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# A TWO-GRID FINITE VOLUME ELEMENT METHOD FOR A NONLINEAR PARABOLIC PROBLEM

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**Abstract.** A two-grid algorithm is presented and discussed for a finite volume element method to a nonlinear parabolic equation in a convex polygonal domain. The two-grid algorithm consists of solving a small nonlinear system on a coarse-grid space with grid size $H$ and then solving a resulting linear system on a fine-grid space with grid size $h$. Error estimates are derived with the $H^1$-norm $O(h + H^2)$ which shows that the two-grid algorithm achieves asymptotically optimal approximation as long as the mesh sizes satisfy $h = O(H^2)$. Numerical examples are presented to validate the usefulness and efficiency of the method.

**Key words.** Two-grid, finite volume element method, nonlinear parabolic equation, error estimates.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain with boundary $\partial\Omega$, and consider the initial-boundary value problem

$$\begin{cases} u_t - \nabla \cdot (A(u)\nabla u) = f(x,t), & x \in \Omega, 0 < t \le T, \\ u(x,t) = 0, & x \in \partial\Omega, 0 < t \le T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $u_t$ denotes $\frac{\partial u}{\partial t}$, $x = (x_1, x_2)$, $f(x,t)$ is a given real-valued function on $\Omega$. We assume that the coefficient $A(u)$ is sufficiently smooth such that there exist constants $C_i$ $(i = 1,2,3)$ satisfying

$$0 < C_1 \le A(u) \le C_2, \quad |A(u)_t| \le C_3, \quad \forall u \in C(\Omega \times [0,T]), \tag{2}$$

and the Lipschitz continuous condition, $\forall u, v \in C(\Omega \times [0,T])$,

$$|A(u) - A(v)| \le L|u - v|, \quad |A(u)_t - A(v)_t| \le L|u - v|, \tag{3}$$

with $L$ a positive constant and $A(u)_t = \frac{\partial}{\partial t}A(u)$.

It is also assumed that the functions $f, u_0$ have enough regularity and they satisfy appropriate compatibility conditions so that the initial-boundary value problem (1) has a unique solution satisfying the regularity results as demanded by our subsequent analysis [1].

We shall study a two-grid algorithm of a nonlinear parabolic equation by using finite volume element method (FVEM). The FVEM is a class of important numerical methods for solving differential equations. It has been widely used in many engineering fields, such as computational fluid mechanics, groundwater hydrology, heat and mass transfer and petroleum engineering, reservoir simulations. Perhaps the most important and attractive property of the FVEM is that it possesses local conservation laws (mass, momentum and energy) which is crucial in many applications. Many researchers have studied this method for linear and nonlinear problems.

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We can refer to [2-11] for general presentation of this method and references therein for details.

On the other hand, two-grid method is a discretization technique for nonlinear equations based on two grids of different sizes. The main idea is to use a coarse-grid space to produce a rough approximation of the solution of nonlinear problems, and then use it as the initial guess on the fine grid. This method involves a nonlinear solve on the coarse grid with grid size $H$ and a linear solve on the fine grid with grid size $h < H$, respectively. Two-grid method was firstly introduced by Xu [12, 13] for linear (nonsymmetric or indefinite) and especially nonlinear elliptic partial differential equations. Later on, two-grid method was further investigated by many authors. We can refer to [14] for finite difference method and to [15, 16, 17, 18, 19, 20, 21] for finite element and mixed finite element method. For finite volume element method, there are also many literatures [22, 23, 24, 25, 26, 27, 28]. For the nonlinear parabolic problem (1), Dawson and Wheeler [14, 15] have constructed the two-grid method by using finite difference method and mixed finite element method. Chen and Liu [21] have studied the two-grid piecewise linear finite element method. Recently, In [24, 25] the two-grid finite volume element method was studied for the semilinear parabolic problem with a nonlinear reaction term, but with a linear diffusion term. Zhang et al. [27, 28] have considered the two-grid finite volume element method for circumcenter based control volumes, with suboptimal estimates in $L^2$ and $H^1$ norms for a nonlinear parabolic equation.

However, as far as we know, there is no convergence analysis of the two-grid FVEM based on barycenter control volumes for the nonlinear parabolic problem (1). In this paper, we consider the two-grid FVEM for barycenter based control volumes for the nonlinear parabolic problem (1). The two-grid FVEM is based on two conforming piecewise linear finite element spaces $S_H$ and $S_h$. Where $S_H$ is the coarse grid with grid size $H$ and $S_h$ is the fine grid with grid size $h$ respectively. With the proposed techniques, solving the nonlinear system on the fine-grid space is reduced to solving a linear system on the fine-grid space and a nonlinear system on a much smaller space. The work for solving the nonlinear system on the coarse-grid space is relatively negligible since $\dim S_H \ll \dim S_h$. This means that solving such a nonlinear problem is not much more difficult than solving one linear problem. A remarkable fact about this simple approach is, as shown in [12], that the coarse grid can be quite coarse and still maintain a good accuracy approximation. The main results in this paper are the error estimates for the considered single-grid and two-grid methods in the Sobolev $H^1$ norm. To get the estimates, we used standard results from the finite volume element convergence analysis which is based upon viewing the finite volume element method as a perturbation of finite element method.

The rest of this paper is organized as follows. In Section 2 we describe the finite volume element scheme for the nonlinear parabolic problem (1). Section 3 contains the error estimates for the semidiscrete finite volume element method. In Section 4 we construct the two-grid finite volume element algorithm and prove its optimal error estimates in the $H^1$ norm. Finally in Section 5 we give the numerical examples to validate the theoretical results.

## 2. Finite volume element method

We will use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ [29] with $1 \leq p \leq \infty$ consisting of functions that have generalized derivatives of order $s$ in the space

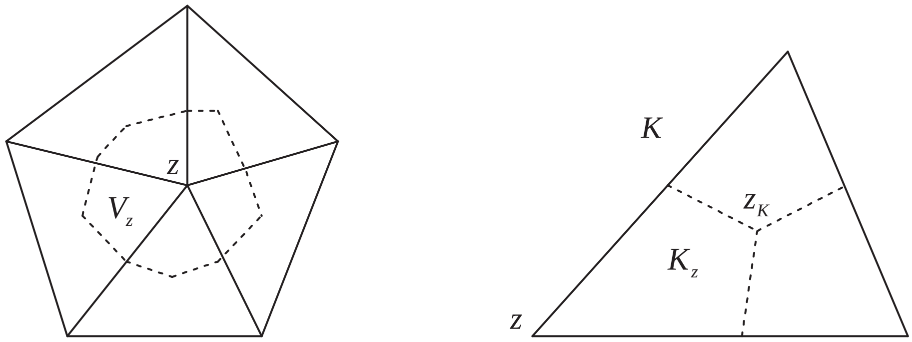


FIGURE 1. *Left*: The dotted line shows the boundary of the corresponding control volume $V_z$ with $z$ a common vertex. *Right*: A triangle $K$ partitioned into three subregions $K_z$.

$L^p(\Omega)$. The norm of $W^{s,p}(\Omega)$ is defined by

$$\|u\|_{s,p,\Omega} = \|u\|_{s,p} = \left( \int_\Omega \sum_{|\alpha| \le s} |D^\alpha u|^p dx \right)^{\frac{1}{p}},$$

with the standard modification for $p = \infty$. In order to simplify the notation, we denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$ and omit the index $p = 2$ and $\Omega$ whenever possible; i.e., $\|u\|_{s,2,\Omega} = \|u\|_{s,2} = \|u\|_s$. We denote by $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ of functions vanishing on the boundary $\partial\Omega$.

The weak formulation of the problem (1) is to find $u \in H_0^1(\Omega)$ such that

$$(u_t, v) + a(u; u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \tag{4}$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$-inner product and the bilinear form $a(\cdot;\cdot,\cdot)$ is defined by

$$a(w; u, v) = \int_\Omega A(w) \nabla u \cdot \nabla v \mathrm{d}x, \ \forall u, v, w \in H_0^1(\Omega).$$

Henceforth, it will be assumed that the problem (4) has a unique solution $u$, and in the appropriate places to follow, additional conditions on the regularity of $u$ which guarantee the convergence results, will be imposed.

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of $\Omega$ with $h = \max h_K$, where $h_K$ is the diameter of the triangle $K \in \mathcal{T}_h$. Based on this triangulation, let $V_h$ be the standard conforming finite element space of piecewise linear functions,

$$V_h = \{v \in C(\Omega) : \ v|_K \text{ is linear}, \forall K \in \mathcal{T}_h; v|_{\partial\Omega} = 0\}.$$

In order to describe the finite volume element method we shall introduce a dual partition $\mathcal{T}_h^*$ based upon the original partition $\mathcal{T}_h$ whose elements are called control volumes. We construct the control volumes in the same way as in [2, 6, 9, 10, 11]. Let $z_K$ be the barycenter of $K \in \mathcal{T}_h$. We connect $z_K$ with line segments to the midpoints of the edges of $K$, thus partitioning $K$ into three quadrilaterals $K_z$, $z \in Z_h(K)$, where $Z_h(K)$ are the vertices of $K$. Then with each vertex $z \in Z_h = \cup_{K \in \mathcal{T}_h} Z_h(K)$ we associate a control volume $V_z$, which consists of the union of the subregions $K_z$, sharing the vertex $z$ (See Figure 1). Thus we obtain a group of control volumes covering the domain $\Omega$, which is called the dual partition $\mathcal{T}_h^*$ of the triangulation $\mathcal{T}_h$. We denote the set of interior vertices of $Z_h$ by $Z_h^0$. We call the partition $\mathcal{T}_h^*$ regular or quasi-uniform, if there exists a positive constant $C$ such that

$$C^{-1}h^2 \leq \text{meas}(V_z) \leq Ch^2, \quad \forall V_z \in \mathcal{T}_h^*.$$

The barycenter-type dual partition can be introduced for any finite element triangulation $\mathcal{T}_h$ and leads to relatively simple calculations. Besides, if the finite element triangulation $\mathcal{T}_h$ is quasi-uniform, then the dual partition $\mathcal{T}_h^*$ is also quasi-uniform [2].

We formulate the finite volume element method for the problem (1) as follows. Given a vertex $z \in Z_h$, integrating (1) over the associated control volume $V_z$ and applying the Greens formula, we obtain

$$\int_{V_z} u_t \mathrm{d}x - \int_{\partial V_z} (A(u)\nabla u) \cdot \mathbf{n} \mathrm{d}s = \int_{V_z} f \mathrm{d}x, \tag{5}$$

where $\mathbf{n}$ denotes the unit outer-normal vector to $\partial V_z$. It should be noted that the above formulation is a way of stating that we have an integral conservation form on the control volume.

The finite volume element approximation of (1) is defined as a solution $u_h(t) \in V_h$, $t \leq T$, such that

$$\int_{V_z} u_{h,t} \mathrm{d}x - \int_{\partial V_z} (A(u_h)\nabla u_h) \cdot \mathbf{n} \mathrm{d}s = \int_{V_z} f \mathrm{d}x. \tag{6}$$

For any $v_h \in H_0^1(\Omega)$, we define an interpolation operator $I_h : H_0^1(\Omega) \to V_h$, such that

$$I_h v_h = \sum_{z \in Z_h^0} v_h(z)\Phi_z,$$

where $\Phi_z$ is the standard nodal basis function associated with the node $z$. By the interpolation theorem in Sobolev space, we have

$$\|v - I_h v\|_m \leq Ch^{2-m}\|v\|_2, \ m = 0, 1. \tag{7}$$

It is easy to see that

$$\|I_h v_h\|_1 \leq C\|v_h\|_1. \tag{8}$$

The finite volume element scheme (6) can be rewritten in a variational form similar to the finite element method with the help of an interpolation operator $I_h^* : V_h \to V_h^*$, defined by

$$I_h^* v_h = \sum_{z \in Z_h^0} v_h(z)\Psi_z,$$

where

$$V_h^* = \{v \in L^2(\Omega) : \ v|_{V_z} \text{ is constant for all } V_z \in \mathcal{T}_h^*; \ v|_{V_z} = 0 \text{ if } z \in \partial\Omega\},$$

and $\Psi_z$ is the characteristic function of the control volume $V_z$. It was shown in [8] that

$$\|v_h - I_h^* v_h\|_{0,p} \leq Ch^s\|v_h\|_{s,p}, \ 0 \leq s \leq 1, \tag{9}$$

and in [9] that

$$\|I_h^* v_h\|_{0,p} \leq C\|v_h\|_{0,p}, \tag{10}$$

for $p > 1$.

We multiply the integral relation in (6) by $v_h(z)$ and sum over all $z \in Z_h^0$. Then we obtain the semidiscrete finite volume element formulation. Find $u_h(t) \in V_h$, for $t \geq 0$, such that

$$\begin{cases} (u_{h,t}, I_h^* v_h) + a_h(u_h; u_h, I_h^* v_h) = (f, I_h^* v_h), \ \forall v_h \in V_h, \\ u_h(0) = R_h u_0, \end{cases} \tag{11}$$

where $R_h$ is defined by (14) in the next section and the bilinear form $a_h(\cdot;\cdot, I_h^*\cdot)$ is defined by, for any $u_h, v_h, w_h \in V_h$,

$$a_h(w_h; u_h, I_h^* v_h) = -\sum_{z \in Z_h^0} v_h(z) \int_{\partial V_z} (A(w_h)\nabla u_h) \cdot \mathbf{n} \mathrm{d}s. \tag{12}$$

This procedure is based on a modification of the finite volume element method which was introduced for nonlinear elliptic problems by Chatzipantelidis, Ginting and Lazarov in[10]. By means of Brouwer a fixed point iteration they have proved the existence and uniqueness of the solution $u_h$ of the corresponding nonlinear elliptic problems. Recently Chatzipantelidis and Ginting [11] have studied the finite volume element method for barycenter based control volumes of problem (1) and shown the existence and uniqueness of the discrete solution and derived error estimates in the $L^2$- and $H^1$-norms. Our analysis follows the results of the finite volume element method for the nonlinear parabolic problems in [11] and we will study the two-grid finite volume element method and provide its error estimates.

Throughout this paper, we use the letter $C$ or with its subscript to denote a generic positive constant. The constant will not depend on the mesh parameters and may represent different values in different places.

## 3. Error analysis for the nonlinear finite volume element method

To describe error estimates for the finite volume element method, we define

$$|||u_h|||_0^2 = (u_h, I_h^* u_h), \forall u_h \in V_h.$$

Further $(u_h, I_h^* v_h)$ is symmetric and positive definite [2] and the corresponding discrete norm is equivalent to the $L^2$-norm, i.e., that there exist two positive constants $C_*, C^*$, independent of $h$ such that

$$C_* \|u_h\| \leq |||u_h|||_0 \leq C^* \|u_h\|, \ \forall u_h \in V_h. \tag{13}$$

In the following paper, we will not declare when we use $\|\cdot\|$ to take place of $|||\cdot|||_0$. Following [1], let $R_h : H_0^1 \longrightarrow V_h$ be the standard Ritz projection such that

$$a(u; u - R_h u, v_h) = 0, \quad \forall v_h \in V_h. \tag{14}$$

In [1] optimal order error estimates were established for the difference $u - R_h u$,

$$\|u - R_h u\| + h\|u - R_h u\|_1 \leq Ch^2 \|u\|_2, \tag{15}$$

$$\|(u - R_h u)_t\| + h\|(u - R_h u)_t\|_1 \leq Ch^2 \|u_t\|_2, \tag{16}$$

under the appropriate regularity assumptions on $u$ for some positive constant $C$ independent of $h$. It is also shown in [1] that there exists a positive constant $M_0$ independent of $h$, such that

$$\|\nabla R_h u\|_\infty + \|\nabla R_h u_t\|_\infty \leq M_0, \ \text{for } t \leq T. \tag{17}$$

In [7] Chou and Li have proved the similar results as the following lemma which shows that $a_h(\cdot;\cdot, I_h^*\cdot)$ is generally not symmetric and how far it is from being symmetric when the case $A(u) = (a_{ij})$ is a symmetric and uniformly elliptic coefficient. Under the assumption (2), we can obtain the similar lemma which shows that there is a high order infinitesimal term difference between $a_h(w_h; u_h, I_h^* v_h)$ and $a_h(w_h; v_h, I_h^* u_h)$.

**Lemma 3.1.** *Suppose that $A(u)$ satisfies the condition (2) and (3). For $h$ sufficiently small, there exists a positive constant $C$ such that*

$$|a_h(w_h; u_h, I_h^* v_h) - a_h(w_h; v_h, I_h^* u_h)| \leq Ch\|u_h\|_1\|v_h\|_1, \forall w_h, u_h, v_h \in V_h. \tag{18}$$

Since the condition (2) and (3) is satisfied, the following lemma could be proved similarly as in [2, 7], which indicates that the bilinear form $a_h(\cdot; \cdot, I_h^* \cdot)$ is coercive and continuous on $V_h$.

**Lemma 3.2.** *Suppose that $A(u)$ satisfies the condition (2) and (3). For $h$ sufficiently small, there exist two positive constants $\alpha, \beta$ such that, for all $u_h, v_h, w_h \in V_h$, the coercive property*

$$a_h(w_h; u_h, I_h^* u_h) \geq \alpha\|u_h\|_1^2$$

*and the boundedness property*

$$|a_h(w_h; u_h, I_h^* v_h)| \leq \beta\|u_h\|_1\|v_h\|_1$$

*hold true.*

For error analysis we introduce two error functions

$$\varepsilon_h(f, \chi) = (f, \chi) - (f, I_h^* \chi), \qquad \forall \chi \in V_h, \tag{19}$$

$$\varepsilon_a(u; \chi, \psi) = a(u; \chi, \psi) - a_h(u; \chi, I_h^* \psi), \ \forall u, \chi, \psi \in V_h. \tag{20}$$

The two error functions are defined in [9, 10] and the bounds for (19) and (20) are shown as the following lemma.

**Lemma 3.3.** *Let $\chi \in V_h$, then*

$$|\varepsilon_h(f, \chi)| \leq Ch^{i+j}\|f\|_i\|\chi\|_j, \quad f \in H^i(\Omega), i, j = 0, 1,$$
$$|\varepsilon_a(u; R_h v, \chi)| \leq Ch^{i+j}\|v\|_{1+i}\|\chi\|_j, \quad v \in H^{1+i}(\Omega) \cap H_0^1(\Omega), i, j = 0, 1.$$

To describe the error estimates for the finite volume element method, we give the error estimates in the $L^2$-norm between the elliptic projection of the exact solution and the finite volume element approximation.

**Lemma 3.4.** *Let $u$ and $u_h$ be the solutions of (1) and (11), respectively. Suppose that $A(u)$ satisfies the condition (2) and (3). Then, there exist $h_0 > 0$ and a positive constant $C$ dependent on the norms of $u$ and $f$ but independent of the discretization parameter $h$, such that for all $h < h_0$ and $t \in [0, T]$,*

$$\|u_h - R_h u\| \leq Ch^2, \tag{21}$$

*with $R_h$ defined by (14).*

**Proof.** In a standard way we split the error $u_h - u$ using the elliptic projection $R_h u$, defined in (14), as

$$u_h - u = (u_h - R_h u) + (R_h u - u) = \xi + \eta.$$

By our definitions we have the following error equation

$$\begin{aligned} &(\xi_t, I_h^* v_h) + a_h(u_h; \xi, I_h^* v_h) \\ &= -(\eta_t, I_h^* v_h) - a_h(u_h; R_h u, I_h^* v_h) + a_h(u; u, I_h^* v_h), \ \forall v_h \in V_h. \end{aligned} \tag{22}$$

By (14), (19) and (20), we have

$$\begin{aligned}
&-a_h(u_h; R_h u, I_h^* v_h) + a_h(u; u, I_h^* v_h) = (f - u_t, I_h^* v_h) - a_h(u_h; R_h u, I_h^* v_h)\\
&= (f - u_t, I_h^* v_h) + a(u_h; R_h u, v_h) - a_h(u_h; R_h u, I_h^* v_h) - a(u_h; R_h u, v_h)\\
&\quad + a(u; R_h u, v_h) - (f - u_t, v_h)\\
&= [(u_t - f, v_h) - (u_t - f, I_h^* v_h)] + [a(u_h; R_h u, v_h) - a_h(u_h; R_h u, I_h^* v_h)]\\
&\quad + [a(u; R_h u, v_h) - a(u_h; R_h u, v_h)]\\
&= \varepsilon_h(u_t - f, v_h) + \varepsilon_a(u_h; R_h u, v_h) + [a(u; R_h u, v_h) - a(u_h; R_h u, v_h)]. \qquad (23)
\end{aligned}$$

Pick $v_h = \xi$ in (22), by (23), we get

$$\begin{aligned}
&(\xi_t, I_h^* \xi) + a_h(u_h; \xi, I_h^* \xi)\\
&= -(\eta_t, I_h^* \xi) + \varepsilon_a(u_h; R_h u, \xi) + \varepsilon_h(u_t - f, \xi) + [a(u; R_h u, \xi) - a(u_h; R_h u, \xi)]. \qquad (24)
\end{aligned}$$

For the left hand side of (24), by Lemma 3.2 and [2], we have

$$\begin{aligned}
(\xi_t, I_h^* \xi) + a_h(u_h; \xi, I_h^* \xi) &= \frac{1}{2}\frac{d}{dt}(\xi, I_h^* \xi) + a_h(u_h; \xi, I_h^* \xi)\\
&\geq \frac{1}{2}\frac{d}{dt}(|||\xi|||_0^2) + \alpha\|\xi\|_1^2. \qquad (25)
\end{aligned}$$

By (13) and (25), integrating (24) from 0 to $t$ and noting that $\xi(0) = 0$, we have

$$\begin{aligned}
&\frac{1}{2}\|\xi\|^2 + \alpha\int_0^t \|\xi\|_1^2 dt\\
&\leq -\int_0^t (\eta_t, I_h^* \xi) dt + \int_0^t \varepsilon_a(u_h; R_h u, \xi) dt + \int_0^t \varepsilon_h(u_t - f, \xi) dt\\
&\quad + \int_0^t [a(u; R_h u, \xi) - a(u_h; R_h u, \xi)] dt \equiv \sum_{i=1}^4 Q_i. \qquad (26)
\end{aligned}$$

We now estimate the right-hand terms of (26). By(16), we have

$$|Q_1| \leq \int_0^t \|\eta_t\| \|I_h^* \xi\| dt \leq C_1 h^4 \int_0^t \|u_t\|_2^2 dt + C_2 \int_0^t \|\xi\|^2 dt, \qquad (27)$$

From Lemma 3.3 and $\epsilon$–Cauchy inequality, we get

$$|Q_2| \leq C\int_0^t h^2\|u\|_2\|\xi\|_1 dt \leq Ch^4\int_0^t \|u\|_2^2 dt + \frac{\alpha}{3}\int_0^t \|\xi\|_1^2 dt, \qquad (28)$$

$$\begin{aligned}
|Q_3| &\leq C\int_0^t h^2\|u_t - f\|_1\|\xi\|_1 dt\\
&\leq Ch^4\int_0^t (\|u_t\|_1^2 + \|f\|_1^2) dt + \frac{\alpha}{3}\int_0^t \|\xi\|_1^2 dt. \qquad (29)
\end{aligned}$$

For $Q_4$, by (3), (15) and (17), we have

$$\begin{aligned}
|Q_4| &\leq C\int_0^t \|A(u) - A(u_h)\| \|\nabla(R_h u)\|_\infty \|\nabla\xi\| dt\\
&\leq C\int_0^t \|\nabla(R_h u)\|_\infty (\|\xi\| + \|\eta\|)\|\nabla\xi\| dt\\
&\leq C\int_0^t (h^4\|u\|_2^2 + \|\xi\|^2) dt + \frac{\alpha}{3}\int_0^t \|\xi\|_1^2 dt. \qquad (30)
\end{aligned}$$

From (26)–(30), we get

$$\frac{1}{2}\|\xi\|^2 \le C_1 h^4 \int_0^t (\|u_t\|_2^2 + \|u\|_2^2 + \|u_t\|_1^2 + \|f\|_1^2)dt + C_2 \int_0^t \|\xi\|^2 dt. \tag{31}$$

Applying the Gronwall lemma, for $t \le T$, we have

$$\frac{1}{2}\|\xi\|^2 \le C_1 h^4 \int_0^T (\|u_t\|_2^2 + \|u\|_2^2 + \|u_t\|_1^2 + \|f\|_1^2)dt. \tag{32}$$

From this argument, for all $h < h_0$ and $t \in [0, T]$ we have

$$\|u_h - R_h u\| = \|\xi\| \le Ch^2,$$

which gives the desired result.

By (15) and Lemma 3.4, we immediately derive the following optimal $L^2$ error estimates for the finite volume element scheme.

**Theorem 3.1.** *Let $u$ and $u_h$ be the solutions of* (1) *and* (11)*, respectively. Consider the same condition as in Lemma 3.4. Then, there exist $h_0 > 0$ and a positive constant $C$ dependent on the norms of $u$ and $f$ but independent of the discretization parameter $h$, such that for all $h < h_0$ and $t \in [0, T]$,*

$$\|u_h - u\| \le Ch^2. \tag{33}$$

For the $H^1$ error estimates of the finite volume element scheme for the nonlinear parabolic equation (1), we have

**Theorem 3.2.** *Let $u$ and $u_h$ be the solutions of* (1) *and* (11)*, respectively. Consider the same condition as in Lemma 3.4. Then, there exist $h_0 > 0$ and a positive constant $C$ dependent on the norms of $u$ and $f$ but independent of the discretization parameter $h$, such that for all $h < h_0$ and $t \in [0, T]$,*

$$\|u_h - u\|_1 \le Ch. \tag{34}$$

**Proof.** From the inverse estimate and (21), we obtain

$$\|u_h - R_h u\|_1 \le Ch^{-1}\|u_h - R_h u\| \le Ch. \tag{35}$$

By (15), (35) and the triangular inequality, we can derive (34) immediately.

## 4. Two-grid finite volume element method and its error analysis

In this section, we shall present two-grid finite volume element algorithm for the nonlinear parabolic problem (1) based on two conforming finite element spaces. The idea of the two-grid method is to reduce the nonlinear system on a fine grid into a linear system on a fine grid by solving a nonlinear system on a coarse grid. The basic mechanisms are two quasi-uniform triangulations of $\Omega$, $\mathcal{T}_H$ and $\mathcal{T}_h$, with two different mesh sizes $H$ and $h$ ($H > h$), and the corresponding finite volume element spaces $V_H$ and $V_h$ which satisfies $V_H \subset V_h$ and will be called the coarse-grid space and the fine-grid space, respectively.

The two-grid finite volume element algorithm is presented as

**Algorithm 1**

Step 1: On the coarse grid $\mathcal{T}_H$, find $u_H(t) \in V_H$, for $t \le T$, such that

$$\begin{cases} (u_{H,t}, I_H^* v_H) + a_H(u_H; u_H, I_H^* v_H) = (f, I_H^* v_H), \ \forall v_H \in V_H, \\ u_H(0) = R_H u_0, \end{cases} \tag{36}$$

where $R_H$ is defined in the same way as $R_h$ defined by (14).
Step 2: On the fine grid $\mathcal{T}_h$, find $u_h(t) \in V_h$, for $t \leq T$, such that

$$\begin{cases} (u_{h,t}, I_h^* v_h) + a_h(u_H; u_h, I_h^* v_h) = (f, I_h^* v_h), \ \forall v_h \in V_h, \\ u_h(0) = R_h u_0. \end{cases} \tag{37}$$

We note that the system in the first step of Algorithm 1 is a nonlinear system on the coarse-grid space. But in the second step it is linear on the fine-grid space. It will be much easier to solve than a nonlinear system solved only on the fine-grid space. Now we will give the error estimates in the $H^1$-norm for the two-grid finite volume element method Algorithm 1.

**Theorem 4.1.** *Let $u$ and $u_h$ be the solutions of* (1) *and Algorithm 1, respectively. Under the assumption conditions* (2) *and* (3)*, there exists a positive constant $C$ dependent on the norms of $u$ and $f$ but independent of $h$, such that*

$$\|u_h(t) - u(t)\|_1 \leq C(h + H^2), \ t \in [0, T]. \tag{38}$$

**Proof.** Once again, we set $u_h(t) - u(t) = (u_h - R_h u) + (R_h u - u) =: \xi + \eta$ and choose $v_h = \xi_t$. Then for Algorithm 1, similarly as in Lemma 3.4 we get the error equation

$$\begin{aligned} (\xi_t, I_h^* \xi_t) + a_h(u_H; \xi, I_h^* \xi_t) = &-(\eta_t, I_h^* \xi_t) + \varepsilon_a(u_H; R_h u, \xi_t) \\ &+ \varepsilon_h(u_t - f, \xi_t) + [a(u; R_h u, \xi_t) - a(u_H; R_h u, \xi_t)]. \end{aligned} \tag{39}$$

Since

$$\frac{d}{dt} a_h(u_H; \xi, I_h^* \xi) = a_h((u_H)_t; \xi, I_h^* \xi) + a_h(u_H; \xi, I_h^* \xi_t) + a_h(u_H; \xi_t, I_h^* \xi), \tag{40}$$

where

$$a_h((u_H)_t; \xi, I_h^* \xi) = -\sum_{z \in Z_h^0} \xi(z) \int_{\partial V_z} (A(u_H)_t \nabla \xi) \cdot \mathbf{n} ds,$$

we have

$$\begin{aligned} &(\xi_t, I_h^* \xi_t) + \frac{1}{2} \frac{d}{dt} a_h(u_H; \xi, I_h^* \xi) \\ &= -(\eta_t, I_h^* \xi_t) + \varepsilon_a(u_H; R_h u, \xi_t) + \varepsilon_h(u_t - f, \xi_t) + \frac{1}{2} a_h((u_H)_t; \xi, I_h^* \xi) \\ &\quad + \frac{1}{2}[a_h(u_H; \xi_t, I_h^* \xi) - a_h(u_H; \xi, I_h^* \xi_t)] + [a(u; R_h u, \xi_t) - a(u_H; R_h u, \xi_t)]. \end{aligned} \tag{41}$$

By Lemma 3.2, (13), integrating (24) from 0 to $t$ and noting that $\xi(0) = 0$, we have

$$\begin{aligned} &\frac{\alpha}{2} \|\xi\|_1^2 + \int_0^t \|\xi_t\|^2 dt \\ &\leq -\int_0^t (\eta_t, I_h^* \xi_t) dt + \int_0^t \varepsilon_a(u_H; R_h u, \xi_t) dt + \int_0^t \varepsilon_h(u_t - f, \xi_t) dt \\ &\quad + \frac{1}{2} \int_0^t a_h((u_H)_t; \xi, I_h^* \xi) dt + \frac{1}{2} \int_0^t [a_h(u_H; \xi_t, I_h^* \xi) - a_h(u_H; \xi, I_h^* \xi_t)] dt \\ &\quad + \int_0^t [a(u; R_h u, \xi_t) - a(u_H; R_h u, \xi_t)] dt \equiv \sum_{i=1}^6 I_i. \end{aligned} \tag{42}$$

We now estimate the right-hand terms of (42). For $I_1$, by(16), Hölder inequality and $\epsilon$–Cauchy inequality we have

$$|I_1| \leq \int_0^t \|\eta_t\| \|I_h^* \xi_t\| dt \leq C h^4 \int_0^t \|u_t\|_2^2 dt + \frac{1}{4} \int_0^t \|\xi_t\|^2 dt, \tag{43}$$

From Lemma 3.2, we get

$$|I_2| \le C\int_0^t h\|u\|_2\|\xi_t\|dt \le Ch^2\int_0^t \|u\|_2^2dt + \frac{1}{4}\int_0^t \|\xi_t\|^2dt, \tag{44}$$

$$|I_3| \le C\int_0^t h\|u_t - f\|_1\|\xi_t\|dt \le Ch^2\int_0^t (\|u_t\|_1^2 + \|f\|_1^2)dt + \frac{1}{4}\int_0^t \|\xi_t\|^2dt. \tag{45}$$

For $I_4$, by (2) and Lemma 3.2, we have

$$|I_4| \le \frac{1}{2}\int_0^t |a_h((u_H)_t;\xi, I_h^*\xi)|dt \le C\int_0^t \|\xi\|_1^2dt. \tag{46}$$

By (18) and the inverse estimate, we have

$$\begin{aligned} |I_5| &\le C\int_0^t h\|\xi\|_1\|\xi_t\|_1dt \\ &\le C\int_0^t \|\xi\|_1\|\xi_t\|_0dt \le C\int_0^t \|\xi\|_1^2dt + \frac{1}{4}\int_0^t \|\xi_t\|^2dt. \end{aligned} \tag{47}$$

In order to estimate $I_6$, by integration by parts, we have

$$\begin{aligned} &\frac{d}{dt}\left[\int_\Omega (A(u) - A(u_H))\nabla R_h u\cdot\nabla\xi dx\right] = \int_\Omega (A(u)_t - A(u_H)_t)\nabla R_h u\cdot\nabla\xi dx \\ &+ \int_\Omega (A(u) - A(u_H))\nabla R_h u_t\cdot\nabla\xi dx + \int_\Omega (A(u) - A(u_H))\nabla R_h u\cdot\nabla\xi_t dx, \end{aligned} \tag{48}$$

Then,

$$\begin{aligned} &a(u;R_hu,\xi_t) - a(u_H;R_hu,\xi_t) = \int_\Omega (A(u) - A(u_H))\nabla R_h u\cdot\nabla\xi_t dx \\ &= \frac{d}{dt}\left[\int_\Omega (A(u) - A(u_H))\nabla R_h u\cdot\nabla\xi dx\right] - \int_\Omega (A(u)_t - A(u_H)_t)\nabla R_h u\cdot\nabla\xi dx \\ &- \int_\Omega (A(u) - A(u_H))\nabla R_h u_t\cdot\nabla\xi dx. \end{aligned} \tag{49}$$

So we get

$$\begin{aligned} I_6 &= \int_\Omega (A(u) - A(u_H))\nabla R_h u\cdot\nabla\xi dx - \int_0^t\int_\Omega (A(u)_t - A(u_H)_t)\nabla R_h u\cdot\nabla\xi dxdt \\ &- \int_0^t\int_\Omega (A(u) - A(u_H))\nabla R_h u_t\cdot\nabla\xi dxdt \equiv \sum_{i=1}^3 I_{6i}. \end{aligned} \tag{50}$$

By (17), (3), and $\epsilon$–Cauchy inequality, we have

$$|I_{61}| \le \|A(u) - A(u_H)\|\|\nabla R_h u\|_\infty\|\nabla\xi\| \le C\|u - u_H\|^2 + \frac{\alpha}{4}\|\xi\|_1^2. \tag{51}$$

$$\begin{aligned} |I_{62}| &\le \int_0^t \|A(u)_t - A(u_H)_t\|\|\nabla R_h u\|_\infty\|\nabla\xi\|dt \\ &\le C\int_0^t \|u - u_H\|^2dt + C\int_0^t \|\xi\|_1^2dt. \end{aligned} \tag{52}$$

$$\begin{aligned} |I_{63}| &\le \int_0^t \|A(u) - A(u_H)\|\|\nabla R_h u_t\|_\infty\|\nabla\xi\|dt \\ &\le C\int_0^t \|u - u_H\|^2dt + C\int_0^t \|\xi\|_1^2dt. \end{aligned} \tag{53}$$

Then, we obtain

$$|I_6| \le C_1\|u-u_H\|^2 + \frac{\alpha}{4}\|\xi\|_1^2 + C_2\int_0^t \|u-u_H\|^2 dt + C_3\int_0^t \|\xi\|_1^2 dt. \tag{54}$$

From (42)–(54), we get

$$\begin{aligned}\frac{\alpha}{4}\|\xi\|_1^2 &\le C_1h^4\int_0^t \|u_t\|_2^2 dt + C_2h^2\int_0^t \|u\|_2^2 dt + C_3h^2\int_0^t (\|u_t\|_1^2 + \|f\|_1^2) dt\\ &+ C_4\|u-u_H\|^2 + C_5\int_0^t \|u-u_H\|^2 dt + C_6\int_0^t \|\xi\|_1^2 dt.\end{aligned} \tag{55}$$

By Theorem 3.1, we have

$$\|u-u_H\| \le CH^2. \tag{56}$$

By (55), (56) and the Gronwall lemma, we have

$$\|\xi\|_1^2 \le C(h^2 + H^4). \tag{57}$$

So we get

$$\|u_h - R_h u\|_1 = \|\xi\|_1 \le C(h + H^2).$$

In view of (15) we have

$$\|u_h - u\|_1 \le \|u_h - R_h u\|_1 + \|R_h u - u\|_1 \le C(h + H^2), \tag{58}$$

which yields the desired result.

We consider the spacial discretization to focus on the two-grid method. Algorithm 1 is only a semidiscrete two-grid finite volume element method. In practical computations, the method should be combined with a time-stepping discretization. We consider a time step $\Delta t$ and approximate the solutions at $t^n = n\Delta t$, $\Delta t = T/N$, $n = 0, 1, \cdots, N$. Denote $u_h^n = u_h(\cdot, t^n)$, $\overline{\partial} u_h^n = \frac{u_h^n - u_h^{n-1}}{\Delta t}$, we can get an implicit backward Euler finite volume element approximations. The fully discrete finite volume element method is to find $u_h^n \in V_h$ $(n = 1, 2, \cdots)$, such that

$$\begin{cases} (\overline{\partial} u_h^n, I_h^* v_h) + a_h(u_h^n; u_h^n, I_h^* v_h) = (f^n, I_h^* v_h), \ \forall v_h \in V_h,\\ u_h^0 = R_h u_0,\end{cases} \tag{59}$$

where $R_h u_0$ is defined by (14). The corresponding fully discrete two-grid finite volume element algorithm is defined as

**Algorithm 1'**

Step 1: On the coarse grid $\mathcal{T}_H$, find $u_H^n \in V_H$, for $n = 1, 2, \cdots$, such that

$$\begin{cases} (\overline{\partial} u_H^n, I_H^* v_H) + a_H(u_H^n; u_H^n, I_H^* v_H) = (f^n, I_H^* v_H), \ \forall v_H \in V_H,\\ u_H^0 = R_H u_0,\end{cases} \tag{60}$$

where $R_H$ is defined in the same way as $R_h$ defined by (14).

Step 2: On the fine grid $\mathcal{T}_h$, find $u_h^n \in V_h$, for $n = 1, 2, \cdots$, such that

$$\begin{cases} (\overline{\partial} u_h^n, I_h^* v_h) + a_h(u_H^n; u_h^n, I_h^* v_h) = (f^n, I_h^* v_h), \ \forall v_h \in V_h,\\ u_h^0 = R_h u_0.\end{cases} \tag{61}$$

Our main goal is to test the usefulness and efficiency of the two-grid method. We will use Algorithm 1' to do the numerical examples for $\Delta t$ small enough. In a practical point of view, we just need to choose space grid $h < H$ to obtain a considerable error reduction in spite of the demanding requirement $h = O(H^2)$.

## 5. Numerical examples

We consider the following nonlinear parabolic problem

$$\begin{cases} u_t - \nabla \cdot (A(u)\nabla u) = f(x,t), \quad x \in \Omega = [0,1]^2,\ t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = x_1 x_2 (1-x_1)(1-x_2), \end{cases} \tag{62}$$

where $x = (x_1, x_2)$, $A = u^2$, the exact solution is $u(x,t) = e^{-t}x_1x_2(1-x_1)(1-x_2)$ and $f(x,t)$ is decided by the exact solution.

Our main interest is to verify the performances of the two-grid finite volume element method. Choose the space step $H$ and obtain the coarse grids. Let $h = H^2$ and then we obtain the fine grids.

In order to prove the efficiency of the two-grid finite volume element method, we compare this method with the standard finite volume element method(FVEM). We choose different $\Delta t$ and the end of time $T = 0.5$. Computational results are shown in Table 1–Table 3.

TABLE 1. $H^1$ error and CPU time of the standard FVEM.

| $h$ | $\Delta t$ | $H_1$ error | CPU time (s) |
|---|---|---|---|
| 1/9 | 0.05 | $5.6259 \times 10^{-3}$ | 21.19 |
| 1/16 | 0.05 | $3.7885 \times 10^{-4}$ | 73.29 |

TABLE 2. $H^1$ error and CPU time of the two-grid FVEM.

| $h$ | $H$ | $\Delta t$ | $H_1$ error | CPU time (s) |
|---|---|---|---|---|
| 1/9 | 1/3 | 0.05 | $5.8762 \times 10^{-3}$ | 0.9673 |
| 1/16 | 1/4 | 0.05 | $3.8303 \times 10^{-4}$ | 1.8744 |

TABLE 3. $H^1$ error and CPU time of the two-grid FVEM with different space grids.

| $h$ | $H$ | $\Delta t$ | $H_1$ error | CPU time (s) |
|---|---|---|---|---|
| 1/4 | 1/2 | 0.05 | $1.3980 \times 10^{-2}$ | 0.158 |
| 1/16 | 1/4 | 0.05 | $3.8303 \times 10^{-4}$ | 1.874 |
| 1/64 | 1/8 | 0.0125 | $1.0692 \times 10^{-4}$ | 47.986 |
| 1/256 | 1/16 | 0.003125 | $0.3028 \times 10^{-4}$ | 6430.348 |

For the case when the diffusion coefficient is $A(u) = (1+u)^2$ and the exact solution is $u(x,t) = 100x_1x_2(1-x_1)(1-x_2)(\cos(1.05\pi t))^2$, computational results are shown in Table 4–Table 6.

TABLE 4. $H^1$ error and CPU time of the standard FVEM.

| $h$ | $\Delta t$ | $H_1$ error | CPU time (s) |
|---|---|---|---|
| 1/9 | 0.05 | $6.4096 \times 10^{-3}$ | 24.14 |
| 1/16 | 0.05 | $4.3162 \times 10^{-4}$ | 83.49 |

From Tables 1–6, we can see that the numerical results coincide with the theoretical analysis, and the two-grid finite volume element method spends less time

Table 5. $H^1$ error and CPU time of the two-grid FVEM.

| $h$ | $H$ | $\Delta t$ | $H_1$ error | CPU time (s) |
|---|---|---|---|---|
| 1/9 | 1/3 | 0.05 | $6.6948 \times 10^{-3}$ | 1.1020 |
| 1/16 | 1/4 | 0.05 | $4.3639 \times 10^{-4}$ | 2.1355 |

Table 6. $H^1$ error and CPU time of the two-grid FVEM with different space grids.

| $h$ | $H$ | $\Delta t$ | $H_1$ error | CPU time (s) |
|---|---|---|---|---|
| 1/4 | 1/2 | 0.05 | $1.5927 \times 10^{-2}$ | 0.182 |
| 1/16 | 1/4 | 0.05 | $4.3639 \times 10^{-4}$ | 2.135 |
| 1/64 | 1/8 | 0.0125 | $1.2181 \times 10^{-4}$ | 54.67 |
| 1/256 | 1/16 | 0.003125 | $0.3450 \times 10^{-4}$ | 7323.12 |

than standard finite volume element method, that is to say, the two-grid algorithm is effective for saving a large amount of computational time and still keeping good accuracy.

## 6. Conclusions

In this paper, we have presented and derived error estimates for the semidiscrete finite volume element method and its two-grid algorithm of the nonlinear parabolic problem (1). Theorem 3.1 and 3.2 present the optimal a priori error estimates in the $L^2$ and $H^1$ norm for the semidiscrete finite volume element method. Theorem 4.1 demonstrates a remarkable fact about the two-grid finite volume element method: the highest possible convergence rate in the $H^1$ norm for the two-grid finite volume element method is $O(H^2)$ if we choose $h = O(H^2)$. It is proved that the coarse grid can be much coarser than the fine grid ($h < H$). We can achieve asymptotically optimal approximation in the $H^1$ norm error estimate as long as the mesh sizes satisfy $h = O(H^2)$. Moreover, in spite of the demanding requirement $h = O(H^2)$, we just need to choose $h < H$ to obtain a considerable reduction of the error in the $H^1$ norm in practical computation.

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