}_h^{k+\frac12},\boldsymbol{K}_h^k)].
\end{aligned}$$

Summing this over $k=1$ to N we obtain,

$$\begin{aligned}
&\tfrac{\epsilon_0}{2}||\boldsymbol{E}_h^N||_0^2+\tfrac{\mu_0}{2}||\boldsymbol{H}_h^{N+\frac12}||_0^2+\tfrac{1}{2\epsilon_0\omega_{pe}^2}||\boldsymbol{J}_h^{N+\frac12}||_0^2+\tfrac{1}{2\mu_0\omega_{pm}^2}||\boldsymbol{K}_h^N||_0^2+\tfrac{\omega_{e0}^2}{2\epsilon_0\omega_{pe}^2}||\boldsymbol{P}_h^N||_0^2\\
&+\tfrac{\omega_{m0}^2}{2\mu_0\omega_{pm}^2}||\boldsymbol{M}_h^{N+\frac12}||_0^2\leqslant\tfrac{\epsilon_0}{2}||\boldsymbol{E}_h^0||_0^2+\tfrac{\mu_0}{2}||\boldsymbol{H}_h^{\frac12}||_0^2+\tfrac{1}{2\epsilon_0\omega_{pe}^2}||\boldsymbol{J}_h^{\frac12}||_0^2+\tfrac{1}{2\mu_0\omega_{pm}^2}||\boldsymbol{K}_h^0||_0^2\\
&\quad+\tfrac{\omega_{e0}^2}{2\epsilon_0\omega_{pe}^2}||\boldsymbol{P}_h^0||_0^2+\tfrac{\omega_{m0}^2}{2\mu_0\omega_{pm}^2}||\boldsymbol{M}_h^{\frac12}||_0^2-\tfrac{\tau}{2}[(\nabla\times\boldsymbol{E}_h^N,\boldsymbol{H}_h^{N+\frac12})-(\nabla\times\boldsymbol{E}_h^0,\boldsymbol{H}_h^{\frac12})]\\
&\quad-\tfrac{\tau\omega_{e0}^2}{2\epsilon_0\omega_{pe}^2}[(\boldsymbol{P}_h^N,\boldsymbol{J}_h^{N+\frac12})-(\boldsymbol{P}_h^0,\boldsymbol{J}_h^{\frac12})]+\tfrac{\tau\omega_{m0}^2}{2\mu_0\omega_{pm}^2}[(\boldsymbol{M}_h^{N+\frac12},\boldsymbol{K}_h^N)-(\boldsymbol{M}_h^{\frac12},\boldsymbol{K}_h^0)],\\
&\qquad+\tfrac{\tau}{2}[(\boldsymbol{J}_h^{N+\frac12},\boldsymbol{E}_h^N)-(\boldsymbol{J}_h^{\frac12},\boldsymbol{E}_h^0)]-\tfrac{\tau}{2}[(\boldsymbol{K}_h^N,\boldsymbol{H}_h^{N+\frac12})-(\boldsymbol{K}_h^0,\boldsymbol{H}_h^{\frac12})],
\end{aligned}\tag{45}$$

and then using the Cauchy-Schwarz inequality and the inverse estimate (42) we have

$$\begin{aligned}
\frac{\tau}{2}(\nabla\times\boldsymbol{E}_h^N,\boldsymbol{H}_h^{N+\frac12})&\leqslant\frac{\tau}{2}\cdot c_{inv}h^{-1}||\boldsymbol{E}_h^N||_0||\boldsymbol{H}_h^{N+\frac12}||_0\\
&=\frac{\tau}{2}\cdot c_vc_{inv}h^{-1}\sqrt{\epsilon_0}||\boldsymbol{E}_h^N||_0\sqrt{\mu_0}||\boldsymbol{H}_h^{N+\frac12}||_0\\
&\leqslant\left(\frac{\tau c_vc_{inv}}{2h}\right)^2\delta_1\epsilon_0||\boldsymbol{E}_h^N||_0^2+\frac{\mu_0}{4\delta_1}||\boldsymbol{H}_h^{N+\frac12}||_0^2
\end{aligned}$$

for all $\delta_1 > 0$. Similarly, for all $\delta_i > 0$ for $i = 2, \dots, 5$, we have first,

$$\begin{aligned}\frac{\tau}{2}(\boldsymbol{J}_h^{N+\frac{1}{2}}, \boldsymbol{E}_h^N) &\leqslant \frac{\tau\omega_{pe}}{2} \cdot \frac{1}{\sqrt{\epsilon_0\omega_{pe}^2}}||\boldsymbol{J}_h^{N+\frac{1}{2}}||_0\sqrt{\epsilon_0}||\boldsymbol{E}_h^N||_0 \\ &\leqslant \left(\frac{\tau\omega_{pe}}{2}\right)^2 \delta_2\epsilon_0||\boldsymbol{E}_h^N||_0^2 + \frac{1}{4\delta_2} \cdot \frac{1}{\epsilon_0\omega_{pe}^2}||\boldsymbol{J}_h^{N+\frac{1}{2}}||_0^2,\end{aligned}$$

second,

$$\begin{aligned}\frac{\tau}{2}(\boldsymbol{K}_h^N, \boldsymbol{H}_h^{N+\frac{1}{2}}) &\leqslant \frac{\tau\omega_{pm}}{2} \cdot \frac{1}{\sqrt{\mu_0\omega_{pm}^2}}||\boldsymbol{K}_h^N||_0\sqrt{\mu_0}||\boldsymbol{H}_h^{N+\frac{1}{2}}||_0 \\ &\leqslant \left(\frac{\tau\omega_{pm}}{2}\right)^2 \frac{\delta_3}{\mu_0\omega_{pm}^2}||\boldsymbol{K}_h^N||_0^2 + \frac{\mu_0}{4\delta_3}||\boldsymbol{H}_h^{N+\frac{1}{2}}||_0^2,\end{aligned}$$

and lastly,

$$\begin{aligned}\frac{\tau\omega_{e0}^2}{2\epsilon_0\omega_{pe}^2}(\boldsymbol{P}_h^N, \boldsymbol{J}_h^{N+\frac{1}{2}}) &\leqslant \frac{1}{4\delta_4} \cdot \frac{1}{\epsilon_0\omega_{pe}^2}||\boldsymbol{J}_h^{N+\frac{1}{2}}||_0^2 + (\frac{\tau\omega_{e0}}{2})^2\delta_4 \cdot \frac{\omega_{e0}^2}{\epsilon_0\omega_{pe}^2}||\boldsymbol{P}_h^N||_0^2, \\ \frac{\tau\omega_{m0}^2}{2\mu_0\omega_{pm}^2}(\boldsymbol{M}_h^{N+\frac{1}{2}}, \boldsymbol{K}_h^N) &\leqslant (\frac{\tau\omega_{m0}}{2})^2\delta_5 \cdot \frac{1}{\mu_0\omega_{pm}^2}||\boldsymbol{K}_h^N||_0^2 + \frac{1}{4\delta_5} \cdot \frac{\omega_{m0}^2}{\mu_0\omega_{pm}^2}||\boldsymbol{M}_h^{N+\frac{1}{2}}||_0^2.\end{aligned}$$

Note that similar estimates can be obtained for the five terms in (45) that involve the initial data, the estimate (44) can then be obtained by substituting the above estimates into (45) and choosing δ_i and τ properly. A simple choice is to select $\delta_1 = \cdots = \delta_5 = 2$ and require (43). This concludes the proof. $\square$

Remark 3.3. *We note that the stability estimate (44) just obtained for the leap-frog scheme has exactly the same form as the stability estimate for the continuous problem, except that the stability coefficient is raised from unity to three. It is easy to see from the proof that this constant of* 3 *in (44) can be reduced, but not to unity.*

Remark 3.4. *Following similar ideas of our early work* [20]*, we can use the ideas above to prove the following error estimate:*

$$\begin{aligned}||\boldsymbol{E}^k - \boldsymbol{E}_h^k||_0 + ||\boldsymbol{H}^{k+\frac{1}{2}} - \boldsymbol{H}_h^{k+\frac{1}{2}}||_0 + ||\boldsymbol{J}^{k+\frac{1}{2}} - \boldsymbol{J}_h^{k+\frac{1}{2}}||_0 + ||\boldsymbol{K}^k - \boldsymbol{K}_h^k||_0 \\ +||\boldsymbol{P}^k - \boldsymbol{P}_h^k||_0 + ||\boldsymbol{M}^{k+\frac{1}{2}} - \boldsymbol{M}_h^{k+\frac{1}{2}}||_0 \leqslant C(h^l + \tau^2)\end{aligned}$$

where the time dependence of C is due to norms of the data and exact solution, but not to an invocation of Gronwall's inequality. This error bound will hold true if the underlying solutions are smooth enough and the errors in the initial data are bounded as $O(h^l + \tau^2)$. Here $l \geqslant 1$ denotes the order of the basis functions in the finite element spaces $\boldsymbol{U}_h$ and $\boldsymbol{V}_h$. The proof of this result with the modified constant is omitted here due to its length. A full example of an error bound with temporally-sharp constants is given later in Theorem 4.6.

4. Investigation of the Lorentz model in four variables

If we solve (16) for $\boldsymbol{P}$ and (18) for $\boldsymbol{M}$ and substitute the results into (15) and (17) we can rewrite the Lorentz model (13)-(18) as

$$\epsilon_0\boldsymbol{E}_t - \nabla \times \boldsymbol{H} + \boldsymbol{J} = 0, \qquad \mu_0\boldsymbol{H}_t + \nabla \times \boldsymbol{E} + \boldsymbol{K} = 0, \tag{46}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}\boldsymbol{J}_t + \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\boldsymbol{J} + \frac{\omega_{e0}^2}{\epsilon_0\omega_{pe}^2}\int_0^t \boldsymbol{J}(s)ds - \boldsymbol{E} = \boldsymbol{f}(0), \tag{47}$$

$$\frac{1}{\mu_0\omega_{pm}^2}\frac{\partial \boldsymbol{K}}{\partial t} + \frac{\Gamma_m}{\mu_0\omega_{pm}^2}\boldsymbol{K} + \frac{\omega_{m0}^2}{\mu_0\omega_{pm}^2}\int_0^t \boldsymbol{K}(s)ds - \boldsymbol{H} = \boldsymbol{g}(0), \tag{48}$$

where

$$\boldsymbol{f}(0) = \frac{1}{\epsilon_0 \omega_{pe}^2}(\boldsymbol{J}_t(0) + \Gamma_e \boldsymbol{J}(0)) - \boldsymbol{E}(0), \quad \boldsymbol{g}(0) = \frac{1}{\mu_0 \omega_{pm}^2}(\boldsymbol{K}_t(0) + \Gamma_m \boldsymbol{K}(0)) - \boldsymbol{H}(0)$$

are time independent and known.

We consider the following weak formulation of (46)-(48): Find $\boldsymbol{E} \in H^1(0,T;\boldsymbol{V}^*) \cap \boldsymbol{L}^2(0,T;\boldsymbol{V})$, and $\boldsymbol{J}, \boldsymbol{H}, \boldsymbol{K} \in H^1(0,T;\boldsymbol{L}^2(\Omega))$ such that

$$\epsilon_0(\boldsymbol{E}_t, \boldsymbol{\phi}) - (\boldsymbol{H}, \nabla \times \boldsymbol{\phi}) + (\boldsymbol{J}, \boldsymbol{\phi}) = 0, \ \mu_0(\boldsymbol{H}_t, \boldsymbol{\psi}) + (\nabla \times \boldsymbol{E} + \boldsymbol{K}, \boldsymbol{\psi}) = 0, \tag{49}$$

$$\left(\frac{1}{\epsilon_0 \omega_{pe}^2}\boldsymbol{J}_t + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2}\boldsymbol{J} + \frac{\omega_{e0}^2}{\epsilon_0 \omega_{pe}^2}\int_0^t \boldsymbol{J}(s)ds - \boldsymbol{E}, \boldsymbol{\phi}_1\right) = (\boldsymbol{f}(0), \boldsymbol{\phi}_1), \tag{50}$$

$$\left(\frac{1}{\mu_0 \omega_{pm}^2}\boldsymbol{K}_t + \frac{\Gamma_m}{\epsilon_0 \omega_{pm}^2}\boldsymbol{K} + \frac{\omega_{m0}^2}{\mu_0 \omega_{pm}^2}\int_0^t \boldsymbol{K}(s)ds - \boldsymbol{H}, \boldsymbol{\psi}_1\right) = (\boldsymbol{g}(0), \boldsymbol{\psi}_1), \tag{51}$$

$$\forall\ \boldsymbol{\phi} \in H_0(\text{curl};\Omega), \quad \forall\ \boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega), \quad \forall\ \boldsymbol{\phi}_1 \in \boldsymbol{L}^2(\Omega) \text{ and } \forall\ \boldsymbol{\psi}_1 \in \boldsymbol{L}^2(\Omega)$$

subject to the boundary condition (10) and initial data ($\boldsymbol{x}$-dependence omitted):

$$\boldsymbol{E}(0) = \boldsymbol{E}_0, \quad \boldsymbol{J}(0) = \boldsymbol{J}_0, \quad \boldsymbol{H}(0) = \boldsymbol{H}_0, \quad \boldsymbol{K}(0) = \boldsymbol{K}_0. \tag{52}$$

Our first result for this formulation demonstrates long-time data stability and provides a comparator for the discrete stability estimate that follows in Theorem 4.4.

Theorem 4.1. *For the solution $(\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{J}, \boldsymbol{K})$ of (49)-(52) and any $t \in (0,T]$, we have the following data-stability estimates:*
(i) If $\omega_{e0}, \omega_{m0} \neq 0$, then

$$\mathcal{E}(t) \leqslant 2\mathcal{E}(0) + \frac{4\epsilon_0 \omega_{pe}^2}{\omega_{e0}^2}||\boldsymbol{f}(0)||_0^2 + \frac{4\mu_0 \omega_{pm}^2}{\omega_{m0}^2}||\boldsymbol{g}(0)||_0^2, \tag{53}$$

where we denote the energy

$$\begin{aligned} \mathcal{E}(t) \quad = \quad & \epsilon_0||\boldsymbol{E}(t)||_0^2 + \mu_0||\boldsymbol{H}(t)||_0^2 + \frac{1}{\epsilon_0 \omega_{pe}^2}||\boldsymbol{J}||_0^2 + \frac{1}{\mu_0 \omega_{pm}^2}||\boldsymbol{K}||_0^2 \\ & + \frac{\omega_{e0}^2}{\epsilon_0 \omega_{pe}^2}\left|\left|\int_0^t \boldsymbol{J}(s)ds\right|\right|_0^2 + \frac{\omega_{m0}^2}{\mu_0 \omega_{pm}^2}\left|\left|\int_0^t \boldsymbol{K}(s)ds\right|\right|_0^2. \end{aligned} \tag{54}$$

Moreover, defining a new energy $\mathcal{E}_1(t)$ as $\mathcal{E}(t)$ but without the integral terms in (54), we also have

$$\mathcal{E}_1(t) \leqslant \mathcal{E}_1(0) + \frac{\epsilon_0 \omega_{pe}^2}{\omega_{e0}^2}||\boldsymbol{f}(0)||_0^2 + \frac{\mu_0 \omega_{pm}^2}{\omega_{m0}^2}||\boldsymbol{g}(0)||_0^2, \tag{55}$$

(ii) For the Lorentz and Drude models where $\Gamma_e, \Gamma_m \neq 0$ we have,

$$\mathcal{E}(t) \leqslant \mathcal{E}(0) + T\left(\frac{\epsilon_0 \omega_{pe}^2}{\Gamma_e}||\boldsymbol{f}(0)||_0^2 + \frac{\mu_0 \omega_{pm}^2}{\Gamma_m}||\boldsymbol{g}(0)||_0^2\right), \tag{56}$$

where $\mathcal{E}(t)$ denotes the same energy as case (i).
(iii) For the cold plasma model where $\Gamma_e = \Gamma_m = \omega_{e0} = \omega_{m0} = 0$ we have,

$$\mathcal{E}(t) \leqslant 2\mathcal{E}(0) + \frac{4}{\epsilon_0 \omega_{pe}^2}||\boldsymbol{J}(0)||_0^2 + \frac{4}{\mu_0 \omega_{pm}^2}||\boldsymbol{K}(0)||_0^2, \tag{57}$$

where in this case the energy is re-defined as,

$$\mathcal{E}(t) = \epsilon_0||\boldsymbol{E}(t)||_0^2 + \mu_0||\boldsymbol{H}(t)||_0^2$$
$$+ \epsilon_0\omega_{pe}^2 \left|\left|\int_0^t \boldsymbol{E}(s)ds\right|\right|_0^2 + \mu_0\omega_{pm}^2 \left|\left|\int_0^t \boldsymbol{H}(s)ds\right|\right|_0^2. \quad (58)$$

Again, defining a new energy $\mathcal{E}_1(t)$ by dropping the integral terms in (58), we also have

$$\mathcal{E}_1(t) \leqslant \mathcal{E}_1(0) + \frac{1}{\epsilon_0\omega_{pe}^2}||\boldsymbol{J}(0)||_0^2 + \frac{1}{\mu_0\omega_{pm}^2}||\boldsymbol{K}(0)||_0^2. \quad (59)$$

Proof. (i) Choosing $\boldsymbol{\phi} = \boldsymbol{E}, \boldsymbol{\psi} = \boldsymbol{H}, \boldsymbol{\phi}_1 = \boldsymbol{J}, \boldsymbol{\psi}_1 = \boldsymbol{K}$ in (49)-(51), respectively, we obtain

$$\frac{1}{2}\frac{d\mathcal{E}}{dt} + \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}||\boldsymbol{J}||_0^2 + \frac{\Gamma_m}{\mu_0\omega_{pm}^2}||\boldsymbol{K}||_0^2 = (\boldsymbol{f}(0), \boldsymbol{J}) + (\boldsymbol{g}(0), \boldsymbol{K}) \quad (60)$$

and then integrating this from 0 to t and using the Cauchy-Schwarz and Young's inequalities, we have

$$\tfrac{1}{2}(\mathcal{E}(t) - \mathcal{E}(0)) \leqslant \tfrac{\epsilon_0\omega_{pe}^2}{\omega_{e0}^2}||\boldsymbol{f}(0)||_0^2 + \tfrac{\mu_0\omega_{pm}^2}{\omega_{m0}^2}||\boldsymbol{g}(0)||_0^2$$
$$+\tfrac{\omega_{e0}^2}{4\epsilon_0\omega_{pe}^2}\left|\left|\int_0^t \boldsymbol{J}(s)ds\right|\right|_0^2 + \tfrac{\omega_{m0}^2}{4\mu_0\omega_{pm}^2}\left|\left|\int_0^t \boldsymbol{K}(s)ds\right|\right|_0^2,$$

which yields (53) easily. The stability estimate in (55) can then be obtained from the following inequality

$$\tfrac{1}{2}(\mathcal{E}(t) - \mathcal{E}(0)) \leqslant \tfrac{\epsilon_0\omega_{pe}^2}{2\omega_{e0}^2}||\boldsymbol{f}(0)||_0^2 + \tfrac{\mu_0\omega_{pm}^2}{2\omega_{m0}^2}||\boldsymbol{g}(0)||_0^2$$
$$+\tfrac{\omega_{e0}^2}{2\epsilon_0\omega_{pe}^2}\left|\left|\int_0^t \boldsymbol{J}(s)ds\right|\right|_0^2 + \tfrac{\omega_{m0}^2}{2\mu_0\omega_{pm}^2}\left|\left|\int_0^t \boldsymbol{K}(s)ds\right|\right|_0^2.$$

(ii) In this case, the proof is completed by substituting the following estimates

$$\begin{aligned}(\boldsymbol{f}(0), \boldsymbol{J}) &\leqslant \frac{\Gamma_e}{2\epsilon_0\omega_{pe}^2}||\boldsymbol{J}||_0^2 + \frac{\epsilon_0\omega_{pe}^2}{2\Gamma_e}||\boldsymbol{f}(0)||_0^2,\\ (\boldsymbol{g}(0), \boldsymbol{K}) &\leqslant \frac{\Gamma_m}{2\mu_0\omega_{pm}^2}||\boldsymbol{K}||_0^2 + \frac{\mu_0\omega_{pm}^2}{2\Gamma_m}||\boldsymbol{g}(0)||_0^2,\end{aligned}$$

into (60).

(iii) When $\Gamma_e = \Gamma_m = \omega_{e0} = \omega_{m0} = 0$, the original governing equations (13)-(18) reduce to

$$\epsilon_0\boldsymbol{E}_t - \nabla\times\boldsymbol{H} + \boldsymbol{J} = 0, \qquad \mu_0\boldsymbol{H}_t + \nabla\times\boldsymbol{E} + \boldsymbol{K} = 0, \quad (61)$$

$$\boldsymbol{J}_t = \epsilon_0\omega_{pe}^2\boldsymbol{E}, \qquad \boldsymbol{K}_t = \mu_0\omega_{pm}^2\boldsymbol{H}. \quad (62)$$

Solving (62) for $\boldsymbol{J}$ and $\boldsymbol{K}$ and substituting into (61) gives,

$$\epsilon_0\boldsymbol{E}_t - \nabla\times\boldsymbol{H} + \epsilon_0\omega_{pe}^2\int_0^t \boldsymbol{E}(s)ds = -\boldsymbol{J}(0), \quad (63)$$

$$\mu_0\boldsymbol{H}_t + \nabla\times\boldsymbol{E} + \mu_0\omega_{pm}^2\int_0^t \boldsymbol{H}(s)ds = -\boldsymbol{K}(0), \quad (64)$$

and then on multiplying (63) and (64) by $\boldsymbol{E}$ and $\boldsymbol{H}$, respectively, integrating over Ω and using the energy definition (58), we obtain

$$\frac{1}{2}\frac{d\mathcal{E}}{dt} = -(\boldsymbol{J}(0), \boldsymbol{E}) - (\boldsymbol{K}(0), \boldsymbol{H}). \quad (65)$$

Integrating (65) from 0 to t, and using the Cauchy-Schwarz inequality, we have

$$\frac{1}{2}(\mathcal{E}(t)-\mathcal{E}(0)) = -\Big(\boldsymbol{J}(0),\textstyle\int_0^t \boldsymbol{E}(s)ds\Big) - \Big(\boldsymbol{K}(0),\textstyle\int_0^t \boldsymbol{H}(s)ds\Big) \leqslant \frac{1}{\epsilon_0\omega_{pe}^2}||\boldsymbol{J}(0)||_0^2$$
$$+\frac{\epsilon_0\omega_{pe}^2}{4}\left|\left|\textstyle\int_0^t \boldsymbol{E}(s)ds\right|\right|_0^2 + \frac{\mu_0\omega_{pm}^2}{4}\left|\left|\textstyle\int_0^t \boldsymbol{H}(s)ds\right|\right|_0^2 + \frac{1}{\mu_0\omega_{pm}^2}||\boldsymbol{K}(0)||_0^2,$$

which easily leads to the proof of (57). To prove the stability bound in (59), we just need to use the following estimate,

$$\begin{aligned}
&\frac{1}{2}(\mathcal{E}(t)-\mathcal{E}(0))\\
= \;& -\Big(\boldsymbol{J}(0),\int_0^t \boldsymbol{E}(s)ds\Big) - \Big(\boldsymbol{K}(0),\int_0^t \boldsymbol{H}(s)ds\Big)\\
\leqslant \;& \frac{1}{2\epsilon_0\omega_{pe}^2}||\boldsymbol{J}(0)||_0^2 + \frac{\epsilon_0\omega_{pe}^2}{2}\left|\left|\int_0^t \boldsymbol{E}(s)ds\right|\right|_0^2 + \frac{\mu_0\omega_{pm}^2}{2}\left|\left|\int_0^t \boldsymbol{H}(s)ds\right|\right|_0^2\\
&+\frac{1}{2\mu_0\omega_{pm}^2}||\boldsymbol{K}(0)||_0^2.
\end{aligned}$$

This completes the proof. □

Remark 4.2. *In Theorem 4.1 we examined the stability in three different cases and used more careful reasoning than is usually encountered in this type of study in order to avoid using the Gronwall inequality. To emphasize this extra effort we remark that from (60), the estimates*

$$(\boldsymbol{f}(0),\boldsymbol{J})+(\boldsymbol{g}(0),\boldsymbol{K}) \leqslant \frac{1}{2\epsilon_0\omega_{pe}^2}||\boldsymbol{J}||_0^2+\frac{\epsilon_0\omega_{pe}^2}{2}||\boldsymbol{f}(0)||_0^2+\frac{1}{2\mu_0\omega_{pm}^2}||\boldsymbol{K}||_0^2+\frac{\mu_0\omega_{pm}^2}{2}||\boldsymbol{g}(0)||_0^2,$$

and the Gronwall inequality, we derive the following general stability estimate that covers all the cases:

$$\mathcal{E}(t) \leqslant C(\mathcal{E}(0)+||\boldsymbol{f}(0)||_0^2+||\boldsymbol{g}(0)||_0^2), \tag{66}$$

where the constant $C>0$ depends on those physical parameters in (3), and exponentially in T due to the usage of the Gronwall inequality. Here the energy $\mathcal{E}$ is defined by

$$\begin{aligned}
\mathcal{E}(t) \;=\;& \epsilon_0||\boldsymbol{E}(t)||_0^2+\mu_0||\boldsymbol{H}(t)||_0^2+\frac{1}{\epsilon_0\omega_{pe}^2}||\boldsymbol{J}||_0^2+\frac{1}{\mu_0\omega_{pm}^2}||\boldsymbol{K}||_0^2\\
&+\frac{\omega_{e0}^2}{\epsilon_0\omega_{pe}^2}\left|\left|\int_0^t \boldsymbol{J}(s)ds\right|\right|_0^2+\frac{\omega_{m0}^2}{\mu_0\omega_{pm}^2}\left|\left|\int_0^t \boldsymbol{K}(s)ds\right|\right|_0^2.
\end{aligned} \tag{67}$$

The main point here is that although one can derive coarse bounds without much effort, extra work is (for this problem) rewarded with sharper estimates.

Theorem 4.1 demonstrates that one can consider the model on a case-by-case basis in terms of the allowed values of the parameters. To save space and keep the arguments simple and demonstrative we will from here on (unless clearly specified otherwise) assume that all parameters in (3) are positive.

To begin with we let $\widetilde{IJ_h}^k$ and $\widetilde{IK_h}^k$ denote trapezoidal rule quadrature approximations that satisfy the following recursive formulas:

$$\widetilde{IJ_h}^0=0,\ \widetilde{IJ_h}^k=\widetilde{IJ_h}^{k-1}+\frac{\tau}{2}(\boldsymbol{J}_h^k+\boldsymbol{J}_h^{k-1}),\ \forall\, k\geqslant 1, \tag{68}$$

$$\widetilde{IK_h}^0=0,\ \widetilde{IK_h}^k=\widetilde{IK_h}^{k-1}+\frac{\tau}{2}(\boldsymbol{K}_h^k+\boldsymbol{K}_h^{k-1}),\ \forall\, k\geqslant 1. \tag{69}$$

Then, using the same spatial finite element discretization framework as earlier, we now consider the following Crank-Nicolson scheme for solving the system (46)-(48): For $k = 1, 2, \cdots, N$, find $\boldsymbol{E}_h^k \in \boldsymbol{V}_h^0, \boldsymbol{J}_h^k \in \boldsymbol{V}_h$ and $\boldsymbol{H}_h^k, \boldsymbol{K}_h^k \in \boldsymbol{U}_h$ such that

$$\epsilon_0(\delta_\tau \boldsymbol{E}_h^k, \boldsymbol{\phi}_h) - (\overline{\boldsymbol{H}}_h^k, \nabla \times \boldsymbol{\phi}_h) + (\overline{\boldsymbol{J}}_h^k, \boldsymbol{\phi}_h) = 0, \tag{70}$$

$$\mu_0(\delta_\tau \boldsymbol{H}_h^k, \boldsymbol{\psi}_h) + (\nabla \times \overline{\boldsymbol{E}}_h^k, \boldsymbol{\psi}_h) + (\overline{\boldsymbol{K}}_h^k, \boldsymbol{\psi}_h) = 0, \tag{71}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}(\delta_\tau \boldsymbol{J}_h^k, \boldsymbol{\phi}_{1h}) + \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}(\overline{\boldsymbol{J}}_h^k, \boldsymbol{\phi}_{1h}) - (\overline{\boldsymbol{E}}_h^k, \boldsymbol{\phi}_{1h}) + \frac{\omega_{e0}^2}{\epsilon_0\omega_{pe}^2}\left(\frac{\widetilde{IJ_h}^k + \widetilde{IJ_h}^{k-1}}{2}, \boldsymbol{\phi}_{1h}\right) = (\boldsymbol{f}(0), \boldsymbol{\phi}_{1h}), \tag{72}$$

$$\frac{1}{\mu_0\omega_{pm}^2}(\delta_\tau \boldsymbol{K}_h^k, \boldsymbol{\psi}_{1h}) + \frac{\Gamma_m}{\mu_0\omega_{pm}^2}(\overline{\boldsymbol{K}}_h^k, \boldsymbol{\psi}_{1h}) - (\overline{\boldsymbol{H}}_h^k, \boldsymbol{\phi}_{1h}) + \frac{\omega_{m0}^2}{\mu_0\omega_{pm}^2}\left(\frac{\widetilde{IK_h}^k + \widetilde{IK_h}^{k-1}}{2}, \boldsymbol{\psi}_{1h}\right) = (\boldsymbol{g}(0), \boldsymbol{\psi}_{1h}), \tag{73}$$

hold true for any $\boldsymbol{\phi}_h \in \boldsymbol{V}_h^0, \boldsymbol{\psi}_h, \boldsymbol{\psi}_{1h} \in \boldsymbol{U}_h, \boldsymbol{\phi}_{1h} \in \boldsymbol{V}_h$, and are subject to the initial approximations

$$\boldsymbol{E}_h^0(\boldsymbol{x}) = \Pi_h \boldsymbol{E}_0(\boldsymbol{x}),\ \boldsymbol{J}_h^0(\boldsymbol{x}) = \Pi_h \boldsymbol{J}_0(\boldsymbol{x}),$$
$$\boldsymbol{H}_h^0(\boldsymbol{x}) = \Pi_2 \boldsymbol{H}_0(\boldsymbol{x}),\ \boldsymbol{K}_h^0(\boldsymbol{x}) = \Pi_2 \boldsymbol{K}_0(\boldsymbol{x}).$$

Next, we need the following identity for the stability analysis of scheme (70)-(69).

Lemma 4.3. *For each $i \in \{1, \ldots, N\}$ we have*

$$2\sum_{j=1}^{N}\sum_{i=1}^{j-1} f_i f_j = \left(\sum_{i=1}^{N} f_i\right)^2 - \sum_{i=1}^{N} f_i^2. \tag{74}$$

Proof. The proof follows from this simple manipulation,

$$\begin{aligned} 2\sum_{j=1}^{N}\sum_{i=1}^{j-1} f_i f_j &= \sum_{j=1}^{N}\sum_{i=1}^{j-1} f_i f_j + \sum_{j=1}^{N}\sum_{i=j+1}^{N} f_i f_j = \sum_{j=1}^{N}\left(\sum_{i=1}^{j-1} f_i f_j + \sum_{i=j+1}^{N} f_i f_j\right) \\ &= \sum_{j=1}^{N}\left(\sum_{i=1}^{N} f_i f_j - f_j^2\right) = \left(\sum_{i=1}^{N} f_i\right)^2 - \sum_{j=1}^{N} f_j^2. \end{aligned}$$

□

With Lemma 4.3 we can now provide the following discrete stability estimates. These are the discrete forms of the estimates for the continuous problem as stated in Theorem 4.1.

Theorem 4.4. *Denote the discrete energy*

$$\mathcal{E}_h(k) = \epsilon_0||\boldsymbol{E}_h^k||_0^2 + \mu_0||\boldsymbol{H}_h^k||_0^2 + \frac{1}{\epsilon_0\omega_{pe}^2}||\boldsymbol{J}_h^k||_0^2 + \frac{1}{\mu_0\omega_{pm}^2}||\boldsymbol{K}_h^k||_0^2$$

then:

(i) If $\omega_{e0}, \omega_{m0} \neq 0$, then for any $1 \leqslant k \leqslant N$, we have

$$\mathcal{E}_h(k) \leqslant \mathcal{E}_h(0) + \frac{\epsilon_0\omega_{pe}^2}{\omega_{e0}^2}||\boldsymbol{f}(0)||_0^2 + \frac{\mu_0\omega_{pm}^2}{\omega_{m0}^2}||\boldsymbol{g}(0)||_0^2. \tag{75}$$

(ii) If $\Gamma_e, \Gamma_m \neq 0$, then for any $1 \leqslant k \leqslant N$, we have

$$\mathcal{E}_h(k) \leqslant \mathcal{E}_h(0) + T\left(\frac{\epsilon_0\omega_{pe}^2}{\Gamma_e}||\boldsymbol{f}(0)||_0^2 + \frac{\mu_0\omega_{pm}^2}{\Gamma_m}||\boldsymbol{g}(0)||_0^2\right). \tag{76}$$

Proof. From the recursive formula (68), we have $\widetilde{IJ_h}^k = \widetilde{IJ_h}^{k-1} + \tau\overline{\boldsymbol{J}}_h^k = \cdots = \tau\sum_{l=1}^k \overline{\boldsymbol{J}}_h^l$, which leads to

$$\left(\frac{\widetilde{IJ_h}^k + \widetilde{IJ_h}^{k-1}}{2}, \tau\overline{\boldsymbol{J}}_h^k\right) = \left(\frac{\tau}{2}\overline{\boldsymbol{J}}_h^k + \tau\sum_{l=1}^{k-1}\overline{\boldsymbol{J}}_h^l, \tau\overline{\boldsymbol{J}}_h^k\right) = \tau^2\left[\frac{1}{2}||\overline{\boldsymbol{J}}_h^k||_0^2 + \sum_{l=1}^{k-1}(\overline{\boldsymbol{J}}_h^l, \overline{\boldsymbol{J}}_h^k)\right].$$

Summing this from $k = 1$ to $k = N$ and using Lemma 4.3, we have

$$\left(\frac{\widetilde{IJ_h}^k + \widetilde{IJ_h}^{k-1}}{2}, \tau\overline{\boldsymbol{J}}_h^k\right) = \frac{\tau^2}{2}\left|\left|\sum_{k=1}^N \overline{\boldsymbol{J}}_h^k\right|\right|_0^2. \tag{77}$$

Choosing $\boldsymbol{\phi}_h = \frac{\tau}{2}(\boldsymbol{E}_h^k + \boldsymbol{E}_h^{k-1}), \boldsymbol{\psi}_h = \frac{\tau}{2}(\boldsymbol{H}_h^k + \boldsymbol{H}_h^{k-1}), \boldsymbol{\phi}_{1h} = \frac{\tau}{2}(\boldsymbol{J}_h^k + \boldsymbol{J}_h^{k-1})$, $\boldsymbol{\psi}_{1h} = \frac{\tau}{2}(\boldsymbol{K}_h^k + \boldsymbol{K}_h^{k-1})$ in (70)-(73), respectively, then summing up the resultants from $k = 1$ to $k = N$ and using (77), we obtain

$$\begin{aligned}
&\tfrac{\epsilon_0}{2}(||\boldsymbol{E}_h^N||_0^2 - ||\boldsymbol{E}_h^0||_0^2) + \tfrac{\mu_0}{2}(||\boldsymbol{H}_h^N||_0^2 - ||\boldsymbol{H}_h^0||_0^2) + \tfrac{1}{2\epsilon_0\omega_{pe}^2}(||\boldsymbol{J}_h^N||_0^2 - ||\boldsymbol{J}_h^0||_0^2)\\
&\quad + \tfrac{1}{2\mu_0\omega_{pm}^2}(||\boldsymbol{K}_h^N||_0^2 - ||\boldsymbol{K}_h^0||_0^2) + \tfrac{\tau\Gamma_e}{\epsilon_0\omega_{pe}^2}\textstyle\sum_{k=1}^N ||\overline{\boldsymbol{J}}_h^k||_0^2\\
&\quad + \tfrac{\tau\Gamma_m}{\mu_0\omega_{pm}^2}\textstyle\sum_{k=1}^N ||\overline{\boldsymbol{K}}_h^k||_0^2 + \tfrac{\tau^2\omega_{e0}^2}{2\epsilon_0\omega_{pe}^2}\left|\left|\sum_{k=1}^N \overline{\boldsymbol{J}}_h^k\right|\right|_0^2 + \tfrac{\tau^2\omega_{m0}^2}{2\mu_0\omega_{pm}^2}\left|\left|\sum_{k=1}^N \overline{\boldsymbol{K}}_h^k\right|\right|_0^2\\
&= \textstyle\sum_{k=1}^N \tau(\boldsymbol{f}(0), \overline{\boldsymbol{J}}_h^k) + \sum_{k=1}^N \tau(\boldsymbol{g}(0), \overline{\boldsymbol{K}}_h^k).
\end{aligned} \tag{78}$$

(i) The proof of (75) is completed by substituting the following estimates into (78):

$$\tau\left(\boldsymbol{f}(0), \sum_{k=1}^N \overline{\boldsymbol{J}}_h^k\right) \leqslant \frac{\epsilon_0\omega_{pe}^2}{2\omega_{e0}^2}||\boldsymbol{f}(0)||_0^2 + \frac{\tau^2\omega_{e0}^2}{2\epsilon_0\omega_{pe}^2}\left|\left|\sum_{k=1}^N \overline{\boldsymbol{J}}_h^k\right|\right|_0^2,$$

$$\tau\left(\boldsymbol{g}(0), \sum_{k=1}^N \overline{\boldsymbol{K}}_h^k\right) \leqslant \frac{\mu_0\omega_{pm}^2}{2\omega_{m0}^2}||\boldsymbol{g}(0)||_0^2 + \frac{\tau^2\omega_{m0}^2}{2\mu_0\omega_{pm}^2}\left|\left|\sum_{k=1}^N \overline{\boldsymbol{K}}_h^k\right|\right|_0^2.$$

(ii) Substituting the following estimates into (78),

$$\tau(\boldsymbol{f}(0), \overline{\boldsymbol{J}}_h^k) \leqslant \frac{\tau\epsilon_0\omega_{pe}^2}{2\Gamma_e}||\boldsymbol{f}(0)||_0^2 + \frac{\tau\Gamma_e}{2\epsilon_0\omega_{pe}^2}||\overline{\boldsymbol{J}}_h^k||_0^2, \tag{79}$$

$$\tau(\boldsymbol{g}(0), \overline{\boldsymbol{K}}_h^k) \leqslant \frac{\tau\mu_0\omega_{pm}^2}{2\Gamma_m}||\boldsymbol{g}(0)||_0^2 + \frac{\tau\Gamma_m}{2\mu_0\omega_{pm}^2}||\overline{\boldsymbol{K}}_h^k||_0^2. \tag{80}$$

then concludes the proof of (76) □

To save space we did not consider the case $\Gamma_e = \Gamma_m = \omega_{e0} = \omega_{m0} = 0$ in Theorem 4.4 although one could examine this case by following the argument given in Theorem 4.1 for the continuous problem.

To give an error analysis for the Crank-Nicolson scheme (70)-(69) we need the following lemma. The error estimate for the trapezoidal rule for numerical quadrature is very standard and included here for completeness.

Lemma 4.5. *Let X be a Banach space. For any function $J \in H^2(0,T;X)$ we have the following step-wise error bound,*

$$\left|\left|\overline{J}^k - \frac{1}{\tau}\int_{t_{k-1}}^{t_k} J(t)dt\right|\right|_X^2 \leqslant \frac{\tau^3}{4}\int_{t_{k-1}}^{t_k} ||J_{tt}(t)||_X^2\, dt,\ \forall\ J \in H^2(0,T), \tag{81}$$

for every $k \in \{1,2,\ldots,N\}$. Furthermore, for the approximation $\widetilde{IJ}^k$ to $\int_0^{t_k} J(t)\,dt$ defined by the recursive trapezoidal rule formula given by (68) we have the error bound,

$$\left|\left|\widetilde{IJ}^k - \int_0^{t_k} J(t)\,dt\right|\right|_X \leqslant \frac{\sqrt{T}\tau^2}{2}\|J_{tt}\|_{L^2(0,T;X)}$$

for every $k \in \{1,2,\ldots,N\}$.

Proof. For (81) we refer to [19, p.3165] and then using the triangle and Cauchy-Schwarz inequalities we obtain,

$$\left|\left|\widetilde{IJ}^k - \int_0^{t_k} J(t)\,dt\right|\right|_X \leqslant \sqrt{t_k}\left(\sum_{l=1}^{k}\tau\left|\left|\overline{J}^l - \frac{1}{\tau}\int_{t_{l-1}}^{t_l} J(t)\,dt\right|\right|_X^2\right)^{1/2}$$

and an application of (81) then completes the proof. □

Theorem 4.6. *For the solution $(\boldsymbol{E}_h^k, \boldsymbol{H}_h^k, \boldsymbol{J}_h^k, \boldsymbol{K}_h^k)$ of (70)-(73), we have the following error estimate: For every time level $k \geqslant 1$,*

$$\begin{aligned}
&\epsilon_0||\boldsymbol{E}^k - \boldsymbol{E}_h^k||_0^2 + \mu_0||\boldsymbol{H}^k - \boldsymbol{H}_h^k||_0^2 + \frac{1}{\epsilon_0\omega_{pe}^2}||\boldsymbol{J}^k - \boldsymbol{J}_h^k||_0^2\\
&+ \frac{1}{\mu_0\omega_{pm}^2}||\boldsymbol{K}^k - \boldsymbol{K}_h^k||_0^2 \leqslant C(1+T^2+T^3)(\tau^4+h^{2l})\\
&+ C(||\boldsymbol{E}^0 - \boldsymbol{E}_h^0||_0^2 + ||\boldsymbol{H}^0 - \boldsymbol{H}_h^0||_0^2 + ||\boldsymbol{J}^0 - \boldsymbol{J}_h^0||_0^2 + ||\boldsymbol{K}^0 - \boldsymbol{K}_h^0||_0^2),
\end{aligned} \tag{82}$$

where $l \geqslant 1$ denotes the degree of the finite element spaces $\boldsymbol{V}_h$ and $\boldsymbol{U}_h$ and C is a constant that depends on time only through the $L_p(0,T)$ norms of the underlying solution that arise from the approximation-error terms.

Remark 4.7. *We do not give full details of the constant C in Theorem 4.6 because it would obscure the main point that the constant governing the error growth (in the indicated norms) is of order $O(T^{3/2})$ rather than $O(e^{cT})$ (for some $c > 0$). For a flavour of the type of terms hidden in C we refer forward to (93).*

Proof. Integrating (49)-(51) with respect to t first from 0 to t_k and then from 0 to t_{k-1}, using these to form finite differences in time and then dividing the resultants by τ, we obtain the following four equalities

$$\epsilon_0(\delta_\tau\boldsymbol{E}^k,\boldsymbol{\phi}) - \left(\tfrac{1}{\tau}\int_{t_{k-1}}^{t_k}\boldsymbol{H}ds, \nabla\times\boldsymbol{\phi}\right) + \left(\tfrac{1}{\tau}\int_{t_{k-1}}^{t_k}\boldsymbol{J}ds,\boldsymbol{\phi}\right) = 0, \tag{83}$$

$$\mu_0(\delta_\tau\boldsymbol{H}^k,\boldsymbol{\psi}) + \left(\tfrac{1}{\tau}\int_{t_{k-1}}^{t_k}\nabla\times\boldsymbol{E}ds + \tfrac{1}{\tau}\int_{t_{k-1}}^{t_k}\boldsymbol{K}ds,\boldsymbol{\psi}\right) = 0, \tag{84}$$

$$\begin{aligned}
&\tfrac{1}{\epsilon_0\omega_{pe}^2}(\delta_\tau\boldsymbol{J}^k,\boldsymbol{\phi}_1) + \tfrac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\left(\tfrac{1}{\tau}\int_{t_{k-1}}^{t_k}\boldsymbol{J}ds,\boldsymbol{\phi}_1\right)\\
&+\tfrac{\omega_{e0}^2}{\epsilon_0\omega_{pe}^2}\left(\tfrac{1}{\tau}\int_{t_{k-1}}^{t_k}\int_0^s\boldsymbol{J}(\chi)d\chi ds,\boldsymbol{\phi}_1\right) = \left(\tfrac{1}{\tau}\int_{t_{k-1}}^{t_k}\boldsymbol{E}ds + \boldsymbol{f}(0),\boldsymbol{\phi}_1\right),
\end{aligned} \tag{85}$$

$$\begin{aligned}
&\tfrac{1}{\mu_0\omega_{pm}^2}(\delta_\tau\boldsymbol{K}^k,\boldsymbol{\psi}_1) + \tfrac{\Gamma_m}{\mu_0\omega_{pm}^2}\left(\tfrac{1}{\tau}\int_{t_{k-1}}^{t_k}\boldsymbol{K}ds,\boldsymbol{\psi}_1\right)\\
&+\tfrac{\omega_{m0}^2}{\mu_0\omega_{pm}^2}\left(\tfrac{1}{\tau}\int_{t_{k-1}}^{t_k}\int_0^s\boldsymbol{K}(\chi)d\chi ds,\boldsymbol{\psi}_1\right) = \left(\tfrac{1}{\tau}\int_{t_{k-1}}^{t_k}\boldsymbol{H}ds + \boldsymbol{g}(0),\boldsymbol{\psi}_1\right).
\end{aligned} \tag{86}$$

Now, to derive error equations we recall the definitions earlier near to (28) and (29) and set $\xi_h^k = \Pi_h \boldsymbol{E}^k - \boldsymbol{E}_h^k \in \boldsymbol{V}_h$, $\eta_h^k = \Pi_2 \boldsymbol{H}^k - \boldsymbol{H}_h^k \in \boldsymbol{U}_h$, $\xi_{1h}^k = \Pi_h \boldsymbol{J}^k - \boldsymbol{J}_h^k \in \boldsymbol{V}_h$ and $\eta_{1h}^k = \Pi_2 \boldsymbol{K}^k - \boldsymbol{K}_h^k \in \boldsymbol{U}_h$. Subtracting (70)-(73) from (83)-(86) and using these definitions then gives four error equations:

$$
\begin{aligned}
(a)\quad & \epsilon_0(\delta_\tau \xi_h^k, \boldsymbol{\phi}_h) - (\overline{\eta}_h^k, \nabla \times \boldsymbol{\phi}_h) + (\overline{\xi}_{1h}^k, \boldsymbol{\phi}_h) = \epsilon_0(\delta_\tau(\Pi_h \boldsymbol{E}^k - \boldsymbol{E}^k), \boldsymbol{\phi}_h) \\
(87)\quad & - \left(\Pi_2 \overline{\boldsymbol{H}}^k - \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \boldsymbol{H} ds, \nabla \times \boldsymbol{\phi}_h \right) + \left(\Pi_h \overline{\boldsymbol{J}}^k - \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \boldsymbol{J} ds, \boldsymbol{\phi}_h \right), \\
(b)\quad & \mu_0(\delta_\tau \eta_h^k, \boldsymbol{\psi}_h) + (\nabla \times \overline{\xi}_h^k, \boldsymbol{\psi}_h) + (\overline{\eta}_{1h}^k, \boldsymbol{\psi}_h) = \mu_0(\delta_\tau(\Pi_2 \boldsymbol{H}^k - \boldsymbol{H}^k), \boldsymbol{\psi}_h) \\
(88)\quad & + \left(\nabla \times \left(\Pi_h \overline{\boldsymbol{E}}^k - \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \boldsymbol{E} ds \right), \boldsymbol{\psi}_h \right) + \left(\Pi_2 \overline{\boldsymbol{K}}^k - \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \boldsymbol{K} ds, \boldsymbol{\psi}_h \right), \\
(c)\quad & \frac{1}{\epsilon_0 \omega_{pe}^2}(\delta_\tau \xi_{1h}^k + \Gamma_e \overline{\xi}_{1h}^k, \boldsymbol{\phi}_{1h}) - (\overline{\xi}_h^k, \boldsymbol{\phi}_{1h}) = \frac{1}{\epsilon_0 \omega_{pe}^2}(\delta_\tau(\Pi_h \boldsymbol{J}^k - \boldsymbol{J}^k), \boldsymbol{\phi}_{1h}) \\
& + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2}\left(\Pi_h \overline{\boldsymbol{J}}^k - \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \boldsymbol{J} ds, \boldsymbol{\phi}_{1h} \right) - \left(\Pi_h \overline{\boldsymbol{E}}^k - \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \boldsymbol{E} ds, \boldsymbol{\phi}_{1h} \right) \\
(89)\quad & + \frac{\omega_{e0}^2}{\epsilon_0 \omega_{pe}^2}\left(\frac{\widetilde{IJ_h}^k + \widetilde{IJ_h}^{k-1}}{2} - \frac{1}{\tau}\int_{t_{k-1}}^{t_k}\int_0^s \boldsymbol{J}(\chi) d\chi ds, \boldsymbol{\phi}_{1h} \right), \\
(d)\quad & \frac{1}{\mu_0 \omega_{pm}^2}(\delta_\tau \eta_{1h}^k + \Gamma_m \overline{\eta}_{1h}^k, \boldsymbol{\psi}_{1h}) - (\overline{\eta}_h^k, \boldsymbol{\psi}_{1h}) = \frac{1}{\mu_0 \omega_{pm}^2}(\delta_\tau(\Pi_2 \boldsymbol{K}^k - \boldsymbol{K}^k), \boldsymbol{\psi}_{1h}) \\
& + \frac{\Gamma_m}{\mu_0 \omega_{pm}^2}\left(\Pi_2 \overline{\boldsymbol{K}}^k - \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \boldsymbol{K} ds, \boldsymbol{\psi}_{1h} \right) - \left(\Pi_2 \overline{\boldsymbol{H}}^k - \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \boldsymbol{H} ds, \boldsymbol{\psi}_{1h} \right) \\
(90)\quad & + \frac{\omega_{m0}^2}{\mu_0 \omega_{pm}^2}\left(\frac{\widetilde{IK_h}^k + \widetilde{IK_h}^{k-1}}{2} - \frac{1}{\tau}\int_{t_{k-1}}^{t_k}\int_0^s \boldsymbol{K}(\chi) d\chi ds, \boldsymbol{\psi}_{1h} \right).
\end{aligned}
$$

In the above we also recalled that $\nabla \times \boldsymbol{V}_h \subset \boldsymbol{U}_h$.

Next we select $\boldsymbol{\phi}_h = \tau \overline{\xi}_h^k$, $\boldsymbol{\psi}_h = \tau \overline{\eta}_h^k$, $\boldsymbol{\phi}_{1h} = \tau \overline{\xi}_{1h}^k$, $\boldsymbol{\psi}_{1h} = \tau \overline{\eta}_{1h}^k$ in (87)-(90) and note that several terms can be altered or eliminated by using the L_2 projection. Specifically, the following replacements can be made,

$$
\begin{gathered}
(\Pi_2 \overline{\boldsymbol{H}}^k, \nabla \times \overline{\xi}_h^k) = (\overline{\boldsymbol{H}}^k, \nabla \times \overline{\xi}_h^k), \qquad (\Pi_2 \overline{\boldsymbol{H}}^k, \overline{\eta}_{1h}^k) = (\overline{\boldsymbol{H}}^k, \overline{\eta}_{1h}^k), \\
(\Pi_2 \overline{\boldsymbol{K}}^k, \overline{\eta}_h^k) = (\overline{\boldsymbol{K}}^k, \overline{\eta}_h^k), \qquad (\Pi_2 \overline{\boldsymbol{K}}^k, \overline{\eta}_{1h}^k) = (\overline{\boldsymbol{K}}^k, \overline{\eta}_{1h}^k),
\end{gathered}
$$

and the eliminations result from $(\delta_\tau(\Pi_2 \boldsymbol{H}^k - \boldsymbol{H}^k), \overline{\eta}_h^k) = (\delta_\tau(\Pi_2 \boldsymbol{K}^k - \boldsymbol{K}^k), \overline{\eta}_{1h}^k) = 0$. These give,

$$
\begin{aligned}
& \frac{\epsilon_0}{2}(||\xi_h^k||_0^2 - ||\xi_h^{k-1}||_0^2) + \frac{\mu_0}{2}(||\eta_h^k||_0^2 - ||\eta_h^{k-1}||_0^2) \\
& + \frac{1}{2\epsilon_0 \omega_{pe}^2}(||\xi_{1h}^k||_0^2 - ||\xi_{1h}^{k-1}||_0^2) + \frac{1}{2\mu_0 \omega_{pm}^2}(||\eta_{1h}^k||_0^2 - ||\eta_{1h}^{k-1}||_0^2) \\
& + \frac{\tau \Gamma_e}{\epsilon_0 \omega_{pe}^2}||\overline{\xi}_{1h}^k||_0^2 + \frac{\tau \Gamma_m}{\mu_0 \omega_{pm}^2}||\overline{\eta}_{1h}^k||_0^2 = \tau \epsilon_0(\delta_\tau(\Pi_h \boldsymbol{E}^k - \boldsymbol{E}^k), \overline{\xi}_h^k) \\
& - \tau \left(\overline{\boldsymbol{H}}^k - \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \boldsymbol{H} ds, \nabla \times \overline{\xi}_h^k \right) + \tau \left(\Pi_h \overline{\boldsymbol{J}}^k - \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \boldsymbol{J} ds, \overline{\xi}_h^k \right)
\end{aligned}
$$

$$+\tau\left(\nabla\times\left(\Pi_h\overline{\boldsymbol{E}}^k-\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\boldsymbol{E}ds\right),\overline{\eta}_h^k\right)+\tau\left(\overline{\boldsymbol{K}}^k-\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\boldsymbol{K}ds,\overline{\eta}_h^k\right)$$
$$+\frac{\tau}{\epsilon_0\omega_{pe}^2}(\delta_\tau(\Pi_h\boldsymbol{J}^k-\boldsymbol{J}^k),\overline{\xi}_{1h}^k)+\frac{\tau\Gamma_e}{\epsilon_0\omega_{pe}^2}\left(\Pi_h\overline{\boldsymbol{J}}^k-\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\boldsymbol{J}ds,\overline{\xi}_{1h}^k\right)$$
$$+\frac{\tau\omega_{e0}^2}{\epsilon_0\omega_{pe}^2}\left(\frac{\widetilde{IJ_h}^k+\widetilde{IJ_h}^{k-1}}{2}-\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\int_0^s\boldsymbol{J}(\chi)d\chi ds,\overline{\xi}_{1h}^k\right)$$
$$+\frac{\tau\Gamma_m}{\mu_0\omega_{pm}^2}\left(\overline{\boldsymbol{K}}^k-\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\boldsymbol{K}ds,\overline{\eta}_{1h}^k\right)-\tau\left(\overline{\boldsymbol{H}}^k-\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\boldsymbol{H}ds,\overline{\eta}_{1h}^k\right)$$
$$+\frac{\tau\omega_{m0}^2}{\mu_0\omega_{pm}^2}\left(\frac{\widetilde{IK_h}^k+\widetilde{IK_h}^{k-1}}{2}-\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\int_0^s\boldsymbol{K}(\chi)d\chi ds,\overline{\eta}_{1h}^k\right)$$
$$-\tau\left(\Pi_h\overline{\boldsymbol{E}}^k-\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\boldsymbol{E}ds,\overline{\xi}_{1h}^k\right)=\sum_{i=1}^{12}Err_i, \tag{91}$$

where these twelve error terms have been introduced for convenience and we deal with them each in turn. For the remainder of this proof C will denote a generic positive constant that is independent of time, h, τ and the exact or approximate solutions.

First, by the Cauchy-Schwarz inequality, the standard interpolation error estimate (29) for $\Pi_h\boldsymbol{E}-\boldsymbol{E}$, and the following estimate [19, p.3165]:

$$||\delta_\tau\boldsymbol{u}^k||_0^2\leqslant\frac{1}{\tau}\int_{t_{k-1}}^{t_k}||\boldsymbol{u}_t(t)||_0^2dt,\ \forall\ \boldsymbol{u}\in H^1(0,T;(L^2(\Omega))^3),$$

we have

$$\begin{aligned}Err_1 &\leqslant \tau\epsilon_0||\delta_\tau(\Pi_h\boldsymbol{E}^k-\boldsymbol{E}^k)||_0||\overline{\xi}_h^k||_0\\ &\leqslant \tau\epsilon_0\cdot\frac{\delta_1}{T}||\xi_h||_{l_\infty}^2+\frac{T\epsilon_0}{4\delta_1}\int_{t_{k-1}}^{t_k}||(\Pi_h\boldsymbol{E}-\boldsymbol{E})_s(s)||_0^2ds\\ &\leqslant \tau\epsilon_0\cdot\frac{\delta_1}{T}||\xi_h||_{l_\infty}^2+\frac{CTh^{2l}}{4\delta_1}\int_{t_{k-1}}^{t_k}||\boldsymbol{E}_t||_{H^l(\mathrm{curl};\Omega)}^2ds,\end{aligned} \tag{92}$$

where here and below δ_i will denote an arbitrary positive number and we define $||\xi_h||_{l_\infty}:=\max_{1\leqslant k\leqslant N}||\xi_h^k||_0$. Summing (92) from $k=1$ to N, we therefore have

$$\sum_{k=1}^N Err_1\leqslant\delta_1\epsilon_0||\xi_h||_{l_\infty}^2+\frac{CTh^{2l}}{\delta_1}||\boldsymbol{E}_t||_{L^2(0,T;H^l(\mathrm{curl};\Omega))}^2.$$

Similarly, by using Lemma 4.5 to obtain the following estimate,

$$\left|\left|\overline{\boldsymbol{u}}^k-\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\boldsymbol{u}(t)dt\right|\right|_0^2\leqslant\frac{\tau^3}{4}\int_{t_{k-1}}^{t_k}||\boldsymbol{u}_{tt}(t)||_0^2dt,\ \forall\ \boldsymbol{u}\in H^2(0,T;(L^2(\Omega))^3),$$

and integrating by parts to move the curl we obtain,

$$\begin{aligned}\sum_{k=1}^{N} Err_2 &\leqslant \sum_{k=1}^{N} \tau \left\| \nabla \times \left(\overline{\boldsymbol{H}}^k - \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \boldsymbol{H} ds \right) \right\|_0 ||\overline{\xi}_h^k||_0 \\ &\leqslant \sum_{k=1}^{N} \tau \left(\frac{T\tau^3}{16\delta_2} \int_{t_{k-1}}^{t_k} ||\nabla \times \boldsymbol{H}_{tt}||_0^2 \, dt + \frac{\delta_2}{T} ||\xi_h||_{l^\infty}^2 \right) \\ &\leqslant \delta_2 ||\xi_h||_{l^\infty}^2 + \frac{T\tau^4}{16\delta_2} ||\nabla \times \boldsymbol{H}_{tt}||_{L^2(0,T;L^2(\Omega))}^2.\end{aligned}$$

By similar techniques, we have the following estimates:

$$\begin{aligned}\sum_{k=1}^{N} Err_3 &\leqslant \sum_{k=1}^{N} \left[\tau \frac{\delta_3}{T} ||\overline{\xi}_h^k||_0^2 + \frac{T\tau}{2\delta_3} \left(||\Pi_h \overline{\boldsymbol{J}}^k - \overline{\boldsymbol{J}}^k||_0^2 + \left\| \overline{\boldsymbol{J}}^k - \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \boldsymbol{J} ds \right\|_0^2 \right) \right] \\ &\leqslant \delta_3 ||\xi_h||_{l^\infty}^2 + \frac{CT}{\delta_3} \Big(\tau^4 ||\boldsymbol{J}_{tt}||_{L^2(0,T;L^2(\Omega))}^2 + Th^{2l} ||\boldsymbol{J}||_{L^\infty(0,T;H^l(\mathrm{curl};\Omega))}^2 \Big), \\ \sum_{k=1}^{N} Err_4 &\leqslant \delta_4 ||\eta_h||_{l^\infty}^2 + \frac{CT}{\delta_4} \Big(\tau^4 ||\nabla \times \boldsymbol{E}_{tt}||_{L^2(0,T;L^2(\Omega))}^2 \\ &\quad + Th^{2l} ||\boldsymbol{E}||_{L^\infty(0,T;H^l(\mathrm{curl};\Omega))}^2 \Big), \\ \sum_{k=1}^{N} Err_5 &\leqslant \delta_5 ||\eta_h||_{l^\infty}^2 + \frac{T\tau^4}{16\delta_5} ||\boldsymbol{K}_{tt}||_{L^2(0,T;L^2(\Omega))}^2, \\ \sum_{k=1}^{N} Err_6 &\leqslant \frac{\delta_6}{\epsilon_0 \omega_{pe}^2} ||\xi_{1h}||_{l^\infty}^2 + \frac{CTh^{2l}}{\delta_6} ||\boldsymbol{J}_t||_{L^2(0,T;H^l(\mathrm{curl};\Omega))}^2, \\ \sum_{k=1}^{N} Err_7 &\leqslant \frac{\delta_7}{\epsilon_0 \omega_{pe}^2} ||\xi_{1h}||_{l^\infty}^2 + \frac{CT}{\delta_7} \Big(\tau^4 ||\boldsymbol{J}_{tt}||_{L^2(0,T;L^2(\Omega))}^2 \\ &\quad + Th^{2l} ||\boldsymbol{J}||_{L^\infty(0,T;H^l(\mathrm{curl};\Omega))}^2 \Big), \\ \sum_{k=1}^{N} Err_9 &\leqslant \delta_9 ||\eta_{1h}||_{l^\infty}^2 + \frac{T\tau^4}{16\delta_9} ||\boldsymbol{K}_{tt}||_{L^2(0,T;L^2(\Omega))}^2, \\ \sum_{k=1}^{N} Err_{10} &\leqslant \delta_{10} ||\eta_{1h}||_{l^\infty}^2 + \frac{T\tau^4}{16\delta_{10}} ||\boldsymbol{H}_{tt}||_{L^2(0,T;L^2(\Omega))}^2, \\ \sum_{k=1}^{N} Err_{12} &\leqslant \delta_{12} ||\xi_{1h}||_{l^\infty}^2 + \frac{CT}{\delta_{12}} \Big(\tau^4 ||\boldsymbol{E}_{tt}||_{L^2(0,T;L^2(\Omega))}^2 \\ &\quad + Th^{2l} ||\boldsymbol{E}||_{L^\infty(0,T;H^l(\mathrm{curl};\Omega))}^2 \Big).\end{aligned}$$

We now move on to investigate the more difficult terms, Err_8 and Err_{11}. Since they are similar we give the detailed working only for Err_8. First, recalling (68) and the introduction of $\widetilde{IJ}^k$ in Lemma 4.5, a simple splitting gives,

$$\begin{aligned}\left(\frac{\epsilon_0\omega_{pe}^2}{\tau\omega_{e0}^2}\right) Err_8 &= \left(\frac{\widetilde{IJ_h}^k + \widetilde{IJ_h}^{k-1}}{2} - \frac{\widetilde{IJ}^k + \widetilde{IJ}^{k-1}}{2}, \overline{\xi}_{1h}^k\right) \\ &\quad + \left(\frac{\widetilde{IJ}^k + \widetilde{IJ}^{k-1}}{2} - \frac{1}{\tau}\int_{t_{k-1}}^{t_k}\int_0^s \boldsymbol{J}(\chi)\,d\chi ds, \overline{\xi}_{1h}^k\right).\end{aligned}$$

Using the recursive formulas (68), and recalling that $\overline{\boldsymbol{J}}^l = (\boldsymbol{J}(t_l) + \boldsymbol{J}(t_{l-1}))/2$ we have,

$$\begin{aligned}&\frac{\widetilde{IJ_h}^k + \widetilde{IJ_h}^{k-1}}{2} - \frac{\widetilde{IJ}^k + \widetilde{IJ}^{k-1}}{2} = \frac{\tau}{2}(\overline{\boldsymbol{J}}_h^k - \overline{\boldsymbol{J}}^k) + \tau\sum_{l=1}^{k-1}(\overline{\boldsymbol{J}}_h^l - \overline{\boldsymbol{J}}^l) \\ &= \frac{\tau}{2}(\overline{\boldsymbol{J}}_h^k - \Pi_h\overline{\boldsymbol{J}}^k + \Pi_h\overline{\boldsymbol{J}}^k - \overline{\boldsymbol{J}}^k) + \tau\sum_{l=1}^{k-1}(\overline{\boldsymbol{J}}_h^l - \Pi_h\overline{\boldsymbol{J}}^l + \Pi_h\overline{\boldsymbol{J}}^l - \overline{\boldsymbol{J}}^l) \\ &= \frac{\tau}{2}(-\overline{\xi}_{1h}^k + \Pi_h\overline{\boldsymbol{J}}^k - \overline{\boldsymbol{J}}^k) + \tau\sum_{l=1}^{k-1}(-\overline{\xi}_{1h}^l + \Pi_h\overline{\boldsymbol{J}}^l - \overline{\boldsymbol{J}}^l)\end{aligned}$$

and with this we can now split Err_8 into three components:

$$\begin{aligned}\left(\frac{\epsilon_0\omega_{pe}^2}{\tau\omega_{e0}^2}\right) Err_8 &= -\left(\frac{\tau}{2}(\overline{\xi}_{1h}^k, \overline{\xi}_{1h}^k) + \tau\sum_{l=1}^{k-1}(\overline{\xi}_{1h}^l, \overline{\xi}_{1h}^k)\right) \\ &\quad + \frac{\tau}{2}\left(\Pi_h\overline{\boldsymbol{J}}^k - \overline{\boldsymbol{J}}^k, \overline{\xi}_{1h}^k\right) + \tau\sum_{l=1}^{k-1}\left(\Pi_h\overline{\boldsymbol{J}}^l - \overline{\boldsymbol{J}}^l, \overline{\xi}_{1h}^k\right) \\ &\quad + \left(\frac{\widetilde{IJ}^k + \widetilde{IJ}^{k-1}}{2} - \frac{1}{\tau}\int_{t_{k-1}}^{t_k}\int_0^s \boldsymbol{J}(\chi)\,d\chi ds, \overline{\xi}_{1h}^k\right) \\ &= \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3.\end{aligned}$$

The next step is to sum this over $k = 1, \ldots, N$ and, first, from Lemma 4.3,

$$\sum_{k=1}^N \mathcal{E}_1 = -\frac{\tau}{2}\left|\left|\sum_{i=1}^N \overline{\xi}_{1h}^i\right|\right|_0^2 \leqslant 0$$

and so can be discarded. For $\mathcal{E}_2$ we get,

$$\begin{aligned}\mathcal{E}_2 &\leqslant \frac{\tau}{2}||\Pi_h\overline{\boldsymbol{J}}^k - \overline{\boldsymbol{J}}^k||_0\,||\overline{\xi}_{1h}^k||_0 + \tau\sum_{i=1}^{k-1}||\Pi_h\overline{\boldsymbol{J}}^i - \overline{\boldsymbol{J}}^i||_0\,||\overline{\xi}_{1h}^k||_0 \\ &\leqslant CTh^l||\boldsymbol{J}||_{L^\infty(0,T;H^l(\mathrm{curl};\Omega))}||\overline{\xi}_{1h}^k||_0\end{aligned}$$

after using (29). Lastly, noting that

$$
\begin{aligned}
&\frac{\widetilde{IJ}^k + \widetilde{IJ}^{k-1}}{2} - \frac{1}{\tau}\int_{t_{k-1}}^{t_k}\int_0^s \boldsymbol{J}(\chi)\,d\chi ds \\
&\quad = \frac{\widetilde{IJ}^k + \widetilde{IJ}^{k-1}}{2} - \frac{1}{2}\left(\int_0^{t_k} \boldsymbol{J}(s)\,ds + \int_0^{t_{k-1}} \boldsymbol{J}(s)\,ds\right) \\
&\qquad + \frac{1}{2}\left(\int_0^{t_k} \boldsymbol{J}(s)\,ds + \int_0^{t_{k-1}} \boldsymbol{J}(s)\,ds\right) - \frac{1}{\tau}\int_{t_{k-1}}^{t_k}\int_0^s \boldsymbol{J}(\chi)\,d\chi ds \\
&\quad = \frac{1}{2}\left(\widetilde{IJ}^k - \int_0^{t_k} \boldsymbol{J}(s)\,ds\right) + \frac{1}{2}\left(\widetilde{IJ}^{k-1} - \int_0^{t_{k-1}} \boldsymbol{J}(s)\,ds\right) \\
&\qquad - \frac{1}{\tau}\left(\int_{t_{k-1}}^{t_k}\int_0^s \boldsymbol{J}(\chi)\,d\chi\,ds - \frac{\tau}{2}\left(\int_0^{t_k} \boldsymbol{J}(\chi)\,d\chi + \int_0^{t_{k-1}} \boldsymbol{J}(\chi)\,d\chi\right)\right).
\end{aligned}
$$

Using this with Lemma 4.5 then gives,

$$
\mathcal{E}_3 \leqslant \frac{1}{2}||\overline{\xi}_{1h}^k||_0 \left(\sqrt{T}\tau^2||\boldsymbol{J}_{tt}||_{L^2(0,t_k;L^2(\Omega))} + \tau^{3/2}||\boldsymbol{J}_t||_{L^2(t_{k-1},t_k;L^2(\Omega))}\right).
$$

Finally, we return to Err_8, sum over $k = 1, \ldots, N$, use these results with two Young's inequalities and infer the analogous result for Err_{11} to obtain,

$$
\begin{aligned}
\sum_{k=1}^{N} Err_8 + \sum_{k=1}^{N} Err_{11} &\leqslant \frac{\tau\Gamma_e}{2\epsilon_0\omega_{pe}^2}\sum_{k=1}^{N}||\overline{\xi}_{1h}^k||_0^2 + \frac{\tau\Gamma_m}{2\mu_0\omega_{pm}^2}\sum_{k=1}^{N}||\overline{\eta}_{1h}^k||_0^2 \\
&\quad + CT^3h^{2l}||\boldsymbol{J}||^2_{L^\infty(0,T;H^l(\mathrm{curl};\Omega))} + C(1+T^2)\tau^4||\boldsymbol{J}_t||^2_{H^1(0,T;L^2(\Omega))} \\
&\quad + CT^3h^{2l}||\boldsymbol{K}||^2_{L^\infty(0,T;H^l(\Omega))} + C(1+T^2)\tau^4||\boldsymbol{K}_t||^2_{H^1(0,T;L^2(\Omega))}.
\end{aligned} \tag{93}
$$

The proof is now completed first by assembling all of these estimates and choosing the δ_i so that each term can be controlled by 'kicking-back' the corresponding left-hand side term. The last two steps are to take the maximum with respect to N and then use triangle inequality along with approximation results to restore the full error terms to the left hand side. □

5. Conclusions

We have presented two practical schemes for the time-domain simulation of Maxwell's equations for Lorenz metamaterials. In each case we have provided data-stability estimates for both the continuous and discrete problems without using Gronwall inequalities. As a result the constants in these estimates do not grow exponentially in T. Instead the constant is either bounded independently of T, as in Theorems 3.1, 4.1 and 4.4 or grows with $T^{1/2}$, as in Theorems 4.1 and 4.4. We also gave one extensive example, Theorem 4.6, to demonstrate how these arguments can be extended to an *a priori* error analysis and this showed that the constant in the error bound is of order $O(T^{3/2})$ in that case. As a result, apart from this moderate (at least, as compared to $\exp(cT)$) growth in the constant the error growth can be expected to be dominated only by the underlying time dependence in the higher derivatives of the exact solution and data.

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Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Xiangtan University, China.

Department of Mathematical Sciences, University of Nevada Las Vegas, NV 89154-4020, USA
E-mail: `jichun@unlv.nevada.edu`

BICOM, Institute of Computational Mathematics, Brunel University, Uxbridge, Middlesex UB8 3PH, UK
E-mail: `simon.shaw@brunel.ac.uk`
URL: `www.brunel.ac.uk/bicom`