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SECOND-ORDER TWO-SCALE ANALYSIS METHOD FOR DYNAMIC THERMO-MECHANICAL PROBLEMS IN PERIODIC STRUCTURE

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Abstract. In this paper, we develop the second-order two-scale (SOTS) analysis method and numerical algorithm for dynamic thermo-mechanical problems of composite materials with 3-D periodic configuration. In the problem considered, there exists a mutual interaction between the displacement and temperature fields. By the asymptotic expansion of temperature and displacement fields, the cell problems, effective thermal and mechanical parameters, homogenized equations and SOTS formulas of temperatures and displacements are obtained. The numerical algorithm based on the SOTS method is given. Finally, some numerical examples are shown. The numerical results show that the SOTS method is feasible and valid to predict the dynamic thermo-mechanical behaviors of periodic composite materials.

Key Words. Composites with 3-D periodic configuration, dynamic thermo-mechanical problems, the SOTS analysis method, numerical algorithm

1. Introduction

With the rapid advance of materials science and technology, composite materials have been widely used to a variety of industrial fields owing to the advantageous physical and mechanical properties. With the appearance of complex and extreme service environments, composite structures usually work under multi-physical fields coupled circumstances. And it is important to understand the thermo-mechanical responses of them in engineering applications. Up to now, some research has been performed on thermo-mechanical problems of periodic composites. However, some studies were devoted to one-way thermo-mechanical coupling problems [1-4], namely, the thermal effects affect the mechanical filed but not vice versa. Other studies have focused on developing different types of micromechanical models with simplified microstructures [5-8] and various numerical modeling approaches [9-11] to obtain the effective thermal and mechanical properties or homogenized behaviors. But in many engineering applications, the understanding of the local fluctuation of temperature and displacement fields is much more important. Besides, in some situations, such as the thermal shock phenomena, the dynamic thermo-mechanical problem should be considered. And the fully coupled analysis will lead to more accurate results. So it is significant and meaningful to study dynamic

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thermo-mechanical problems of periodic composites. In this field, Francfort [12] and Parnell [13] have given the homogenized procedure for the dynamic problems with different periodic configurations.

The dynamic thermo-mechanical problem is strongly coupled by hyperbolic and parabolic equations, so the transient displacement and temperature fields must be solved simultaneously. Thus, it is difficult to apply analytical methods to study it. As for numerical solutions, due to the sharply varying of material coefficients, in order to capture the local fluctuation behaviors of temperature and displacement fields and their derivatives, the mesh size must be very small while employing the traditional finite element method (FEM) and finite difference method (FDM). So it leads to tremendous amount of computer memory and CPU time. It is needed to develop new effective method for predicting the physical and mechanical performances.

Based on the homogenization methods [14-16], various multi-scale methods have been proposed [22, 23]. They only considered the first-order asymptotic expansions. In recent years, Cui et al. introduced the Second-Order Two-Scale (SOTS) analysis method [17-19] to predict the physical and mechanical behaviors of composites. By the second-order correctors, the microscopic fluctuation of physical and mechanical behaviors inside the materials can be captured more accurately. Feng et al. [20] studied the two-scale analysis for the static thermo-mechanical coupling problem of periodic composites. After that, Wan [21] studied the dynamic thermo-mechanical problem by two-scale analysis method and gave the numerical results of 1-D. In this paper, we study the SOTS's numerical method on dynamic thermo-mechanical problems of 3-D periodic composites (Fig.1) which is much more popular in engineering practice. In 3-D case, temperatures, displacements, temperature gradients and stresses are calculated.

The reminder of this paper is outlined as follows. In section 2, the SOTS asymptotic analysis for the dynamic thermo-mechanical problem is presented briefly. Section 3 describes the algorithm procedure. And some numerical results are shown in section 4. Finally the conclusions are given.

For convenience, we use the Einstein summation convention on repeated indices throughout the paper. For simplicity, we do not give the definitions of the associated Sobolev spaces in this paper, and we refer the reader to some classical books [26, 27].

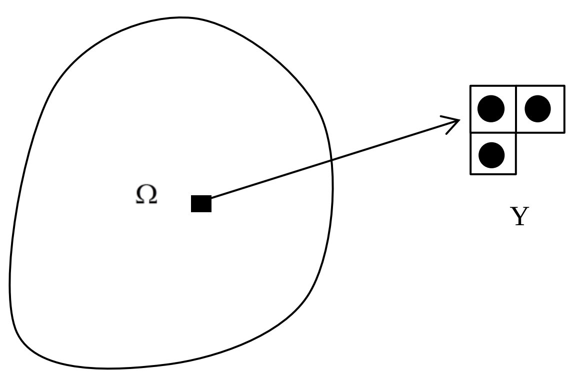


FIGURE 1. Macroscopic and Microscopic structure

2. Setting of the problem and second-order two-scale formulation

Consider the dynamic thermo-mechanical problems with mixed initial-boundary conditions for the composite structure with periodic configuration as follows

(1)
$$
\begin{cases}
\rho^\varepsilon(x)c^\varepsilon(x)\dfrac{\partial T^\varepsilon(x,t)}{\partial t}+\tilde{T}\beta^\varepsilon_{ij}(x)\dfrac{\partial}{\partial t}\left(\dfrac{\partial u^\varepsilon_i(x,t)}{\partial x_j}\right)-\dfrac{\partial}{\partial x_i}\left(k^\varepsilon_{ij}(x)\dfrac{\partial T^\varepsilon(x,t)}{\partial x_j}\right)=h \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad in\ \Omega\times(0,\breve{t}\,) \\
\rho^\varepsilon(x)\dfrac{\partial^2 u^\varepsilon_i(x,t)}{\partial t^2}-\dfrac{\partial}{\partial x_j}\left(C^\varepsilon_{ijkl}(x)\dfrac{\partial u^\varepsilon_k(x,t)}{\partial x_l}-\beta^\varepsilon_{ij}(x)\left(T^\varepsilon(x,t)-\tilde{T}\right)\right)=f_i \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad in\ \Omega\times(0,\breve{t}\,) \\
T^\varepsilon(x,t)=\hat{T}(x,t) \qquad on\ \Gamma_T\times(0,\breve{t}\,) \\
\mathbf{u}^\varepsilon(x,t)=\hat{\mathbf{u}}(x,t) \qquad on\ \Gamma_u\times(0,\breve{t}\,) \\
n_i k^\varepsilon_{ij}(x)\dfrac{\partial T^\varepsilon}{\partial x_j}=\bar{q}(x,t) \qquad on\ \Gamma_q\times(0,\breve{t}\,) \\
n_j\left(C^\varepsilon_{ijkl}(x)\dfrac{\partial u^\varepsilon_k(x,t)}{\partial x_l}-\beta^\varepsilon_{ij}(x)\left(T^\varepsilon(x,t)-\tilde{T}\right)\right)=\bar{\sigma}_i(x,t) \qquad on\ \Gamma_t\times(0,\breve{t}\,) \\
T^\varepsilon(x,0)=\tilde{T},\mathbf{u}^\varepsilon(x,0)=\tilde{\mathbf{u}}(x),\dfrac{\partial \mathbf{u}^\varepsilon}{\partial t}(x,0)=\breve{\mathbf{u}}(x) \qquad in\ \Omega
\end{cases}
$$

where Ω is a bounded domain with Lipschitz continuous boundary as shown in Fig.1; $i,j,k,l=1,2,3$; $T^\varepsilon(x,t)$ and $\mathbf{u}^\varepsilon(x,t)$ denote the temperature and displacement vector; $\rho^\varepsilon(x)$ and $c^\varepsilon(x)$ are the mass density and specific heat, $\beta^\varepsilon_{ij}(x)$, $C^\varepsilon_{ijkl}(x)$ and $k^\varepsilon_{ij}(x)$ are the thermal modulus, elastic tensor and thermal conductivity, and they are periodic functions with small period ε; $h(x,t)$ and $f_i(x,t)$ are the internal heat source and the body force; $\hat{\mathbf{u}}(x,t)$ is the prescribed displacements on the boundary Γ_u, $\hat{T}(x,t)$ is the prescribed temperature on the boundary Γ_T; $\bar{q}(x,t)$ is the heat flux prescribed normal to the boundary Γ_q, $\bar{\sigma}_i(x,t)$ is the prescribed tractions on the boundary Γ_t, n_i and n_j are the normal direction cosine of Γ_q and Γ_t; $\tilde{T}(x)$, $\tilde{\mathbf{u}}(x)$ and $\breve{\mathbf{u}}(x)$ are the initial temperature, displacement and velocity field. Besides, there holds

$$\Gamma_T\cap\Gamma_q=\emptyset,\ \ \Gamma_u\cap\Gamma_t=\emptyset,\ \ \Gamma_T\cup\Gamma_q=\Gamma_u\cup\Gamma_t=\partial\Omega$$

Let $y=x/\varepsilon$ be the local coordinates of unit cell Y, and then

$$\rho^\varepsilon(x)=\rho(x/\varepsilon),\ \ c^\varepsilon(x)=c(x/\varepsilon),\ \ k^\varepsilon_{ij}(x)=k_{ij}(x/\varepsilon)$$
$$C^\varepsilon_{ijkl}(x)=C_{ijkl}(x/\varepsilon),\ \ \beta^\varepsilon_{ij}(x)=\beta_{ij}(x/\varepsilon)$$

where $\rho(y)$, $c(y)$, $k_{ij}(y)$, $C_{ijkl}(y)$ and $\beta_{ij}(y)$ are 1-periodic functions, respectively.

At first, we make following assumptions:

(A) $\rho(y)$, $c(y)$, $k_{ij}(y)$, $C_{ijkl}(y)$ and $\beta_{ij}(y)$ are bounded measurable functions, and ρ, c, β_{ij}, C_{ijkl}, $k_{ij}\in L^\infty(Y)$.

(B) $k_{ij}(y)$, $C_{ijkl}(y)$ and $\beta_{ij}(y)$ are symmetric, and there exist two positive constant τ_1 and τ_2 independent of ε such that

$$
\begin{aligned}
&k_{ij}=k_{ji},\ \tau_1\gamma_i\gamma_i\leqslant k_{ij}(y)\gamma_i\gamma_j\leqslant\tau_2\gamma_i\gamma_i \\
&C_{ijkl}=C_{jikl}=C_{klij},\ \tau_1\eta_{ij}\eta_{ij}\leqslant C_{ijkl}(y)\eta_{ij}\eta_{kl}\leqslant\tau_2\eta_{ij}\eta_{ij} \\
&\beta_{ij}=\beta_{ji},\ \tau_1\gamma_i\gamma_i\leqslant\beta_{ij}(y)\gamma_i\gamma_j\leqslant\tau_2\gamma_i\gamma_i
\end{aligned}
$$

where $\{\eta_{ij}\}$ is an arbitrary symmetric matrix and $\{\gamma_i\}$ is an arbitrary vector with real elements.

Now we briefly derive the SOTS computation formula for the dynamic thermo-mechanical problem of periodic composites. Firstly we assume that $T^\varepsilon(x,t)$ and $\mathbf{u}^\varepsilon(x,t)$ can be formally expanded as follows

$$\begin{aligned} T^\varepsilon(x,t) &= T_0(x,y,t) + \varepsilon T_1(x,y,t) + \varepsilon^2 T_2(x,y,t) + \cdots \\ \mathbf{u}^\varepsilon(x,t) &= \mathbf{u}_0(x,y,t) + \varepsilon \mathbf{u}_1(x,y,t) + \varepsilon^2 \mathbf{u}_2(x,y,t) + \cdots \end{aligned} \tag{2}$$

where $\mathbf{u}_i(x,y,t) = (u_{i1}, u_{i2}, u_{i3})$. Inserting (2) into (1), taking account of the chain rule

$$\frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}$$

and equating the coefficients of the same powers ε, we have

$$O(\varepsilon^{-2}): \quad \begin{cases} \dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\dfrac{\partial T_0}{\partial y_j}\right) = 0 \\ \dfrac{\partial}{\partial y_j}\left(C_{ijkl}(y)\dfrac{\partial u_{0k}}{\partial y_l}\right) = 0 \end{cases} \tag{3}$$

$$O(\varepsilon^{-1}): \quad \begin{cases} \tilde{T}\beta_{ij}(y)\dfrac{\partial^2 u_{0i}}{\partial t \partial y_j} - \dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\left(\dfrac{\partial T_1}{\partial y_j} + \dfrac{\partial T_0}{\partial x_j}\right)\right) \\ -\dfrac{\partial}{\partial x_i}\left(k_{ij}(y)\dfrac{\partial T_0}{\partial y_j}\right) = 0 \\ \dfrac{\partial}{\partial y_j}\left(\beta_{ij}(y)T_0\right) - \dfrac{\partial}{\partial y_j}\left(\beta_{ij}(y)\tilde{T}\right) - \dfrac{\partial}{\partial x_j}\left(C_{ijkl}(y)\dfrac{\partial u_{0k}}{\partial y_l}\right) \\ -\dfrac{\partial}{\partial y_j}\left(C_{ijkl}(y)\left(\dfrac{\partial u_{0k}}{\partial x_l} + \dfrac{\partial u_{1k}}{\partial y_l}\right)\right) = 0 \end{cases} \tag{4}$$

$$O(\varepsilon^{0}): \quad \begin{cases} \tilde{T}\beta_{ij}(y)\left(\dfrac{\partial^2 u_{1i}}{\partial t \partial y_j} + \dfrac{\partial^2 u_{0i}}{\partial t \partial x_j}\right) - \dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\left(\dfrac{\partial T_2}{\partial y_j} + \dfrac{\partial T_1}{\partial x_j}\right)\right) \\ +\rho(y)c(y)\dfrac{\partial T_0}{\partial t} - \dfrac{\partial}{\partial x_i}\left(k_{ij}(y)\left(\dfrac{\partial T_1}{\partial y_j} + \dfrac{\partial T_0}{\partial x_j}\right)\right) = h \\ \rho(y)\dfrac{\partial^2 u_{0i}}{\partial t^2} + \dfrac{\partial}{\partial y_j}\left(\beta_{ij}(y)T_1\right) - \dfrac{\partial}{\partial x_j}\left(C_{ijkl}(y)\left(\dfrac{\partial u_{0k}}{\partial x_l} + \dfrac{\partial u_{1k}}{\partial y_l}\right)\right) \\ -\dfrac{\partial}{\partial y_j}\left(C_{ijkl}(y)\left(\dfrac{\partial u_{1k}}{\partial x_l} + \dfrac{\partial u_{2k}}{\partial y_l}\right)\right) + \dfrac{\partial}{\partial x_j}\left(\beta_{ij}(y)T_0 - \beta_{ij}(y)\tilde{T}\right) = f_i \end{cases} \tag{5}$$

From (3) we can acquire that T_0 and $\mathbf{u}_0$ are independent of the microscale y, namely

$$T_0 = T_0(x,t), \ \mathbf{u}_0 = \mathbf{u}_0(x,t) \tag{6}$$

Refer to [20, 21], from (4) and (6), T_1 and $\mathbf{u}_1$ can be defined as follows

$$T_1(x,y,t) = M_{\alpha_1}(y)\frac{\partial T_0}{\partial x_{\alpha_1}}(x,t) \tag{7}$$

$$\mathbf{u}_1(x,y,t) = \mathbf{N}_{\alpha_1}(y)\frac{\partial \mathbf{u}_0}{\partial x_{\alpha_1}}(x,t) - \mathbf{P}_0(y)\left(T_0(x,t) - \tilde{T}\right) \tag{8}$$

where $\mathbf{N}_{\alpha_1}(y)$ are matrix-valued functions and $\mathbf{P}_0(y)$ are vector-valued functions defined on unit cell Y. $M_{\alpha_1}(y)$, $\mathbf{N}_{\alpha_1}(y)$ and $\mathbf{P}_0(y)$ are the solutions of following cell problems

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\dfrac{\partial M_{\alpha_1}(y)}{\partial y_j}\right) = \dfrac{\partial}{\partial y_i}\left(k_{i\alpha_1}(y)\right) \quad y \in Y \\ \displaystyle\int_Y M_{\alpha_1}(y)dy = 0, \quad M_{\alpha_1}(y) \in H^1_{per}(Y) \end{cases} \tag{9}$$

$$\begin{cases} -\dfrac{\partial}{\partial y_j}\left(C_{ijkl}(y)\dfrac{\partial \mathbf{N}_{\alpha_1 km}(y)}{\partial y_l}\right) = \dfrac{\partial}{\partial y_j}\left(C_{ijm\alpha_1}(y)\right) \quad y \in Y \\ \displaystyle\int_Y \mathbf{N}_{\alpha_1 m}(y)dy = 0, \quad \mathbf{N}_{\alpha_1 m}(y) \in H^1_{per}(Y) \end{cases} \tag{10}$$

$$\begin{cases} -\dfrac{\partial}{\partial y_j}\left(C_{ijkl}(y)\dfrac{\partial \mathbf{P}_{0k}(y)}{\partial y_l}\right) = \dfrac{\partial}{\partial y_j}\left(\beta_{ij}(y)\right) \quad y \in Y \\ \displaystyle\int_Y \mathbf{P}_0(y)dy = 0, \quad \mathbf{P}_0(y) \in H^1_{per}(Y) \end{cases} \tag{11}$$

where

$$H^1_{per}(Y) = \{v | v \in H^1(Y),\ v\ is\ Y - periodic\} \tag{12}$$

Introducing (7)-(8) into (5), integrating over both sides of each equation of (5) in Y and using (9)-(11), following equations are obtained

$$\begin{aligned} &\int_Y \frac{\partial}{\partial y_i}\left(k_{ij}(y)\frac{\partial T_2}{\partial y_j}\right)dy + \frac{\partial}{\partial x_i}\left[\frac{1}{|Y|}\int_Y\left(k_{ij}(y) + k_{ik}(y)\frac{\partial M_j(y)}{\partial y_k}\right)dy\frac{\partial T_0}{\partial x_j}\right] \\ &= \frac{1}{|Y|}\int_Y\left(\rho(y)c(y) - \tilde{T}\beta_{ij}(y)\frac{\partial \mathbf{P}_{0i}(y)}{\partial y_j}\right)dy\frac{\partial T_0}{\partial t} \\ &+\tilde{T}\frac{1}{|Y|}\int_Y\left(\beta_{ij}(y) + \beta_{kl}(y)\frac{\partial \mathbf{N}_{jki}(y)}{\partial y_l}\right)dy\frac{\partial^2 u_{0i}}{\partial t \partial x_j} - h \end{aligned}$$

$$\begin{aligned} &\frac{1}{|Y|}\int_Y \rho(y)dy\frac{\partial^2 u_{0i}}{\partial t^2} + \frac{1}{|Y|}\int_Y\left(\beta_{ij}(y) + C_{ijkl}(y)\frac{\partial \mathbf{P}_{0k}(y)}{\partial y_l}\right)dy\frac{\partial}{\partial x_j}\left(T_0 - \tilde{T}\right) \\ &-\frac{\partial}{\partial x_j}\left[\frac{1}{|Y|}\int_Y\left(C_{ijkl}(y) + C_{ijm\alpha_1}(y)\frac{\partial \mathbf{N}_{lmk}(y)}{\partial y_{\alpha_1}}\right)dy\frac{\partial u_{0k}}{\partial x_l}\right] \\ &= f_i + \int_Y \frac{\partial}{\partial y_j}\left(C_{ijkl}(y)\frac{\partial u_{2k}}{\partial y_l}\right)dy \end{aligned}$$

Thus the homogenized specific heat capacity $\hat{S}$, thermal modulus $\hat{\beta}_{ij}$ and $\hat{\varphi}_{ij}$, thermal conductivity $\hat{k}_{ij}$, mass density $\hat{\rho}$, and elastic tensor $\hat{C}_{ijkl}$ can be defined as follows

$$\hat{S} = \frac{1}{|Y|}\int_Y\left(\rho(y)c(y) - \tilde{T}\beta_{ij}(y)\frac{\partial \mathbf{P}_{0i}(y)}{\partial y_j}\right)dy \tag{13}$$

$$\hat{\beta}_{ij} = \frac{1}{|Y|}\int_Y\left(\beta_{ij}(y) + \beta_{kl}(y)\frac{\partial \mathbf{N}_{jki}(y)}{\partial y_l}\right)dy \tag{14}$$

$$\hat{k}_{ij} = \frac{1}{|Y|}\int_Y\left(k_{ij}(y) + k_{ik}(y)\frac{\partial M_j(y)}{\partial y_k}\right)dy \tag{15}$$

$$\hat{\rho} = \frac{1}{|Y|}\int_Y \rho(y)dy \tag{16}$$

$$\hat{C}_{ijkl} = \frac{1}{|Y|}\int_Y \left(C_{ijkl}(y) + C_{ijm\alpha_1}(y)\frac{\partial \mathbf{N}_{lmk}(y)}{\partial y_{\alpha_1}}\right)dy \tag{17}$$

$$\hat{\varphi}_{ij} = \frac{1}{|Y|}\int_Y \left(\beta_{ij}(y) + C_{ijkl}(y)\frac{\partial \mathbf{P}_{0k}(y)}{\partial y_l}\right)dy \tag{18}$$

According to (10)-(11), it is easy to prove that $\hat{\beta}_{ij} = \hat{\varphi}_{ij}$ [21]. Moreover, the homogenized problem associated with the original problem (1) can be defined as follows

(19)
$$\begin{cases}
\hat{S}\dfrac{\partial T_0(x,t)}{\partial t} + \tilde{T}\hat{\beta}_{ij}\dfrac{\partial^2 u_{0i}(x,t)}{\partial t\partial x_j} - \dfrac{\partial}{\partial x_i}\left(\hat{k}_{ij}\dfrac{\partial T_0(x,t)}{\partial x_j}\right) = h & in\ \Omega\times(0,\breve{t}\,) \\
\hat{\rho}\dfrac{\partial^2 u_{0i}(x,t)}{\partial t^2} - \dfrac{\partial}{\partial x_j}\left(\hat{C}_{ijkl}\dfrac{\partial u_{0k}(x,t)}{\partial x_l} - \hat{\beta}_{ij}\left(T_0(x,t)-\tilde{T}\right)\right) = f_i & in\ \Omega\times(0,\breve{t}\,) \\
T_0(x,t) = \hat{T}(x,t) \qquad on\ \Gamma_T\times(0,\breve{t}\,) \\
\mathbf{u}_0(x,t) = \hat{\mathbf{u}}(x,t) \qquad on\ \Gamma_u\times(0,\breve{t}\,) \\
n_i\hat{k}_{ij}\dfrac{\partial T_0(x,t)}{\partial x_j} = \bar{q}(x,t) \qquad on\ \Gamma_q\times(0,\breve{t}\,) \\
n_i\left(\hat{C}_{ijkl}\dfrac{\partial u_{0k}}{\partial x_l} - \hat{\beta}_{ij}\left(T_0-\tilde{T}\right)\right) = \bar{\sigma}_j(x,t) \qquad on\ \Gamma_t\times(0,\breve{t}\,) \\
T_0(x,0) = \breve{T}, \mathbf{u}_0(x,0) = \tilde{\mathbf{u}}(x), \dfrac{\partial \mathbf{u}_0}{\partial t}(x,0) = \breve{\mathbf{u}}(x) \qquad in\ \Omega
\end{cases}$$

According to [12], it can be obtained

$$\begin{aligned}\breve{T} = \tilde{T} + \frac{1}{\hat{S}}\Bigg\{&\left(T^\varepsilon(x,0)-\tilde{T}\right)\cdot\frac{1}{|Y|}\int_Y \rho(y)c(y)d\,y \\ &+ \frac{\partial \tilde{u}_i(x)}{\partial x_j}\cdot\frac{1}{|Y|}\int_Y \beta_{kl}(y)\frac{\partial \mathbf{N}_{jki}(y)}{\partial y_l}dy\Bigg\}\end{aligned} \tag{20}$$

Furthermore, introducing (7), (8) and (19) into (5) and applying (6), following equations are obtained

$$\begin{aligned}
&-\frac{\partial}{\partial y_i}\left(k_{ij}(y)\frac{\partial T_2}{\partial y_j}\right) = \left(\tilde{T}\beta_{ij}(y)\frac{\partial \mathbf{P}_{0i}(y)}{\partial y_j} - \rho(y)c(y) + \hat{S}\right)\frac{\partial T_0}{\partial t} \\
&+\tilde{T}\left(\hat{\beta}_{\alpha_1\alpha_2} - \beta_{\alpha_1\alpha_2}(y) - \beta_{ij}(y)\frac{\partial \mathbf{N}_{\alpha_2 i\alpha_1}(y)}{\partial y_j}\right)\frac{\partial^2 u_{0\alpha_1}}{\partial t\partial x_{\alpha_2}} \\
&+\Bigg(-\hat{k}_{\alpha_1\alpha_2} + k_{\alpha_1\alpha_2}(y) + k_{\alpha_2 j}(y)\frac{\partial M_{\alpha_1}(y)}{\partial y_j} \\
&+\frac{\partial}{\partial y_i}\left(k_{i\alpha_2}(y)M_{\alpha_1}(y)\right)\Bigg)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}
\end{aligned} \tag{21}$$

$$
\begin{aligned}
&\frac{\partial}{\partial y_j}\left(C_{ijkl}(y)\frac{\partial u_{2k}}{\partial y_l}\right)=\Big(\hat{C}_{i\alpha_1 m\alpha_2}-C_{i\alpha_1 m\alpha_2}(y)\\
&-C_{i\alpha_1 kj}(y)\frac{\partial \mathbf{N}_{\alpha_2 km}(y)}{\partial y_j}-\frac{\partial}{\partial y_j}\left(C_{ijk\alpha_2}(y)\mathbf{N}_{\alpha_1 km}(y)\right)\Big)\frac{\partial^2 u_{0m}}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\\
&-\Big(\hat{\beta}_{i\alpha_1}-\beta_{i\alpha_1}(y)-\frac{\partial}{\partial y_j}\left(\beta_{ij}(y)M_{\alpha_1}(y)\right)-C_{i\alpha_1 kl}(y)\frac{\partial \mathbf{P}_{0k}(y)}{\partial y_l}\\
&-\frac{\partial}{\partial y_l}\left(C_{ilk\alpha_1}(y)\mathbf{P}_{0k}(y)\right)\Big)\frac{\partial T_0}{\partial x_{\alpha_1}}-\left(\hat{\rho}-\rho(y)\right)\frac{\partial^2 u_{0i}}{\partial t^2}
\end{aligned}
\tag{22}
$$

Then following [21] T_2 and $\mathbf{u}_2$ can be defined as follows

$$
T_2(x,y,t)=M_{\alpha_1\alpha_2}(y)\frac{\partial^2 T_0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}+R_{\alpha_1\alpha_2}(y)\frac{\partial^2 u_{0\alpha_1}(x,t)}{\partial t\partial x_{\alpha_2}}+Q_2(y)\frac{\partial T_0(x,t)}{\partial t}
\tag{23}
$$

$$
\mathbf{u}_2(x,y,t)=\mathbf{N}_{\alpha_1\alpha_2}(y)\frac{\partial^2 \mathbf{u}_0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}-\mathbf{P}_{\alpha_1}(y)\frac{\partial T_0(x,t)}{\partial x_{\alpha_1}}-\mathbf{F}_2(y)\frac{\partial^2 \mathbf{u}_0(x,t)}{\partial t^2}
\tag{24}
$$

where $\mathbf{N}_{\alpha_1\alpha_2}(y)$, $\mathbf{F}_2(y)$ are matrix-valued functions and $\mathbf{P}_{\alpha_1}(y)$ are vector-valued functions defined on unit cell Y. $M_{\alpha_1\alpha_2}(y)$, $R_{\alpha_1\alpha_2}(y)$, $Q_2(y)$, $\mathbf{N}_{\alpha_1\alpha_2}(y)$, $\mathbf{P}_{\alpha_1}(y)$ and $\mathbf{F}_2(y)$ are the solutions of following cell problems

$$
\begin{cases}
-\dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\dfrac{\partial M_{\alpha_1\alpha_2}(y)}{\partial y_j}\right)=-\hat{k}_{\alpha_1\alpha_2}+k_{\alpha_1\alpha_2}(y)+k_{\alpha_2 j}(y)\dfrac{\partial M_{\alpha_1}(y)}{\partial y_j}\\
+\dfrac{\partial}{\partial y_i}\left(k_{i\alpha_2}(y)M_{\alpha_1}(y)\right),\quad y\in Y\\
\displaystyle\int_Y M_{\alpha_1\alpha_2}(y)dy=0,\quad M_{\alpha_1\alpha_2}(y)\in H^1_{per}(Y)
\end{cases}
\tag{25}
$$

$$
\begin{cases}
-\dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\dfrac{\partial R_{\alpha_1\alpha_2}(y)}{\partial y_j}\right)=\tilde{T}\Big(\hat{\beta}_{\alpha_1\alpha_2}-\beta_{\alpha_1\alpha_2}(y)\\
-\beta_{ij}(y)\dfrac{\partial \mathbf{N}_{\alpha_2 i\alpha_1}(y)}{\partial y_j}\Big),\quad y\in Y\\
\displaystyle\int_Y R_{\alpha_1\alpha_2}(y)dy=0,\quad R_{\alpha_1\alpha_2}(y)\in H^1_{per}(Y)
\end{cases}
\tag{26}
$$

$$
\begin{cases}
-\dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\dfrac{\partial Q_2(y)}{\partial y_j}\right)=\tilde{T}\beta_{ij}(y)\dfrac{\partial \mathbf{P}_{0i}(y)}{\partial y_j}-\rho(y)c(y)+\hat{S},\quad y\in Y\\
\displaystyle\int_Y Q_2(y)dy=0,\quad Q_2(y)\in H^1_{per}(Y)
\end{cases}
\tag{27}
$$

$$
\begin{cases}
\dfrac{\partial}{\partial y_j}\left(C_{ijkl}(y)\dfrac{\partial \mathbf{N}_{\alpha_1\alpha_2 km}(y)}{\partial y_l}\right)=\hat{C}_{i\alpha_1 m\alpha_2}-C_{i\alpha_1 m\alpha_2}(y)\\
-C_{i\alpha_1 kj}(y)\dfrac{\partial \mathbf{N}_{\alpha_2 km}(y)}{\partial y_j}-\dfrac{\partial}{\partial y_j}\left(C_{ijk\alpha_2}(y)\mathbf{N}_{\alpha_1 km}(y)\right),\quad y\in Y\\
\displaystyle\int_Y \mathbf{N}_{\alpha_1\alpha_2 m}(y)dy=0,\quad \mathbf{N}_{\alpha_1\alpha_2 m}(y)\in H^1_{per}(Y)
\end{cases}
\tag{28}
$$

$$
\left\{
\begin{aligned}
&\frac{\partial}{\partial y_j}\left(C_{ijkl}(y)\frac{\partial \mathbf{P}_{\alpha_1 k}(y)}{\partial y_l}\right)=\hat{\beta}_{i\alpha_1}-\beta_{i\alpha_1}(y)-\frac{\partial}{\partial y_j}\left(\beta_{ij}(y)M_{\alpha_1}(y)\right)\\
&-C_{i\alpha_1 kl}(y)\frac{\partial \mathbf{P}_{0k}(y)}{\partial y_l}-\frac{\partial}{\partial y_l}\left(C_{ilk\alpha_1}(y)\mathbf{P}_{0k}(y)\right),\quad y\in\ Y\\
&\int_Y \mathbf{P}_{\alpha_1 k}(y)dy=0,\quad \mathbf{P}_{\alpha_1 k}(y)\in H^1_{per}(Y)
\end{aligned}
\right. \tag{29}
$$

$$
\left\{
\begin{aligned}
&\frac{\partial}{\partial y_j}\left(C_{ijkl}(y)\frac{\partial \mathbf{F}_{2km}(y)}{\partial y_l}\right)=\delta_{mi}\left(\hat{\rho}-\rho(y)\right),\quad y\in\ Y\\
&\int_Y \mathbf{F}_{2m}(y)dy=0,\quad \mathbf{F}_{2m}(y)\in H^1_{per}(Y)
\end{aligned}
\right. \tag{30}
$$

Remark 2.1. Existence and uniqueness of the cell problems (9)-(11) and (25)-(30) can be established based on suppositions (A)-(B), Lax-Milgram lemma and Korn's Inequalities [16]. The notation δ_{mi} is the Kronecker delta function, and if $m=i$, $\delta_{mi}=1$, or $\delta_{mi}=0$. According to (23)-(30) there holds

$$
\int_Y \frac{\partial}{\partial y_i}\left(k_{ij}(y)\frac{\partial T_2}{\partial y_j}\right)dy=0,\ \int_Y \frac{\partial}{\partial y_j}\left(C_{ijkl}(y)\frac{\partial u_{2k}}{\partial y_l}\right)dy=0
$$

Now we can define the two-scale approximate solutions of the problem (1) as follows

$$
\begin{aligned}
&\widehat{T}^{\varepsilon}_1(x,t)=T_0(x,t)+\varepsilon T_1(x,y,t)\\
&\widehat{\mathbf{u}}^{\varepsilon}_1(x,t)=\mathbf{u}_0(x,t)+\varepsilon \mathbf{u}_1(x,y,t)\\
&\widehat{T}^{\varepsilon}_2(x,t)=T_0(x,t)+\varepsilon T_1(x,y,t)+\varepsilon^2 T_2(x,y,t)\\
&\widehat{\mathbf{u}}^{\varepsilon}_2(x,t)=\mathbf{u}_0(x,t)+\varepsilon \mathbf{u}_1(x,y,t)+\varepsilon^2 \mathbf{u}_2(x,y,t)
\end{aligned} \tag{31}
$$

where T_0, $\mathbf{u}_0$ are the solutions of problem (19); T_1, $\mathbf{u}_1$ and T_2, $\mathbf{u}_2$ are defined by (7), (8) and (23), (24), respectively. $\widehat{T}^{\varepsilon}_1(x,t)$, $\widehat{\mathbf{u}}^{\varepsilon}_1(x,t)$ are called as first-order two-scale (FOTS) approximate solutions, $\widehat{T}^{\varepsilon}_2(x,t)$, $\widehat{\mathbf{u}}^{\varepsilon}_2(x,t)$ second-order two-scale (SOTS) approximate solutions. Moreover, let

$$
T^{\varepsilon 1}_{\Delta}(x,t)=T^{\varepsilon}(x,t)-\widehat{T}^{\varepsilon}_1(x,t),\quad \mathbf{u}^{\varepsilon 1}_{\Delta}(x,t)=\mathbf{u}^{\varepsilon}(x,t)-\widehat{\mathbf{u}}^{\varepsilon}_1(x,t) \tag{32}
$$

$$
T^{\varepsilon 2}_{\Delta}(x,t)=T^{\varepsilon}(x,t)-\widehat{T}^{\varepsilon}_2(x,t),\quad \mathbf{u}^{\varepsilon 2}_{\Delta}(x,t)=\mathbf{u}^{\varepsilon}(x,t)-\widehat{\mathbf{u}}^{\varepsilon}_2(x,t) \tag{33}
$$

To compare $\widehat{T}^{\varepsilon}_s(x,t)$, $\widehat{\mathbf{u}}^{\varepsilon}_s(x,t)$ $(s=1,2)$ with the original solutions $T^{\varepsilon}(x,t)$, $\mathbf{u}^{\varepsilon}(x,t)$, taking $T^{\varepsilon 1}_{\Delta}(x,t)$, $\mathbf{u}^{\varepsilon 1}_{\Delta}(x,t)$ into (1), according to assumption (B) and using (9)-(11), (19), we have

$$
\begin{aligned}
&\rho(y)c(y)\frac{\partial T^{\varepsilon 1}_{\Delta}(x,t)}{\partial t}+\tilde{T}\beta_{ij}(y)\frac{\partial}{\partial t}\left(\frac{\partial u^{\varepsilon 1}_{\Delta i}(x,t)}{\partial x_j}\right)\\
&-\frac{\partial}{\partial x_i}\left(k_{ij}(y)\frac{\partial T^{\varepsilon 1}_{\Delta}(x,t)}{\partial x_j}\right)=\mathbb{F}_0+\varepsilon\mathbb{F}_1\\
&\rho(y)\frac{\partial^2 u^{\varepsilon 1}_{\Delta i}(x,t)}{\partial t^2}+\frac{\partial}{\partial x_j}\left(\beta_{ij}(y)T^{\varepsilon 1}_{\Delta}(x,t)\right)\\
&-\frac{\partial}{\partial x_j}\left(C_{ijkl}(y)\frac{\partial u^{\varepsilon 1}_{\Delta k}(x,t)}{\partial x_l}\right)=\mathbb{S}_{0i}+\varepsilon\mathbb{S}_{1i}
\end{aligned} \tag{34}
$$

where

$$\begin{aligned}\mathbb{F}_0 =& \left(\hat{S} - \rho(y)c(y) + \tilde{T}\beta_{ij}(y)\frac{\partial P_{0i}(y)}{\partial y_j}\right)\frac{\partial T_0(x,t)}{\partial t}\\&+\tilde{T}\left(\hat{\beta}_{ij} - \beta_{ij}(y) - \beta_{kl}(y)\frac{\partial N_{jki}(y)}{\partial y_l}\right)\frac{\partial^2 u_{0i}(x,t)}{\partial t\partial x_j}\\&+\left(k_{ij}(y) + k_{i\alpha_1}(y)\frac{\partial M_j(y)}{\partial y_{\alpha_1}} - \hat{k}_{ij}\right)\frac{\partial^2 T_0(x,t)}{\partial x_i\partial x_j}\\&+\frac{\partial}{\partial y_{\alpha_1}}\left(k_{\alpha_1 i}(y)M_j(y)\right)\frac{\partial^2 T_0(x,t)}{\partial x_i\partial x_j}\end{aligned}$$

$$\begin{aligned}\mathbb{F}_1 =& \left(\tilde{T}\beta_{ij}(y)P_{0i}(y) - \rho(y)c(y)M_j(y)\right)\frac{\partial^2 T_0(x,t)}{\partial t\partial x_j}\\&+k_{ij}(y)M_{\alpha_1}(y)\frac{\partial^3 T_0(x,t)}{\partial x_i\partial x_j\partial x_{\alpha_1}} - \tilde{T}\beta_{ij}(y)N_{\alpha_1 ik}(y)\frac{\partial^3 u_{0k}(x,t)}{\partial t\partial x_j\partial x_{\alpha_1}}\end{aligned}$$

$$\begin{aligned}\mathbb{S}_{0i} =& (\hat{\rho} - \rho(y))\frac{\partial^2 u_{0i}(x,t)}{\partial t^2} + \frac{\partial}{\partial y_{\alpha_1}}\left(C_{\alpha_2\alpha_1 il}(y)N_{j\alpha_2 k}(y)\right)\frac{\partial^2 u_{0k}(x,t)}{\partial x_j\partial x_l}\\&+\left(\hat{\beta}_{ij} - \beta_{ij}(y) - C_{ijkl}(y)\frac{\partial P_{0k}(y)}{\partial y_l}\right)\frac{\partial T_0(x,t)}{\partial x_j}\\&-\frac{\partial}{\partial y_l}\left(C_{ijkl}(y)P_{0k}(y) + \beta_{ij}(y)M_l(y)\right)\frac{\partial T_0(x,t)}{\partial x_j}\\&+\left(C_{ijkl}(y) + C_{\alpha_2\alpha_1 il}(y)\frac{\partial N_{j\alpha_2 k}(y)}{\partial y_{\alpha_1}} - \hat{C}_{ijkl}\right)\frac{\partial^2 u_{0k}(x,t)}{\partial x_j\partial x_l}\end{aligned}$$

$$\begin{aligned}\mathbb{S}_{1i} =& -\left(\beta_{ij}(y)M_l(y) + C_{ijkl}(y)P_{0k}(y)\right)\frac{\partial^2 T_0(x,t)}{\partial x_j\partial x_l} + \rho(y)P_{0i}(y)\frac{\partial^2 T_0(x,t)}{\partial t^2}\\&+C_{ijkl}(y)N_{\alpha_1}(y)\frac{\partial^3 u_{0k}(x,t)}{\partial x_j\partial x_l\partial x_{\alpha_1}} - \rho(y)N_{\alpha_1}(y)\frac{\partial^3 u_{0i}(x,t)}{\partial t^2\partial x_{\alpha_1}}\end{aligned}$$

From (34), we obtain that the residuals of $T_\Delta^{\varepsilon 1}(x,t)$ and $\mathbf{u}_\Delta^{\varepsilon 1}(x,t)$ are of order $O(1)$ nearly everywhere inside Ω. So the FOTS approximate solutions are not accepted by the engineers since they are not approximate enough to capture the micro-scale fluctuations of temperatures, displacements and their gradients.

Then taking $T_\Delta^{\varepsilon 2}(x,t)$, $\mathbf{u}_\Delta^{\varepsilon 2}(x,t)$ into (1) and using assumption (B) and (9)-(11), (19), (25)-(30) we have

$$(35)\quad \begin{aligned}&\rho(y)\frac{\partial^2 u_{\Delta i}^{\varepsilon 2}(x,t)}{\partial t^2} + \frac{\partial}{\partial x_j}\left(\beta_{ij}(y)T_\Delta^{\varepsilon 2}(x,t)\right)\\&-\frac{\partial}{\partial x_j}\left(C_{ijkl}(y)\frac{\partial u_{\Delta k}^{\varepsilon 2}(x,t)}{\partial x_l}\right) = \varepsilon\mathbb{H}_i + \varepsilon\frac{\partial}{\partial x_j}\mathbb{E}_{ij}\\&\rho(y)c(y)\frac{\partial T_\Delta^{\varepsilon 2}(x,t)}{\partial t} + \tilde{T}\beta_{ij}(y)\frac{\partial}{\partial t}\left(\frac{\partial u_{\Delta i}^{\varepsilon 2}(x,t)}{\partial x_j}\right)\\&-\frac{\partial}{\partial x_i}\left(k_{ij}(y)\frac{\partial T_\Delta^{\varepsilon 2}(x,t)}{\partial x_j}\right) = \varepsilon\mathbb{G}_0 + \varepsilon\frac{\partial}{\partial x_i}\mathbb{G}_i\end{aligned}$$

where

$$
\begin{aligned}
\mathbb{H}_i = & -\big(\beta_{ij}(y)M_l(y) + C_{ijkl}(y)P_{0k}(y,\omega)\big)\,\frac{\partial^2 T_0(x,t)}{\partial x_j \partial x_l} \\
& + \rho(y)P_{0i}(y)\frac{\partial^2 T_0(x,t)}{\partial t^2} + C_{ijkl}(y)N_{\alpha_1 km}(y,\omega)\frac{\partial^3 u_{0m}(x,t)}{\partial x_j \partial x_l \partial x_{\alpha_1}} \\
& - \rho(y)N_{\alpha_1 im}(y)\frac{\partial^3 u_{0m}(x,t)}{\partial t^2 \partial x_{\alpha_1}} + C_{ijkl}(y)\frac{\partial N_{\alpha_1\alpha_2 km}(y)}{\partial y_l}\frac{\partial^3 u_{0m}(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_j} \\
& - C_{ijkl}(y)\frac{\partial P_{\alpha_1 k}(y)}{\partial y_l}\frac{\partial^2 T_0(x,t)}{\partial x_{\alpha_1}\partial x_j} - C_{ijkl}(y)\frac{\partial F_{2km}(y)}{\partial y_l}\frac{\partial^3 u_{0m}(x,t)}{\partial t^2 \partial x_j} \\
& - \varepsilon\left[\rho(y)N_{\alpha_1\alpha_2 im}(y)\frac{\partial^4 u_{0m}(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial t^2} - \rho(y)P_{\alpha_1 i}(y)\frac{\partial^3 T_0(x,t)}{\partial x_{\alpha_1}\partial t^2}\right. \\
& \left. - \rho(y)F_{2im}(y)\frac{\partial^4 u_{0m}(x,t)}{\partial t^4}\right]
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
\mathbb{E}_{ij} = & -\varepsilon\beta_{ij}(y)M_{\alpha_1\alpha_2}(y)\frac{\partial^2 T_0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} - \varepsilon\beta_{ij}(y)R_{\alpha_1\alpha_2}(y)\frac{\partial^2 u_{0\alpha_1}(x,t)}{\partial t \partial x_{\alpha_2}} \\
& - \varepsilon\beta_{ij}(y)Q_2(y)\frac{\partial T_0(x,t)}{\partial t} + \varepsilon C_{ijkl}(y)N_{\alpha_1\alpha_2 km}(y)\frac{\partial^3 u_{0m}(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_l} \\
& - \varepsilon C_{ijkl}(y)P_{\alpha_1 k}(y)\frac{\partial^2 T_0(x,t)}{\partial x_{\alpha_1}\partial x_l} - \varepsilon C_{ijkl}(y)F_{2km}(y)\frac{\partial^3 u_{0m}(x,t)}{\partial t^2 \partial x_l}
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
\mathbb{G}_0 = & -\tilde{T}\beta_{ij}(y)N_{\alpha_1 ik}(y)\frac{\partial^3 u_{0k}(x,t)}{\partial t \partial x_j \partial x_{\alpha_1}} \\
& + \Big(\tilde{T}\beta_{ij}(y)P_{0i}(y) - \rho(y)c(y)M_j(y)\Big)\,\frac{\partial^2 T_0(x,t)}{\partial t \partial x_j} \\
& + k_{ij}(y)M_{\alpha_1}(y,\omega)\frac{\partial^3 T_0(x,t)}{\partial x_i \partial x_j \partial x_{\alpha_1}} - \tilde{T}\beta_{ij}(y)\frac{\partial N_{\alpha_1\alpha_2 im}(y)}{\partial y_j}\frac{\partial^3 u_{0m}(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial t} \\
& + \tilde{T}\beta_{ij}(y)\frac{\partial P_{\alpha_1 i}(y)}{\partial y_j}\frac{\partial^2 T_0(x,t)}{\partial x_{\alpha_1}\partial t} + \tilde{T}\beta_{ij}(y)\frac{\partial F_{2im}(y)}{\partial y_j}\frac{\partial^3 u_{0m}(x,t)}{\partial t^3} \\
& + k_{ij}(y)\frac{\partial M_{\alpha_1\alpha_2}(y)}{\partial y_j}\frac{\partial^3 T_0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i} + k_{ij}(y)\frac{\partial R_{\alpha_1\alpha_2}(y)}{\partial y_j}\frac{\partial^3 u_{0\alpha_1}(x,t)}{\partial t \partial x_{\alpha_2}\partial x_i} \\
& + k_{ij}(y)\frac{\partial Q_2(y)}{\partial y_j}\frac{\partial^2 T_0(x,t)}{\partial t \partial x_i} - \varepsilon\rho(y)c(y)M_{\alpha_1\alpha_2}(y)\frac{\partial^3 T_0(x,t)}{\partial t \partial x_{\alpha_1}\partial x_{\alpha_2}} \\
& - \varepsilon\rho(y)c(y)R_{\alpha_1\alpha_2}(y)\frac{\partial^3 u_{0\alpha_1}(x,t)}{\partial t^2 \partial x_{\alpha_2}} - \varepsilon\rho(y)c(y)Q_2(y)\frac{\partial^2 T_0(x,t)}{\partial t^2} \\
& - \varepsilon\tilde{T}\beta_{ij}(y)N_{\alpha_1\alpha_2 im}(y)\frac{\partial^4 u_{0m}(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_j \partial t} + \varepsilon\tilde{T}\beta_{ij}(y)P_{\alpha_1 i}(y)\frac{\partial^3 T_0(x,t)}{\partial x_{\alpha_1}\partial x_j \partial t} \\
& + \varepsilon\tilde{T}\beta_{ij}(y)F_{2im}(y)\frac{\partial^4 u_{0m}(x,t)}{\partial t^3 \partial x_j}
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
\mathbb{G}_i = & \ \varepsilon k_{ij}(y)M_{\alpha_1\alpha_2}(y)\frac{\partial^3 T_0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_j} + \varepsilon k_{ij}(y)R_{\alpha_1\alpha_2}(y)\frac{\partial^3 u_{0\alpha_1}(x,t)}{\partial t \partial x_{\alpha_2}\partial x_j} \\
& + \varepsilon k_{ij}(y)Q_2(y)\frac{\partial^2 T_0(x,t)}{\partial t \partial x_j}
\end{aligned} \tag{39}
$$

From (35), it is easy to see that the residuals of $T_\Delta^{\varepsilon 2}(x,t)$ and $\mathbf{u}_\Delta^{\varepsilon 2}(x,t)$ are of order $O(\varepsilon)$. It means that the SOTS solutions are equivalent to the solutions of original problem (1) with order $O(\varepsilon)$ in nearly pointwise sense. This is the reason for seeking second-order two-scale expansions.

Summing up, one obtains following theorem

Theorem 2.1. The dynamic thermo-mechanical problem (1) of composite materials with periodic configuration has SOTS approximate solutions as follows

$$
\begin{aligned}
T^\varepsilon(x,t) &\cong T_0(x,t) + \varepsilon M_{\alpha_1}(y)\frac{\partial T_0(x,t)}{\partial x_{\alpha_1}} \\
&+\varepsilon^2\left(M_{\alpha_1\alpha_2}(y)\frac{\partial^2 T_0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} + R_{\alpha_1\alpha_2}(y)\frac{\partial^2 u_{0\alpha_1}(x,t)}{\partial t\partial x_{\alpha_2}} + Q_2(y)\frac{\partial T_0(x,t)}{\partial t}\right)
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
\mathbf{u}^\varepsilon(x,t) &\cong \mathbf{u}_0(x,t) + \varepsilon\left(\mathbf{N}_{\alpha_1}(y)\frac{\partial \mathbf{u}_0(x,t)}{\partial x_{\alpha_1}} - \mathbf{P}_0(y)\left(T_0(x,t)-\tilde{T}\right)\right) \\
&+\varepsilon^2\left(\mathbf{N}_{\alpha_1\alpha_2}(y)\frac{\partial^2 \mathbf{u}_0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} - \mathbf{P}_{\alpha_1}(y)\frac{\partial T_0(x,t)}{\partial x_{\alpha_1}} - \mathbf{F}_2(y)\frac{\partial^2 \mathbf{u}_0(x,t)}{\partial t^2}\right)
\end{aligned} \tag{41}
$$

where $T_0(x,t)$ and $\mathbf{u}_0(x,t)$ are the solutions of the homogenized problem (19), $M_{\alpha_1}(y)$, $\mathbf{N}_{\alpha_1}(y)$ and $\mathbf{P}_0(y)$ are one-order auxiliary functions defined by (9)-(11), $M_{\alpha_1\alpha_2}(y)$, $R_{\alpha_1\alpha_2}(y)$, $Q_2(y)$, $\mathbf{N}_{\alpha_1\alpha_2}(y)$, $\mathbf{P}_{\alpha_1}(y)$ and $\mathbf{F}_2(y)$ are second-order auxiliary functions defined by (25)-(30), respectively.

And then the strains and temperature gradient can be evaluated by the formulas

$$
\begin{aligned}
\varepsilon_{hk}^\varepsilon(x,t) &= \frac{1}{2}\left(\frac{\partial u_h^\varepsilon(x)}{\partial x_k} + \frac{\partial u_k^\varepsilon(x)}{\partial x_h}\right) = \frac{1}{2}\left(\frac{\partial u_{0h}(x)}{\partial x_k} + \frac{\partial u_{0k}(x)}{\partial x_h}\right) \\
&+\frac{1}{2}\left(\frac{\partial N_{\alpha_1 hm}(y)}{\partial y_k} + \frac{\partial N_{\alpha_1 km}(y)}{\partial y_h}\right)\frac{\partial u_{0m}(x)}{\partial x_{\alpha_1}} \\
&+\varepsilon\frac{1}{2}\left(N_{\alpha_1 hm}(y)\frac{\partial^2 u_{0m}(x)}{\partial x_{\alpha_1}\partial x_k} + N_{\alpha_1 km}(y)\frac{\partial^2 u_{0m}(x)}{\partial x_{\alpha_1}\partial x_h}\right) \\
&-\frac{1}{2}\left(\frac{\partial P_{0h}(y)}{\partial y_k} + \frac{\partial P_{0k}(y)}{\partial y_h}\right)\left(T_0(x)-\tilde{T}\right) \\
&-\varepsilon\frac{1}{2}\left(P_{0h}(y)\frac{\partial T_0(x)}{\partial x_k} + P_{0k}(y)\frac{\partial T_0(x)}{\partial x_h}\right) \\
&+\varepsilon\frac{1}{2}\left(\frac{\partial N_{\alpha_1\alpha_2 hm}(y)}{\partial y_k} + \frac{\partial N_{\alpha_1\alpha_2 km}(y)}{\partial y_h}\right)\frac{\partial^2 u_{0m}(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} \\
&-\varepsilon\frac{1}{2}\left(\frac{\partial P_{\alpha_1 h}(y)}{\partial y_k} + \frac{\partial P_{\alpha_1 k}(y)}{\partial y_h}\right)\frac{\partial T_0(x)}{\partial x_{\alpha_1}} \\
&+\varepsilon^2\frac{1}{2}\left(N_{\alpha_1\alpha_2 hm}(y)\frac{\partial^3 u_{0m}(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_k} + N_{\alpha_1\alpha_2 km}(y)\frac{\partial^3 u_{0m}(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_h}\right) \\
&-\varepsilon^2\frac{1}{2}\left(P_{\alpha_1 h}(y)\frac{\partial^2 T_0(x)}{\partial x_{\alpha_1}\partial x_k} + P_{\alpha_1 k}(y)\frac{\partial^2 T_0(x)}{\partial x_{\alpha_1}\partial x_h}\right) \\
&-\varepsilon\frac{1}{2}\left(\frac{\partial F_{2hm}(y)}{\partial y_k} + \frac{\partial F_{2km}(y)}{\partial y_h}\right)\frac{\partial^2 u_{0m}(x)}{\partial t^2} \\
&-\varepsilon^2\frac{1}{2}\left(F_{2hm}(y)\frac{\partial^3 u_{0m}(x)}{\partial x_k\partial t^2} + F_{2km}(y)\frac{\partial^3 u_{0m}(x)}{\partial x_h\partial t^2}\right)
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
\frac{\partial T^{\varepsilon}(x,t)}{\partial x_i} &= \frac{\partial T_0(x)}{\partial x_i} + \frac{\partial M_{\alpha_1}(y)}{\partial y_i}\frac{\partial T_0(x)}{\partial x_{\alpha_1}} + \varepsilon M_{\alpha_1}(y)\frac{\partial^2 T_0(x)}{\partial x_{\alpha_1}\partial x_i} \\
&+\varepsilon\frac{\partial M_{\alpha_1\alpha_2}(y)}{\partial y_i}\frac{\partial^2 T_0(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} + \varepsilon^2 M_{\alpha_1\alpha_2}(y)\frac{\partial^3 T_0(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i} + \varepsilon\frac{\partial Q_2(y)}{\partial y_i}\frac{\partial T_0(x)}{\partial t} \\
&+\varepsilon^2\frac{\partial^2 T_0(x)}{\partial x_i \partial t} + \varepsilon\frac{\partial R_{\alpha_1\alpha_2}(y)}{\partial y_i}\frac{\partial^2 u_{0\alpha_1}(x)}{\partial t\partial x_{\alpha_2}} + \varepsilon^2 R_{\alpha_1\alpha_2}(y)\frac{\partial^3 u_{0\alpha_1}(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i}
\end{aligned}
\tag{43}
$$

According to Hooker's Law, the stresses are calculated by the formula

$$
\sigma_{ij}^{\varepsilon}(x,t) = C_{ijhk}(y)\varepsilon_{hk}^{\varepsilon}(x,t) - \beta_{ij}(y)\left(T^{\varepsilon}(x,t) - \tilde{T}\right)
\tag{44}
$$

3. Algorithms for SOTS analysis method

In this section, the algorithm procedure for the SOTS analysis method, based on the finite difference method in time direction and finite element method in spatial region, for predicting the dynamic thermo-mechanical behaviors is presented.

It can be observed that cell problems (9)-(11) and (25)-(30) are all elliptic boundary value problems, and they can be solved by the standard finite element method to get the FE solutions of the first-order and second-order auxiliary functions. And then the homogenized parameters are evaluated by (13)-(17) based on the FE solutions of first-order auxiliary functions. The homogenized problem (19) is dynamic and strongly coupled by hyperbolic and parabolic equations. So the spatial region Ω is divided by using the FE mesh first, then the temporal domain $(0, \breve{t})$ is divided by using the finite difference. And the heat equation is integrated in time using the backward difference scheme [24], and the Newmark difference scheme [24] is used for the dynamical equation. The algorithm procedure is presented as follows

1. Determine the composite materials and verify the material parameters of basic configuration.
2. Solve problem (9)-(11) by the FEM to get the FE solutions of $M_{\alpha_1}(y)$, $\mathbf{N}_{\alpha_1}(y)$ and $\mathbf{P}_0(y)$, respectively. And the homogenized parameters $\hat{S}$, $\hat{\beta}_{ij}$, $\hat{k}_{ij}$, $\hat{\rho}$ and $\hat{C}_{ijkl}$ can be calculated through (13)-(17), respectively.
3. With the homogenized parameters obtained in step 2, the homogenized solutions $T_0(x,t)$ and $\mathbf{u}_0(x,t)$ can be obtained by solving problem (19) using the FEM and FDM.
4. Solve problems (25)-(30) by using the same FE meshes as in step 2 to get the FE solutions of $M_{\alpha_1\alpha_2}(y)$, $R_{\alpha_1\alpha_2}(y)$, $Q_2(y)$, $\mathbf{N}_{\alpha_1\alpha_2}(y)$, $\mathbf{P}_{\alpha_1}(y)$ and $\mathbf{F}_2(y)$, respectively.
5. Solve the derivatives of the homogenized solutions $T_0(x,t)$ and $\mathbf{u}_0(x,t)$ with respect to spatial and temporal variables. The derivatives with respect to spatial variable are evaluated by the average technique on relative elements [25] and the derivatives with respect to temporal variable are evaluated using the difference schemes in step 3.
6. The displacement and temperature fields can be calculated through (40)-(41), and then from (42)-(44), the strain tensor, temperature gradient and stress tensor are evaluated, respectively.

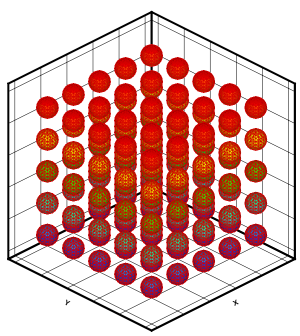
4. Numerical results

To illustrate to the effectiveness of the SOTS method for studying the dynamic thermo-mechanical problem, some numerical results are given here. A brick domain Ω with a 10mm side is considered here shown in Fig.2(a), the periodic cell Y is shown in Fig.2(b), and $\varepsilon = 1/5$. The material properties are listed in Table 1. The brick is clamped on its bottom surface, and the temperature at the bottom surface is kept at 100°C. The initial temperature is at 100°C and the internal heat source h is taken as $500J/cm^3s$. The time step is chosen as $\Delta t = 0.1$.

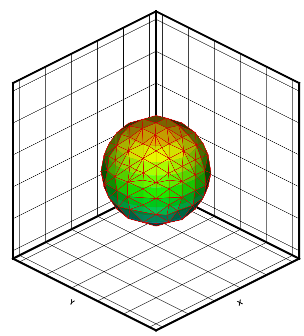
Since it is difficult to find the exact solutions of above problem, we have to take $T^\varepsilon(x,t)$ and $\mathbf{u}^\varepsilon(x,t)$ to be their FE solutions in the very fine mesh for comparison. The tetrahedron partition is implemented and the information of the FE meshes is listed in Table 2. Set

$$e_{T0} = T_e - T_{0e},\ \ e_{T1} = T_e - T_{1e},\ \ e_{T2} = T_e - T_{2e},$$
$$e_{u0} = \mathbf{u}_e - \mathbf{u}_{0e},\ \ e_{u1} = \mathbf{u}_e - \mathbf{u}_{1e},\ \ e_{u2} = \mathbf{u}_e - \mathbf{u}_{2e}$$
$$Terror0 = \frac{|e_{T0}|_{H^1}}{|T_e|_{H^1}},\ Terror1 = \frac{|e_{T1}|_{H^1}}{|T_e|_{H^1}},\ Terror2 = \frac{|e_{T2}|_{H^1}}{|T_e|_{H^1}},$$
$$uerror0 = \frac{|e_{u0}|_{H^1}}{|u_e|_{H^1}},\ uerror1 = \frac{|e_{u1}|_{H^1}}{|u_e|_{H^1}},\ uerror2 = \frac{|e_{u2}|_{H^1}}{|u_e|_{H^1}}$$

Here $|\cdot|_{H^1}$ denotes the semi-norm, T_e, $\mathbf{u}_e$ are the FE solutions of problem (1) in the very fine mesh, T_{0e}, $\mathbf{u}_{0e}$ are the FE solutions of the homogenized equations (19), and T_{1e}, $\mathbf{u}_{1e}$ and T_{2e}, $\mathbf{u}_{2e}$ are the FOTS and SOTS FE solutions, respectively.



(a)



(b)

FIGURE 2. (a) The whole domain Ω (b) The unit cell Y

TABLE 1. Material properties of matrix and particles

Property	Particles	Matrix
Young's modulus(GPa)	117.0	66.2
Poisson's ration	0.333	0.321
Thermal expansion coefficient$(1/K)$	7.11×10^{-6}	10.3×10^{-6}
Mass density(kg/m^3)	5600	4410
Thermal conductivity(W/mK)	2.036	18.1
Specific heat(J/KgK)	615.6	808.3

Fig.3(a)-(c) show the evolution of relative errors in semi-norm between different approximate solutions and FE solutions in the very fine mesh for temperatures and displacements. Fig.4(a)-(h) show the results for T_{0e}, T_{1e}, T_{2e}, T_e and $\mathbf{u}_{0e}$, $\mathbf{u}_{1e}$, $\mathbf{u}_{2e}$, $\mathbf{u}_e$ at the intersection $z = 0.5$ at time $t = 1.0$, respectively. Fig.5(a)-(c) show the numerical results for $\partial T_{1e}/\partial z$, $\partial T_{2e}/\partial z$, $\partial T_e/\partial z$ at the intersection $x = 0.5$, and

TABLE 2. Mesh information

	Original equation	Unit cell	Homogenized equation
Elements	793500	6348	93750
Nodes	141651	1403	17576

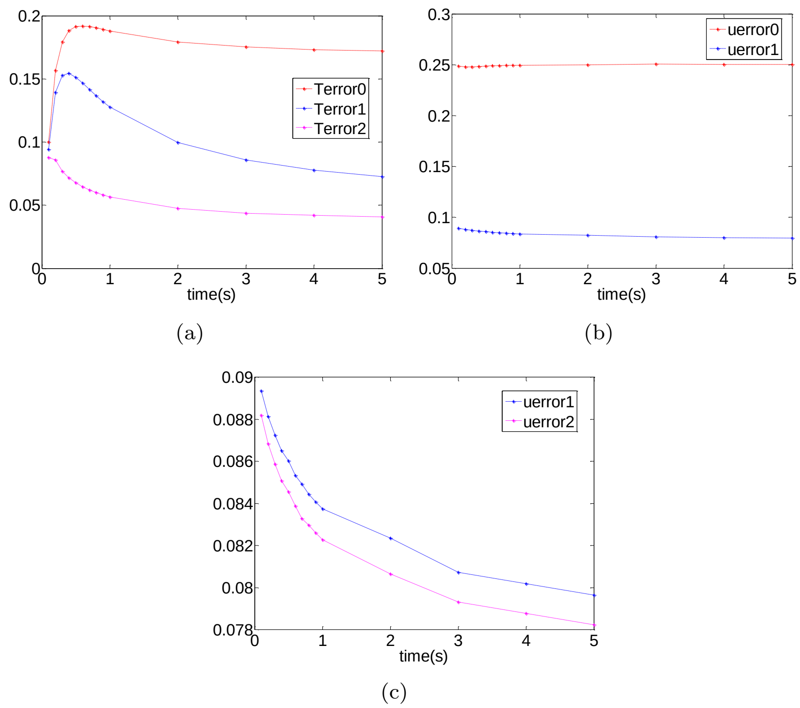


FIGURE 3. (a) $Terror0$, $Terror1$ and $Terror2$ (b) $uerror0$ and $uerror1$ (c) $uerror1$ and $uerror2$

Fig.5(d)-(f) show the results for the axial stresses σ_{1z}, σ_{2z}, σ_{ez} corresponding to $\mathbf{u}_{1e}$, $\mathbf{u}_{2e}$ and $\mathbf{u}_e$ at the intersection $z = 0.5$ at time $t = 1.0$, respectively.

From Table 2, it is clear that the mesh partition numbers of SOTS approximate solutions are much less than that of refined FE solutions. It means that the SOTS method can greatly save computer memory and CPU time, which is very important in engineering computation. From Fig.4 and Fig.5, we can see that the homogenized, FOST and SOTS approximate solutions are in accordance with the FE solutions in the very fine mesh. But Fig.3, Fig.4 and Fig.5 demonstrate that the SOTS approximate solutions are much better than the homogenized solutions and FOTS approximation solutions for temperatures, displacements and their gradient. All the results show that the SOTS method is effective to predict the dynamic thermo-mechanical behaviors of composite materials with periodic configurations.

5. Conclusions

This paper discussed the SOTS method and related numerical algorithm for the dynamic thermo-mechanical problem of periodic composites. The SOTS solutions

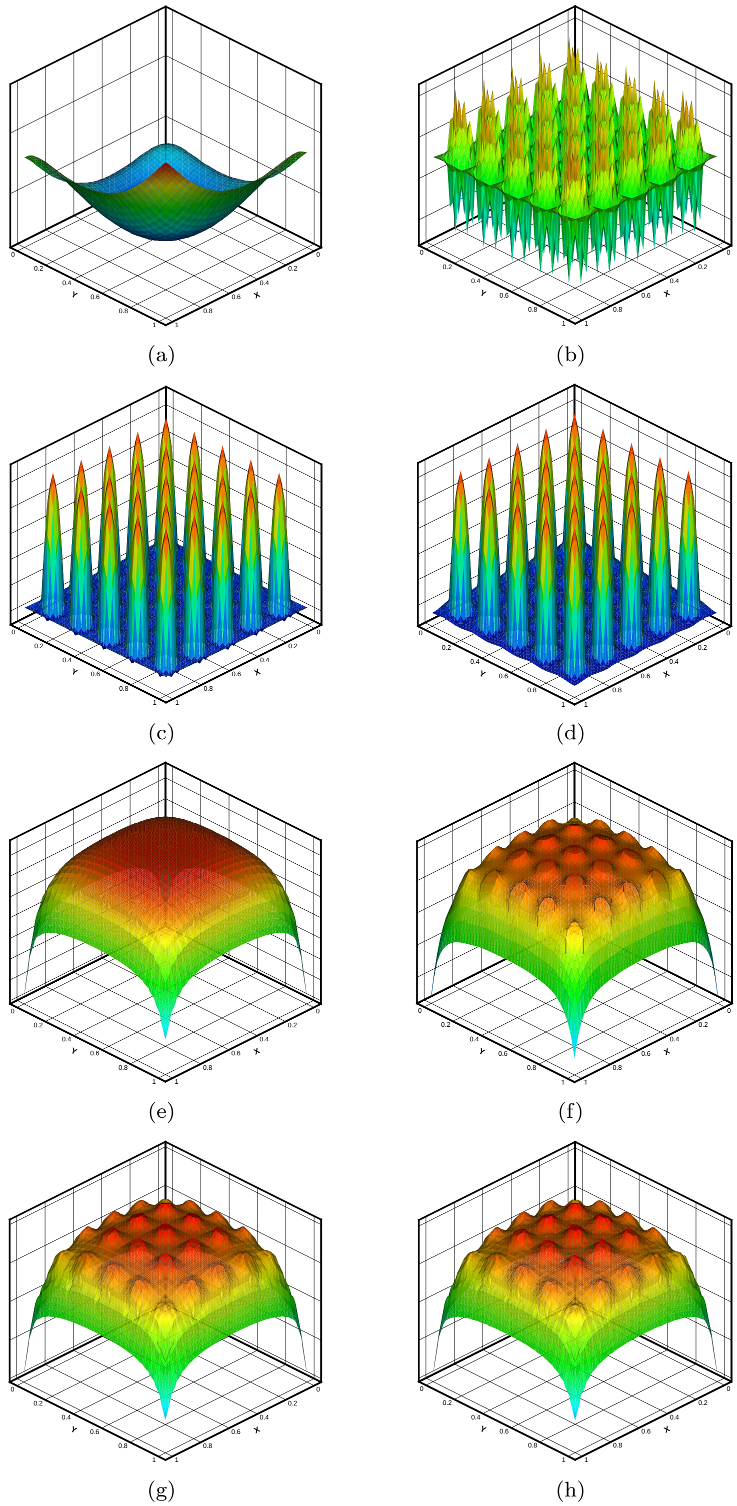


FIGURE 4. (a) T_{0e} (b) T_{1e} (c) T_{2e} (d) T_e (e) $\mathbf{u}_{0e}$ (f) $\mathbf{u}_{1e}$ (g) $\mathbf{u}_{2e}$ (h) $\mathbf{u}_e$

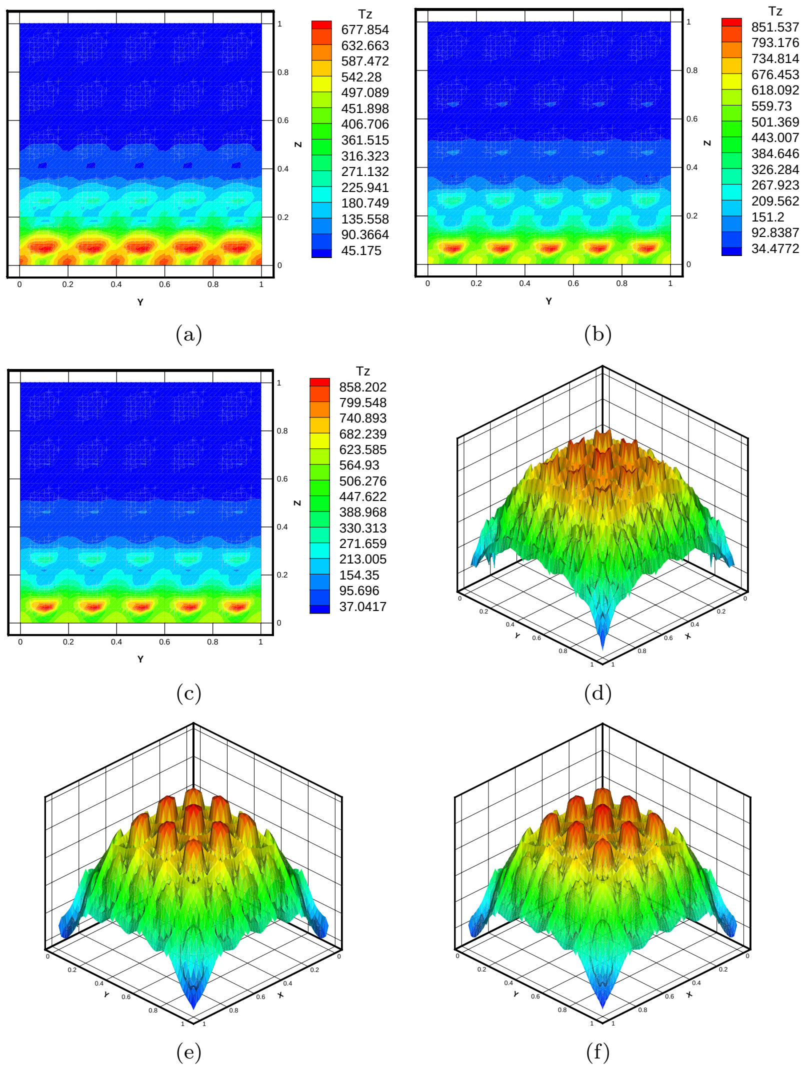


FIGURE 5. (a) $\partial T_{1e}/\partial z$ (b) $\partial T_{2e}/\partial z$ (c) $\partial T_e/\partial z$ (d) σ_{1z} (e) σ_{2z} (f) σ_{ez}

for the dynamic thermo-mechanical problems are presented. Numerical results indicate that the local fluctuation of temperatures, displacements and their gradient can be captured more precisely by considering the second-order correctors. And it also concluded that the SOTS method is not only feasible, but also accurate and efficient to study the dynamic thermo-mechanical problems of composite materials with periodic configurations.

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