

## A HYBRIDIZED WEAK GALERKIN FINITE ELEMENT METHOD FOR THE BIHARMONIC EQUATION

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**Abstract.** This paper presents a hybridized formulation for the weak Galerkin finite element method for the biharmonic equation based on the discrete weak Hessian recently proposed by the authors. The hybridized weak Galerkin scheme is based on the use of a Lagrange multiplier defined on the element interfaces. The Lagrange multiplier is verified to provide a numerical approximation for certain derivatives of the exact solution. An error estimate of optimal order is established for the numerical approximations arising from the hybridized weak Galerkin finite element method. The paper also derives a computational algorithm (Schur complement) by eliminating all the unknowns associated with the interior variables on each element, yielding a significantly reduced system of linear equations for unknowns on the element interfaces.

**Key words.** Weak Galerkin, hybridized weak Galerkin, finite element methods, weak Hessian, biharmonic problems.

### 1. Introduction

In this paper, we are concerned with new developments of weak Galerkin finite element methods for partial differential equations. In particular, we shall employ the usual hybridization technique [7, 1, 6] to the weak Galerkin finite element method for the biharmonic equations proposed and analyzed in [13].

For simplicity, we consider the following biharmonic equation with Dirichlet and Neumann boundary conditions: Find  $u \in H^2(\Omega)$  such that  $u = \xi$  and  $\frac{\partial u}{\partial \mathbf{n}} = \eta$  on the boundary of the domain, and satisfying

$$(1) \quad (1 - \nu) \int_{\Omega} \nabla^2 u : \nabla^2 v d\Omega + \nu \int_{\Omega} \Delta u \Delta v d\Omega = \int_{\Omega} f v d\Omega, \quad \forall v \in H_0^2(\Omega).$$

Here  $\Omega$  is an open bounded domain in the Euclidean space  $\mathbb{R}^d$  ( $d = 2, 3$ ) with Lipschitz continuous boundary  $\partial\Omega$ ,  $\nabla^2 v$  is the Hessian tensor of  $v$ ,  $\Delta v$  is the Laplacian of  $v$ , and  $\nu \in [0, \frac{1}{2}]$  is the Poisson ratio of the plate. For simplicity, we consider the case of  $\nu = 0$  so that the weak form is reduced to

$$(2) \quad \int_{\Omega} \nabla^2 u : \nabla^2 v d\Omega = \int_{\Omega} f v d\Omega, \quad \forall v \in H_0^2(\Omega).$$

The weak Galerkin method is a finite element technique that approximates differential operators (e.g., gradient, divergence, curl, Laplacian, Hessian, etc) as distributions. The method has been successfully applied to several classes of partial differential equations, such as the second order elliptic equation [15, 8, 14], the Stokes equation [16], the Maxwell's equations [10], and the biharmonic equation [9, 13]. For example, in [13], a weak Galerkin finite element method was developed for the biharmonic equation (1) by using polynomials of degree  $P_k/P_{k-2}/P_{k-2}$  for

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any  $k \geq 2$ , where  $P_k$  was used to approximate the function  $u$  on each element and  $P_{k-2}$  was employed to approximate the trace of  $u$  and  $\nabla u$  on the boundary of each element. The objective of this paper is to exploit the use of hybridization techniques in the weak Galerkin methods that shall further relax the connection of the finite element functions among elements.

Hybridization is a useful technique in the finite element methods. The key to hybridization is to identify a Lagrange multiplier which can be used to relax certain constraints (e.g., continuity) imposed on the finite element function across element interfaces. Hybridization has been employed in the mixed finite element methods to yield hybridized mixed finite element formulations suitable for efficient implementation in practical computation [1, 3, 4, 5, 7, 11, 12]. The idea of hybridization was also used in discontinuous Galerkin methods [2] for deriving hybridized discontinuous Galerkin (HDG) finite element methods [6].

We shall show in this paper that hybridization is a natural approach for the weak Galerkin finite element methods. For illustrative purpose, we demonstrate how hybridization can be accomplished for the weak Galerkin finite element scheme of [13]. We shall also establish a theoretical foundation to address critical issues such as stability and convergence for the hybridized weak Galerkin (HWG) finite element method. The hybridized weak Galerkin is further used as a tool to derive a Schur complement problem for variables defined on element boundaries. Therefore, the Schur complement involves the solution of a linear system with significantly less number of unknowns than the original WG or HWG formulation. We believe the hybridization technique is widely applicable in weak Galerkin family for various partial differential equations, and would like to encourage interested readers to conduct some independent study along this direction.

Throughout the paper,  $C$  appearing in different places denotes different constant. Let  $T$  be a polygonal or polyhedral domain with boundary  $\partial T$ . Denote by  $(\cdot, \cdot)_T$  and  $\langle \cdot, \cdot \rangle_{\partial T}$  the usual inner products in  $L^2(T)$  and  $L^2(\partial T)$ .  $\|\cdot\|_{m,T}$  denotes the norm in the Sobolev space  $H^m(T)$ .  $|\cdot|_{m,T}$  stands for the semi-norm of order  $m$ . For simplicity,  $\|\cdot\|_{m,\Omega}$ ,  $|\cdot|_{m,\Omega}$ ,  $(\cdot, \cdot)_\Omega$  and  $\langle \cdot, \cdot \rangle_\Omega$  are denoted as  $\|\cdot\|_m$ ,  $|\cdot|_m$ ,  $(\cdot, \cdot)$  and  $\langle \cdot, \cdot \rangle$ , respectively.  $\|\cdot\|_{0,T}$  and  $\|\cdot\|_{0,\partial T}$  are simply denoted by  $\|\cdot\|_T$  and  $\|\cdot\|_{\partial T}$ , respectively.

The paper is organized as follows. In Section 2, we introduce a weak Hessian and a discrete weak Hessian by using polynomial approximations. In Section 3, we present a HWG finite element algorithm for the biharmonic problem (1). In Section 4, we verify all the stability conditions in Brezzi's theorem [3] for the HWG scheme. In Section 5, we derive an error equation for the HWG approximation. In Section 6, we establish an error estimate of optimal order for the numerical approximation. Finally in Section 7, we present a Schur complement by eliminating all the variables on the element, yielding a system of linear equations with significantly reduced number of unknowns defined on the element boundary.

## 2. Weak Hessian and Discrete Weak Hessian

By a weak function on  $T$ , we mean a triplet  $v = \{v_0, v_b, \mathbf{v}_g\}$  such that  $v_0 \in L^2(T)$ ,  $v_b \in L^2(\partial T)$  and  $\mathbf{v}_g \in [L^2(\partial T)]^d$ . Let  $\mathcal{W}(T)$  be the space of all weak functions on  $T$ ; i.e.,

$$(3) \quad \mathcal{W}(T) = \{v = \{v_0, v_b, \mathbf{v}_g\} : v_0 \in L^2(T), v_b \in L^2(\partial T), \mathbf{v}_g \in [L^2(\partial T)]^d\}.$$

For classical functions, the Hessian is a square matrix of second order partial derivatives if they all exist. If  $f(x_1, \dots, x_d)$  stands for the function, then the

Hessian of  $f$  is

$$H(f) = \{\partial_{ij}^2 f\}_{d \times d},$$

where  $\partial_{ij}^2$  is the second order partial derivative along the directions  $x_i$  and  $x_j$ . The goal of this section is to introduce weak Hessian for weak functions defined on  $T$ .

For any  $v \in \mathcal{W}(T)$ , the weak second order partial derivative  $\partial_{ij}^2$  of  $v$  is defined as a bounded linear functional  $\partial_{ij,w}^2 v$  in  $H^2(T)$  such that the action of  $\partial_{ij,w}^2 v$  on  $\varphi \in H^2(T)$  is given by

$$(4) \quad \langle \partial_{ij,w}^2 v, \varphi \rangle_T = (v_0, \partial_{ji}^2 \varphi)_T - \langle v_b n_i, \partial_j \varphi \rangle_{\partial T} + \langle v_{gi}, \varphi n_j \rangle_{\partial T},$$

where  $\mathbf{n} = (n_1, \dots, n_d)$  is the outward normal direction of  $T$  on its boundary. The weak Hessian is then defined as

$$H_{w,T}(v) = \{\partial_{ij,w}^2 v\}_{d \times d}, \quad v \in \mathcal{W}(T).$$

A discrete version of  $\partial_{ij,w}^2$  is an approximation  $\partial_{ij,w,r,T}^2$  in the space of polynomials of degree  $r$ , denoted by  $P_r(T)$ , such that

$$(5) \quad (\partial_{ij,w,r,T}^2 v, \varphi)_T = (v_0, \partial_{ji}^2 \varphi)_T - \langle v_b n_i, \partial_j \varphi \rangle_{\partial T} + \langle v_{gi}, \varphi n_j \rangle_{\partial T}, \quad \forall \varphi \in P_r(T).$$

Analogously, the discrete Hessian is given by

$$H_{w,r,T}(v) = \{\partial_{ij,w,r,T}^2 v\}_{d \times d}, \quad v \in \mathcal{W}(T).$$

**Remark 2.1.** Let  $v = \{v_0, v_b, \mathbf{v}_g\} \in \mathcal{W}(T)$  be a weak function on  $T$  such that  $v_0$  is twice differentiable on  $T$ . By applying the usual integration by parts to the first term on the right-hand side of (5), we obtain

$$(6) \quad (\partial_{ij,w,r,T}^2 v, \varphi)_T = (\partial_{ij}^2 v_0, \varphi)_T + \langle (v_0 - v_b) n_i, \partial_j \varphi \rangle_{\partial T} - \langle (\partial_i v_0 - v_{gi}) n_j, \varphi \rangle_{\partial T}$$

for all  $\varphi \in P_r(T)$ .

### 3. A Hybridized Weak Galerkin Formulation

Let  $\mathcal{T}_h$  be a partition of the domain  $\Omega$  into polygons in 2D or polyhedra in 3D. Denote by  $\mathcal{E}_h$  the set of all edges or flat faces in  $\mathcal{T}_h$  and  $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$  the set of all interior edges or flat faces. Assume that  $\mathcal{T}_h$  is shape regular as described in [14]. Denote by  $h_T$  the diameter of  $T \in \mathcal{T}_h$  and  $h = \max_{T \in \mathcal{T}_h} h_T$  the meshsize for the partition  $\mathcal{T}_h$ . Assume that the mesh is quasi-uniform in the sense that there exists a constant  $C$  such that  $h \leq Ch_T$  for all  $T \in \mathcal{T}_h$ .

For each element  $T \in \mathcal{T}_h$ , the trace of  $\mathcal{W}(T)$  on the boundary  $\partial T$  is the usual Sobolev space  $L^2(\partial T) \times [L^2(\partial T)]^d$ . Define the spaces  $\mathcal{W}$  and  $\Lambda$  by

$$(7) \quad \mathcal{W} = \prod_{T \in \mathcal{T}_h} \mathcal{W}(T), \quad \Lambda = \prod_{T \in \mathcal{T}_h} L^2(\partial T) \times [L^2(\partial T)]^d.$$

We emphasize that on each edge or flat face  $e \subset \mathcal{E}_h^0$  shared by two adjacent elements  $T_1$  and  $T_2$ , the function  $v \in \mathcal{W}$  assumes two independent values: one is taken as the trace from  $T_1$ , and the other as the trace taken from  $T_2$ . Define the jump of  $v \in \mathcal{W}$  on  $e \subset \mathcal{E}_h$  by

$$(8) \quad [v]_e = \begin{cases} \{v_b, \mathbf{v}_g\}|_{\partial T_1} - \{v_b, \mathbf{v}_g\}|_{\partial T_2}, & e \in \mathcal{E}_h^0, \\ \{v_b, \mathbf{v}_g\}, & e \subset \partial\Omega, \end{cases}$$

where  $\{v_b, \mathbf{v}_g\}|_{\partial T_i}$  denotes the value of  $\{v_b, \mathbf{v}_g\}$  on  $e$  as seen from the element  $T_i$ ,  $i = 1, 2$ . The order of  $T_1$  and  $T_2$  is non-essential in (8) as long as the difference is taken in a consistent way in all the formulas. We shall also use the notation  $\{v_b, \mathbf{v}_g\}_L$  for  $\{v_b, \mathbf{v}_g\}|_{\partial T_1}$  and  $\{v_b, \mathbf{v}_g\}_R$  for  $\{v_b, \mathbf{v}_g\}|_{\partial T_2}$  in the rest of the paper.

For any function  $\lambda \in \Lambda$ , define its similarity on  $e \in \mathcal{E}_h$  by

$$(9) \quad \langle\langle \lambda \rangle\rangle_e = \begin{cases} \{\lambda_b, \boldsymbol{\lambda}_g\}_L + \{\lambda_b, \boldsymbol{\lambda}_g\}_R, & e \in \mathcal{E}_h^0, \\ \{\lambda_b, \boldsymbol{\lambda}_g\}, & e \subset \partial\Omega. \end{cases}$$

Denote by  $\langle\langle \lambda \rangle\rangle$  the similarity of  $\lambda$  in  $\mathcal{E}_h$ .

For any given integer  $k \geq 2$ , denote by  $\mathcal{W}_k(T)$  the discrete weak function space given by

$$\mathcal{W}_k(T) = \{ \{v_0, v_b, \mathbf{v}_g\} : v_0 \in P_k(T), v_b \in P_{k-2}(e), \mathbf{v}_g \in [P_{k-2}(e)]^d, e \subset \partial T \}.$$

Denote by  $\Lambda_k(\partial T)$  the trace of  $\mathcal{W}_k(T)$  on the boundary  $\partial T$ ; i.e.,

$$(10) \quad \Lambda_k(\partial T) = \{ \lambda = \{\lambda_b, \boldsymbol{\lambda}_g\} : \lambda_b|_e \in P_{k-2}(e), \boldsymbol{\lambda}_g|_e \in [P_{k-2}(e)]^d, e \subset \partial T \}.$$

By patching  $\mathcal{W}_k(T)$  and  $\Lambda_k(\partial T)$  over all the elements  $T \in \mathcal{T}_h$ , we obtain two weak Galerkin finite element spaces  $\mathcal{W}_h$  and  $\Lambda_h$  as follows

$$(11) \quad \mathcal{W}_h = \prod_{T \in \mathcal{T}_h} \mathcal{W}_k(T), \quad \Lambda_h = \prod_{T \in \mathcal{T}_h} \Lambda_k(\partial T).$$

Denote by  $\mathcal{W}_h^0$  the subspace of  $\mathcal{W}_h$  consisting of functions with vanishing boundary values

$$\mathcal{W}_h^0 = \{ v \in \mathcal{W}_h : v_b|_e = 0, \mathbf{v}_g|_e = \mathbf{0}, e \subset \partial\Omega \}.$$

Furthermore, let  $\mathcal{V}_h$  be the subspace of  $\mathcal{W}_h$  consisting of functions which are continuous across each interior edge or flat face

$$\mathcal{V}_h = \{ v \in \mathcal{W}_h : \llbracket v \rrbracket_e = \{0, \mathbf{0}\}, e \in \mathcal{E}_h^0 \}.$$

Denote by  $\mathcal{V}_h^0$  the subspace of  $\mathcal{V}_h$  consisting of functions with vanishing boundary values

$$\mathcal{V}_h^0 = \{ v \in \mathcal{V}_h : v_b|_e = 0, \mathbf{v}_g|_e = \mathbf{0}, e \subset \partial\Omega \}.$$

Let  $\Xi_h$  be the subspace of  $\Lambda_h$  consisting of functions with similarity zero across each edge or flat face; i.e.,

$$\Xi_h = \left\{ \lambda \in \Lambda_h : \langle\langle \lambda \rangle\rangle_e = \{0, \mathbf{0}\}, e \in \mathcal{E}_h \right\}.$$

The functions in the space  $\Xi_h$  will serve as Lagrange multipliers in hybridization methods.

Denote by  $\partial_{ij,w,k-2}^2$  the discrete weak partial derivative, which is computed by using (5) on each element  $T$  as follows

$$(\partial_{ij,w,k-2}^2 v)|_T = \partial_{ij,w,k-2,T}^2(v|_T), \quad v \in \mathcal{W}.$$

For simplicity of notation, we shall drop the subscript  $k-2$  from the notation  $\partial_{ij,w,k-2}^2$  in the rest of the paper. We also introduce the following notation

$$(\partial_w^2 u, \partial_w^2 v)_h = \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (\partial_{ij,w}^2 u, \partial_{ij,w}^2 v)_T, \quad \forall u, v \in \mathcal{W}.$$

On each element  $T$ , denote by  $Q_0$  the  $L^2$  projection onto  $P_k(T)$ . Similarly, for each edge or face  $e \subset \partial T$ , denote by  $Q_b$  the  $L^2$  projection onto  $P_{k-2}(e)$  or  $[P_{k-2}(e)]^d$ , as appropriate. For any  $q \in H^2(\Omega)$ , define the  $L^2$  projection  $Q_h q \in \mathcal{V}_h$  so that on each element  $T$ , one has

$$Q_h q = \{Q_0 q, Q_b q, Q_b(\nabla q)\}.$$

**3.1. Algorithm.** For any  $w = \{w_0, w_b, \mathbf{w}_g\} \in \mathcal{W}_k(T)$  and  $v = \{v_0, v_b, \mathbf{v}_g\} \in \mathcal{W}_k(T)$  and  $\lambda \in \Lambda_k(\partial T)$ , set

$$\begin{aligned} a_T(w, v) &= \sum_{i,j=1}^d (\partial_{ij,w}^2 w, \partial_{ij,w}^2 v)_T, \\ s_T(w, v) &= h_T^{-1} \langle Q_b(\nabla w_0) - \mathbf{w}_g, Q_b(\nabla v_0) - \mathbf{v}_g \rangle_{\partial T} \\ &\quad + h_T^{-3} \langle Q_b w_0 - w_b, Q_b v_0 - v_b \rangle_{\partial T}, \\ b_T(v, \lambda) &= \langle v, \lambda \rangle_{\partial T} \\ &= \langle v_b, \lambda_b \rangle_{\partial T} + \langle \mathbf{v}_g, \boldsymbol{\lambda}_g \rangle_{\partial T}. \end{aligned}$$

Define

$$a_{s,T}(w, v) = a_T(w, v) + s_T(w, v).$$

Summing over all the elements  $T \in \mathcal{T}_h$  yields four bilinear forms

$$\begin{aligned} a(w, v) &= \sum_{T \in \mathcal{T}_h} a_T(w, v), & w, v \in \mathcal{W}_h, \\ s(w, v) &= \sum_{T \in \mathcal{T}_h} s_T(w, v), & w, v \in \mathcal{W}_h, \\ b(v, \lambda) &= \sum_{T \in \mathcal{T}_h} b_T(v, \lambda), & v \in \mathcal{W}_h, \lambda \in \Lambda_h, \\ a_s(w, v) &= \sum_{T \in \mathcal{T}_h} a_{s,T}(w, v), & w, v \in \mathcal{W}_h. \end{aligned}$$

Since  $\lambda \in \Xi_h$  implies  $\lambda_L + \lambda_R = 0$  on each interior edge and  $\lambda = 0$  on the boundary edge, then for any  $v \in \mathcal{W}_h$  and  $\lambda \in \Xi_h$ , we have

$$(12) \quad b(v, \lambda) = \sum_{e \in \mathcal{E}_h^0} \langle \llbracket v \rrbracket_e, \lambda_L \rangle_e.$$

The following weak Galerkin finite element scheme for the biharmonic equation (1) was introduced and analyzed in [13].

**Algorithm 1.** (Weak Galerkin) Find  $\bar{u}_h = \{\bar{u}_0, \bar{u}_b, \bar{\mathbf{u}}_g\} \in \mathcal{V}_h$  such that  $\bar{u}_b = Q_b \xi$ ,  $\bar{\mathbf{u}}_g \cdot \mathbf{n} = Q_b \eta$ ,  $\bar{\mathbf{u}}_g \cdot \boldsymbol{\tau} = Q_b(\nabla \xi \cdot \boldsymbol{\tau})$  on  $\partial \Omega$  and satisfying

$$(13) \quad a_s(\bar{u}_h, v) = (f, v_0), \quad \forall v = \{v_0, v_b, \mathbf{v}_g\} \in \mathcal{V}_h^0,$$

where  $\boldsymbol{\tau} \in \mathbb{R}^d$  is the tangential direction to the edges or faces on the boundary of  $\Omega$ .

Next, we proposed a hybridized formulation for (13) by using a Lagrange multiplier.

**Algorithm 2.** (Hybridized Weak Galerkin) Find  $(u_h; \lambda_h) \in \mathcal{W}_h \times \Xi_h$  with  $u_h = \{u_0, u_b, \mathbf{u}_g\}$  such that  $u_b = Q_b \xi$ ,  $\mathbf{u}_g \cdot \mathbf{n} = Q_b \eta$ ,  $\mathbf{u}_g \cdot \boldsymbol{\tau} = Q_b(\nabla \xi \cdot \boldsymbol{\tau})$  on  $\partial \Omega$  and satisfying the following equations

$$(14) \quad a_s(u_h, v) - b(v, \lambda_h) = (f, v_0), \quad \forall v = \{v_0, v_b, \mathbf{v}_g\} \in \mathcal{W}_h^0,$$

$$(15) \quad b(u_h, \rho) = 0, \quad \forall \rho \in \Xi_h.$$

**3.2. Equivalence.** The HWG scheme (14)-(15) is in fact equivalent to the WG scheme (13) in that the solution  $u_h$  of (14)-(15) and  $\bar{u}_h$  of (13) are identical. But the HWG scheme (14)-(15) is expected to be advantageous over WG for biharmonic interface problems.

For any  $v \in \mathcal{V}_h^0$ , let

$$(16) \quad \|v\| = a_s^{\frac{1}{2}}(v, v).$$

It has been verified in [13] that (16) defines a norm in the linear space  $\mathcal{V}_h^0$ .

**Theorem 3.1.** *Let  $u_h \in \mathcal{W}_h$  be the first component of the solution of the hybridized WG algorithm (14)-(15). Then, we have  $\llbracket u_h \rrbracket_e = 0$  on each interior edge or flat face  $e \in \mathcal{E}_h^0$ ; i.e.,  $u_h \in \mathcal{V}_h$ . Furthermore, we have  $u_b = Q_b \xi$ ,  $\mathbf{u}_g \cdot \mathbf{n} = Q_b \eta$ ,  $\mathbf{u}_g \cdot \boldsymbol{\tau} = Q_b(\nabla \xi \cdot \boldsymbol{\tau})$  on  $\partial\Omega$  and  $u_h$  satisfies the equation (13). Thus, one has  $u_h = \bar{u}_h$ .*

*Proof.* Let  $e$  be an interior edge or flat face shared by two elements  $T_1$  and  $T_2$ . By letting  $\rho = \llbracket u_h \rrbracket_e$  on  $e$  as seen from  $T_1$  (i.e.,  $\rho = -\llbracket u_h \rrbracket_e$  on  $e$  as seen from  $T_2$ ) and  $\rho = 0$  otherwise in (15), we obtain from (12) that

$$0 = b(u_h, \rho) = \sum_{T \in \mathcal{T}_h} \langle u_h, \rho \rangle_{\partial T} = \int_e \llbracket u_h \rrbracket_e^2 ds,$$

which implies that  $\llbracket u_h \rrbracket_e = 0$  for each interior edge or flat face  $e \in \mathcal{E}_h^0$ .

Now by restricting  $v \in \mathcal{V}_h^0$  in the equation (14) and using the fact that  $b(v, \lambda_h) = 0$ , we arrive at

$$a_s(u_h, v) = (f, v_0), \quad \forall v \in \mathcal{V}_h^0,$$

which is the same as (13). It follows from the solution uniqueness for (13) that  $u_h \equiv \bar{u}_h$ . This completes the proof.  $\square$

#### 4. Stability Conditions for HWG

It is easy to see that the following defines a norm in the finite element space  $\Xi_h$

$$(17) \quad \|\lambda_h\|_{\Xi_h} = \left( \sum_{e \in \mathcal{E}_h^0} h_e^3 \|\lambda_b\|_e^2 + h_e \|\boldsymbol{\lambda}_g\|_e^2 \right)^{\frac{1}{2}}.$$

For any  $v = \{v_0, v_b, \mathbf{v}_g\} \in \mathcal{W}_h^0$ , let

$$(18) \quad \|v\|_{\mathcal{W}_h^0} = \left( \|v\|^2 + \sum_{e \in \mathcal{E}_h^0} h_e^{-3} \|\llbracket v_b \rrbracket_e\|_e^2 + h_e^{-1} \|\llbracket \mathbf{v}_g \rrbracket_e\|_e^2 \right)^{\frac{1}{2}}.$$

We claim that  $\|\cdot\|_{\mathcal{W}_h^0}$  defines a norm in  $\mathcal{W}_h^0$ . In fact, if  $\|v\|_{\mathcal{W}_h^0} = 0$ , then  $\llbracket v_b \rrbracket_e = 0$  and  $\llbracket \mathbf{v}_g \rrbracket_e = \mathbf{0}$  on each interior edge or flat face  $e \in \mathcal{E}_h^0$ , and hence  $v \in \mathcal{V}_h^0$ . Since  $\|\cdot\|$  defines a norm in the linear space  $\mathcal{V}_h^0$ , then  $v = 0$ . This verifies the positivity property of  $\|\cdot\|_{\mathcal{W}_h^0}$ . The other properties for a norm can be checked trivially.

**Lemma 4.1.** ([14]) (*Trace Inequality*) *Let  $\mathcal{T}_h$  be a partition of the domain  $\Omega$  into polygons in 2D or polyhedra in 3D. Assume that the partition  $\mathcal{T}_h$  satisfies the assumptions (P1), (P2), and (P3) as specified in [14]. Let  $p > 1$  be any real number. Then, there exists a constant  $C$  such that for any  $T \in \mathcal{T}_h$  and edge/face  $e \in \partial T$ , we have*

$$(19) \quad \|\theta\|_{L^p(e)}^p \leq Ch_T^{-1} (\|\theta\|_{L^p(T)}^p + h_T^p \|\nabla \theta\|_{L^p(T)}^p),$$

where  $\theta \in W^{1,p}(T)$  is any function.

This paper will make use of the trace inequality (19) with  $p = 2$ :

$$(20) \quad \|\theta\|_e^2 \leq Ch_T^{-1} \|\theta\|_T^2 + Ch_T \|\nabla \theta\|_T^2.$$

**Lemma 4.2.** (*boundedness*) *There exists a constant  $C > 0$  such that*

$$(21) \quad |a_s(u, v)| \leq C \|u\|_{\mathcal{W}_h^0} \|v\|_{\mathcal{W}_h^0}, \quad \forall u, v \in \mathcal{W}_h^0,$$

$$(22) \quad |b(v, \lambda)| \leq C \|v\|_{\mathcal{W}_h^0} \|\lambda\|_{\Xi_h}, \quad \forall v \in \mathcal{W}_h^0, \lambda \in \Xi_h.$$

*Proof.* To prove (21), we use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} |a_s(u, v)| &= \left| \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (\partial_{ij,w}^2 u, \partial_{ij,w}^2 v)_T + h_T^{-1} \langle Q_b(\nabla u_0) - \mathbf{u}_g, Q_b(\nabla v_0) - \mathbf{v}_g \rangle_{\partial T} \right. \\ &\quad \left. + h_T^{-3} \langle Q_b u_0 - u_b, Q_b v_0 - v_b \rangle_{\partial T} \right| \\ &\leq \left( \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \|\partial_{ij,w}^2 u\|_T^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \|\partial_{ij,w}^2 v\|_T^2 \right)^{\frac{1}{2}} \\ &\quad + \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_b(\nabla u_0) - \mathbf{u}_g\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_b(\nabla v_0) - \mathbf{v}_g\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ &\quad + \left( \sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_b u_0 - u_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_b v_0 - v_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ &\leq C \|u\|_{\mathcal{W}_h^0} \|v\|_{\mathcal{W}_h^0}. \end{aligned}$$

As to (22), it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} |b(v, \lambda)| &= \left| \sum_{T \in \mathcal{T}_h} \langle v_b, \lambda_b \rangle_{\partial T} + \langle \mathbf{v}_g, \boldsymbol{\lambda}_g \rangle_{\partial T} \right| \\ &= \left| \sum_{e \in \mathcal{E}_h^0} \langle \llbracket v_b \rrbracket, \lambda_b \rangle_e + \langle \llbracket \mathbf{v}_g \rrbracket, \boldsymbol{\lambda}_g \rangle_e \right| \\ &\leq \left( \sum_{e \in \mathcal{E}_h^0} h_e^{-3} \|\llbracket v_b \rrbracket\|_e^2 \right)^{\frac{1}{2}} \left( \sum_{e \in \mathcal{E}_h^0} h_e^3 \|\lambda_b\|_e^2 \right)^{\frac{1}{2}} \\ &\quad + \left( \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \|\llbracket \mathbf{v}_g \rrbracket\|_e^2 \right)^{\frac{1}{2}} \left( \sum_{e \in \mathcal{E}_h^0} h_e \|\boldsymbol{\lambda}_g\|_e^2 \right)^{\frac{1}{2}} \\ &\leq C \|v\|_{\mathcal{W}_h^0} \|\lambda\|_{\Xi_h}, \end{aligned}$$

which ends the proof.  $\square$

**Lemma 4.3.** (*coercivity*) *There exists a constant  $C > 0$ , such that*

$$(23) \quad a_s(v, v) \geq C \|v\|_{\mathcal{W}_h^0}^2, \quad \forall v \in \mathcal{V}_h^0.$$

*Proof.* For any  $v \in \mathcal{V}_h^0$ , we have  $\|v\|_{\mathcal{W}_h^0} = \|v\|$ . Thus, the estimate (23) holds true with  $C = 1$ .  $\square$

**Lemma 4.4.** (*inf-sup condition*) *There exists a constant  $C > 0$  such that*

$$(24) \quad \sup_{v \in \mathcal{W}_h^0} \frac{b(v, \sigma)}{\|v\|_{\mathcal{W}_h^0}} \geq C \|\sigma\|_{\Xi_h}, \quad \forall \sigma \in \Xi_h.$$

*Proof.* For any  $\sigma \in \Xi_h$ , we have  $\langle\langle \sigma \rangle\rangle_e = 0$  or equivalently  $\sigma^L + \sigma^R = 0$  on each interior edge/face  $e \in \mathcal{E}_h^0$  and  $\sigma = 0$  on all boundary edges/faces. By letting  $v = \{0, h_e^3 \sigma_b, h_e \sigma_g\} \in \mathcal{W}_h^0$  in  $b(v, \sigma)$  and  $s(v, v)$ , we obtain

$$\begin{aligned}
(25) \quad b(v, \sigma) &= \sum_{e \in \mathcal{E}_h^0} \langle v_b^L, \sigma_b^L \rangle_e + \langle v_b^R, \sigma_b^R \rangle_e + \langle \mathbf{v}_g^L, \sigma_g^L \rangle_e + \langle \mathbf{v}_g^R, \sigma_g^R \rangle_e \\
&= \sum_{e \in \mathcal{E}_h^0} \langle v_b^L - v_b^R, \sigma_b^L \rangle_e + \langle \mathbf{v}_g^L - \mathbf{v}_g^R, \sigma_g^L \rangle_e \\
&= 2 \sum_{e \in \mathcal{E}_h^0} h_e^3 \|\sigma_b\|_e^2 + h_e \|\sigma_g\|_e^2,
\end{aligned}$$

and

$$\begin{aligned}
(26) \quad s(v, v) &= \sum_{e \in \mathcal{E}_h^0} h_e^{-1} h_e^2 \|\sigma_g^L\|_e^2 + h_e^{-3} h_e^6 \|\sigma_b^L\|_e^2 \\
&\quad + h_e^{-1} h_e^2 \|\sigma_g^R\|_e^2 + h_e^{-3} h_e^6 \|\sigma_b^R\|_e^2 \\
&= 2 \sum_{e \in \mathcal{E}_h^0} h_e \|\sigma_g\|_e^2 + h_e^3 \|\sigma_b\|_e^2.
\end{aligned}$$

It follows from (5), Cauchy-Schwarz inequality, the trace inequality (20) and the inverse inequality that

$$\begin{aligned}
(27) \quad & (\partial_{ij,w}^2 v, \partial_{ij,w}^2 v)_T \\
&= \sum_{e \subset \partial T} -\langle v_b^*, \partial_j(\partial_{ij,w}^2 v) \cdot n_i \rangle_e + \langle v_{gi}^* \cdot n_j, \partial_{ij,w}^2 v \rangle_e \\
&\leq \sum_{e \subset \partial T} h_e^3 \|\sigma_b^*\|_e \|\partial_j(\partial_{ij,w}^2 v)\|_e + h_e \|\sigma_{gi}^*\|_e \|\partial_{ij,w}^2 v\|_e \\
&\leq C \sum_{e \subset \partial T} h_e^3 \|\sigma_b^*\|_e h_e^{-3/2} \|\partial_{ij,w}^2 v\|_T + h_e \|\sigma_{gi}^*\|_e h_e^{-1/2} \|\partial_{ij,w}^2 v\|_T \\
&= C \sum_{e \subset \partial T} \|\partial_{ij,w}^2 v\|_T \left( h_e^{\frac{3}{2}} \|\sigma_b^*\|_e + h_e^{\frac{1}{2}} \|\sigma_{gi}^*\|_e \right),
\end{aligned}$$

where  $v_b^*$  is chosen to be  $v_b^L$  or  $v_b^R$  according to the relative position of  $v_b$  and  $e$ , and the same to  $v_{gi}^*$ ,  $\sigma_b^*$ ,  $\sigma_{gi}^*$ , which implies that

$$(28) \quad \|\partial_{ij,w}^2 v\|_T \leq C \sum_{e \subset \partial T} h_e^{\frac{3}{2}} \|\sigma_b^*\|_e + h_e^{\frac{1}{2}} \|\sigma_{gi}^*\|_e.$$

Summing over all element  $T$  yields

$$(29) \quad (\partial_w^2 v, \partial_w^2 v)_h \leq C \sum_{e \in \mathcal{E}_h^0} \sum_{i=1}^d \left( h_e^3 \|\sigma_b^*\|_e^2 + h_T \|\sigma_{gi}^*\|_e^2 \right).$$

It follows from (26) and (29) that

$$(30) \quad \|v\|^2 \leq C \sum_{e \in \mathcal{E}_h^0} h_e^3 \|\sigma_b\|_e^2 + h_e \|\sigma_g\|_e^2 = C \|\sigma\|_{\Xi_h}^2.$$

Recall that  $\sigma^L + \sigma^R = 0$ . Thus,

$$\begin{aligned}
(31) \quad h_e^{-3} \|[v_b]\|_e^2 + h_e^{-1} \|\llbracket \mathbf{v}_g \rrbracket\|_e^2 &= h_e^{-3} \|v_b^L - v_b^R\|_e^2 + h_e^{-1} \|\mathbf{v}_g^L - \mathbf{v}_g^R\|_e^2 \\
&= h_e^{-3} \|h_e^3 \sigma_b^L - h_e^3 \sigma_b^R\|_e^2 + h_e^{-1} \|h_e \sigma_g^L - h_e \sigma_g^R\|_e^2 \\
&= 2h_e^3 \|\sigma_b\|_e^2 + 2h_e \|\sigma_g\|_e^2.
\end{aligned}$$

Combining (25), (30), (31) and (18) gives

$$(32) \quad \sup_{v \in \mathcal{W}_h^0} \frac{b(v, \sigma)}{\|v\|_{\mathcal{W}_h^0}} \geq C \frac{\sum_{e \in \mathcal{E}_h^0} h_e^3 \|\sigma_b\|_e^2 + h_e \|\sigma_g\|_e^2}{(\sum_{e \in \mathcal{E}_h^0} h_e^3 \|\sigma_b\|_e^2 + h_e \|\sigma_g\|_e^2)^{\frac{1}{2}}} \geq C \|\sigma\|_{\Xi_h},$$

which completes the proof.  $\square$

## 5. Error Equations

The goal of this section is to derive an error equation for the hybridized WG Algorithm (14)-(15). The error equations shall play an important role in the forthcoming error analysis.

**Lemma 5.1.** [13] *On each element  $T \in \mathcal{T}_h$ , let  $\mathcal{Q}_h$  be the local  $L^2$  projection onto  $P_{k-2}(T)$ . Then, the  $L^2$  projections  $Q_h$  and  $\mathcal{Q}_h$  satisfy the following commutative property:*

$$(33) \quad \partial_{ij,w}^2(Q_h w) = \mathcal{Q}_h(\partial_{ij}^2 w), \quad \forall i, j = 1, \dots, d,$$

for all  $w \in H^2(T)$ .

Let  $u$  and  $(u_h; \lambda_h) \in \mathcal{W}_h \times \Xi_h$  be the solutions of (1) and (14)-(15), respectively. Let  $\lambda = \{\lambda_b, \lambda_g\}$  be given by

$$\lambda_b = \partial_n(\Delta u), \quad \lambda_g = -\partial_n(\nabla u) \quad \text{on } \partial T.$$

Define error functions by

$$(34) \quad e_h = Q_h u - u_h, \quad \epsilon_h = Q_h \lambda - \lambda_h.$$

**Lemma 5.2.** *Let  $u$  and  $(u_h; \lambda_h) \in \mathcal{W}_h \times \Xi_h$  be the solutions of (1) and (14)-(15), respectively. Then, the error functions  $e_h$  and  $\epsilon_h$  satisfy the following equations*

$$(35) \quad a_s(e_h, v) + b(v, \epsilon_h) = \ell_u(v), \quad \forall v \in \mathcal{W}_h^0$$

$$(36) \quad b(\epsilon_h, \rho) = 0, \quad \forall \rho \in \Xi_h,$$

where

$$(37) \quad \begin{aligned} \ell_u(v) &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_{ij}^2 u - \mathcal{Q}_h(\partial_{ij}^2 u), (\partial_i v_0 - v_{gi}) \cdot n_j \rangle_{\partial T} \\ &\quad - \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_j(\partial_{ij}^2 u - \mathcal{Q}_h \partial_{ij}^2 u) \cdot n_i, v_0 - v_b \rangle_{\partial T} \\ &\quad + s(Q_h u, v). \end{aligned}$$

*Proof.* The equation (36) is obvious from the definition of  $\epsilon_h$ . It remains to verify (35). To this end, from (6) we have for any  $\varphi \in P_{k-2}(T)$ ,

$$(\varphi, \partial_{ij,w}^2 v)_T = (\partial_{ij}^2 v_0, \varphi)_T + \langle v_0 - v_b, \partial_j \varphi \cdot n_i \rangle_{\partial T} - \langle (\partial_i v_0 - v_{gi}) \cdot n_j, \varphi \rangle_{\partial T}.$$

By substituting  $\varphi$  by  $\partial_{ij,w}^2 Q_h u$  and then using Lemma 5.1, we obtain

$$\begin{aligned} &(\partial_{ij,w}^2 Q_h u, \partial_{ij,w}^2 v)_T \\ &= (\partial_{ij}^2 v_0, \mathcal{Q}_h(\partial_{ij}^2 u))_T + \langle v_0 - v_b, \partial_j(\mathcal{Q}_h(\partial_{ij}^2 u)) \cdot n_i \rangle_{\partial T} \\ &\quad - \langle (\partial_i v_0 - v_{gi}) \cdot n_j, \mathcal{Q}_h(\partial_{ij}^2 u) \rangle_{\partial T} \\ &= (\partial_{ij}^2 v_0, \partial_{ij}^2 u)_T + \langle v_0 - v_b, \partial_j(\mathcal{Q}_h(\partial_{ij}^2 u)) \cdot n_i \rangle_{\partial T} \\ &\quad - \langle (\partial_i v_0 - v_{gi}) \cdot n_j, \mathcal{Q}_h(\partial_{ij}^2 u) \rangle_{\partial T}, \end{aligned}$$

which can be rewritten as

$$(38) \quad \begin{aligned} (\partial_{ij}^2 u, \partial_{ij}^2 v_0)_T &= (\partial_{ij,w}^2(Q_h u), \partial_{ij,w}^2 v)_T - \langle v_0 - v_b, \partial_j(\mathcal{Q}_h(\partial_{ij}^2 u)) \cdot n_i \rangle_{\partial T} \\ &\quad + \langle (\partial_i v_0 - v_{gi}) \cdot n_j, \mathcal{Q}_h(\partial_{ij}^2 u) \rangle_{\partial T}. \end{aligned}$$

With  $\lambda_b = \partial_n(\Delta u)$  and  $\lambda_g = -\partial_n(\nabla u)$ , we have

$$\begin{aligned} b(Q_h \lambda, v) &= \sum_{T \in \mathcal{T}_h} \langle Q_h \lambda, v \rangle_{\partial T} = \sum_{T \in \mathcal{T}_h} \langle \lambda, v \rangle_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} \langle \lambda_g, \mathbf{v}_g \rangle_{\partial T} + \sum_{T \in \mathcal{T}_h} \langle \lambda_b, v_b \rangle_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle -\partial_{ij}^2 u \cdot n_j, v_{gi} \rangle_{\partial T} + \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_j(\partial_{ij}^2 u) \cdot n_i, v_b \rangle_{\partial T}. \end{aligned}$$

In addition, from the integration by parts,

$$(\partial_{ij}^2 u, \partial_{ij}^2 v_0)_T = ((\partial_{ij}^2)^2 u, v_0)_T + \langle \partial_{ij}^2 u, \partial_i v_0 \cdot n_j \rangle_{\partial T} - \langle \partial_j(\partial_{ij}^2 u) \cdot n_i, v_0 \rangle_{\partial T}.$$

Summing over all  $T \in \mathcal{T}_h$  and then using the fact that  $(\Delta^2 u, v_0) = (f, v_0)$ , we obtain

$$\begin{aligned} b(Q_h \lambda, v) &+ \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (\partial_{ij}^2 u, \partial_{ij}^2 v_0)_T = (f, v_0) \\ &+ \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_{ij}^2 u, (\partial_i v_0 - v_{gi}) \cdot n_j \rangle_{\partial T} \\ &- \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_j(\partial_{ij}^2 u) \cdot n_i, v_0 - v_b \rangle_{\partial T}. \end{aligned}$$

Combining the above equation with (38) yields

$$\begin{aligned} &b(Q_h \lambda, v) + (\partial_w^2 Q_h u, \partial_w^2 v)_h \\ &= (f, v_0) + \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_{ij}^2 u - \mathcal{Q}_h(\partial_{ij}^2 u), (\partial_i v_0 - v_{gi}) \cdot n_j \rangle_{\partial T} \\ &\quad - \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_j(\partial_{ij}^2 u - \mathcal{Q}_h \partial_{ij}^2 u) \cdot n_i, v_0 - v_b \rangle_{\partial T}. \end{aligned}$$

Adding  $s(Q_h u, v)$  to both sides of the above equation gives

$$(39) \quad \begin{aligned} &(\partial_w^2 Q_h u, \partial_w^2 v)_h + s(Q_h u, v) + b(Q_h \lambda, v) = (f, v_0) \\ &\quad + \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_{ij}^2 u - \mathcal{Q}_h(\partial_{ij}^2 u), (\partial_i v_0 - v_{gi}) \cdot n_j \rangle_{\partial T} \\ &\quad - \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_j(\partial_{ij}^2 u - \mathcal{Q}_h \partial_{ij}^2 u) \cdot n_i, v_0 - v_b \rangle_{\partial T} + s(Q_h u, v). \end{aligned}$$

Subtracting (14) from (39) gives the desired equation (35). This completes the proof.  $\square$

## 6. Error Estimates

The goal of this section is to establish some error estimates for the hybridized WG finite element solution  $(u_h; \lambda_h)$  arising from (14)-(15). The error equations (35)-(36) imply

$$\begin{aligned} a_s(Q_h u - u_h, v) + b(v, Q_h \lambda - \lambda_h) &= \ell_u(v), \quad \forall v \in \mathcal{W}_h^0, \\ b(Q_h u - u_h, \rho) &= 0, \quad \forall \rho \in \Xi_h, \end{aligned}$$

where  $\ell_u(v)$  is given by (37). The above is a saddle point problem for which the Brezzi's theorem [4] can be applied for an analysis on its stability and solvability. Note that all the conditions of Brezzi's theorem have been verified in Section 4 (see Lemmas 4.2-4.4).

**Theorem 6.1.** *Let  $u$  and  $(u_h; \lambda_h) \in \mathcal{W}_h \times \Xi_h$  be the solutions of (1) and (14)-(15) respectively. Then, there exists a constant  $C$  such that*

$$(40) \quad \|Q_h u - u_h\|_{\mathcal{W}_h^0} + \|Q_h \lambda - \lambda_h\|_{\Xi_h} \leq Ch^{k-1} \left( \|u\|_{k+1} + \delta_{k,2} \|u\|_4 \right),$$

where  $\delta_{i,j}$  is the Kronecker's delta with value 1 for  $i = j$  and 0 otherwise.

*Proof.* From the Brezzi's theorem [4], we have

$$(41) \quad \|Q_h u - u_h\|_{\mathcal{W}_h^0} + \|Q_h \lambda - \lambda_h\|_{\Xi_h} \leq C \|\ell_u\|_{\mathcal{W}_h^{0'}}.$$

For any  $v \in \mathcal{W}_h^0$ , it has been shown in [13] that

$$|\ell_u(v)| \leq Ch^{k-1} \left( \|u\|_{k+1} + \delta_{k,2} \|u\|_4 \right) \|v\|.$$

Thus, we have

$$(42) \quad \|\ell_u\|_{\mathcal{W}_h^{0'}} = \sup_{v \in \mathcal{W}_h^0} \frac{\ell_u(v)}{\|v\|_{\mathcal{W}_h^0}} \leq \sup_{v \in \mathcal{W}_h^0} \frac{\ell_u(v)}{\|v\|} \leq Ch^{k-1} \left( \|u\|_{k+1} + \delta_{k,2} \|u\|_4 \right).$$

Substituting (42) into (41) yields the desired estimate (40), which completes the proof.  $\square$

**Theorem 6.2.** *Let  $u$  and  $\lambda_h = \{\lambda_{h,b}, \lambda_{h,g}\} \in \Xi_h$  be the solution of (1) and part of the solution of (14)-(15), respectively. On the set of interior edges  $\mathcal{E}_h^0$ , let  $\lambda = \{\lambda_b, \lambda_g\}$  be given by*

$$\lambda_b = \partial_n(\Delta u), \quad \lambda_g = -\partial_n(\nabla u).$$

*Then, the following estimate holds true*

$$(43) \quad \|\lambda - \lambda_h\|_{\Xi_h} \leq Ch^{k-1} \left( \|u\|_{k+1} + \delta_{k,2} \|u\|_4 \right).$$

*Proof.* From the triangle inequality, we have

$$(44) \quad \|\lambda - \lambda_h\|_{\Xi_h} \leq \|\lambda - Q_h \lambda\|_{\Xi_h} + \|Q_h \lambda - \lambda_h\|_{\Xi_h}.$$

The second term on the right-hand side of (44) can be handled by (40). The first term is simply the error between  $\lambda$  and its  $L^2$  projection, and can be rewritten as

$$(45) \quad \begin{aligned} \|\lambda - Q_h \lambda\|_{\Xi_h}^2 &= \sum_{e \in \mathcal{E}_h^0} h_e^3 \|\lambda_b - Q_b \lambda\|_e^2 + h_e \|\lambda_g - Q_b \lambda_g\|_e^2 \\ &= \sum_{e \in \mathcal{E}_h^0} h_e^3 \|\partial_n \Delta u - Q_b(\partial_n \Delta u)\|_e^2 + h_e \|\partial_n \nabla u - Q_b(\partial_n \nabla u)\|_e^2. \end{aligned}$$

Let  $e$  be an edge/face of the element  $T$  and denote by  $Q_{k-1}$  the  $L^2$  projection onto  $P_{k-1}(T)$ . From the trace inequality (20), we obtain

$$\begin{aligned}
& \|\partial_n \Delta u - Q_b(\partial_n \Delta u)\|_e^2 \\
(46) \quad & \leq \|\partial_n \Delta u - \partial_n(Q_{k-1} \Delta u)\|_e^2 \\
& \leq Ch^{-1} \|\Delta u - Q_{k-1} \Delta u\|_{1,T}^2 + Ch \|\Delta u - Q_{k-1} \Delta u\|_{2,T}^2 \\
& \leq Ch^{2k-5} \|u\|_{k+1,T}^2 + Ch \delta_{k,2} \|u\|_{4,T}^2.
\end{aligned}$$

Analogously,

$$\begin{aligned}
& \|\partial_n \nabla u - Q_b(\partial_n \nabla u)\|_e^2 \\
(47) \quad & \leq \|\partial_n \nabla u - \partial_n(Q_{k-1} \nabla u)\|_e^2 \\
& \leq Ch^{-1} \|\nabla u - Q_{k-1} \nabla u\|_{1,T}^2 + Ch \|\nabla u - Q_{k-1} \nabla u\|_{2,T}^2 \\
& \leq Ch^{2k-3} \|u\|_{k+1,T}^2.
\end{aligned}$$

Substituting (46) and (47) into (45) yields

$$(48) \quad \|\lambda - Q_h \lambda\|_{\Xi_h}^2 \leq Ch^{2k-2} (\|u\|_{k+1}^2 + Ch^2 \delta_{k,2} \|u\|_4^2).$$

This completes the proof of the theorem.  $\square$

## 7. Efficient Implementation via Variable Reduction

The degrees of freedom in the WG algorithm (13) can be divided into two classes: (1) the interior variables  $u_0$ , and (2) the interface variables  $\{u_b, \mathbf{u}_g\}$ . For the hybridized WG algorithm (14)-(15), more unknowns must be added to the picture from the Lagrange multiplier  $\lambda_h$ . Thus, the size of the discrete system arising from either (13) or (14)-(15) is enormously large.

The goal of this section is to present a Schur complement formulation for the WG algorithm (13) based on the hybridized formulation (14)-(15). The method shall eliminate all the unknowns associated with  $u_0$ , and produce a much smaller system of linear equations involving only the unknowns of the interface variables  $\{u_b, \mathbf{u}_g\}$ .

**7.1. Theory of variable reduction.** Denote by  $\mathcal{B}_h$  the interface finite element space defined as the restriction of the finite element space  $\mathcal{V}_h$  on the set of edges  $\mathcal{E}_h$ ; i.e.,

$$\mathcal{B}_h = \{\{\mu_b, \boldsymbol{\mu}_g\} : \mu_b \in P_{k-2}(e), \boldsymbol{\mu}_g \in [P_{k-2}(e)]^d, e \in \mathcal{E}_h\}.$$

$\mathcal{B}_h$  is a Hilbert space equipped with the following inner product

$$\langle \{w_b, \mathbf{w}_g\}, \{q_b, \mathbf{q}_g\} \rangle_{\mathcal{E}_h} = \sum_{e \in \mathcal{E}_h} \langle w_b, q_b \rangle_e + \langle \mathbf{w}_g, \mathbf{q}_g \rangle_e, \quad \forall \{w_b, \mathbf{w}_g\}, \{q_b, \mathbf{q}_g\} \in \mathcal{B}_h.$$

Denote by  $\mathcal{B}_h^0$  the subspace of  $\mathcal{B}_h$  consisting of functions with vanishing boundary value.

We introduce an operator  $S_f : \mathcal{B}_h \rightarrow \mathcal{B}_h^0$  as follows. For any  $\{w_b, \mathbf{w}_g\} \in \mathcal{B}_h$ , the image  $S_f(\{w_b, \mathbf{w}_g\})$  is obtained as follows:

**Step 1:** On each element  $T \in \mathcal{T}_h$ , compute  $w_0$  in terms of  $\{w_b, \mathbf{w}_g\}$  by solving the following local equations

$$(49) \quad a_{s,T}(w_h, v) = (f, v_0)_T, \quad \forall v = \{v_0, 0, \mathbf{0}\} \in \mathcal{W}_k(T),$$

where  $w_h = \{w_0, w_b, \mathbf{w}_g\} \in \mathcal{W}_k(T)$ . We denote the solution by  $w_0 = D_f(\{w_b, \mathbf{w}_g\})$ .

**Step 2:** Compute  $\zeta_{h,T} \in \Lambda_k(\partial T)$  on each element  $T \in \mathcal{T}_h$  such that

$$(50) \quad b_T(v, \zeta_{h,T}) = a_{s,T}(w_h, v), \quad \forall v = \{0, v_b, \mathbf{v}_g\} \in \mathcal{W}_k(T).$$

This provides a function  $\zeta_h \in \Lambda_h$ . Denote  $\zeta_h$  by  $\zeta_h = L_f(\{w_b, \mathbf{w}_g\})$ .

**Step 3:** Set  $S_f(\{w_b, \mathbf{w}_g\})$  as the similarity of  $\zeta_h$  on interior edges /faces and zero on boundary edges/faces; i.e.,

$$(51) \quad S_f(\{w_b, \mathbf{w}_g\}) = \begin{cases} \zeta_{hL} + \zeta_{hR}, & \text{on } e \in \mathcal{E}_h^0, \\ 0, & \text{on } e \subset \partial\Omega. \end{cases}$$

By adding the two equations (49) and (50), we obtain the following identity

$$(52) \quad b_T(v, \zeta_{h,T}) = a_{s,T}(w_h, v) - (f, v_0)_T, \quad \forall v = \{v_0, v_b, \mathbf{v}_g\} \in \mathcal{W}_k(T).$$

From the superposition principle, one has the following result.

**Lemma 7.1.** *For any  $\{w_b, \mathbf{w}_g\} \in \mathcal{B}_h$ , we have*

$$(53) \quad S_f(\{w_b, \mathbf{w}_g\}) = S_0(\{w_b, \mathbf{w}_g\}) + S_f(\{0, \mathbf{0}\}).$$

Here  $S_0$  is the operator corresponding to the case  $f = 0$ .

It is clear that  $S_0$  is a linear map from  $\mathcal{B}_h$  into  $\mathcal{B}_h^0$ . Moreover, the following result can be verified for  $S_0$ .

**Theorem 7.2.** *For any  $\{w_b, \mathbf{w}_g\}, \{q_b, \mathbf{q}_g\} \in \mathcal{B}_h^0$ , we have*

$$(54) \quad \langle S_0(\{w_b, \mathbf{w}_g\}), \{q_b, \mathbf{q}_g\} \rangle_{\mathcal{E}_h^0} = a_s(w_h, q_h),$$

where  $w_h = \{D_0(\{w_b, \mathbf{w}_g\}), w_b, \mathbf{w}_g\}$  and  $q_h = \{D_0(\{q_b, \mathbf{q}_g\}), q_b, \mathbf{q}_g\}$ . In other words, the linear map  $S_0$ , when restricted to the subspace  $\mathcal{B}_h^0$ , is symmetric and positive definite.

*Proof.* For any  $\{w_b, \mathbf{w}_g\}, \{q_b, \mathbf{q}_g\} \in \mathcal{B}_h^0$ , let

$$\begin{aligned} w_h &= \{D_0(\{w_b, \mathbf{w}_g\}), w_b, \mathbf{w}_g\}, & \zeta_h &= L_0(\{w_b, \mathbf{w}_g\}), \\ q_h &= \{D_0(\{q_b, \mathbf{q}_g\}), q_b, \mathbf{q}_g\}, & \eta_h &= L_0(\{q_b, \mathbf{q}_g\}). \end{aligned}$$

Using (52) with  $f = 0$ , we arrive at

$$\begin{aligned} \langle S_0(\{w_b, \mathbf{w}_g\}), \{q_b, \mathbf{q}_g\} \rangle_{\mathcal{E}_h^0} &= \sum_{e \in \mathcal{E}_h^0} \langle \langle \zeta_h \rangle \rangle_e, \{q_b, \mathbf{q}_g\} \rangle_e \\ &= \sum_{T \in \mathcal{T}_h} \langle \zeta_{h,T}, \{q_b, \mathbf{q}_g\} \rangle_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} b_T(q_h, \zeta_{h,T}) \\ &= \sum_{T \in \mathcal{T}_h} a_{s,T}(w_h, q_h), \end{aligned}$$

which completes the proof.  $\square$

**Lemma 7.3.** *Let  $(u_h; \lambda_h) = (\{u_0, u_b, \mathbf{u}_g\}; \lambda_h) \in \mathcal{W}_h \times \Xi_h$  be the unique solution of the hybridized WG algorithm (14)-(15). Then,  $u_h \in \mathcal{V}_h$  and  $\{u_b, \mathbf{u}_g\}$  is well defined in the space  $\mathcal{B}_h$ . Moreover, it satisfies the following equation*

$$(55) \quad S_f(\{u_b, \mathbf{u}_g\}) = \{0, \mathbf{0}\}.$$

*Proof.* Since  $(u_h; \lambda_h)$  is the unique solution of the hybridized WG algorithm (14)-(15), then we have from Lemma 3.1 that  $[[u_h]]_e = 0$  on each interior edge or flat face  $e \in \mathcal{E}_h^0$ . Furthermore, on each boundary edge, we have  $u_b = Q_b \xi$ ,  $\mathbf{u}_g \cdot \mathbf{n} = Q_b \eta$ ,  $\mathbf{u}_g \cdot \boldsymbol{\tau} = Q_b (\nabla \xi \cdot \boldsymbol{\tau})$ . Thus,  $u_h \in \mathcal{V}_h$  and its restriction on  $\mathcal{E}_h$  is a well defined function in the space  $\mathcal{B}_h$ .

Now in (14), choose  $v = \{v_0, 0, \mathbf{0}\} \in \mathcal{W}_k(T)$  on  $T$  and zero elsewhere. Then,

$$a_{s,T}(u_h, v) = (f, v_0)_T, \quad \forall v = \{v_0, 0, \mathbf{0}\} \in \mathcal{W}_k(T).$$

This implies that  $u_h$  satisfies the local equation (49).

Next in (14), choose  $v = \{0, v_b, \mathbf{v}_g\} \in \mathcal{W}_k(T)$  on  $T$  and zero elsewhere. Then,

$$b_T(\lambda_{h,T}, v) = a_{s,T}(u_h, v), \quad \forall v = \{0, v_b, \mathbf{v}_g\} \in \mathcal{W}_k(T),$$

where  $\lambda_{h,T}$  is the restriction of  $\lambda_h$  on the boundary of  $T$ . This means that  $\lambda_h$  satisfies (50).

From the definition of the operator  $S_f$ , we have on interior edges/faces

$$S_f(\{u_b, \mathbf{u}_g\}) = \langle\langle \lambda_h \rangle\rangle.$$

The fact that  $\lambda_h \in \Xi_h$  implies  $\langle\langle \lambda_h \rangle\rangle = \{0, \mathbf{0}\}$ , and hence  $S_f(\{u_b, \mathbf{u}_g\}) = \{0, \mathbf{0}\}$ . This completes the proof of the theorem.  $\square$

**Lemma 7.4.** *Let  $\{\bar{u}_b, \bar{\mathbf{u}}_g\} \in \mathcal{B}_h$  satisfy  $\bar{u}_b = Q_b \xi$  and  $\bar{\mathbf{u}}_g \cdot \mathbf{n} = Q_b \eta$ ,  $\bar{\mathbf{u}}_g \cdot \boldsymbol{\tau} = Q_b (\nabla \xi \cdot \boldsymbol{\tau})$  on  $\partial\Omega$  and the following operator equation*

$$(56) \quad S_f(\{\bar{u}_b, \bar{\mathbf{u}}_g\}) = \{0, \mathbf{0}\}.$$

*Then,  $\bar{u}_h = \{\bar{u}_0, \bar{u}_b, \bar{\mathbf{u}}_g\} \in \mathcal{V}_h$  is the solution of the WG algorithm (13). Here  $\bar{u}_0$  is the solution of the following local problems on each element  $T \in \mathcal{T}_h$ ,*

$$(57) \quad a_{s,T}(\bar{u}_h, v) = (f, v_0)_T, \quad \forall v = \{v_0, 0, \mathbf{0}\} \in \mathcal{W}_k(T).$$

*Proof.* Let  $\{\bar{u}_b, \bar{\mathbf{u}}_g\} \in \mathcal{B}_h$  satisfy the operator equation (56) and the given boundary condition. Let  $\bar{u}_0$  be given by the local equations (57). Now on each element  $T$ , we compute  $\bar{\lambda}_{h,T} \in \Lambda_k(\partial T)$  by solving the local problem

$$(58) \quad b_T(v, \bar{\lambda}_{h,T}) = a_{s,T}(\bar{u}_h, v), \quad \forall v = \{0, v_b, \mathbf{v}_g\} \in \mathcal{W}_k(T).$$

This defines a function  $\bar{\lambda}_h \in \Lambda_h$  given by  $\bar{\lambda}|_{\partial T} = \bar{\lambda}_{h,T}$  with modification  $\bar{\lambda}|_{\partial\Omega} = 0$ . From the definition of the operator  $S_f$ , on each interior edge  $e \in \mathcal{E}_h^0$ , we have

$$S_f(\{\bar{u}_b, \bar{\mathbf{u}}_g\}) = \langle\langle \bar{\lambda}_h \rangle\rangle,$$

which, together with (56) leads to

$$(59) \quad \langle\langle \bar{\lambda}_h \rangle\rangle = \{0, \mathbf{0}\}$$

on each interior edge/face. Thus,  $\bar{\lambda}_h \in \Xi_h$ .

Subtracting (58) from (57) gives

$$a_{s,T}(\bar{u}_h, v) - b_T(v, \bar{\lambda}_{h,T}) = (f, v_0)_T, \quad \forall v = \{v_0, v_b, \mathbf{v}_g\} \in \mathcal{W}_k(T).$$

Summing up the above equation over all elements  $T \in \mathcal{T}_h$  gives

$$(60) \quad a_s(\bar{u}_h, v) - b(v, \bar{\lambda}_h) = (f, v_0), \quad \forall v = \{v_0, v_b, \mathbf{v}_g\} \in \mathcal{W}_h^0.$$

Note that the above equation holds true only for test functions  $v$  with vanishing boundary value since  $\lambda_h$  was modified from  $\lambda_{h,T}$  on the boundary of the domain.

For any  $\sigma$  in the finite element space  $\Xi_h$ , we have from (12) that

$$(61) \quad b(\bar{u}_h, \sigma) = \sum_{e \in \mathcal{E}_h^0} \langle\langle [[\bar{u}_h]]_e, \sigma_L \rangle\rangle_e = 0.$$

The equations (60) and (61) indicate that  $(\bar{u}_h; \bar{\lambda}_h)$  is a solution to the hybridized WG scheme (14)-(15). Recall that on the boundary  $\partial\Omega$ , we have  $\bar{u}_b = Q_b\xi$  and  $\bar{\mathbf{u}}_g \cdot \mathbf{n} = Q_b\eta$ ,  $\bar{\mathbf{u}}_g \cdot \boldsymbol{\tau} = Q_b(\nabla\xi \cdot \boldsymbol{\tau})$ . Thus, using Theorem 3.1 we see that  $\bar{u}_h$  is the WG solution defined by the formulation (13). This completes the proof of the theorem.  $\square$

The results developed in Lemmas 7.3 -7.4 can be summarized as follows.

**Theorem 7.5.** *Let  $\{\bar{u}_b, \bar{\mathbf{u}}_g\} \in \mathcal{B}_h$  be any function such that  $\bar{u}_b = Q_b\xi$  and  $\bar{\mathbf{u}}_g \cdot \mathbf{n} = Q_b\eta$ ,  $\bar{\mathbf{u}}_g \cdot \boldsymbol{\tau} = Q_b(\nabla\xi \cdot \boldsymbol{\tau})$  on  $\partial\Omega$ . Define  $\bar{u}_0$  as the solution of (57). Then,  $\bar{u}_h = \{\bar{u}_0, \bar{u}_b, \bar{\mathbf{u}}_g\}$  is the solution of (13) if and only if  $\{\bar{u}_b, \bar{\mathbf{u}}_g\}$  satisfies the following operator equation*

$$(62) \quad S_f(\{\bar{u}_b, \bar{\mathbf{u}}_g\}) = \{0, \mathbf{0}\}.$$

**7.2. Computational algorithm with reduced variables.** From (53), the operator equation (62) can be rewritten as

$$(63) \quad S_0(\{\bar{u}_b, \bar{\mathbf{u}}_g\}) = -S_f(\{0, \mathbf{0}\}).$$

Let  $\{G_b, \mathbf{G}_g\} \in \mathcal{B}_h$  be a finite element function satisfying  $G_b = Q_b\xi$ ,  $\mathbf{G}_g \cdot \mathbf{n} = Q_b\eta$  and  $\mathbf{G}_g \cdot \boldsymbol{\tau} = Q_b(\nabla\xi \cdot \boldsymbol{\tau})$  on  $\partial\Omega$  and zero elsewhere. It follows from the linearity of  $S_0$  that

$$S_0(\{\bar{u}_b, \bar{\mathbf{u}}_g\}) = S_0(\{\bar{u}_b, \bar{\mathbf{u}}_g\} - \{G_b, \mathbf{G}_g\}) + S_0(\{G_b, \mathbf{G}_g\}).$$

Substituting the above into (63) yields

$$S_0(\{\bar{u}_b, \bar{\mathbf{u}}_g\} - \{G_b, \mathbf{G}_g\}) = -S_f(\{0, \mathbf{0}\}) - S_0(\{G_b, \mathbf{G}_g\}).$$

Note that the function  $\{p_b, \mathbf{p}_g\} = \{\bar{u}_b, \bar{\mathbf{u}}_g\} - \{G_b, \mathbf{G}_g\}$  has vanishing boundary value. By setting  $\{r_b, \mathbf{r}_g\} = -S_f(\{0, \mathbf{0}\}) - S_0(\{G_b, \mathbf{G}_g\})$ , we have

$$(64) \quad S_0(\{p_b, \mathbf{p}_g\}) = \{r_b, \mathbf{r}_g\}.$$

The reduced system of linear equations (64) is actually a Schur complement formulation for the WG algorithm (13). Note that (64) involves only the variables representing the value of the function on  $\mathcal{E}_h^0$ . This is clearly a significant reduction on the size of the linear system that has to be solved in the WG finite element method.

**Variable Reduction Algorithm 1.** *The solution  $u_h = \{u_0, u_b, \mathbf{u}_g\}$  to the WG algorithm (13) can be obtained step-by-step as follows:*

- (1) *On each element  $T$ , compute*

$$r_h = -S_f(\{0, \mathbf{0}\}) - S_0(\{G_b, \mathbf{G}_g\}).$$

*This task requires the inversion of local stiffness matrices and can be accomplished in parallel. The computational complexity is linear with respect to the number of unknowns.*

- (2) *Compute  $\{p_b, \mathbf{p}_g\} \in \mathcal{B}_h^0$  by solving the system of linear equations (64). This step requires an efficient linear solver.*
- (3) *Compute  $\{u_b, \mathbf{u}_g\} = \{p_b, \mathbf{p}_g\} + \{G_b, \mathbf{G}_g\}$  to get the solution on the element interfaces. Then, on each element  $T$ , compute  $u_0 = D_f(\{u_b, \mathbf{u}_g\})$  by solving the local problem (49). This task can be accomplished in parallel, and the computational complexity is proportional to the number of unknowns.*

Step (2) in the VARIABLE REDUCTION ALGORITHM 1 is the only computation-extensive part of the implementation. Note that, due to Theorem 7.2, the reduced system (64) is symmetric and positive definite. Preconditioning techniques should be applied for an efficient solving of (64). This is left to interested readers for investigation.

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