THE CLIQUE AND COCLIQUE NUMBERS' BOUNDS BASED ON THE H-EIGENVALUES OF UNIFORM HYPERGRAPHS

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Abstract. In this paper, some inequality relations between the Laplacian/signless Laplacian H-eigenvalues and the clique/coclique numbers of uniform hypergraphs are presented. For a connected uniform hypergraph, some tight lower bounds on the largest Laplacian H$^+$-eigenvalue and signless Laplacian H-eigenvalue related to the clique/coclique numbers are given. And some upper and lower bounds on the clique/coclique numbers related to the largest Laplacian/signless Laplacian H-eigenvalues are obtained. Also some bounds on the sum of the largest/smallest adjacency/Laplacian/signless Laplacian H-eigenvalues of a hypergraph and its complement hypergraph are showed. All these bounds are consistent with what we have known when k is equal to 2.

Key words. H-eigenvalue, clique, coclique, hypergraph, tensor, signless Laplacian, Laplacian, adjacency.

1. Introduction

In the current combinatorics and graph theory associative literatures, a growing number of them studied hypergraphs and their applications in various fields [1, 3, 7] because hypergraphs can be the better mathematical models in many practical cases and higher order structures than graphs. On the other hand, tensor is well known as an important tool in applied mathematics and virtually every discipline in the engineering and physical sciences that makes some use of it. So it is a natural thought to study properties of hypergraphs by using the tool of tensor. In 2005, the definition of eigenvalue of a tensor was independently proposed by Lim [16] and Qi [23]. At the same time, several kinds of eigenvalues for tensors had been proposed, such as H-eigenvalues, Z-eigenvalues, E-eigenvalues and N-eigenvalues. In 2007, by Lim [17] the study of hypergraph via its adjacency tensor and its eigenvalues was initiated. Then in 2009, Rota Bulò and Pelillo [26–28] gave new bounds on the clique number of a uniform hypergraph based on analysis of the largest eigenvalue of the adjacency tensor. As we know, the problem to find the clique number of a 2-uniform hypergraph(i.e. graph) is the NP-complete problem [8], and turns out to be even intractable to a k-uniform hypergraph for $k \geq 3$. However, we have a good algorithm for calculating the largest H-eigenvalues of an irreducible nonnegative tensor [20]. Therefore, it is significant to us depict the bounds on the clique number related to the largest H-eigenvalues for k-uniform hypergraphs.

In this paper, we study some relations between the Laplacian/signless Laplacian H-eigenvalues and the clique/coclique numbers of uniform hypergraphs. The Laplacian/signless Laplacian H-eigenvalues of a uniform hypergraph refer to respectively

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the H-eigenvalues of the Laplacian/signless Laplacian tensors of this uniform hypergraph. This work is motivated by the classic results for graphs [2,9,18,19,21,29], the results of Rota Bulò and Pelillo [28] and the latest results of Yi and Chang [34]. Recently, several papers appeared on nonnegative tensors and spectral hypergraph theory via tensors [4, 6, 10–17, 22–28, 30–34]. Among them, Cooper et al [6] and Qi [24] respectively systematically studied the adjacency tensors, Laplacian and signless Laplacian tensors of uniform hypergraphs. These three notions of tensors are more natural and simpler than those in the literature, so we follow these three notions of tensors throughout the sequel discussion.

The rest of this paper is organized as follows. In the next section, we restatement some definitions on eigenvalues of tensors and uniform hypergraphs. Also we give the definitions and some known results on clique and coclique numbers of a uniform hypergraph. We discuss in Section 3 some inequality relations between the Laplacian/signless Laplacian H-eigenvalues and the clique number of a uniform hypergraph. In Section 4, we present some inequality relations between the Laplacian/ signless Laplacian H-eigenvalues and the coclique number of a uniform hypergraph. Also we give some bounds on the sum of the largest/smallest adjacency/Laplacian/signless Laplacian H-eigenvalues of a hypergraph and its complement hypergraph.

2. Preliminaries

Some definitions of eigenvalues of tensors and uniform hypergraphs are presented in this section.

2.1. H-Eigenvalues of tensors. In this subsection, some basic definitions on H-eigenvalues of tensors are reviewed. For comprehensive references, see [10, 23] and references therein. Especially, for spectral hypergraph theory oriented facts on H-eigenvalues of tensors, please see [12, 24].

Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}^n$ the n-dimensional real space. $\mathbb{R}^n_+$ denotes the nonnegative orthant of $\mathbb{R}^n$. $\mathbb{R}^n_{++}$ denotes the positive orthant of $\mathbb{R}^n$. For integers $k \geq 3$ and $n \geq 2$, a real tensor $\mathcal{T} = (t_{i_1 \dots i_k})$ of order k and dimension n refers to a multiway array (also called hypermatrix) with entries $t_{i_1 \dots i_k}$ such that $t_{i_1 \dots i_k} \in \mathbb{R}$ for all $i_j \in [n] := \{1, \dots, n\}$ and $j \in [k]$. Tensors are always referred to k-th order real tensors in this paper, and the dimensions will be clear from the content. Given a vector $\mathbf{x} \in \mathbb{R}^n$, $\mathcal{T}\mathbf{x}^k$ is defined as $\displaystyle\sum_{i_1,i_2,\dots,i_k \in [n]} t_{i_1 i_2 \dots i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ and $\mathcal{T}\mathbf{x}^{k-1}$ is defined as an n-dimensional vector such that its i-th element being $\displaystyle\sum_{i_2,\dots,i_k \in [n]} t_{i i_2 \dots i_k} x_{i_2} \cdots x_{i_k}$ for all $i \in [n]$. Let $\mathcal{I}$ be the identity tensor of appropriate dimension, e.g., $i_{i_1 \dots i_k} = 1$ if and only if $i_1 = \cdots = i_k \in [n]$, and zero otherwise when the dimension is n. The following definition was introduced by Qi [23].

Definition 2.1. *Let $\mathcal{T}$ be a k-th order n-dimensional real tensor. For some $\lambda \in \mathbb{R}$, if eigenvalue equation $(\lambda\mathcal{I} - \mathcal{T})\mathbf{x}^{k-1} = 0$ has a solution $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, then λ is called an H-eigenvalue and $\mathbf{x}$ an H-eigenvector associated to $\lambda = \frac{\mathcal{T}\mathbf{x}^k}{\|\mathbf{x}\|_k^k}$. Furthermore, if $\mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, then we say that λ is an H^+-eigenvalue of $\mathcal{T}$.*

It is seen that H-eigenvalues are real numbers [23]. By [10, 23], we have that the number of H-eigenvalues of a real tensor is finite. By [24], we have that all the tensors considered in this paper have at least one H-eigenvalue. Hence, we can denote by $\lambda(\mathcal{T})$ (respectively $\mu(\mathcal{T})$) as the largest (respectively smallest) H-eigenvalue of a real tensor $\mathcal{T}$.

For a subset $S \subseteq [n]$, we denoted by $|S|$ its cardinality, and $\sup(\mathbf{x}) := \{i \in [n] \mid x_i \neq 0\}$ its *support*.

2.2. Uniform hypergraphs. In this subsection, we present some essential concepts of uniform hypergraphs which will be used in the sequel. Please refer to $[1, 3, 5, 12, 24]$ for comprehensive references.

In this paper, a hypergraph means an undirected simple k-uniform hypergraph G with vertex set V, which is labeled as $[n] = \{1, \ldots, n\}$, and edge set $E = \{e_1, \ldots, e_m\}$ with $e_p \subseteq V$ for $p = 1, \ldots, m$. By k-uniformity, we mean that for every edge $e \in E$, the cardinality $|e|$ of e is equal to k. A 2-uniform hypergraph is typically called *graph*. Throughout this paper, we focus on $k \geq 3$ and $n \geq k$. Moreover, since the trivial hypergraph (i.e., $E = \emptyset$) is of less interest, we consider only hypergraphs having at least one edge (i.e., nontrivial) in this paper.

For a subset $S \subset [n]$, we denoted by E_S the set of edges $\{e \in E \mid S \cap e \neq \emptyset\}$. For a vertex $i \in V$, we simplify $E_{\{i\}}$ as E_i. It is the set of edges containing the vertex i, i.e., $E_i := \{e \in E \mid i \in e\}$. The cardinality $|E_i|$ of the set E_i is defined as the *degree* of the vertex i, which is denoted by d_i. Two different vertices i and j are *connected* to each other (or the pair i and j is connected), if there is a sequence of edges $(e_1, \ldots, e_m)$ such that $i \in e_1$, $j \in e_m$ and $e_r \cap e_{r+1} \neq \emptyset$ for all $r \in [m-1]$. A hypergraph is called *connected*, if every pair of different vertices of G is connected. Let $S \subseteq V$, the hypergraph G_S with vertex set S and edge set $\{e \in E \mid e \subseteq S\}$ is called the *sub-hypergraph* of G induced by S. We denote the maximum degree, the minimum degree and the average degree of G by d_{max}, d_{min} and $\bar{d}$ respectively. If $d_{max} = d_{min} = \bar{d}$, then G is a *regular* graph, called a $\bar{d}$-regular hypergraph. An n-vertex k-uniform hypergraph G is *complete* if G is d-regular with $d = \binom{n-1}{k-1}$. Here we denote an n-vertex k-uniform complete hypergraph by K_n^k. The *complement hypergraph* of a k-uniform hypergraph G is given by $\bar{G} = (V, \bar{E})$ where $\bar{E} = \binom{V}{k} \setminus E$. In the sequel, unless stated otherwise, all the notations introduced above are reserved for the specific meanings.

For the sake of simplicity, we mainly consider connected hypergraphs in the subsequent analysis. By the techniques in $[12, 24]$, the conclusions on connected hypergraphs can be easily generalized to general hypergraphs.

The following definition for the adjacency tensor was proposed by Cooper and Dutle [6], which differs from Lim [17] by a constant multiple $\frac{1}{(k-1)!}$.

Definition 2.2. *Let $G = (V, E)$ be a k-uniform hypergraph. The* adjacency tensor *of G is defined as the k-th order n-dimensional tensor $\mathcal{A}$ whose $(i_1 \ldots i_k)$-entry is:*

$$a_{i_1 \ldots i_k} := \begin{cases} \frac{1}{(k-1)!} & if \ \{i_1, \ldots, i_k\} \in E, \\ 0 & otherwise. \end{cases}$$

The normalization factor $\frac{1}{(k-1)!}$ in Definition 2.2 is included essentially for aesthetic reasons. It could easily be absorbed into λ in the eigenvalue equation of Definition 2.1 without altering any of the calculations. Normalization allows the adjacency tensor to faithfully generalize the adjacency matrix of a graph while removing a factor of $\frac{1}{(k-1)!}$ that would otherwise make an appearance in some of the results below [6].

The following definition for the Laplacian tensor and signless Laplacian tensor was proposed by Qi [24].

Definition 2.3. *Let $G = (V, E)$ be a k-uniform hypergraph and $\mathcal{A}$ be its adjacency tensor. Let $\mathcal{D}$ be a k-th order n-dimensional diagonal tensor with its diagonal*

element $d_{i...i}$ being d_i, the degree of vertex i in G, for all $i \in [n]$. Then $\mathcal{L} := \mathcal{D} - \mathcal{A}$ is the Laplacian tensor *of the hypergraph G, and $\mathcal{Q} := \mathcal{D} + \mathcal{A}$ is the* signless Laplacian tensor *of the hypergraph G.*

Using Definition 2.2 and Definition 2.3, a number of results from [27] and [34] can be also generalized to the k-uniform hypergraph case in a more natural way. Indeed, some results need only slight modifications of their standard proofs. Others require the use of new techniques.

Definition 2.4. *The* clique *of a k-uniform hypergraph G is a set of vertices such that any of its k vertex subsets is an edge of G, and the largest cardinality of a clique of G is called the* clique number *of G, denoted by $\omega(G)$ or simply by ω.*

Definition 2.5. *The* coclique *(or* independence set*) of a k-uniform hypergraph G is a set of vertices such that any of its k vertex subsets is not an edge of G, and the largest cardinality of a coclique of G is called the* coclique number *(or* independence number*) of G, denoted by $\bar{\omega}(G)$ or simply by $\bar{\omega}$.*

In the following discussion, we denote by $\lambda(\mathcal{A})$, $\lambda(\mathcal{L})$, $\lambda(\mathcal{Q})$ and $\mu(\mathcal{Q})$ respectively as the largest adjacency/Laplacian/ signless Laplacian H-eigenvalues and the smallest signless Laplacian H-eigenvalue of a k-uniform hypergraph G. we denote by $\lambda^+(\mathcal{A})$, $\lambda^+(\mathcal{L})$, $\lambda^+(\mathcal{Q})$ and $\mu^+(\mathcal{Q})$ respectively as the largest adjacency/Laplacian/ signless Laplacian H$^+$-eigenvalues and the smallest signless Laplacian H$^+$-eigenvalue of a k-uniform hypergraph G. From [24], we know that $\lambda^+(\mathcal{A}) = \lambda(\mathcal{A}) \leq \lambda^+(\mathcal{L}) = d_{max} \leq \lambda(\mathcal{L}) \leq \lambda(\mathcal{Q}) = \lambda^+(\mathcal{Q})$ and $\mu(\mathcal{Q}) \leq \mu^+(\mathcal{Q})$.

2.3. Some known results. We show the latest three results of Yi and Chang in [34] first.

Theorem 2.1. *Given a k-uniform hypergraph G, let $\mathcal{A}$ be its adjacency tensor, ω its clique number. Then we have*

$$\lambda(\mathcal{A}) \geq \binom{\omega - 1}{k - 1}$$

and equality holds if and only if G is a k-uniform complete hypergraph.

Theorem 2.2. *Given a k-uniform hypergraph G, let $\mathcal{A}$ be its adjacency tensor, ω its clique number. Then we have*

$$\omega \leq \sqrt[k-1]{\lambda(\mathcal{A})(k-1)!} + (k-1)$$

and equality holds if and only if G is a 2-uniform complete hypergraph(i.e. complete graph).

Theorem 2.3. *Given a k-uniform hypergraph $G = (V, E)$ with $|V| = n$, let $\mathcal{A}$ be its adjacency tensor, ω its clique number. If any sub-hypergraph of G is either a complete hypergraph or a hypergraph where exists two different vertices v_i and v_j not contained in the same edge, then we have*

$$\omega \geq \frac{n}{n - \sqrt[k-1]{\lambda(\mathcal{A})(k-1)!}}.$$

3. The H-eigenvalues and the clique number

This section presents some inequality relations between the Laplacian/ signless Laplacian H-eigenvalues and the clique number of a k-uniform hypergraph.

Now we provide some new upper and lower bounds about the clique number of a uniform hypergraph based on the largest Laplacian and signless Laplacian H-eigenvalues.

Theorem 3.1. *Given a k-uniform hypergraph G, let $\mathcal{Q}$ be its signless Laplacian tensor, ω its clique number. Then we have*

$$\lambda(\mathcal{Q}) \geq 2\binom{\omega-1}{k-1}$$

and equality holds if and only if G is a k-uniform complete hypergraph.

Proof. Assume that S be a maximum clique of G, $\mathbf{y}$ be a vector such that $y_i = \frac{1}{\sqrt[k]{\omega}}$ for $i \in S$ and $y_i = 0$ for otherwise, it is obvious that $\|\mathbf{y}\|_k^k = 1$. By Definition 2.1,

$$
\begin{aligned}
\lambda(\mathcal{Q}) &= \max_{\mathbf{x}\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\mathcal{Q}\mathbf{x}^k}{\|\mathbf{x}\|_k^k} \\
&\geq \mathcal{Q}\mathbf{y}^k = \sum_{i_1,i_2,\ldots,i_k\in[n]} q_{i_1i_2\ldots i_k} y_{i_1} y_{i_2} \cdots y_{i_k} \\
&= \sum_{p=1}^n d_p y_p^k + \sum_{\{i_1,i_2,\ldots,i_k\}=e_i\in E} q_{i_1i_2\ldots i_k} y_{i_1} y_{i_2} \cdots y_{i_k} \\
&\geq \sum_{p=1}^n \binom{\omega-1}{k-1} y_p^k + \binom{\omega}{k} \frac{1}{(k-1)!} k! \left(\frac{1}{\sqrt[k]{\omega}}\right)^k \\
&= \binom{\omega-1}{k-1} + \binom{\omega-1}{k-1} \\
&= 2\binom{\omega-1}{k-1}.
\end{aligned}
$$

If $\lambda(\mathcal{Q}) = 2\binom{\omega-1}{k-1}$, we have $\mathcal{Q}\mathbf{y}^k = \max_{\mathbf{x}\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\mathcal{Q}\mathbf{x}^k}{\|\mathbf{x}\|_k^k}$, i.e., $\mathbf{y}$ is the largest H-eigenvector corresponding to $\lambda(\mathcal{Q})$. Then by Theorem 6 in [24], we known that $\mathbf{y} \in \mathbb{R}_{++}^n$, i.e., all the entries of $\mathbf{y}$ are greater than 0. So all the vertices belong to S, it also means that $\omega = n$, i.e., G is a complete hypergraph. On the other hand, if G is a complete hypergraph, then$\omega = n$. By Proposition 4.2 in [14], $\lambda(\mathcal{Q}) = 2\binom{n-1}{k-1} = 2\binom{\omega-1}{k-1}$. Therefore, the theorem follows. $\square$

Proposition 3.1. *Given a k-uniform hypergraph G, let $\mathcal{L}$ be its Laplacian tensor, ω its clique number. Then we have*

$$\lambda^+(\mathcal{L}) \geq \binom{\omega-1}{k-1}$$

and equality holds if and only if G is a k-uniform complete hypergraph.

Proof. By Theorem 10 in [24], we have that $\lambda^+(\mathcal{L}) = d_{max}$. On the other hand, since ω is the clique number, we have $d_{max} \geq \binom{\omega-1}{k-1}$ and equality holds if and only if G is a k-uniform complete hypergraph. Thus, $\lambda^+(\mathcal{L}) \geq \binom{\omega-1}{k-1}$ and equality holds if and only if G is a k-uniform complete hypergraph. $\square$

Since $\lambda(\mathcal{L}) \geq \lambda^+(\mathcal{L})$, by Proposition 3.1, we can easily get the following corollary.

Corollary 3.1. *Given a k-uniform hypergraph G, let $\mathcal{L}$ be its Laplacian tensor, ω its clique number. Then we have*

$$\lambda(\mathcal{L}) \geq \binom{\omega-1}{k-1}.$$

Theorem 3.2. *Given a k-uniform hypergraph G, let $\mathcal{Q}$ be its signless Laplacian tensor, ω its clique number. Then we have*

$$\omega \leq \sqrt[k-1]{\lambda(\mathcal{Q})(k-1)!/2} + (k-1)$$

and equality holds if and only if G is a 2-uniform complete hypergraph.

Proof. Note that

$$\binom{\omega - 1}{k - 1} = \frac{(\omega - 1)(\omega - 2) \cdots [\omega - (k - 1)]}{(k - 1)!} \geq \frac{[\omega - (k - 1)]^{k-1}}{(k - 1)!}.$$

By Theorem 3.1, we get that $\lambda(\mathcal{Q}) \geq 2\frac{[\omega-(k-1)]^{k-1}}{(k-1)!}$. Then, one can have

$$[\omega - (k - 1)]^{k-1} \leq \lambda(\mathcal{Q})(k - 1)!/2.$$

Since $\omega \geq k$, we have that

$$\omega \leq \sqrt[k-1]{\lambda(\mathcal{Q})(k - 1)!/2} + (k - 1)$$

and equality holds if and only if $k = 2$ and G is a k-uniform complete hypergraph, i.e., G is a 2-uniform complete hypergraph. $\qquad\square$

Proposition 3.2. *Given a k-uniform hypergraph G, let $\mathcal{L}$ be its Laplacian tensor, ω its clique number. Then we have*

$$\omega \leq \sqrt[k-1]{\lambda^+(\mathcal{L})(k - 1)!} + (k - 1)$$

and equality holds if and only if G is a 2-uniform complete hypergraph.

Proof. According to Proposition 3.1, similar to the proof of Theorem 3.2, we obtain this proposition. $\qquad\square$

Since $\lambda(\mathcal{L}) \geq \lambda^+(\mathcal{L})$, by Proposition 3.2, we also easily get the following corollary.

Corollary 3.2. *Given a k-uniform hypergraph G, let $\mathcal{L}$ be its Laplacian tensor, ω its clique number. Then we have*

$$\omega \leq \sqrt[k-1]{\lambda(\mathcal{L})(k - 1)!} + (k - 1).$$

Theorem 3.3. *Given a k-uniform hypergraph $G = (V, E)$ with $|V| = n$, let $\mathcal{L}$, $\mathcal{Q}$, ω and d_{max} be its Laplacian tensor, signless Laplacian tensor, clique number and maximum degree respectively. If any sub-hypergraph of G is either a complete hypergraph or a hypergraph where exists two different vertices v_i and v_j not contained in the same edge, then we have*

$$\omega \geq \frac{n}{n - \sqrt[k-1]{(\lambda(\mathcal{Q}) - d_{max})(k - 1)!}}, \qquad \omega \geq \frac{n}{n - \sqrt[k-1]{(\lambda(\mathcal{L}) - d_{max})(k - 1)!}}.$$

Proof. According to Proposition 14 in [24] and Proposition 4.1 in [14], we have

$$0 \leq \lambda(\mathcal{L}) - d_{max} \leq \lambda(\mathcal{Q}) - d_{max} \leq \lambda(\mathcal{A}).$$

On the other hand, $\frac{n}{n - \sqrt[k-1]{\lambda(k-1)!}}$ is a monotone increasing function on $\lambda \geq 0$. By Theorem 2.3, this theorem is easily obtained. $\qquad\square$

At last of this section, we give an obvious proposition as follows.

Proposition 3.3. *Given an n-vertex k-uniform hypergraph G with at least one edge, let ω be its clique number. Then we have*

$$k \leq \omega \leq n.$$

4. The H-eigenvalues and the coclique number

This section presents some inequality relations between the Laplacian/ signless Laplacian H-eigenvalues and the coclique number of a k-uniform hypergraph, and gives some bounds on the sum of the largest/smallest adjacency/Laplacian/signless Laplacian H-eigenvalues of a hypergraph and its complement hypergraph.

First we show some properties of the coclique number of a k-uniform hypergraph. Since the proofs of Lemmas 4.1 and 4.2 need only slight modifications of their standard proofs when $k = 2$, we just give Lemmas 4.1 and 4.2 and omit their proofs here.

Lemma 4.1. *Given a k-uniform hypergraph G, let $\bar{G}$ be its complement hypergraph. Then we have*

$$\bar{\omega}(G) = \omega(\bar{G}).$$

Lemma 4.2. *Given a k-uniform hypergraph G of n vertices, let $\bar{\omega}$ be its coclique number. Then we have*

$$k - 1 \leq \bar{\omega} \leq n.$$

By Lemma 4.1, it is not difficult to know that $\bar{\omega}(\bar{G}) = \omega(G)$. Hence, according to Theorems 3.1, 3.2, 3.3 and Proposition 3.3, we have the following four corollaries.

Corollary 4.1. *Given a k-uniform hypergraph G, let $\bar{G}$ and $\mathcal{Q}$ be its complement hypergraph and signless Laplacian tensor, $\bar{\omega}$ its coclique number. Then we have*

$$\lambda(\mathcal{Q}) \geq 2\binom{\bar{\omega}(\bar{G}) - 1}{k - 1}$$

and equality holds if and only if G is a k-uniform complete hypergraph.

Corollary 4.2. *Given a k-uniform hypergraph G, let $\bar{G}$ and $\mathcal{Q}$ be its complement hypergraph and signless Laplacian tensor, $\bar{\omega}$ its coclique number. Then we have*

$$\bar{\omega}(\bar{G}) \leq \sqrt[k-1]{\lambda(\mathcal{Q})(k - 1)!/2} + (k - 1)$$

and equality holds if and only if G is a 2-uniform complete hypergraph.

Corollary 4.3. *Given a k-uniform hypergraph $G = (V, E)$ with $|V| = n$, let $\mathcal{L}$, $\mathcal{Q}$, $\bar{\omega}$ and d_{max} be its Laplacian tensor, signless Laplacian tensor, coclique number and maximum degree respectively. If any sub-hypergraph of G is either a complete hypergraph or a hypergraph where exists two different vertices v_i and v_j not contained in the same edge, then we have*

$$\bar{\omega}(\bar{G}) \geq \frac{n}{n - \sqrt[k-1]{(\lambda(\mathcal{Q}) - d_{max})(k - 1)!}}, \ \bar{\omega}(\bar{G}) \geq \frac{n}{n - \sqrt[k-1]{(\lambda(\mathcal{L}) - d_{max})(k - 1)!}}.$$

Corollary 4.4. *Given an n-vertex k-uniform hypergraph G with at least one edge, let $\bar{G}$ be its complement hypergraph, $\bar{\omega}$ its coclique number. Then we have*

$$k \leq \bar{\omega}(\bar{G}) \leq n.$$

Then we show some bounds on the sum of the largest adjacency/Laplacian/signless Laplacian H-eigenvalues of a hypergraph and its complement hypergraph.

Theorem 4.1. *Given an n-vertex k-uniform hypergraph G, let $\mathcal{A}$, $\mathcal{L}$, $\mathcal{Q}$, $\bar{G}$, d_{max} and d_{min}, be its adjacency tensor, Laplacian tensor, signless Laplacian tensor, complement hypergraph, maximum degree and minimum degree respectively. Then we have*

$$\binom{n - 1}{k - 1} \leq \lambda(\mathcal{A}(G)) + \lambda(\mathcal{A}(\bar{G})) \leq \frac{2n - k - 1}{k - 1}\binom{n - 2}{k - 2},$$

$$\binom{n-1}{k-1} \le \lambda(\mathcal{L}(G)) + \lambda(\mathcal{L}(\bar{G})) \le \lambda(\mathcal{Q}(G)) + \lambda(\mathcal{Q}(\bar{G})) \le \frac{2(2n-k-1)}{k-1}\binom{n-2}{k-2}.$$

Proof. By [24] and Proposition 4.1 of [14], we know that

$$d_{min}(G) \le \lambda(\mathcal{A}(G)) \le d_{max}(G) \le \lambda(\mathcal{L}(G)) \le \lambda(\mathcal{Q}(G)) \le \lambda(\mathcal{A}(G)) + d_{max}(G).$$

So we have

$$d_{min}(\bar{G}) \le \lambda(\mathcal{A}(\bar{G})) \le d_{max}(\bar{G}) \le \lambda(\mathcal{L}(\bar{G})) \le \lambda(\mathcal{Q}(\bar{G})) \le \lambda(\mathcal{A}(\bar{G})) + d_{max}(\bar{G}).$$

On the other hand, similar to the proof of Proposition 4.1 of [14], it is not difficult to prove that

$$\lambda(\mathcal{T}(K_n^k)) \le \lambda(\mathcal{T}(G)) + \lambda(\mathcal{T}(\bar{G}))$$

for $\mathcal{T} = \mathcal{A}$, $\mathcal{L}$ and $\mathcal{Q}$.

Hence, we have

$$\binom{n-1}{k-1} = d_{max}(K_n^k) = \lambda(\mathcal{A}(K_n^k)) \le \lambda(\mathcal{A}(G)) + \lambda(\mathcal{A}(\bar{G}))$$

and

$$\lambda(\mathcal{A}(G)) + \lambda(\mathcal{A}(\bar{G})) \le d_{max}(G) + d_{max}(\bar{G}) \le \binom{n-1}{k-1} + \binom{n-2}{k-1} \le \frac{2n-k-1}{k-1}\binom{n-2}{k-2}.$$

Furthermore, we have

$$
\begin{aligned}
\lambda(\mathcal{Q}(G)) + \lambda(\mathcal{Q}(\bar{G})) &\le \left(\lambda(\mathcal{A}(G)) + d_{max}(G)\right) + \left(\lambda(\mathcal{A}(\bar{G})) + d_{max}(\bar{G})\right) \\
&\le \left(\lambda(\mathcal{A}(G)) + \lambda(\mathcal{A}(\bar{G}))\right) + \left(d_{max}(G) + d_{max}(\bar{G})\right) \\
&\le \frac{2(2n-k-1)}{k-1}\binom{n-2}{k-2}
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda(\mathcal{Q}(G)) + \lambda(\mathcal{Q}(\bar{G})) &\ge \lambda(\mathcal{L}(G)) + \lambda(\mathcal{L}(\bar{G})) \\
&\ge d_{max}(G) + d_{max}(\bar{G}) \\
&\ge d_i(G) + d_i(\bar{G}) \\
&= \binom{n-1}{k-1}.
\end{aligned}
$$

Consequently, this theorem follows.   □

At last, we show some bounds on the sum of the smallest adjacency/Laplacian/signless Laplacian H-eigenvalues of a hypergraph and its complement hypergraph.

Theorem 4.2. *Given an n-vertex k-uniform hypergraph G, let $\mathcal{A}$, $\mathcal{L}$, $\mathcal{Q}$, $\bar{G}$ and d_{min}, be its adjacency tensor, Laplacian tensor, signless Laplacian tensor, complement hypergraph and minimum degree respectively. Then we have*

$$-\frac{2n-k-1}{k-1}\binom{n-2}{k-2} \le \mu(\mathcal{A}(G)) + \mu(\mathcal{A}(\bar{G})) \le -1,$$

$$0 \le \mu(\mathcal{L}(G)) + \mu(\mathcal{L}(\bar{G})) \le \mu(\mathcal{Q}(G)) + \mu(\mathcal{Q}(\bar{G})) \le \binom{n-1}{k-1}.$$

Proof. By [24] and Proposition 4.1 of [14], we know that

$$-\lambda(\mathcal{A}(G)) \leq \mu(\mathcal{A}(G)) \leq 0 \leq \mu(\mathcal{L}(G)) \leq \mu(\mathcal{Q}(G)) \leq d_{min}(G).$$

So we have

$$-\lambda(\mathcal{A}(\bar{G})) \leq \mu(\mathcal{A}(\bar{G})) \leq 0 \leq \mu(\mathcal{L}(\bar{G})) \leq \mu(\mathcal{Q}(\bar{G})) \leq d_{min}(\bar{G}).$$

On the other hand, similar to the proof of Proposition 4.1 of [14], it is not difficult to prove that

$$\mu(\mathcal{T}(K_n^k)) \geq \mu(\mathcal{T}(G)) + \mu(\mathcal{T}(\bar{G}))$$

for $\mathcal{T} = \mathcal{A}$, $\mathcal{L}$ and $\mathcal{Q}$.

Hence, we have

$$\mu(\mathcal{A}(G)) + \mu(\mathcal{A}(\bar{G})) \leq \mu(\mathcal{A}(K_n^k)) \leq -1$$

and

$$\mu(\mathcal{A}(G)) + \mu(\mathcal{A}(\bar{G})) \geq -\Big(\lambda(\mathcal{A}(G)) + \lambda(\mathcal{A}(\bar{G}))\Big) \geq -\frac{2n-k-1}{k-1}\binom{n-2}{k-2}.$$

Furthermore, we have

$$\mu(\mathcal{Q}(G)) + \mu(\mathcal{Q}(\bar{G})) \geq \mu(\mathcal{L}(G)) + \mu(\mathcal{L}(\bar{G})) \geq \mu(\mathcal{L}(K_n^k)) \geq 0$$

and

$$\mu(\mathcal{Q}(G)) + \mu(\mathcal{Q}(\bar{G})) \leq d_{min}(G) + d_{min}(\bar{G}) \leq d_i(G) + d_i(\bar{G}) = \binom{n-1}{k-1}.$$

Consequently, this theorem follows.   □

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