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CONVERGENT FINITE DIFFERENCE SCHEME FOR 1D FLOW OF COMPRESSIBLE MICROPOLAR FLUID

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Abstract. In this paper we define a finite difference method for the nonstationary 1D flow of the compressible viscous and heat-conducting micropolar fluid, assuming that it is in the thermodynamical sense perfect and polytropic. The homogeneous boundary conditions for velocity, microrotation and heat flux are proposed. The sequence of approximate solutions for our problem is constructed by using the defined finite difference approximate equations system. We investigate the properties of these approximate solutions and establish their convergence to the strong solution of our problem globally in time, which is the main results of the paper. A numerical experiment is performed by solving the defined approximate ordinary differential equations system using strong-stability preserving (SSP) Runge-Kutta scheme for time discretization.

Key words. micropolar fluid flow, initial-boundary value problem, finite difference approximations, strong and weak convergence.

1. Introduction

The theory of micropolar fluid was introduced by A. C. Eringen in 1960, [8]. Eringen suggested many possible applications of the micropolar fluid, but from the mathematical point of view the theory is still in the early stage of development. The results for incompressible flow are very well systematized in the book of Lukaszewicz,[11] but the theory for compressible flows, especially for the flows involving temperature, is still in the beginning.

In this paper we focus on the compressible flow of the isotropic, viscous, and heat conducting micropolar fluid, which is in thermodynamical sense perfect and polytropic. The model for this type of flow was first considered by Mujaković in [12] where she developed a one-dimensional model. The model is quite complex from numerical point of view, as well as from theoretical standpoint. It consists of four partial differential equations - one of which is a differential equation of the first order, and the other three are non-linear parabolic equations of second order. In the work [13] the local existence and uniqueness of the solution, which is called generalized, for our model with the homogeneous boundary conditions for velocity, microrotation and heat flux were proved, while in [13] Mujaković proved the existence of global in time solution for the described problem. So far, the numerical analysis of this model was done only by Faedo-Galerkin method [12, 7, 15] that is unsuitable for wider application.

The main goal of this paper is to propose a numerical method for solving a given model using the finite difference approach, which is more acceptable in practical applications. We define the semidiscrete finite difference approximate equations system and investigate the properties of the sequence of the approximate solutions. We prove that the limit of this sequence is the solution to our problem and that it has the same properties as the solution in [12]. In this way the convergence

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of the corresponding numerical scheme is established and furthermore, the global existence of the solution for the considered problem, already proved in [13], verified. In our work we follow some ideas of [3, 4].

Other authors who have discussed various models of fluid by using finite differences mainly don't analyze the problem of convergence of approximate solutions from a theoretical point of view. The approach used here can be applied in other research models based on similar systems of partial differential equations.

The paper is organized as follows. In the second section we introduce the mathematical formulation of our problem. In the third section we derive the finite difference approximate equations system and in the fourth section present the main result. In Sections 5-8, we prove uniform a priori estimates for the approximate solutions. Proof of convergence of a sequence of approximate solutions to a solution of our problem is given in the ninth section. Finally, in the tenth section we perform the numerical experiment.

2. Mathematical model

We are dealing with the one-dimensional flow of the compressible viscous and heat-conducting micropolar fluid flow, which is thermodynamically perfect and polytropic. Let ρ, v, w and θ denote, respectively, the mass density, velocity, microrotation velocity and temperature in the Lagrangian description. The motion of the fluid under consideration is described by the following system of four equations (see, for example, [12]):

$$\partial_t\rho + \rho^2\partial_x v = 0, \tag{2.1}$$

$$\partial_t v = \partial_x(\rho\,\partial_x v) - K\partial_x(\rho\,\theta), \tag{2.2}$$

$$\rho\,\partial_t\omega = A\left[\rho\,\partial_x(\rho\,\partial_x\omega) - \omega\right], \tag{2.3}$$

$$\rho\,\partial_t\theta = -K\rho^2\theta\,\partial_x v + \rho^2(\partial_x v)^2 + \rho^2(\partial_x\omega)^2 + \omega^2 + D\rho\,\partial_x(\rho\,\partial_x\theta)\,. \tag{2.4}$$

The system is considered in the domain $Q_T = (0,1)\times(0,T)$, where $T > 0$ is arbitrary; K, A and D are positive constants. Equations (2.1)-(2.4) are, respectively, local forms of the conservation laws for the mass, momentum, momentum moment and energy. We take the following non-homogeneous initial conditions:

$$\rho(x,0) = \rho_0(x),\ v(x,0) = v_0(x),\ \omega(x,0) = \omega_0(x),\ \theta(x,0) = \theta_0(x), \tag{2.5}$$

and homogeneous boundary conditions:

$$v(0,t) = v(1,t) = 0, \qquad \omega(0,t) = \omega(1,t) = 0 \tag{2.6}$$

$$\partial_x\theta(0,t) = \partial_x\theta(1,t) = 0, \tag{2.7}$$

for $x \in (0,1)$ and $t \in (0,T)$. Here ρ_0, v_0, ω_0 and θ_0 are given functions. We assume that there exists a constant $m \in \mathrm{R}^+$ such that

$$\rho_0(x) \geq m, \quad \theta_0(x) \geq m \quad \text{for } x \in (0,1). \tag{2.8}$$

Let the initial data (2.5) have the following properties of smoothness

$$\rho_0, \theta_0 \in H^1((0,1)) \quad \text{and} \quad v_0, \omega_0 \in H_0^1((0,1)). \tag{2.9}$$

Because of embedding $H^1((0,1))$ into $C([0,1])$, it is easy to check that there exists $M \in \mathrm{R}^+$ such that

$$\rho_0(x), |v_0(x)|, |\omega_0(x)|, \theta_0(x) \leq M, \quad \text{for } x \in [0,1]. \tag{2.10}$$

Under the stated assumptions (2.8)-(2.9) in the previous papers [12, 13] is proven that problem (2.1)-(2.7) has unique solution $(\rho, v, \omega, \theta)$ in the domain Q_T, for every

$T > 0$, with the following properties:

$$\rho \in L^\infty(0,T;H^1((0,1))) \cap H^1(Q_T), \tag{2.11}$$

$$v,\omega,\theta \in L^\infty(0,T;H^1((0,1))) \cap H^1(Q_T) \cap L^2(0,T;H^2((0,1))), \tag{2.12}$$

$$\rho > 0, \theta > 0 \quad \text{on } \bar{Q}_T. \tag{2.13}$$

These results were obtained by using the Faedo–Galerkin method for a local existence theorem and the principle of extension for a global existence theorem. From embedding and interpolation theorems (e.g. see [5, 6]) we can conclude that from (2.11) and (2.12) it follows:

$$\rho \in L^\infty(0,T;C([0,1])) \cap C([0,T];L^2((0,1))), \tag{2.14}$$

$$v,\omega,\theta \in L^2(0,T;C^1([0,1])) \cap C([0,T];H^1((0,1))), \tag{2.15}$$

$$v,\omega,\theta \in C(\bar{Q}_T). \tag{2.16}$$

Also, in article [14] we have a proof that the solution $(\rho, v, \omega, \theta)$ converges to the stationary constant solution $(\alpha^{-1}, 0, 0, E_1)$ in the space $(H_1((0,1)))^4$ (when $t \to \infty$), where

$$\alpha = \int_0^1 \frac{1}{\rho_0(x)} dx, \quad E_1 = \frac{1}{2}\|v_0\|^2 + \frac{1}{2A}\|\omega_0\|^2 + \|\theta_0\|^2_{L^1((0,1))}. \tag{2.17}$$

$(\| \| = \| \|_{L^2((0,1))})$.

3. Finite-difference spatial discretization and approximate solutions

In this section we introduce a space discrete difference scheme in order to obtain appropriate approximate system of the equation system (2.1)-(2.7). We construct semi-discrete finite difference approximate solutions on a uniform staggered grid. In making a discrete scheme we use some ideas from [4] and [3].

Let h be an increment in x such that $Nh = 1$ for $N \in \mathrm{Z}^+$. The staggered grid points are denoted with $x_k = kh$, $k \in \{0, 1, \dots, N\}$ and $x_j = jh$, $j \in \left\{\frac{1}{2}, \dots, N - \frac{1}{2}\right\}$. For each integer N, we construct the following time dependent functions

$$\rho_j(t), \theta_j(t), \quad j = \tfrac{1}{2}, \dots, N - \tfrac{1}{2}, \tag{3.1}$$

$$v_k(t), \omega_k(t), \quad k = 0, 1, \dots, N, \tag{3.2}$$

that form a discrete approximation to the solution at defined grid points

$$\rho(x_j,t), \theta(x_j,t), \quad j = \tfrac{1}{2}, \dots, N - \tfrac{1}{2},$$

$$v(x_k,t), \omega(x_k,t), \quad k = 0, 1, \dots, N.$$

First, the functions $\rho_j(t), v_k(t), \omega_k(t), \theta_j(t)$, $j = \frac{1}{2}, \dots, N - \frac{1}{2}$, $k = 1, \dots, N-1$, are determined by using appropriate spatial discretization of equation system (2.1)-(2.4):

$$\dot{\rho}_j = -\rho_j^2 \delta v_j \tag{3.3}$$

$$\dot{v}_k = \delta(\rho\delta v)_k - K\delta(\rho\theta)_k \tag{3.4}$$

$$\rho_k \dot{\omega}_k = A\left[\rho_k \delta(\rho\delta\omega)_k - \omega_k\right] \tag{3.5}$$

$$\rho_j \dot{\theta}_j = -K\rho_j^2 \theta_j \delta v_j + \rho_j^2 (\delta v_j)^2 + \rho_j^2 (\delta\omega_j)^2 + \omega_j^2 + D\rho_j \delta(\rho\delta\theta)_j \tag{3.6}$$

where $j = \frac{1}{2}, \dots, N - \frac{1}{2}$ and $k = 1, \dots, N-1$. δ is the operator defined with

$$\delta g_l = \frac{g_{l+\frac{1}{2}} - g_{l-\frac{1}{2}}}{h}, \tag{3.7}$$

for $l=j$ or $l=k$. For $k\in\{1,\ldots,N\}$ and $j\in\{\frac{1}{2},\ldots,N-\frac{1}{2}\}$, the functions ρ_k, θ_k and v_j, ω_j we define by

$$\rho_k=\rho_{k-\frac{1}{2}},\quad \theta_k=\theta_{k-\frac{1}{2}}\quad\text{and}\quad v_j=v_{j+\frac{1}{2}},\quad \omega_j=\omega_{j+\frac{1}{2}}. \tag{3.8}$$

Equations (3.3)-(3.6) are ordinary differential equations.

Taking into account the boundary conditions (2.6)-(2.7), we define

$$v_0(t)=v_N(t)=0,\ \omega_0(t)=\omega_N(t)=0, \tag{3.9}$$

$$\delta\theta_0(t)=\delta\theta_N(t)=0. \tag{3.10}$$

Now the system (3.3)-(3.6) with (3.9)-(3.10) contains $4N+2$ equations for $4N+2$ unknown functions.

The initial conditions are defined in accordance with given initial conditions (2.5) as:

(3.11)

$$(\rho_j,\theta_j)(0)=\left(\frac{1}{h}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}\rho_0(x)dx,\frac{1}{h}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}\theta_0(x)dx\right),\ j\in\left\{\tfrac{1}{2},\ldots,N-\tfrac{1}{2}\right\},$$

(3.12)

$$(v_k,\omega_k)(0)=\left(\frac{1}{h}\int_{(k-1)h}^{kh}v_0(x)dx,\frac{1}{h}\int_{(k-1)h}^{kh}\omega_0(x)dx\right),\ k\in\{1,\ldots,N-1\},$$

and

$$v_0(0)=v_N(0)=0\quad \omega_0(0)=\omega_N(0)=0,\quad\text{and}\quad \delta\theta_0(0)=\delta\theta_N(0)=0. \tag{3.13}$$

It is easy to see, that from (2.8) and (2.10) it follows

$$m\le\rho_j(0),\ \theta_j(0)\le M,\quad j=\tfrac{1}{2},\ldots,N-\tfrac{1}{2}. \tag{3.14}$$

and

$$\frac{1}{M}\le\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\frac{h}{\rho_j(0)}\le\frac{1}{m}. \tag{3.15}$$

Averaging of the initial conditions (3.11) and (3.12) was necessary for obtaining the following estimates based on the smoothness properties (2.9). Using (2.9) one can conclude that

$$\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\left|(\rho_j,\theta_j)\right|^2(0)h\le C,\quad \sum_{k=0}^{N}\left|(v_k,\omega_k)\right|^2(0)h\le C, \tag{3.16}$$

$$\sum_{k=1}^{N-1}\left|(\delta\rho_k,\delta\theta_k)\right|^2(0)h\le C,\quad \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\left|(\delta v_j,\delta\omega_j)\right|^2(0)h\le C, \tag{3.17}$$

and

$$\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\left|(\rho_j,\theta_j)\right|^4(0)h\le C,\quad \sum_{k=1}^{N}\left|(v_k,\omega_k)\right|^4(0)h\le C, \tag{3.18}$$

where $C>0$ is a constant, which depends on initial functions and not on step h (or N).

¿From the basic theory of differential equations and the local existence theorem, it is known that there exists a smooth solution of the Cauchy problem (3.3)-(3.6), (3.9)-(3.10) with the initial conditions (3.11)-(3.13) locally on some time interval $[0,T)$, $T>0$ ([1, 16]). Because of positivity of the initial conditions (see (3.14)) and smoothness of the solution on the considered interval, we can choose such T so that

$$0<\rho_j(t),\theta_j(t)<\infty,\quad j=\tfrac{1}{2},\dots,N-\tfrac{1}{2} \tag{3.19}$$

$$|v_k(t)|,\ |\omega_k(t)|<\infty,\quad k=0,\dots,N \tag{3.20}$$

for $t\in[0,T)$. Let $[0,T_{max})$ be the maximal time interval on which the smooth solution satisfying (3.19) and (3.20) exists. Our first goal is to show that the solution is globally defined on $[0,\infty)$, i.e., that $T_{max}=\infty$. We will achieve this by showing, for fixed $h>0$, the boundedness of the mass density, the velocity, the microrotation velocity and the temperature, as well as the lower boundedness of the density and the temperature away from zero (see Section 5). ¿From that, we conclude that the solution $(\rho_j,v_k,\omega_k,\theta_j)$, $j=\frac{1}{2},\dots,N-\frac{1}{2},k=0,\dots,N$ can be defined globally in time.

Now, using the solution of the Cauchy problem (3.3)-(3.6), (3.9)-(3.13) we construct for $t\geq 0$ the following approximate functions.

For each fixed N, $x\in(\frac{1}{N}[xN],\frac{1}{N}([xN]+1)]$, we define

$$v^N(x,t) = v_{[xN]}(t)+(xN-[xN])(v_{[xN]+1}(t)-v_{[xN]}(t)), \tag{3.21}$$

$$\omega^N(x,t) = \omega_{[xN]}(t)+(xN-[xN])(\omega_{[xN]+1}(t)-\omega_{[xN]}(t)) \tag{3.22}$$

and similarly for $x\in(\frac{1}{N}([xN+\frac{1}{2}]-\frac{1}{2}),\frac{1}{N}([xN+\frac{1}{2}]+\frac{1}{2})]$, we define

$$\rho^{N-\frac{1}{2}}(x,t) = \rho_{[xN+\frac{1}{2}]-\frac{1}{2}}(t)+(xN-([xN+\tfrac{1}{2}]-\tfrac{1}{2}))(\rho_{[xN+\frac{1}{2}]+\frac{1}{2}}(t)-\rho_{[xN+\frac{1}{2}]-\frac{1}{2}}(t)), \tag{3.23}$$

$$\theta^{N-\frac{1}{2}}(x,t) = \theta_{[xN+\frac{1}{2}]-\frac{1}{2}}(t)+(xN-([xN+\tfrac{1}{2}]-\tfrac{1}{2}))(\theta_{[xN+\frac{1}{2}]+\frac{1}{2}}(t)-\theta_{[xN+\frac{1}{2}]-\frac{1}{2}}(t)). \tag{3.24}$$

For $x\in[0,\frac{1}{2N}]$ we take

$$\rho^{N-\frac{1}{2}}(x,t)=\rho_{\frac{1}{2}}(t),\quad \theta^{N-\frac{1}{2}}(x,t)=\theta_{\frac{1}{2}}(t)$$

and for $x\in(1-\frac{1}{2N},1]$

$$\rho^{N-\frac{1}{2}}(x,t)=\rho_{N-\frac{1}{2}}(t),\quad \theta^{N-\frac{1}{2}}(x,t)=\theta_{N-\frac{1}{2}}(t).$$

We also introduce the corresponding step functions:

$$(v_h,\omega_h)(x,t)=(v_{[xN]},\omega_{[xN]})(t),\ \ x\in(\tfrac{1}{N}[xN],\tfrac{1}{N}([xN]+1)], \tag{3.25}$$

$$(\rho_{h-\frac{1}{2}},\theta_{h-\frac{1}{2}})(x,t) = (\rho_{[xN+\frac{1}{2}]-\frac{1}{2}},\theta_{[xN+\frac{1}{2}]-\frac{1}{2}})(t), \quad x\in(\tfrac{1}{N}([xN+\tfrac{1}{2}]-\tfrac{1}{2}),\tfrac{1}{N}([xN+\tfrac{1}{2}]+\tfrac{1}{2})], \tag{3.26}$$

$$(\rho_{h-\frac{1}{2}},\theta_{h-\frac{1}{2}})(x,t)=(\rho_{\frac{1}{2}},\theta_{\frac{1}{2}})(t),\ \ x\in[0,\tfrac{1}{2N}], \tag{3.27}$$

$$(\rho_{h-\frac{1}{2}},\theta_{h-\frac{1}{2}})(x,t)=(\rho_{N-\frac{1}{2}},\theta_{N-\frac{1}{2}})(t),\ \ x\in(1-\tfrac{1}{2N},1]. \tag{3.28}$$

In this section the semi-discrete finite difference scheme resulting with the system of ordinary differential equations is defined. In what follows the convergence of this scheme will be proved. For determining the solution of the system (3.3)-(3.6), (3.9)-(3.10) numerically, the time discretization should be performed. The time discretization algorithm used in the considered numerical experiment is briefly described in Section 10.

4. The main result

The aim of this paper is to prove the following statements.

Theorem 4.1. *Suppose that the initial data* $(\rho_0, v_0, \omega_0, \theta_0)$ *satisfy properties (2.8)-(2.9). Then there exist subsequences of approximate solutions (still denoted)* $\{(\rho^{N-\frac{1}{2}}, v^N, \omega^N, \theta^{N-\frac{1}{2}})\}$ *and* $\{(\rho_{h-\frac{1}{2}}, v_h, \omega_h, \theta_{h-\frac{1}{2}})\}$ *in the domain* Q_T *(for each* $T \in \mathrm{R}^+$*) such that, as* $N \to \infty$ *(or* $h \to 0$*),*

$$(\rho^{N-\frac{1}{2}}, v^N, \omega^N, \theta^{N-\frac{1}{2}}) \to (\rho, v, \omega, \theta) \quad \text{strongly in } (C(\overline{Q}_T))^4, \tag{4.1}$$

$$\text{* weakly in } (L^\infty(0,T; H^1((0,1))))^4, \tag{4.2}$$

$$\text{weakly in } (H^1(Q_T))^4, \tag{4.3}$$

$$(v^N, \omega^N, \theta^{N-1}) \to (v, \omega, \theta) \quad \text{weakly in } (L^2(0,T; H^2((0,1))))^3, \tag{4.4}$$

$$(\rho_{h-\frac{1}{2}}, v_h, \omega_h, \theta_{h-\frac{1}{2}}) \to (\rho, v, \omega, \theta) \quad \text{strongly in } (L^\infty(0,T; L^2((0,1))))^4. \tag{4.5}$$

The function $(\rho, v, \omega, \theta)$ *satisfies equations (2.1)-(2.4) a.e. in* Q_T*, conditions (2.5)-(2.7) in the sense of traces and* ρ *and* θ *have the properties*

$$\inf_{Q_T} \rho > 0, \quad \inf_{Q_T} \theta > 0. \tag{4.6}$$

Notice that the function $(\rho, v, \omega, \theta)$ introduced in Theorem 4.1 is the solution of the same problem as the function introduced in Section 2. Moreover, the properties of the function $(\rho, v, \omega, \theta)$ coincide with the properties (2.11)-(2.16). Indeed, our goal was to obtain the same solution, but using the different approach.

The proof of Theorem 4.1 is essentially based on a careful examination of a priori estimates and limit procedure. We first study, for each N, the approximate problem (3.3)-(3.6), (3.9)-(3.13) and derive the a priori estimates for its solution independent of N (or h) by utilizing a technique of articles [4] and [3]. Using the obtained a priori estimates and results of weak and strong compactness [5, 6], we extract the subsequences of approximate solutions, which, when N tends to infinity (or $h \to 0$), has the limit in the strong or weak sense on the domain $(0,1) \times (0,T)$, where $T > 0$ is arbitrary. Finally, we show that this limit is the solution to our problem. In this article we use some ideas from [2].

The proof of our theorem is a direct consequence of the results that we obtain in the following sections.

5. Basic estimates and global construction of the difference scheme

Throughout this paper, we denote by $C > 0$ or $C_i > 0$ $(i = 1, 2, \ldots)$ generic constants independent of N, having possibly different values at different places.

In order to construct a global differential scheme, in this section we make some key estimations for $(\rho_j, v_k, \omega_k, \theta_j)(t)$, $j = \frac{1}{2}, \ldots, N - \frac{1}{2}$, $k = 1, \ldots, N-1$.

Lemma 5.1. *There exists $C > 0$ such that, for $T > 0$ arbitrary and for all $t \in [0,T]$, it holds*

$$K\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\Phi(\frac{1}{\rho_j})h+\frac{1}{2}\sum_{k=1}^{N-1}v_k^2h+\frac{1}{2A}\sum_{k=1}^{N-1}\omega_k^2h+\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\Phi\left(\theta_j\right)h$$
$$+\int_0^t\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\frac{\rho_j}{\theta_j}(\delta v_j)^2h\ d\tau+\int_0^t\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\frac{\rho_j}{\theta_j}(\delta\omega_j)^2h\ d\tau+ \tag{5.1}$$
$$+\int_0^t\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\frac{\omega_j^2}{\rho_j\theta_j}h\ d\tau+D\int_0^t\sum_{k=1}^{N-1}\frac{\rho_k(\delta\theta_k)^2}{\theta_{k-\frac{1}{2}}\theta_{k+\frac{1}{2}}}h\ d\tau\le C,$$

where $\Phi(w)=w-1-\ln w$ is a nonnegative convex function.

Proof. Multiplying (3.3), (3.4), (3.5) and (3.6), respectively, by $K(\rho_j-1)\rho_j^{-2}h$, v_kh, $A^{-1}\rho_k^{-1}\omega_kh$ and $\rho_j^{-1}(1-\frac{1}{\theta_j})h$, summing over $j=\frac{1}{2},\dots,N-\frac{1}{2}$ and $k=1,\dots,N-1$ and adding the obtained equations, we get

$$\frac{d}{dt}\left[K\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\Phi(\frac{1}{\rho_j})h+\frac{1}{2}\sum_{k=1}^{N-1}v_k^2h+\frac{1}{2A}\sum_{k=1}^{N-1}\omega_k^2h+\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\Phi\left(\theta_j\right)h\right] \tag{5.2}$$
$$+\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\frac{\rho_j}{\theta_j}(\delta v_j)^2h+\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\frac{\rho_j}{\theta_j}(\delta\omega_j)^2h+\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\frac{\omega_j^2}{\rho_j\theta_j}h+D\sum_{k=1}^{N-1}\frac{(\delta\theta_k)^2\rho_k}{\theta_{k-\frac{1}{2}}\theta_{k+\frac{1}{2}}}h=0,$$

Taking into account (3.14) we easily see that

$$\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\Phi(\frac{1}{\rho_j(0)})h\le C,\quad \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\Phi(\theta_j(0))h\le C, \tag{5.3}$$

uniformly by N. Integrating (5.2) over $[0,t]$, $t\le T$, and using (3.16) and (5.3), we immediately get (5.1). □

In the same way as in [4], this results verifies the existence of solution to the Cauchy problem (3.3)-(3.6), (3.9)-(3.13) for all time, that is $T_{max}=\infty$. Indeed, for fixed $h>0$, the estimation

$$\Phi(\frac{1}{\rho_j(t)})+v_k(t)^2+\frac{1}{2A}\omega_k(t)^2+\Phi\left(\theta_j(t)\right)\le\frac{C}{h},$$

implies the global bounds of the functions $(\rho_j,v_k,\omega_k,\theta_j)$:

$$0<\Phi_-^{-1}(\frac{C}{h})\le\rho_j^{-1}(t),\ \theta_j(t)\le\Phi_+^{-1}(\frac{C}{h})<\infty,$$

and

$$|v_k(t)|<\frac{C}{\sqrt{h}},\quad |\omega_k(t)|<\frac{C}{\sqrt{h}},$$

where $\Phi_\pm^{-1}$ denote the two branches of the inverse function of Φ defined on $(0,1]$ and $[1,\infty)$, respectively. In this case, $(\rho_j,v_k,\omega_k,\theta_j)$ can be locally extended beyond the maximal time interval $[0,T_{max})$, that is a contradiction unless $T_{max}=\infty$.

Hence, we have our construction of the difference scheme $(\rho_j,v_k,\omega_k,\theta_j)(t)$ and the corresponding approximate solutions

$$(\rho^{N-\frac{1}{2}},v^N,\omega^N,\theta^{N-\frac{1}{2}})(x,t)\quad\text{and}\quad(\rho_{h-\frac{1}{2}},v_h,\omega_h,\theta_{h-\frac{1}{2}})(x,t)$$

defined on $[0,T]$, for each $T>0$.

The estimations obtained in the next three lemmas do not depend of N and T also.

Lemma 5.2. *For each $t\in[0,T]$, $T>0$ arbitrary, it holds*

$$\frac{1}{M}\leq\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\frac{h}{\rho_j(t)}\leq\frac{1}{m} \tag{5.4}$$

(where $m,M\in\mathrm{R}^+$ are as in (3.14)).

Proof. Multiplying (3.3) by $\rho_j^{-2}h$ and summing over j, we have

$$\frac{d}{dt}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\frac{h}{\rho_j}=\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\delta v_j\ h=0. \tag{5.5}$$

Integrating (5.5) over $[0,t]$ and using (3.15) we easily get (5.4). □

Lemma 5.3. *There exists a constant $C>0$ such that, for $t\in[0,T]$, it holds*

$$\frac{1}{2}\sum_{k=1}^{N-1}v_k^2(t)h+\frac{1}{2A}\sum_{k=1}^{N-1}\omega_k^2(t)h+\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\theta_j(t)h\leq C. \tag{5.6}$$

Proof. Multiplying (3.4), (3.5) and (3.6), respectively, by v_kh, $\rho_k^{-1}A^{-1}\omega_kh$ and $\rho_k^{-1}h$, summing over j and k, taking into account (3.8)-(3.10) after addition of the obtained equations, we get the following equality

$$\frac{d}{dt}\left(\frac{1}{2}\sum_{k=1}^{N-1}v_k^2h+\frac{1}{2A}\sum_{k=1}^{N-1}\omega_k^2h+\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\theta_jh\right)=0. \tag{5.7}$$

Integrating over $[0,t]$ and using (3.16), from (5.7) follows (5.6). □

Lemma 5.4. *There exists $C>0$ such that, for $t\in[0,T]$, it holds*

$$\frac{1}{2A}\sum_{k=1}^{N-1}\omega_k^2h+\int_0^t\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\rho_j(\delta\omega_j)^2hd\tau+\int_0^t\sum_{k=1}^{N-1}\frac{\omega_k^2}{\rho_k}hd\tau\leq C, \tag{5.8}$$

where ρ_k is defined by (3.8).

Proof. We multiply (3.5) by $\rho_k^{-1}A^{-1}\omega_kh$ and sum over k. After integration over $[0,t]$ and using (3.16), we get immediately (5.8). □

6. Boundedness of the density

In the following sections we make estimates for the difference scheme at some fixed interval $[0,T]\subset[0,\infty)$. Now, we establish the uniform bounds for the density, which are essential for our proof of the main results.

Lemma 6.1. *There exist constants $C_1,C_2\in\mathrm{R}^+$ such that, for all $t\in[0,T]$*

$$C_1\leq\rho_j(t)\leq C_2. \tag{6.1}$$

Proof. ¿From (5.1) we have

$$\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \Phi(\frac{1}{\rho_j(t)}) + \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \Phi\left(\theta_j(t)\right) \leq \frac{C}{h} = C\,N,$$

which implies that there exists at least one $a_t \in \{\frac{1}{2}, \dots, N-\frac{1}{2}\}$ such that, for all $t \in [0,T]$

$$C^{-1} \leq \rho_{a_t}(t) \leq C, \quad C^{-1} \leq \theta_{a_t}(t) \leq C, \tag{6.2}$$

where $C \in \mathrm{R}^+$. Inserting (3.3) into (3.4) we get

$$\dot{v}_k = -\frac{d}{dt}(\delta \ln \rho)_k - K\delta(\rho\theta)_k, \quad k = 1, \dots, N-1.$$

After integration over $[0,t]$, multiplying by h and summation, it takes the form

$$\begin{aligned}\sum_{m=a_t+\frac{1}{2}}^{k}(v_m(t) - v_m(0))h &= -\ln\rho_{k+\frac{1}{2}}(t) + \ln\rho_{a_t}(t) + \ln\rho_{k+\frac{1}{2}}(0) - \ln\rho_{a_t}(0) - \\ &- K\int_0^t \rho_{k+\frac{1}{2}}\theta_{k+\frac{1}{2}}d\tau + K\int_0^t \rho_{a_t}\theta_{a_t}d\tau,\end{aligned}$$

i.e.,

$$\begin{aligned}&\exp\left\{-\sum_{m=a_t+\frac{1}{2}}^{k} v_m(t)h + \sum_{m=a_t+\frac{1}{2}}^{k} v_m(0)h\right\} = \\ &= \frac{\rho_{k+\frac{1}{2}}(t)\rho_{a_t}(0)}{\rho_{a_t}(t)\rho_{k+\frac{1}{2}}(0)}\exp\left\{K\int_0^t \rho_{k+\frac{1}{2}}\theta_{k+\frac{1}{2}}d\tau - K\int_0^t \rho_{a_t}\theta_{a_t}d\tau\right\},\end{aligned} \tag{6.3}$$

where we have used a convention notation $\sum_{m=x}^{y} = -\sum_{m=y}^{x}$ for the case $y < x$.

We define the discrete Kazhikov-Shelukhin type of functions by

$$Y(t) = \exp\left\{K\int_0^t \rho_{a_t}(\tau)\theta_{a_t}(\tau)d\tau\right\}, \tag{6.4}$$

$$B_k(t) = \exp\left\{-\sum_{m=a_t+\frac{1}{2}}^{k} v_m(t)h + \sum_{m=a_t+\frac{1}{2}}^{k} v_m(0)h\right\} \tag{6.5}$$

Inserting (6.4)–(6.5) into (6.3) and multiplying by $K\rho_{k+\frac{1}{2}}(t)\theta_{k+\frac{1}{2}}(t)$ we get, for $k = 1, \dots, N-1$, that

$$KB_k(t)Y(t)\frac{\rho_{a_t}(t)\rho_{k+\frac{1}{2}}(0)\theta_{k+\frac{1}{2}}(t)}{\rho_{a_t}(0)} = \frac{d}{dt}\left(\exp\left\{K\int_0^t \rho_{k+\frac{1}{2}}\theta_{k+\frac{1}{2}}d\tau\right\}\right).$$

Integrating over $[0,t]$ and using (6.3) again, we obtain

$$K\frac{\rho_{k+\frac{1}{2}}(0)}{\rho_{a_t}(0)}\int_0^t B_k(\tau)Y(\tau)\rho_{a_t}(\tau)\theta_{k+\frac{1}{2}}(\tau)d\tau = \frac{\rho_{a_t}(t)\rho_{k+\frac{1}{2}}(0)}{\rho_{k+\frac{1}{2}}(t)\rho_{a_t}(0)}B_k(t)Y(t) - 1. \tag{6.6}$$

Notice that, because of (5.1) and (3.16), it holds

$$\Big| -\sum_{a_t+\frac{1}{2}}^{k} v_m(t)h + \sum_{a_t+\frac{1}{2}}^{k} v_m(0)h\Big| \leq \left(\sum_{k=1}^{N-1} v_k^2(t)h\right)^{1/2} + \left(\sum_{k=1}^{N-1} v_k^2(0)h\right)^{1/2} \leq C,$$

thus, we conclude that there exists $C \in \mathrm{R}^+$ such that

$$C^{-1} \leq B_k(t) \leq C, \tag{6.7}$$

for all $t \in [0,T]$ and $k = 1, \dots, N-1$. We can easily see that $Y(t) > 1$, $t \in [0,T]$. Let be

$$\hat{Y}(t) = \frac{Y(t)}{\rho_{a_t}(0)}.$$

Then equation (6.6), with the help of (6.7) and (6.2) gives

$$\frac{\hat{Y}(t)}{\rho_{k+\frac{1}{2}}(t)} \leq \frac{C}{\rho_{k+\frac{1}{2}}(0)} + \int_0^t \theta_{k+\frac{1}{2}}(\tau)\hat{Y}(\tau)d\tau.$$

Multiplying the above inequality by h, summing up for $k = 0, \dots, N-1$, and using estimates (5.4), (3.15) and (5.6), we obtain

$$\hat{Y}(t) \leq C\left(1 + \int_0^t \hat{Y}(\tau)d\tau\right),$$

from which, after applying the Gronwall inequality, follows $\hat{Y}(t) \leq C$, $t \in [0,T]$. Therefore, we have

$$1 \leq Y(t) \leq \rho_{a_t}(0)C \leq C. \tag{6.8}$$

Then, using (6.3)-(6.4) and estimates (3.14), (6.2), (6.7) and (6.8) we get

$$\begin{aligned} \rho_{k+\frac{1}{2}}(t) &\leq \rho_{k+\frac{1}{2}}(t)\exp\left\{K\int_0^t \rho_{k+\frac{1}{2}}(\tau)\theta_{k+\frac{1}{2}}(\tau)d\tau\right\} = \\ &= \frac{\rho_{a_t}(t)\rho_{k+\frac{1}{2}}(0)}{\rho_{a_t}(0)}B_k(t)Y(t) \leq C, \end{aligned} \tag{6.9}$$

for $t \in [0,T]$ and $k = 0, \dots, N-1$. Notice that from (6.6) follows the inequality

$$\frac{1}{\rho_{k+\frac{1}{2}}(t)} \leq C\left(1 + \int_0^t \theta_{k+\frac{1}{2}}(\tau)d\tau\right). \tag{6.10}$$

Taking into account that, for a_t defined by (6.2), we have

$$\begin{aligned} \theta_{k+\frac{1}{2}}(t) &= \left(\sqrt{\theta_{a_t}(t)} + \sum_{r=a_t+\frac{1}{2}}^{k} \delta(\sqrt{\theta})_r h\right)^2 \leq \\ &\leq C\left(1 + \sum_{r=a_t+\frac{1}{2}}^{k} \frac{(\delta\theta)_r^2\rho_{r-\frac{1}{2}}h}{\theta_{r-\frac{1}{2}}\theta_{r+\frac{1}{2}}} \sum_{r=a_t+\frac{1}{2}}^{k} \frac{\theta_{r+\frac{1}{2}}}{\rho_{r-\frac{1}{2}}}h\right) \end{aligned} \tag{6.11}$$

and inserting (6.11) into (6.10) we obtain

$$\frac{1}{\rho_{k+\frac{1}{2}}(t)} \leq C\left(1 + \int_0^t \max_{1\leq r\leq N} \frac{1}{\rho_{r-\frac{1}{2}}(\tau)} \sum_{r=a_t+\frac{1}{2}}^{k} \theta_{r+\frac{1}{2}}h \sum_{r=a_t+\frac{1}{2}}^{k} \frac{(\delta\theta)_r^2\rho_{r-\frac{1}{2}}h}{\theta_{r-\frac{1}{2}}\theta_{r+\frac{1}{2}}}\,d\tau\right). \tag{6.12}$$

Using (5.6) from (6.12) we get immediately

$$\max_{0\leq k\leq N-1}\frac{1}{\rho_{k+\frac{1}{2}}(t)} \leq C\left(1 + \int_0^t \max_{0\leq k\leq N-1} \frac{1}{\rho_{k+\frac{1}{2}}(\tau)} \sum_{r=a_t+\frac{1}{2}}^{k} \frac{(\delta\theta)_r^2\rho_{r-\frac{1}{2}}h}{\theta_{r-\frac{1}{2}}\theta_{r+\frac{1}{2}}}\,d\tau\right),$$

from which, after applying the Gronwall inequality and estimate (5.1), follows the boundedness

$$\frac{1}{\rho_{k+\frac{1}{2}}(t)} \leq C, \tag{6.13}$$

for each $k = 0, \dots, N-1$ and $t \in [0,T]$. The estimates (6.9) and (6.13) give the proof of (6.1). □

7. Boundedness of the energy density and its consequences

Denote the energy density with

$$W_k(t) = \frac{1}{2}v_k^2(t) + \frac{1}{2A}\omega_k^2(t) + \theta_{k-\frac{1}{2}}(t), \tag{7.1}$$

for $k = 1, \dots, N$ and $t \in [0,T]$. It is easy to see that $W_k(t) > 0$, for each k.

Multiply equations (3.4), (3.5) and (3.6), respectively by $v_k W_k h$, $A^{-1}\rho_k^{-1}\omega_k W_k h$ and $\rho_j^{-1} W_{j+\frac{1}{2}} h$, $j = k - \frac{1}{2}$ respectively, and sum up the resulting equality for $k = 1, \dots, N$. Then with help of

$$(\delta v^2)_{k-\frac{1}{2}} = (\delta v)_{k-\frac{1}{2}}(v_k + v_{k-1}). \tag{7.2}$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{k=1}^{N} W_k^2 h + \frac{D}{6}\sum_{k=1}^{N}\rho_{k-\frac{1}{2}}(\delta W_{k-\frac{1}{2}})^2 h = \sum_{m=1}^{7} I_m(t), \tag{7.3}$$

where

$$\begin{aligned}
I_1(t) &= \left(\tfrac{D}{12} - 1\right)\sum_{k=1}^{N}\rho_{k-\frac{1}{2}}\delta v_{k-\frac{1}{2}} v_{k-1}\delta W_{k-\frac{1}{2}} h,\\
I_2(t) &= \left(\tfrac{D}{12A} - 1\right)\sum_{k=1}^{N}\rho_{k-\frac{1}{2}}\delta \omega_{k-\frac{1}{2}} \omega_{k-1}\delta W_{k-\frac{1}{2}} h,\\
I_3(t) &= K\sum_{k=1}^{N}\rho_{k-\frac{1}{2}}\theta_{k-\frac{1}{2}} v_k\delta W_{k-\frac{1}{2}} h,\\
I_4(t) &= -D\sum_{k=1}^{N-1}\rho_k\delta\theta_k\delta W_{k+\frac{1}{2}} h,\\
I_5(t) &= \tfrac{D}{12}\sum_{k=1}^{N}\rho_{k-\frac{1}{2}}\delta v_{k-\frac{1}{2}} v_k\delta W_{k-\frac{1}{2}} h,\\
I_6(t) &= \tfrac{D}{12A}\sum_{k=1}^{N}\rho_{k-\frac{1}{2}}\delta \omega_{k-\frac{1}{2}} \omega_k\delta W_{k-\frac{1}{2}} h,\\
I_7(t) &= \tfrac{D}{6}\sum_{k=1}^{N}\rho_{k-\frac{1}{2}}\delta\theta_{k-1}\delta W_{k-\frac{1}{2}} h.
\end{aligned}$$

Taking into account

$$\delta W_{k-\frac{1}{2}} = \frac{1}{2}(\delta v^2)_{k-\frac{1}{2}} + \frac{1}{2A}(\delta\omega^2)_{k-\frac{1}{2}} + \delta\theta_{k-1},$$

(7.2) and (6.1), and applying the Young inequality, we obtain the estimates of the functions $I_m(t)$, $m = 1, \ldots, 7$. For instance,

(7.4)

$$\begin{aligned} I_1(t) &= \left(\tfrac{D}{12} - 1\right) \sum_{k=1}^{N} \rho_{k-\frac{1}{2}} \delta v_{k-\frac{1}{2}} v_{k-1} \delta W_{k-\frac{1}{2}} h = \tfrac{1}{2}\left(\tfrac{D}{12} - 1\right) \sum_{k=1}^{N} \rho_{k-\frac{1}{2}} \delta v_{k-\frac{1}{2}} v_{k-1} (\delta v^2)_{k-\frac{1}{2}} h \\ &\quad + \tfrac{1}{2A}\left(\tfrac{D}{12} - 1\right) \sum_{k=1}^{N} \rho_{k-\frac{1}{2}} \delta v_{k-\frac{1}{2}} v_{k-1} (\delta \omega^2)_{k-\frac{1}{2}} h + \left(\tfrac{D}{12} - 1\right) \sum_{k=1}^{N} \rho_{k-\frac{1}{2}} \delta v_{k-\frac{1}{2}} v_{k-1} \delta\theta_{k-1} h \\ &\leq C \sum_{k=1}^{N} (\delta v_{k-\frac{1}{2}})^2 (v_k^2 + v_{k-1}^2) h + C \sum_{k=1}^{N} |\delta v_{k-\frac{1}{2}}|\, |v_k|\, |\delta \omega_{k-\frac{1}{2}}|\, |\omega_k + \omega_{k-1}| h \\ &\quad + C \sum_{k=1}^{N} |\delta v_{k-\frac{1}{2}}|\, |v_{k-1}|\, |\delta\theta_{k-1}| h \\ &\leq C \sum_{k=1}^{N} (\delta v_{k-\frac{1}{2}})^2 (v_k^2 + v_{k-1}^2) h + C \sum_{k=1}^{N} (\delta \omega_{k-\frac{1}{2}})^2 (\omega_k^2 + \omega_{k-1}^2) h + \epsilon \sum_{k=1}^{N-1} \rho_{k+\frac{1}{2}} (\delta\theta_k)^2 h, \end{aligned}$$

where $\epsilon > 0$ is arbitrary. In an analogous way one obtains the inequalities:

$$\begin{aligned} (7.5)\quad I_2(t) &\leq C \sum_{k=0}^{N-1} (\delta \omega_{k+\frac{1}{2}})^2 (\omega_{k+1}^2 + \omega_k^2) h + C \sum_{k=0}^{N-1} (\delta v_{k+\frac{1}{2}})^2 (v_{k+1}^2 + v_k^2) h \\ &\quad + \epsilon \sum_{k=0}^{N-1} \rho_{k+\frac{1}{2}} (\delta\theta_{k+1})^2 h, \end{aligned}$$

$$\begin{aligned} (7.6)\quad I_3(t) &\leq C \sum_{k=0}^{N-1} (\delta v_{k+\frac{1}{2}})^2 (v_{k+1}^2 + v_k^2) h + C \sum_{k=0}^{N-1} (\delta \omega_{k+\frac{1}{2}})^2 (\omega_{k+1}^2 + \omega_k^2) h \\ &\quad + \epsilon \sum_{k=0}^{N-1} \rho_{k+\frac{1}{2}} (\delta\theta_k)^2 h + C \sum_{k=0}^{N-1} \theta_{k+\frac{1}{2}}^2 v_k^2 h, \end{aligned}$$

$$\begin{aligned} (7.7)\, I_4(t) &\leq -(D - 2\epsilon) \sum_{k=0}^{N-1} \rho_{k+\frac{1}{2}} (\delta\theta_k)^2 h + \\ &\quad + C \sum_{k=0}^{N-1} (\delta v_{k+\frac{1}{2}})^2 (v_{k+1}^2 + v_k^2) h + C \sum_{k=0}^{N-1} (\delta \omega_{k+\frac{1}{2}})^2 (\omega_{k+1}^2 + \omega_k^2) h, \end{aligned}$$

$$\begin{aligned} (7.8)\quad I_5(t) &\leq C \sum_{k=0}^{N-1} (\delta v_{k+\frac{1}{2}})^2 (v_{k+1}^2 + v_k^2) h + C \sum_{k=0}^{N-1} (\delta \omega_{k+\frac{1}{2}})^2 (\omega_{k+1}^2 + \omega_k^2) h \\ &\quad + \epsilon \sum_{k=0}^{N-1} \rho_{k+\frac{1}{2}} (\delta\theta_k)^2 h, \end{aligned}$$

$$\begin{aligned} (7.9)\quad I_6(t) &\leq C \sum_{k=0}^{N-1} (\delta v_{k+\frac{1}{2}})^2 (v_{k+1}^2 + v_k^2) h + C \sum_{k=0}^{N-1} (\delta \omega_{k+\frac{1}{2}})^2 (\omega_{k+1}^2 + \omega_k^2) h \\ &\quad + \epsilon \sum_{k=0}^{N-1} \rho_{k+\frac{1}{2}} (\delta\theta_k)^2 h, \end{aligned}$$

$$(7.10)\quad I_7(t) \leq C\sum_{k=0}^{N-1}(\delta v_{k+\frac{1}{2}})^2(v_{k+1}^2+v_k^2)h + C\sum_{k=0}^{N-1}(\delta\omega_{k+\frac{1}{2}})^2(\omega_{k+1}^2+\omega_k^2)h$$
$$+2\epsilon\sum_{k=0}^{N-1}\rho_{k+\frac{1}{2}}(\delta\theta_k)^2h + \tfrac{D}{6}\sum_{k=0}^{N-1}\rho_{k+\frac{1}{2}}(\delta\theta_k)^2h.$$

Inserting (7.4)-(7.10) into (7.3) and integrating over $[0,t]$ we conclude (for sufficiently small $\epsilon>0$) that

$$(7.11)\quad \sum_{k=1}^{N}W_k^2h + C\int_0^t\sum_{k=1}^{N}(\delta W_{k-\frac{1}{2}})^2h\ d\tau + C\int_0^t\sum_{k=1}^{N-1}(\delta\theta_k)^2h\ d\tau \leq$$
$$\leq C\left(1+\int_0^t\sum_{k=0}^{N-1}(\delta v_{k+\frac{1}{2}})^2(v_{k+1}^2+v_k^2)h\ d\tau + \right.$$
$$\left. + \int_0^t\sum_{k=0}^{N-1}(\delta\omega_{k+\frac{1}{2}})^2(\omega_{k+1}^2+\omega_k^2)h\ d\tau + \int_0^t\sum_{k=0}^{N-1}\theta_{k+\frac{1}{2}}^2v_k^2h\ d\tau\right).$$

Lemma 7.1. *There exist $C_1, C_2, C\in \mathrm{R}^+$ such that, for $t\in[0,T]$, it holds*

$$(7.12)\quad \sum_{k=1}^{N}W_k^2h + \int_0^t\sum_{k=1}^{N}(\delta W_{k-\frac{1}{2}})^2h\ d\tau + \int_0^t\sum_{k=1}^{N-1}(\delta\theta_k)^2h\ d\tau +$$
$$+C_1\sum_{k=1}^{N}v_k^4h + C_2\sum_{k=1}^{N}\omega_k^4h + \int_0^t\sum_{k=1}^{N}\omega_k^4hd\tau \leq C\left(1+\int_0^t\sum_{k=1}^{N}\theta_{k-\frac{1}{2}}^2(v_k^2+v_{k-1}^2)h\ d\tau\right).$$

Proof. Multiplying (3.4) and (3.5), respectively, by v_k^3h and $A^{-1}\rho_k^{-1}\omega_k^3h$, summing over k, we get

$$(7.13)\quad \frac{1}{4}\frac{d}{dt}\sum_{k=1}^{N}v_k^4h = -\sum_{k=1}^{N}\rho_{k-\frac{1}{2}}\delta v_{k-\frac{1}{2}}(\delta v^3)_{k-\frac{1}{2}}h + K\sum_{k=1}^{N}\rho_{k-\frac{1}{2}}\theta_{k-\frac{1}{2}}(\delta v^3)_{k-\frac{1}{2}}h,$$

$$(7.14)\quad \frac{1}{4A}\frac{d}{dt}\sum_{k=1}^{N}\omega_k^4h + \sum_{k=1}^{N}\frac{\omega_k^4}{\rho_k}h + \sum_{k=1}^{N}\rho_{k-\frac{1}{2}}\delta\omega_{k-\frac{1}{2}}(\delta\omega^3)_{k-\frac{1}{2}}h = 0.$$

We use the equality

$$(7.15)\quad (\delta v^3)_{k-\frac{1}{2}} = (v_k^2+v_{k-1}^2)\delta v_{k-\frac{1}{2}} + v_kv_{k-1}\delta v_{k-\frac{1}{2}},$$

that satisfies the function $(\delta\omega^3)_{k-\frac{1}{2}}$ also. Applying the Young inequality with the parameter $\epsilon>0$ and estimation (3.18) to (7.13), after integration of the obtained inequality, we have

$$(7.16)\quad \frac{1}{4}\sum_{k=1}^{N}v_k^4 + (\frac{1}{2}-2\epsilon)\int_0^t\sum_{k=1}^{N}\rho_{k-\frac{1}{2}}(\delta v_{k-\frac{1}{2}})^2(v_k^2+v_{k-1}^2)h\ d\tau$$
$$\leq C\left(1+\int_0^t\sum_{k=1}^{N}\rho_{k-\frac{1}{2}}\theta_{k-\frac{1}{2}}^2(v_k^2+v_{k-1}^2)h\ d\tau\right),$$

where we take $\epsilon = \frac{1}{8}$. From (7.14) it follows

(7.17)
$$\frac{1}{4A}\sum_{k=1}^{N}\omega_k^4 h + \int_0^t \sum_{k=1}^{N}\frac{\omega_k^4}{\rho_k} h d\tau + \frac{1}{2}\int_0^t \sum_{k=1}^{N}\rho_{k-\frac{1}{2}}(\delta\omega_{k-\frac{1}{2}})^2(\omega_k^2+\omega_{k-1}^2) h d\tau \le C.$$

We take into account (6.1) and then multiply (7.16) and (7.17) with constants determined in such a way that after adding the obtained inequalities to (7.11), the parts consisting of $(\delta\omega_{k-\frac{1}{2}})^2(\omega_k^2+\omega_{k-1}^2)$ and $(\delta v_{k-\frac{1}{2}})^2(v_k^2+v_{k-1}^2)$ cancel each other out. So, we get (7.12). □

Lemma 7.2. *For each $t \in [0,T]$, it holds*

$$\sum_{k=1}^{N}(\theta_{k-\frac{1}{2}}^2 + v_k^4 + \omega_k^4) h \le C. \tag{7.18}$$

Proof. Inserting $\theta_{k+\frac{1}{2}}$ defined by (6.11) into the right hand side of inequality (7.12) and integrating over $[0,t]$, we obtain

$$\begin{aligned}(7.19)\qquad &\int_0^t \sum_{k=1}^{N}\theta_{k-\frac{1}{2}}^2(v_k^2+v_{k-1}^2) h d\tau \\ \le\ & C\int_0^t \sum_{k=1}^{N}\left(1+\sum_{r=1}^{N-1}\frac{(\delta\theta_r)^2\rho_{r-\frac{1}{2}}h}{\theta_{r-\frac{1}{2}}\theta_{r+\frac{1}{2}}}\sum_{r=1}^{N-1}\frac{\theta_{r+\frac{1}{2}}h}{\rho_{r-\frac{1}{2}}}\right)\theta_{k-\frac{1}{2}}(v_k^2+v_{k-1}^2) h d\tau \\ =\ & C\int_0^t \left(1+\sum_{r=1}^{N-1}\frac{(\delta\theta_r)^2\rho_{r-\frac{1}{2}}h}{\theta_{r-\frac{1}{2}}\theta_{r+\frac{1}{2}}}\sum_{r=1}^{N-1}\frac{\theta_{r+\frac{1}{2}}h}{\rho_{r-\frac{1}{2}}}\right)\sum_{k=1}^{N}\theta_{k-\frac{1}{2}}(v_k^2+v_{k-1}^2) h d\tau.\end{aligned}$$

Using estimates (6.1), (5.6) and the Young inequality from (7.19), we get

$$\begin{aligned}(7.20)\qquad &\int_0^t \sum_{k=1}^{N}\theta_{k-\frac{1}{2}}^2(v_k^2+v_{k-1}^2) h d\tau \\ \le\ & C\int_0^t \left(1+\sum_{r=1}^{N-1}\frac{(\delta\theta_r)^2\rho_{r-\frac{1}{2}}h}{\theta_{r-\frac{1}{2}}\theta_{r+\frac{1}{2}}}\right)\sum_{k=1}^{N}(\theta_{k-\frac{1}{2}}^2+v_k^4+v_{k-1}^4) h d\tau.\end{aligned}$$

Inserting (7.20) into (7.12) we have the inequality

$$\begin{aligned}(7.21)\qquad &\sum_{k=1}^{N}(\theta_{k-\frac{1}{2}}^2+v_k^4+v_{k-1}^4+\omega_k^4) h \le \\ \le\ & C + C\int_0^t \left(1+\sum_{r=1}^{N-1}\frac{(\delta\theta_r)^2\rho_{r-\frac{1}{2}}h}{\theta_{r-\frac{1}{2}}\theta_{r+\frac{1}{2}}}\right)\sum_{k=1}^{N}(\theta_{k-\frac{1}{2}}^2+v_k^4+v_{k-1}^4+\omega_k^4) h d\tau\end{aligned}$$

Taking into account estimation (5.1) and applying the Gronwall inequality, from (7.21) it follows that the boundedness of the function

$$\sum_{k=1}^{N}(\theta_{k-\frac{1}{2}}^2+v_k^4+v_{k-1}^4+\omega_k^4) h$$

for all $t \in [0,T]$ and (7.18) is obtained. □

Now with the help of (7.18) and (5.1), from (7.20) we get immediately

$$\int_0^t \sum_{k=1}^{N} \theta_{k-\frac{1}{2}}^2 (v_k^2 + v_{k-1}^2) h d\tau \le C, \tag{7.22}$$

and then from (7.12) we easily conclude that for all $t \in [0,T]$ it holds

$$\int_0^t \sum_{k=1}^{N} (\delta v_{k-\frac{1}{2}})^2 (v_k^2 + v_{k-1}^2) h d\tau \le C, \tag{7.23}$$

$$\int_0^t \sum_{k=1}^{N} (\delta \omega_{k-\frac{1}{2}})^2 (\omega_k^2 + \omega_{k-1}^2) h d\tau \le C, \tag{7.24}$$

$$\int_0^t \sum_{k=1}^{N-1} (\delta \theta_k)^2 h d\tau \le C. \tag{7.25}$$

Lemma 7.3. *There exists $C \in \mathrm{R}^+$ such that, for all $t \in [0,T]$, it holds*

$$\sum_{k=1}^{N} v_k^2 h + \int_0^t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta v_j)^2 h d\tau \le C. \tag{7.26}$$

Proof. Multiplying (3.4) by $v_k h$, summing over $k = 1, \dots, N-1$ and applying the Young inequality with a parameter $\epsilon > 0$, we get

$$\frac{1}{2}\frac{d}{dt} \sum_{k=1}^{N-1} v_k^2 h + \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \rho_j (\delta v_j)^2 h \le \epsilon \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \rho_j (\delta v_j)^2 h + C \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \rho_j \theta_j^2 h. \tag{7.27}$$

Integrating (7.27) over $[0,t]$ and using (3.16), (6.1) and (7.18) (for ϵ small enough), we obtain (7.26). □

With the help of (7.15) we have

$$\begin{aligned} \int_0^t \max_{0 \le k \le N} |v_k| d\tau &\le C \left(1 + \int_0^t \max_{0 \le k \le N} |v_k|^3 d\tau\right) \\ &= C \left(1 + \int_0^t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} |(\delta v^3)_j| h d\tau\right) \le C \left(1 + \int_0^t \sum_{k=1}^{N} |\delta v_{k-\frac{1}{2}}| (v_k^2 + v_{k-1}^2) h d\tau\right) \\ &\le C \left(1 + \int_0^t \sum_{k=1}^{N} (\delta v_{k-\frac{1}{2}})^2 h d\tau + \int_0^t \sum_{k=1}^{N} (v_k^4 + v_{k-1}^4) h d\tau\right) \end{aligned} \tag{7.28}$$

and using (7.18) and (7.26) we get

$$\int_0^t \max_{0 \le k \le N} |v_k| d\tau \le C \quad \text{for } t \in [0,T]. \tag{7.29}$$

In the same way we conclude that

$$\int_0^t \max_{0 \le k \le N} |\omega_k| d\tau \le C \quad \text{for } t \in [0,T]. \tag{7.30}$$

Now, for $j = \frac{1}{2}, \dots, N - \frac{1}{2}$, we have

$$\theta_j^2 = \theta_{a_t}^2 + \sum_{i=a_t}^{j} (\delta \theta^2)_{i+\frac{1}{2}} h, \tag{7.31}$$

where θ_{a_t} is defined by (6.2). Using the equality

$$(\delta\theta^2)_{i+\frac{1}{2}} = \delta\theta_{i+\frac{1}{2}}(\theta_{i+1} + \theta_i),$$

the estimation (6.2) and applying the Young inequality from (7.31) we obtain

$$\theta_j^2 \leq C + \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \theta_j^2\, h + \sum_{k=1}^{N} (\delta\theta_k)^2 h. \tag{7.32}$$

Taking into account (6.1), (7.18) and (7.25) we conclude that from (7.32) it follows

$$\int_0^t \max_{\frac{1}{2}\leq j\leq N-\frac{1}{2}} \theta_j^2 d\tau \leq C \quad \text{for } t \in [0,T]. \tag{7.33}$$

The consequence of (7.33) is

$$\int_0^t \max_{\frac{1}{2}\leq j\leq N-\frac{1}{2}} \theta_j d\tau \leq C \quad \text{for } t \in [0,T]. \tag{7.34}$$

8. Further bounds for the density, velocity, microrotation velocity and temperature

We proceed with the further bounds for the variables of the system, needed for proving the main theorem.

Lemma 8.1. *There exists a constant $C \in R^+$ such that, for all $t \in [0,T]$, it holds*

$$\sum_{k=1}^{N-1} (\delta\rho_k)^2(t) h \leq C. \tag{8.1}$$

Proof. ¿From (6.6) we can easily conclude that

$$\rho_{k+\frac{1}{2}}(t) = F_{k+\frac{1}{2}}(t) \cdot G^{-1}_{k+\frac{1}{2}}(t) \tag{8.2}$$

where

$$F_{k+\frac{1}{2}}(t) = \frac{\rho_{k+\frac{1}{2}}(0)\rho_{a_t}(t)}{\rho_{a_t}(0)} B_k(t)Y(t), \tag{8.3}$$

$$G_{k+\frac{1}{2}}(t) = 1 + K\frac{\rho_{k+\frac{1}{2}}(0)}{\rho_{a_t}(0)} \int_0^t B_k(\tau)Y(\tau)\rho_{a_t}(\tau)\theta_{k+\frac{1}{2}}(\tau)d\tau \tag{8.4}$$

(B_k and Y are defined by (6.5) and (6.4)). For estimating

$$\delta\rho_k = \frac{\delta F_k\, G_{k-\frac{1}{2}} - F_{k-\frac{1}{2}}\, \delta G_k}{G_{k+\frac{1}{2}}\, G_{k-\frac{1}{2}}} \tag{8.5}$$

we need the estimates of the functions (8.3), (8.4), δF_k and δG_k. Using (6.1), (3.3), (6.7), (6.8) and (7.34), we find that there exist $C_1, C_2 \in \mathrm{R}^+$ such that

$$C_1^{-1} \leq G_{k+\frac{1}{2}}(t) \leq C_1, \tag{8.6}$$

$$C_2^{-1} \leq F_{k+\frac{1}{2}}(t) \leq C_2, \tag{8.7}$$

for $k = 0, \ldots, N-1$ and all $t \in [0,T]$. By the Taylor development we obtain

$$\begin{aligned} \delta B_{k-\frac{1}{2}}(t) &= B_k(t)\left(\frac{1-\exp\{(v_k(t)-v_k(0))h\}}{h}\right) \\ &= -B_k(t)\left(v_k(t)-v_k(0)+\frac{h}{2}(v_k(t)-v_k(0))^2\exp\{\lambda(v_k(t)-v_k(0))h\}\right), \end{aligned} \tag{8.8}$$

for some $0 < \lambda < 1$. Using (6.7) and (7.26) for (8.8) we get

$$|\delta B_{k-\frac{1}{2}}(t)| \leq C(|v_k(0)| + |v_k(t)| + 1). \tag{8.9}$$

With the help of (3.14), (6.2), (6.7), (6.8), (7.34) and (8.9) for the functions δF_k and δG_k, $k = 1, \ldots, N-1$, we have

$$\begin{aligned} |\delta F_k| &= \Big|\frac{\rho_{a_t}(t)}{\rho_{a_t}(0)} Y(t)(\delta\rho_k(0)B_k(t) + \rho_{k-\frac{1}{2}}(0)\delta B_{k-\frac{1}{2}}(t))\Big| \\ &\leq C(|\delta\rho_k(0)| + |v_k(t)| + |v_k(0)| + 1), \end{aligned} \tag{8.10}$$

$$\begin{aligned} |\delta G_k| &= \Big|\frac{K}{\rho_{a_t}(0)} \left(\delta\rho_k(0) \int_0^t \rho_{a_t}(\tau) B_k(\tau) Y(\tau) \theta_{k+\frac{1}{2}}(\tau) d\tau + \right. \\ &+ \left. \rho_{k-\frac{1}{2}}(0) \int_0^t \rho_{a_t}(\tau) Y(\tau) \left(\delta B_{k-\frac{1}{2}}(\tau)\theta_{k+\frac{1}{2}}(\tau) + B_k(\tau)\delta\theta_k(\tau)\right) d\tau\right) \Big| \\ &\leq C\left(1 + |\delta\rho_k(0)| + \int_0^t (|v_k(t)| + |v_k(0)|)\theta_{k+\frac{1}{2}} d\tau + \int_0^t |\delta\theta_k(\tau)| d\tau\right). \end{aligned} \tag{8.11}$$

Using (8.6), (8.7) and inequalities (8.10), (8.11) , from (8.5) we obtain

$$\begin{aligned} \sum_{k=1}^{N} (\delta\rho_k(t))^2 h \leq C &\left(\sum_{k=1}^{N-1} (\delta\rho_k(0))^2 h + \sum_{k=1}^{N-1} v_k^2(t) h + \sum_{k=1}^{N-1} v_k^2(0) h + \sum_{k=1}^{N-1} h + \right. \\ &+ \left. \int_0^t \max_{0 \leq k \leq N-1} \theta_{k+\frac{1}{2}}^2 \sum_{k=1}^{N-1} (v_k^2(0) + v_k^2(t)) h d\tau + \int_0^t \sum_{k=1}^{N-1} (\delta\theta_k)^2 h d\tau\right). \end{aligned} \tag{8.12}$$

Taking into account (3.16), (3.17), (5.6), (7.25) and (7.33), from (8.12), it follows (8.1). □

Lemma 8.2. *There exists* $C \in \mathrm{R}^+$ *such that, for* $j = \frac{1}{2}, \ldots, N - \frac{1}{2}$*, and all* $t \in [0, T]$*, it holds*

$$\theta_j(t) \geq C. \tag{8.13}$$

Proof. Multiplying (3.6) by $\rho_j^{-1}\theta_j^{-2}$ we obtain

$$\frac{d}{dt}\left(\frac{1}{\theta_j}\right) + \frac{\rho_j}{\theta_j^2}(\delta v_j)^2 + \frac{\rho_j}{\theta_j^2}(\delta\omega_j)^2 + \frac{\omega_j^2}{\rho_j\theta_j^2} = K\frac{\rho_j}{\theta_j}(\delta v_j) - D\delta(\rho\delta\theta)_j \frac{1}{\theta_j^2}. \tag{8.14}$$

Applying on the first part of the right-hand side the Young inequality, with a parameter $\epsilon > 0$, we get inequality

$$\frac{d}{dt}\left(\frac{1}{\theta_j}\right) + \frac{\rho_j}{\theta_j^2}(\delta v_j)^2 \leq \epsilon\frac{\rho_j}{\theta_j^2}(\delta v_j)^2 + C_\epsilon\rho_j - D\delta(\rho\delta\theta)_j \frac{1}{\theta_j^2},$$

which for $\epsilon = 1$ reads

$$\frac{d}{dt}\left(\frac{1}{\theta_j}\right) \leq C_\epsilon\rho_j - D\delta(\rho\delta\theta)_j \frac{1}{\theta_j^2}. \tag{8.15}$$

Using the inequality

$$\frac{1}{\theta_j^2}\delta(\rho\delta\theta)_j \geq -\delta\left(\rho\delta(\frac{1}{\theta})\right)_j,$$

multiplying (8.15) by $2q(\frac{1}{\theta_j})^{2q-1}h$, $q \in \mathrm{N}\backslash\{1\}$, and summing up for $j = \frac{1}{2}, \ldots, N-\frac{1}{2}$, we obtain for $\Psi_j = \frac{1}{\theta_j}$, the following inequality

$$\frac{d}{dt}\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \Psi_j^{2q} h\right) \leq 2qC \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \rho_j \Psi_j^{2q-1} h + 2qD \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \delta(\rho\,\delta\Psi)_j \Psi_j^{2q-1} h. \tag{8.16}$$

Taking into account

$$\begin{aligned} & 2qD \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \delta(\rho\,\delta\Psi)_j \Psi_j^{2q-1} h = \\ = \quad & -2qD \sum_{k=1}^{N-1} \rho_k (\delta\Psi_k)^2 (\Psi_{k+\frac{1}{2}}^{2q-2} + \Psi_{k+\frac{1}{2}}^{2q-3} \cdot \Psi_{k-\frac{1}{2}} + \ldots + \Psi_{k-\frac{1}{2}}^{2q-2}) \leq 0 \end{aligned}$$

and applying the Hölder inequality on the first part of the right-hand side of (8.16), we conclude that

$$\frac{d}{dt}\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \Psi_j^{2q} h\right) \leq 2qC \left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \Psi_j^{2q} h\right)^{\frac{2q-1}{2q}} \left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \rho_j^{2q} h\right)^{\frac{1}{2q}}. \tag{8.17}$$

Using (6.1), from (8.17) we obtain the following differential inequality

$$D'(t) \leq C, \tag{8.18}$$

where the function $D(t)$ is defined by

$$D(t) = \left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \Psi_j^{2q} h\right)^{\frac{1}{2q}}. \tag{8.19}$$

¿From (8.18) it follows

$$D(t) \leq Ct + D(0), \tag{8.20}$$

for $t \in [0,T]$, i.e., for $\Psi = (\Psi_{\frac{1}{2}}, \ldots, \Psi_{N-\frac{1}{2}})$, we have

$$\|\Psi(t)\|_{L^{2q}} \leq CT + D(0). \tag{8.21}$$

Notice that because of (3.14), $D(0) \leq \frac{1}{m}$. If $q \to \infty$ in (8.21), we obtain

$$\|\Psi(t)\|_{L^\infty} \leq CT + \frac{1}{m},$$

which implies

$$\theta_j(t) \geq \left(CT + \frac{1}{m}\right)^{-1}$$

for $j = \frac{1}{2}, \ldots, N - \frac{1}{2}$ and all $t \in [0,T]$. □

Lemma 8.3. *There exists $C \in \mathrm{R}^+$ such that, for all $t \in [0,T]$ and $j \in \{\frac{1}{2}, \ldots, N - \frac{1}{2}\}$, the functions δv_j and $\delta\omega_j$ satisfy following inequalities:*

$$|\delta v_j| \leq C \left(\sum_{r=\frac{1}{2}}^{N-\frac{3}{2}} (\delta^2 v_{r+\frac{1}{2}})^2 h\right)^{1/4} \left(\sum_{r=\frac{1}{2}}^{N-\frac{1}{2}} (\delta v_r)^2 h\right)^{1/4}, \tag{8.22}$$

$$|\delta\omega_j| \leq C\left(\sum_{r=\frac{1}{2}}^{N-\frac{3}{2}} (\delta^2\omega_{r+\frac{1}{2}})^2 h\right)^{1/4}\left(\sum_{r=\frac{1}{2}}^{N-\frac{1}{2}} (\delta\omega_r)^2 h\right)^{1/4}. \tag{8.23}$$

Proof. Using (2.6) we get immediately

$$\delta v_{\frac{1}{2}} + \delta v_{\frac{3}{2}} + \ldots + \delta v_{N-\frac{1}{2}} = 0 \tag{8.24}$$

for all $t \in [0, T]$. It means that, for each $t \in [0, T]$, there exists at least one pair of indices $\alpha(t), \beta(t) \in \{\frac{1}{2}, \ldots, N - \frac{1}{2}\}$ such that

$$\delta v_{\alpha(t)} \geq 0, \quad \delta v_{\beta(t)} \leq 0. \tag{8.25}$$

Suppose that $\alpha(t), \beta(t)$ are the smallest indices for which (8.25) is valid. If $\delta v_j(t) \geq 0$, then we use $\delta v_{\beta(t)}$ and (for $j < \beta(t)$) we obtain the inequality

$$\big|\delta v_j|\delta v_j| - \delta v_{\beta(t)}|\delta v_{\beta(t)}|\big| \leq \sum_{r=j}^{\beta(t)-1} |\delta^2 v_{r+\frac{1}{2}}|\,(|\delta v_{r+1}| + |\delta v_r|)h, \tag{8.26}$$

from which, using the Hölder inequality, it follows

$$\begin{aligned} |\delta v_j|^2 &\leq \left(\sum_{r=j}^{\beta(t)-1} |\delta^2 v_{r+\frac{1}{2}}|^2 h\right)^{\frac{1}{2}} \left(\sum_{r=j}^{\beta(t)-1} (|\delta v_{r+1}| + |\delta v_r|)^2 h\right)^{\frac{1}{2}} \\ &\leq C\left(\sum_{r=\frac{1}{2}}^{N-\frac{3}{2}} |\delta^2 v_{r+\frac{1}{2}}|^2 h\right)^{\frac{1}{2}} \left(\sum_{r=\frac{1}{2}}^{N-\frac{1}{2}} (\delta v_r)^2 h\right)^{\frac{1}{2}}. \end{aligned} \tag{8.27}$$

The same estimate can be obtained for $j > \beta(t)$. On the other side, if $\delta v_j(t) \leq 0$, we use $\delta v_{\alpha(t)}$ and (for $j < \alpha(t)$ or $j > \alpha(t)$), we have

$$\begin{aligned} |\delta v_j|^2 &\leq \big|\delta v_{\alpha(t)}|\delta v_{\alpha(t)}| - \delta v_j|\delta v_j|\big| \\ &\leq C\left(\sum_{r=\frac{1}{2}}^{N-\frac{3}{2}} |\delta^2 v_{r+\frac{1}{2}}|^2 h\right)^{\frac{1}{2}} \left(\sum_{r=\frac{1}{2}}^{N-\frac{1}{2}} (\delta v_r)^2 h\right)^{\frac{1}{2}}. \end{aligned} \tag{8.28}$$

Inequality (8.23) is obtained analogously. □

Lemma 8.4. *There exists $C \in \mathrm{R}^+$ such that*

$$\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta v_j)^2 h + \int_0^t \sum_{k=1}^{N-1} (\delta^2 v_k)^2 h d\tau \leq C, \tag{8.29}$$

$$\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta \omega_j)^2 h + \int_0^t \sum_{k=1}^{N-1} (\delta^2 \omega_k)^2 h d\tau \leq C, \tag{8.30}$$

$$|v_k(t)| \leq C, \quad |\omega_k(t)| \leq C, \tag{8.31}$$

for all $t \in [0, T]$ and $k \in \{1, \ldots, N-1\}$.

Proof. Multiplying (3.4) and (3.5), respectively, by $\delta^2 v_k h$ and $A^{-1}\rho_k^{-1}\delta^2\omega_k h$ and summing up for $k = 1, \dots, N-1$, we get

$$\frac{1}{2}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\frac{d}{dt}(\delta v_j)^2 h + \sum_{k=1}^{N-1}\rho_{k-\frac{1}{2}}(\delta^2 v_k)^2 h = \tag{8.32}$$

$$= K\sum_{k=1}^{N-1}\delta\rho_k\theta_{k+\frac{1}{2}}\delta^2 v_k h + K\sum_{k=1}^{N-1}\rho_{k-\frac{1}{2}}\delta\theta_k\delta^2 v_k h - \sum_{k=1}^{N-1}\delta\rho_k\delta v_{k+\frac{1}{2}}\delta^2 v_k h,$$

$$\frac{1}{2A}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\frac{d}{dt}(\delta\omega_j)^2 h + \sum_{k=1}^{N-1}\rho_{k-\frac{1}{2}}(\delta^2\omega_k)^2 h = -\sum_{k=1}^{N-1}\delta\rho_k\delta\omega_{k+\frac{1}{2}}\delta^2\omega_k h - \sum_{k=1}^{N-1}\frac{\omega_k}{\rho_k}\,\delta^2\omega_k h. \tag{8.33}$$

With the help of (6.1), (8.22), (8.23) and using the Hölder inequality and the Young inequality with a parameter $\epsilon > 0$, for the terms on the right-hand side of (8.32) and (8.33) we find estimates on $[0, T]$ as follows:

$$\Big|K\sum_{k=1}^{N-1}\delta\rho_k\theta_{k+\frac{1}{2}}\delta^2 v_k h\Big| \le \epsilon\sum_{k=1}^{N-1}(\delta^2 v_k)^2 h + C\max_{\frac{1}{2}\le j\le N-\frac{1}{2}}\theta_j^2\sum_{k=1}^{N-1}(\delta\rho_k)^2 h,$$

$$\Big|K\sum_{k=1}^{N-1}\rho_{k-\frac{1}{2}}\delta\theta_k\delta^2 v_k h\Big| \le \epsilon\sum_{k=1}^{N-1}(\delta^2 v_k)^2 h + C\sum_{k=1}^{N-1}(\delta\theta_k)^2 h,$$

$$\begin{aligned}\Big|\sum_{k=1}^{N-1}\delta\rho_k\delta v_{k+\frac{1}{2}}\delta^2 v_k h\Big| &\le C\left(\sum_{k=1}^{N-1}(\delta^2 v_k)^2 h\right)^{\frac{3}{4}}\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta v_j)^2 h\right)^{\frac{1}{4}}\left(\sum_{k=1}^{N-1}(\delta\rho_k)^2 h\right)^{\frac{1}{2}}\\ &\le \epsilon\sum_{k=1}^{N-1}(\delta^2 v_k)^2 h + C\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta v_j)^2 h\right)\left(\sum_{k=1}^{N-1}(\delta\rho_k)^2 h\right)^2,\end{aligned}$$

$$\begin{aligned}\Big|\sum_{k=1}^{N-1}\delta\rho_k\delta\omega_{k+\frac{1}{2}}\delta^2\omega_k h\Big| &\le C\left(\sum_{k=1}^{N-1}(\delta^2\omega_k)^2 h\right)^{\frac{3}{4}}\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta\omega_j)^2 h\right)^{\frac{1}{4}}\left(\sum_{k=1}^{N-1}(\delta\rho_k)^2 h\right)^{\frac{1}{2}}\\ &\le \epsilon\sum_{k=1}^{N-1}(\delta^2\omega_k)^2 h + C\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta\omega_j)^2 h\right)\left(\sum_{k=1}^{N-1}(\delta\rho_k)^2 h\right)^2,\end{aligned}$$

$$\Big|\sum_{k=1}^{N-1}\frac{\omega_k}{\rho_k}\,\delta^2\omega_k h\Big| \le \epsilon\sum_{k=1}^{N-1}(\delta^2\omega_k)^2 h + C\sum_{k=1}^{N-1}\omega_k^2 h.$$

Inserting these inequalities into (8.32) and (8.33), integrating over $[0, t]$, using (6.1), (3.17), (5.6), (7.25), (7.33), (8.1), (7.26), (5.8) and ϵ small enough, we conclude that (8.29) and (8.30) are true. From the inequality

$$|v_k(t)| \le \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}|\delta v_j|h \le \left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta v_j)^2 h\right)^{\frac{1}{2}}\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}h\right)^{\frac{1}{2}}$$

that satisfies the function ω_k, $k = 1, \dots, N-1$, also, follows (8.31). □

Lemma 8.5. *There exists $C \in \mathrm{R}^+$ such that*

$$\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\dot{\rho}_j(t))^2 h \leq C, \tag{8.34}$$

$$\int_0^t \sum_{k=1}^{N-1} (\dot{v}_k(t))^2 h d\tau \leq C, \tag{8.35}$$

$$\int_0^t \sum_{k=1}^{N-1} (\dot{\omega}_k(t))^2 h d\tau \leq C, \tag{8.36}$$

for all $t \in [0,T]$.

Proof. Squaring (3.3), (3.4) and (3.5), multiplying by h, summing up for $j = \frac{1}{2}, \ldots, N-\frac{1}{2}$ and $k = 1, \ldots, N-1$, and using (6.1), (8.29), (8.30), (8.22), (8.1), (7.33), (7.25) and (5.8), we get

$$\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\dot{\rho}_j(t))^2 h = \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \rho_j^4 (\delta v_j)^2 h \leq \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta v_j)^2 h \leq C,$$

$$\begin{aligned}
\int_0^t \sum_{k=1}^{N-1} (\dot{v}_k(t))^2 h \quad &\leq \quad C \int_0^t \left(\sum_{k=1}^{N-1} (\delta^2 v_k)^2 h \right)^{\frac{1}{2}} \left(\sum_{k=1}^{N-1} (\delta v_k)^2 h \right)^{\frac{1}{2}} \sum_{k=1}^{N-1} (\delta \rho_k)^2 h\tau + \\
&+ \quad \int_0^t \sum_{k=1}^{N-1} \rho_{k-1}^2 (\delta^2 v_k)^2 h d\tau + \int_0^t \max_{\frac{1}{2} \leq j \leq N-\frac{1}{2}} \theta_j^2 \sum_{k=1}^{N-1} (\delta \rho_k)^2 h d\tau + \\
&+ \quad \int_0^t \sum_{k=1}^{N-1} \rho_{k-\frac{1}{2}}^2 (\delta \theta_k)^2 h d\tau \leq C,
\end{aligned}$$

$$\begin{aligned}
\int_0^t \sum_{k=1}^{N-1} (\dot{\omega}_k(t))^2 h \quad &\leq \quad C \int_0^t \left(\sum_{k=1}^{N-1} (\delta^2 \omega_k)^2 h \right)^{\frac{1}{2}} \left(\sum_{k=1}^{N-1} (\delta \omega_k)^2 h \right)^{\frac{1}{2}} \sum_{k=1}^{N-1} (\delta \rho_k)^2 h\tau + \\
&+ \quad \int_0^t \sum_{k=1}^{N-1} \rho_{k-1}^2 (\delta^2 \omega_k)^2 h d\tau + \int_0^t \sum_{k=1}^{N-1} \frac{\omega_k^2}{\rho_k} h d\tau \leq C.
\end{aligned}$$

□

In what follows we make the estimates for the functions $\delta\theta_k$ and $\delta^2\theta_k$. Notice that due to $\delta\theta_0 = 0$ we have the inequality

$$(\delta\theta)_k^2 \leq \left(\sum_{r=1}^{N} (\delta^2 \theta_{r-\frac{1}{2}})^2 h \right)^{\frac{1}{2}} \left(\sum_{r=1}^{N-1} (\delta \theta_r)^2 h \right)^{\frac{1}{2}} \tag{8.37}$$

for each $k = 1, \ldots, N-1$.

Lemma 8.6. *There exists $C \in \mathrm{R}^+$ such that*

$$\sum_{k=1}^{N-1} (\delta\theta_k)^2 h + \int_0^t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta^2 \theta_j)^2 h d\tau \leq C, \tag{8.38}$$

$$|\theta_j(t)| \leq C, \tag{8.39}$$

for all $t \in [0,T]$ *and* $j = \frac{1}{2}, \ldots, N - \frac{1}{2}$.

Proof. Multiplying (3.6) by $\rho_j^{-1}\delta^2\theta_j h$, summing up for $j = \frac{1}{2}, \ldots, N - \frac{1}{2}$, and using (6.1), (8.22), (8.23) and (8.37) and the Young inequality we obtain

$$\frac{1}{2}\sum_{k=1}^{N-1}\frac{d}{dt}(\delta\theta_k)^2 h + C_1\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta^2\theta_j)^2 h \leq \tag{8.40}$$

$$\begin{aligned}
\leq \quad & 5\epsilon\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta^2\theta_j)^2 h + C\max_{\frac{1}{2}\leq j\leq N-\frac{1}{2}}\theta_j^2\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta v_j)^2 h \\
+ \quad & C\left(\sum_{k=1}^{N-1}(\delta^2 v_k)^2 h\right)^{\frac{1}{2}}\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta v_j)^2 h\right)^{\frac{3}{2}} + C\left(\sum_{k=1}^{N-1}(\delta^2\omega_k)^2 h\right)^{\frac{1}{2}}\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta\omega_j)^2 h\right)^{\frac{3}{2}} + \\
+ \quad & C(\max|\omega_j|)^4 + C\left(\sum_{k=1}^{N-1}(\delta\theta_k)^2 h\right)\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta\rho_j)^2 h\right)^2 .
\end{aligned}$$

Integrating over $[0,t]$ from (8.40), for ϵ small enough, we get

$$\begin{aligned}
& \sum_{k=1}^{N-1}(\delta\theta_k)^2 h + \int_0^t\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta^2\theta_j)^2 h \leq \\
\leq \quad & C\int_0^t\max_{\frac{1}{2}\leq j\leq N-\frac{1}{2}}\theta_j^2\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta v_j)^2 h d\tau + C\int_0^t\sum_{k=1}^{N-1}(\delta^2 v_k)^2 h d\tau + C\int_0^t\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta v_j)^2 h\right)^3 d\tau \\
+ \quad & C\int_0^t\sum_{k=1}^{N-1}(\delta^2\omega_k)^2 h d\tau + C\int_0^t\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta\omega_j)^2 h\right)^3 d\tau + C\int_0^t(\max|\omega_j|)^4 d\tau + \\
+ \quad & C\int_0^t\left(\sum_{k=1}^{N-1}(\delta\theta_k)^2 h\right)\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta\rho_j)^2 h\right)^2 d\tau + \sum_{k=1}^{N-1}(\delta\theta_k)^2(0)h.
\end{aligned}$$

Taking into account (3.17), (8.29), (8.30), (8.31), (8.1) and (7.25), we easily conclude that (8.38) is true.

The function θ_j satisfies the inequality

$$|\theta_j(t)| \leq \sum_{k=1}^{N-1}|\delta\theta_k(t)|h + \theta_{a_t}(t) \leq \left(\sum_{k=1}^{N-1}(\delta\theta_k(t))^2 h\right)^{\frac{1}{2}} + \theta_{a_t}(t)$$

for $j = \frac{1}{2}, \ldots, N - \frac{1}{2}$ and $t \in [0,T]$, where θ_{a_t} is introduced by (6.2). Using (8.38) we get (8.39). □

It remains to prove the following estimation.

Lemma 8.7. *There exists* $C \in \mathrm{R}^+$ *such that for each* $t \in [0,T]$ *it holds*

$$\int_0^t\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\dot\theta_j(t))^2 h d\tau \leq C. \tag{8.41}$$

Proof. Squaring (3.6), multiplying by h, summing up for $j = \frac{1}{2}, \dots, N - \frac{1}{2}$ and using (6.1), (8.22), (8.23), (8.37) and the inequality

$$\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta v_j)^2 h \leq C \sum_{k=1}^{N-1} (\delta^2 v_k)^2 h,$$

that satisfy the functions $\delta\omega_j$ and $\delta\theta_k$ also, we find that

$$\begin{aligned}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\dot{\theta}_j(t))^2 h \quad \leq \quad & C\left(\sum_{k=1}^{N-1} (\delta^2 v_k)^2 h\right)\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \theta_j^2 h\right) + C\left(\sum_{k=1}^{N-1} (\delta^2 v_k)^2 h\right)\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta v_j)^2 h\right) + \\ + \quad & C\left(\sum_{k=1}^{N-1} (\delta^2 \omega_k)^2 h\right)\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta\omega_j)^2 h\right) + C(\max|\omega_j|)^4 \\ + \quad & C\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta^2 \theta_j)^2 h\right)\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta\rho_j)^2 h\right) + C\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta^2 \theta_j)^2 h.\end{aligned}$$

Integrating over $[0, t]$ and taking into account (8.29), (8.30), (8.31), (8.38), (7.18) and (8.1) we get immediately (8.41). □

9. Convergence of approximate solutions to a solution of (2.1)-(2.7)

In this section we show the compactness of sequences of approximate solutions $(\rho^{N-\frac{1}{2}}, v^N, \omega^N, \theta^{N-\frac{1}{2}})$ and $(\rho_{h-\frac{1}{2}}, v_h, \omega_h, \theta_{h-\frac{1}{2}})$ which are defined by (3.21)-(3.28) and their convergence to a solution $(\rho, v, \omega, \theta)$ of (2.1)-(2.7).

With the help of (6.1), (8.1) and (8.34) we conclude that there exists $C \in \mathrm{R}^+$ (independent of N), such that

$$(9.1) \qquad |\rho^{N-\frac{1}{2}}(x,t)| + \int_0^1 (\partial_x \rho^{N-\frac{1}{2}})^2(x,t)dx + \int_0^1 (\partial_t \rho^{N-\frac{1}{2}})^2(x,t)dx \leq C,$$

which implies the following statements.

Lemma 9.1. *There exists a function*

$$(9.2) \qquad \rho \in C(\overline{Q}_T) \cap H^1(Q_T) \cap L^\infty(0,T; H^1((0,1)))$$

and a subsequence of $\{\rho^{N-\frac{1}{2}}\}$ *(for simplicity denoted again as* $\{\rho^{N-\frac{1}{2}}\}$*), such that*

$$(9.3) \qquad \rho^{N-\frac{1}{2}} \longrightarrow \rho \qquad \text{strongly in } C(\overline{Q}_T),$$

$$(9.4) \qquad \qquad * \text{ weakly in } L^\infty(0,T; H^1((0,1))),$$

$$(9.5) \qquad \qquad \text{weakly in } H^1(Q_T)$$

(when $N \to \infty$ *or* $h \to 0$*). There exists a subsequence of* $\{\rho_{h-\frac{1}{2}}\}$ *(still denoted* $\{\rho_{h-\frac{1}{2}}\}$*) such that*

$$(9.6) \qquad \rho_{h-\frac{1}{2}} \longrightarrow \rho \quad \text{strongly in } L^\infty(0,T; L^2((0,1))).$$

The function ρ *satisfies the condition*

$$(9.7) \qquad C_1 \leq \rho(x,t) \leq C_2 \ \text{for}\ (x,t) \in \overline{Q}_T,$$

where $C_1, C_2 \in \mathrm{R}^+$.

Proof. The conclusions (9.3)–(9.5) follow immediately from (9.1). Notice that using (9.1), we have

$$\int_0^1 (\rho^{N-\frac{1}{2}} - \rho_{h-\frac{1}{2}})^2(x,t)dx \leq h^2 \int_0^1 (\partial_x \rho^{N-\frac{1}{2}})^2(x,t)dx \leq Ch^2, \tag{9.8}$$

so (9.6) is satisfied. From (6.1) we can easily conclude (9.7). □

Taking into account estimates (8.31), (8.29), (8.35) for the function v^N, (8.31), (8.30), (8.36) for the function ω^N and estimates (8.38), (8.39) and (8.41) for the function $\theta^{N-\frac{1}{2}}$, we can conclude that there exists $C \in \mathrm{R}^+$ (independent of N), such that

$$|v^N(x,t)| + \int_0^1 (\partial_x v^N)^2(x,t)dx + \int_0^T \int_0^1 \big[(\partial_t v^N)^2 + (\partial_{xx} v^N)^2\big](x,t)dxd\tau \leq C, \tag{9.9}$$

(9.10)

$$|\omega^N(x,t)| + \int_0^1 (\partial_x \omega^N)^2(x,t)dx + \int_0^T \int_0^1 \big[(\partial_t \omega^N)^2 + (\partial_{xx} \omega^N)^2\big](x,t)dxd\tau \leq C,$$

(9.11)

$$|\theta^{N-\frac{1}{2}}(x,t)| + \int_0^1 (\partial_x \theta^{N-\frac{1}{2}})^2(x,t)dx + \int_0^T \int_0^1 \Big[(\partial_t \theta^{N-\frac{1}{2}})^2 + (\partial_{xx} \theta^{N-\frac{1}{2}})^2\Big](x,t)dxd\tau \leq C,$$

which implies the following statements.

Lemma 9.2. *There exist functions*

$$v, \omega, \theta \in C(\overline{Q}_T) \cap H^1(Q_T) \cap L^\infty(0,T; H^1((0,1))) \cap L^2(0,T; H^2((0,1))) \tag{9.12}$$

and a subsequence of $\{(v^N, \omega^N, \theta^{N-\frac{1}{2}})\}$ *(denoted again as* $\{(v^N, \omega^N, \theta^{N-\frac{1}{2}})\}$*) such that*

(9.13) $(v^N, \omega^N, \theta^{N-\frac{1}{2}}) \longrightarrow (v, \omega, \theta)$ *strongly in* $(C(\overline{Q}_T))^3$,

(9.14) ** weakly in* $(L^\infty(0,T; H^1((0,1))))^3$,

(9.15) *weakly in* $(L^2(0,T; H^2((0,1))))^3$,

(9.16) *weakly in* $(H^1(Q_T))^3$,

(when $N \to \infty$ *or* $h \to 0$*). There exists a subsequence of* $\{(v_h, \omega_h, \theta_{h-\frac{1}{2}})\}$ *(still denoted* $\{(v_h, \omega_h, \theta_{h-\frac{1}{2}})\}$*) such that*

$$(v_h, \omega_h, \theta_{h-\frac{1}{2}}) \longrightarrow (v, \omega, \theta) \text{ strongly in } (L^\infty(0,T; L^2((0,1))))^3 \tag{9.17}$$

The function θ *has the property*

$$\theta(x,t) > C, \quad (x,t) \in \overline{Q}_T, \tag{9.18}$$

where $C \in \mathrm{R}^+$.

Proof. The conclusions follow from (9.9)-(9.11) and inequality (9.8) that satisfy couples of functions v^N, v_h, ω^N, ω_h and $\theta^{N-\frac{1}{2}}$, $\theta_{h-\frac{1}{2}}$. Using the estimation (8.13) we can easily conclude that (9.18) is correct. □

Notice that, because of (3.11) and (3.26)-(3.28), we have

$$\int_0^t [\rho_{h-\frac{1}{2}}(x,0) - \rho_0(x)]^2 dx \to 0, \text{ as } h \to 0. \tag{9.19}$$

Furthermore, the sequences $\{v_h(x,0)\}$, $\{\omega_h(x,0)\}$, $\{\theta_{h-\frac{1}{2}}(x,0)\}$ converge, respectively, to v_0, ω_0 and θ_0 in $L^2((0,1))$ (ρ_0, v_0, ω_0 and θ_0 are introduced by (2.9)).

Lemma 9.3. *The functions ρ, v, ω, θ defined by Lemmas 9.1 and 9.2 satisfy equations (2.1)-(2.4) a.e. in Q_T.*

Proof. Equation (3.3) can be written in the from

$$\partial_t \rho_h(x,t) = -\rho_h^2(x,t)\partial_x v^N(x,t) \text{ on } Q_T. \tag{9.20}$$

For any test function $\varphi \in \mathcal{D}(Q_T)$ from (9.20) we obtain

$$\int_0^T\!\!\int_0^1 \rho_h(x,\tau)\partial_t\varphi(x,\tau)\,dx\,d\tau - \int_0^T\!\!\int_0^1 \rho_h^2(x,\tau)\partial_x v^N(x,\tau)\varphi(x,\tau)\,dx\,d\tau = 0. \tag{9.21}$$

Using the convergence $\rho_h(x,t) \to \rho(x,t)$ strongly and $\partial_x v^N(x,t) \to \partial_x v(x,t)$ weakly, from (9.21) immediately follows

$$\int_0^T\!\!\int_0^1 \partial_t\rho(x,\tau)\varphi(x,\tau)\,dx\,d\tau + \int_0^T\!\!\int_0^1 \rho^2(x,\tau)\partial_x v(x,\tau)\varphi(x,\tau)\,dx\,d\tau = 0.$$

for all $\varphi \in \mathcal{D}(Q_T)$.

Now, we choose $N = \frac{1}{h}$ large enough so that the support of the test function φ is away enough from the boundaries, that is $\operatorname{supp}\varphi \subset (h, 1-h) \times (0,T) = (\frac{1}{N}, 1-\frac{1}{N}) \times (0,T)$. Define

$$\varphi_k(t) = \varphi_h(x,t) \equiv \varphi([xN]h, t), \;\; kh \le x < (k+1)h, \tag{9.22}$$

$$\varphi_j(t) = \varphi_{h-\frac{1}{2}}(x,t) \equiv \varphi(([xN+\tfrac{1}{2}]h - \tfrac{1}{2})h, t), \;\; jh \le x < (j+1)h. \tag{9.23}$$

We can see that

$$\varphi_k(t) = 0, \;\text{ for }\; k = 0, 1, N-1, N, \tag{9.24}$$

$$\varphi_j(t) = 0, \;\text{ for }\; j = \tfrac{1}{2}, N - \tfrac{1}{2}. \tag{9.25}$$

Multiply equations (3.4) and (3.5) by $\varphi_k h$, sum it up for $k = 1, \dots, N-1$ and integrate over $[0,T]$ to get

$$\int_0^T \sum_{k=1}^{N-1} \partial_t v_k \varphi_k h\,d\tau + K\int_0^T \sum_{k=1}^{N-1} \delta(\rho\theta)_k \varphi_k h\,d\tau - \int_0^T \sum_{k=1}^{N-1} \delta(\rho\delta v)_k \varphi_k h\,d\tau = 0, \tag{9.26}$$

$$\frac{1}{A}\int_0^T \sum_{k=1}^{N-1} \partial_t \omega_k \varphi_k h\,d\tau - \int_0^T \sum_{k=1}^{N-1} \delta(\rho\delta\omega)_k \varphi_k h\,d\tau + \int_0^T \sum_{k=1}^{N-1} \frac{\omega_k}{\rho_k}\varphi_k h\,d\tau = 0. \tag{9.27}$$

Since $\varphi_h \to \varphi$, $\delta\varphi_{h-\frac{1}{2}} \to \partial_x\varphi$ $\partial_t\varphi_h \to \partial_t\varphi$ strongly converge as $h \to 0$, we can write equalities (9.26) and (9.27) as follows

$$\int_0^T\!\!\int_0^1 v_h\partial_t\varphi\,dx\,d\tau + K\int_0^T\!\!\int_0^1 \rho_{h-\frac{1}{2}}\theta_{h-\frac{1}{2}}\partial_x\varphi\,dx\,d\tau - \int_0^T\!\!\int_0^1 \rho_{h-\frac{1}{2}}\partial_x v^N \partial_x\varphi\,dx\,d\tau = \mathcal{O}(h), \tag{9.28}$$

$$\frac{1}{A}\int_0^T\!\!\int_0^1 \omega_h\partial_t\varphi\,dx\,d\tau - \int_0^T\!\!\int_0^1 \rho_{h-\frac{1}{2}}\partial_x\omega^N \partial_x\varphi\,dx\,d\tau - \int_0^T\!\!\int_0^1 \frac{\omega_h}{\rho_{h-\frac{1}{2}}}\,\varphi\,dx\,d\tau = \mathcal{O}(h), \tag{9.29}$$

where $\mathcal{O}(h) \to 0$ as $h \to 0$.

Using convergence (9.17), (9.6) and (9.16) we get that the functions ρ, v and ω satisfy equations (2.2) and (2.3) a.e. in Q_T. Similarly, multiplying (3.4)-(3.6), respectively, by $v_k\varphi_k h$, $A^{-1}\rho_k^{-1}\omega_k\varphi_k h$ and $\rho_j^{-1}\varphi_j h$, summing up for $k = 1, \ldots, N-1$ and $j = \frac{1}{2}, \ldots, N-\frac{1}{2}$ and integrating over $[0,T]$ we obtain

$$\begin{aligned}
&(9.30)\\
&-\frac{1}{2}\int_0^T\int_0^1 v_h^2\partial_t\varphi\,dx\,d\tau + \int_0^T\int_0^1 \rho_{h-\frac{1}{2}}(\partial_x v^N)^2\varphi\,dx\,d\tau + \int_0^T\int_0^1 \rho_{h-\frac{1}{2}}\partial_x v^N\, v_h\partial_x\varphi\,dx\,d\tau\\
&-K\int_0^T\int_0^1 \rho_{h-\frac{1}{2}}\theta_{h-\frac{1}{2}}\partial_x v^N\varphi\,dx\,d\tau - K\int_0^T\int_0^1 \rho_{h-\frac{1}{2}}\theta_{h-\frac{1}{2}}v_h\partial_x\varphi\,dx\,d\tau = \mathcal{O}(h),
\end{aligned}$$

$$\begin{aligned}
&(9.31)\\
&-\frac{1}{2A}\int_0^T\int_0^1 \omega_h^2\partial_t\varphi\,dx\,d\tau + \int_0^T\int_0^1 \rho_{h-\frac{1}{2}}(\partial_x \omega^N)^2\varphi\,dx\,d\tau + \int_0^T\int_0^1 \rho_{h-\frac{1}{2}}\partial_x \omega^N\, \omega_h\partial_x\varphi\,dx\,d\tau\\
&+\int_0^T\int_0^1 \frac{\omega_h^2}{\rho_{h-\frac{1}{2}}}\,\varphi\,dx\,d\tau = \mathcal{O}(h),
\end{aligned}$$

$$\begin{aligned}
&(9.32)\\
&-\int_0^T\int_0^1 \theta_{h-\frac{1}{2}}\partial_t\varphi\,dx\,d\tau + K\int_0^T\int_0^1 \rho_{h-\frac{1}{2}}\theta_{h-\frac{1}{2}}\partial_x v^N\varphi\,dx\,d\tau - \int_0^T\int_0^1 \rho_{h-\frac{1}{2}}(\partial_x v^N)^2\varphi\,dx\,d\tau\\
&-\int_0^T\int_0^1 \rho_{h-\frac{1}{2}}(\partial_x \omega^N)^2\varphi\,dx\,d\tau - \int_0^T\int_0^1 \frac{\omega_h^2}{\rho_{h-\frac{1}{2}}}\,\varphi\,dx\,d\tau + D\int_0^T\int_0^1 \rho_{h-\frac{1}{2}}\partial_x\theta^N\partial_x\varphi\,dx\,d\tau\\
&= \mathcal{O}(h),
\end{aligned}$$

where $\mathcal{O}(h) \to 0$ as $h \to 0$. After summing the above equations we get

$$\begin{aligned}
&(9.33)\\
&\quad -\int_0^T\int_0^1 \left(\tfrac{1}{2}v_h^2 + \tfrac{1}{2A}\omega_h^2 + \theta_{h-\frac{1}{2}}\right)\partial_t\varphi\,dx\,d\tau + \int_0^T\int_0^1 \rho_{h-\frac{1}{2}}\partial_x v^N\, v_h\partial_x\varphi\,dx\,d\tau\\
&\quad -K\int_0^T\int_0^1 \rho_{h-\frac{1}{2}}\theta_{h-\frac{1}{2}}v_h\partial_x\varphi\,dx\,d\tau + \int_0^T\int_0^1 \rho_{h-\frac{1}{2}}\partial_x\omega^N\,\omega_h\partial_x\varphi\,dx\,d\tau\\
&\quad +D\int_0^T\int_0^1 \rho_{h-\frac{1}{2}}\partial_x\theta^N\partial_x\varphi\,dx\,d\tau = \mathcal{O}(h).
\end{aligned}$$

Because of (9.6), (9.17) and (9.16), from (9.33), for $h \to 0$, follows

$$\begin{aligned}
(9.34)\quad &\int_0^T\int_0^1 \left(v\partial_t v + \tfrac{1}{A}\omega\partial_t\omega + \partial_t\theta\right)\varphi\,dx\,d\tau - \int_0^T\int_0^1 \partial_x(\rho v\partial_x v)\varphi\,dx\,d\tau\\
&+K\int_0^T\int_0^1 \partial_x(\rho\theta v)\varphi\,dx\,d\tau - \int_0^T\int_0^1 \partial_x(\rho\omega\partial_x\omega)\varphi\,dx\,d\tau - D\int_0^T\int_0^1 \partial_x(\rho\partial_x\theta)\varphi\,dx\,d\tau = 0.
\end{aligned}$$

Now, already proven equations (2.2) and (2.3) we multiply, respectively, by $v\varphi$ and $A^{-1}\rho^{-1}\omega\varphi$, integrate over $[0,1]\times[0,T]$ and add up to (9.34). So we get that (2.4) is satisfied. □

Lemma 9.4. *The functions ρ, v, ω and θ satisfy the following conditions*

$$\rho(x,0) = \rho_0(x),\ \ v(x,0) = v_0(x),\ \ \omega(x,0) = \omega_0(x),\ \ \theta(x,0) = \theta_0(x), \tag{9.35}$$

$$v(0,t) = v(1,t) = 0,\ \ \omega(0,t) = \omega(1,t) = 0,\ \ \partial_x\theta(0,t) = \partial_x\theta(1,t) = 0, \tag{9.36}$$

for $x \in (0,1)$ and $t \in (0,T)$ (ρ_0, v_0, ω_0 and θ_0 are introduced by (2.9)).

Proof. We choose $\varphi \in C^\infty([0,T])$ that is equal to zero at some neighborhood of point T, while $\varphi(0) \neq 0$, and $u \in L^2((0,1))$. We can apply the Green's formula for the functions v^N and v and find out

$$\int_0^T\int_0^1 \partial_t v^N \varphi u \, dx \, d\tau + \int_0^T\int_0^1 v^N \varphi' u \, dx \, d\tau = -\varphi(0) \int_0^1 v^N(x,0)u(x)\, dx, \tag{9.37}$$

$$\int_0^T\int_0^1 \partial_t v \varphi u \, dx \, d\tau + \int_0^T\int_0^1 v \varphi' u \, dx \, d\tau = -\varphi(0) \int_0^1 v(x,0)u(x)\, dx. \tag{9.38}$$

Taking into account the strong convergence of $v^N(x,t) \to v(x,t)$, $v^N(x,0) \to v_0(x)$ and the weak convergence of $\partial_t v^N \to \partial_t v$ and comparing (9.37) and (9.38) we get

$$v(x,0) = v_0(x), \text{ a.e. in } (0,1).$$

In the same way we obtain $\omega(x,0) = \omega_0(x)$, $\rho(x,0) = \rho_0(x)$ and $\theta(x,0) = \theta_0(x)$ for $x \in (0,1)$.

Now we take $\varphi \in C^\infty([0,1])$ with the property $\varphi(0) \neq 0$ and that is equal to zero at some neighborhood of point 1. Let be $u \in L^2((0,T))$. We apply the Green's formula for the functions v^N and v again. It holds

$$\int_0^T\int_0^1 \partial_x v^N \varphi u \, dx \, d\tau + \int_0^T\int_0^1 v^N \varphi' u \, dx \, d\tau = -\varphi(0) \int_0^T v^N(0,t)u(t)\, dt, \tag{9.39}$$

$$\int_0^T\int_0^1 \partial_x v \varphi u \, dx \, d\tau + \int_0^T\int_0^1 v \varphi' u \, dx \, d\tau = -\varphi(0) \int_0^T v(0,t)u(t)\, dt. \tag{9.40}$$

Taking into account $v^N(0,t) = 0$, from (9.39) and (9.40), when $N \to \infty$, we obtain

$$v(0,t) = 0, \text{ a.e. in } (0,T).$$

In the same way we get $\omega(0,t) = 0$. For the functions $\partial_x \theta^N$ and $\partial_x \theta$ we have

$$\int_0^T\int_0^1 \partial_{xx} \theta^N \varphi u \, dx \, d\tau + \int_0^T\int_0^1 \partial_x \theta^N \varphi' u \, dx \, d\tau = -\varphi(0) \int_0^T \partial_x \theta^N(0,t)u(t)\, dt, \tag{9.41}$$

$$\int_0^T\int_0^1 \partial_{xx} \theta \varphi u \, dx \, d\tau + \int_0^T\int_0^1 \partial_x \theta \varphi' u \, dx \, d\tau = -\varphi(0) \int_0^T \partial_x \theta(0,t)u(t)\, dt, \tag{9.42}$$

Comparing (9.41) and (9.42), when $N \to \infty$, and using the property $\partial_x \theta^N(0,t) = \delta\theta_0(t) = 0$, we get easily that

$$\partial_x \theta(0,t) = 0, \text{ a.e. in } (0,T).$$

Finally, taking $\varphi \in C^\infty([0,1])$ with the property $\varphi(1) \neq 0$ and that is equal to zero at some neighborhood of point 0, we conclude as above, that

$$v(1,t) = \omega(1,t) = \partial_x \theta(1,t) = 0 \text{ a.e. in } (0,T)$$

is true. □

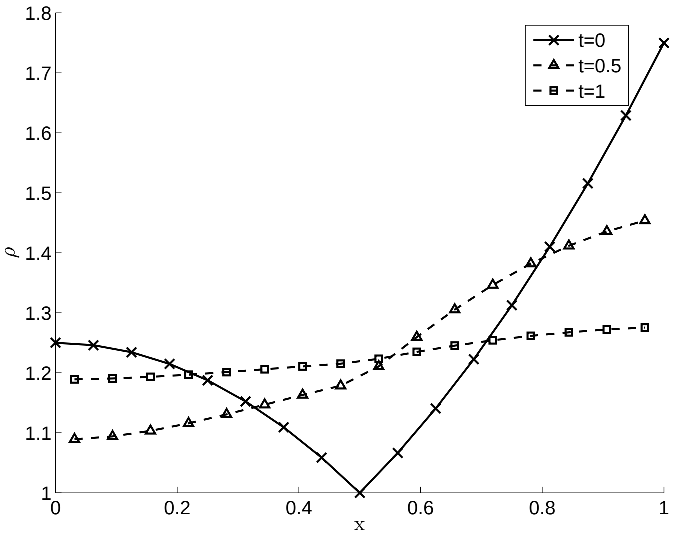

FIGURE 1. Numerical results at different time moments - density (ρ). The initial values at discretization points are denoted with $\times$.

10. Numerical example

In this section we consider the numerical solutions obtained by using finite difference approach on the chosen test example.

In order to determine numerical solutions of the system (2.1)-(2.7), the temporal discretization of the system (3.3)-(3.13), which was obtained by using the semi-discrete approach combined with the described spatial discretization, should be performed. The system (3.3)-(3.6) is actually the ordinary differential equation system of the first order in time variable that can be written in the form

$$\dot{u}(t) = F(u(t)), \tag{10.1}$$

where vector u consists of $4N-2$ unknown functions, i.e., $u = (\rho_j, v_k, \omega_k, \theta_j)$, $j = \frac{1}{2}, \ldots, N - \frac{1}{2}$, $k = 1, \ldots, N-1$. The corresponding boundary conditions are given with (3.9)-(3.10), while the initial conditions are defined with (3.11)-(3.13). In this work the temporal discretization is obtained by approximating numerically the system (10.1) using the second-order strongly stable explicit Runge-Kutta method (see, for example, [10, 9]) given by:

$$\begin{aligned} u^{(1)} &= u^n + \Delta t F(u^n) \\ u^{n+1} &= \frac{1}{2}u^n + \frac{1}{2}u^{(1)} + \frac{1}{2}\Delta t F(u^{(1)}), \end{aligned}$$

Here u^n denotes the numerical solution of system (10.1) at time moment $t^n = n\Delta t$ for the chosen time step Δt. For stability reasons of obtained numerical scheme, we choose $\Delta t = \mathcal{O}(h^2)$.

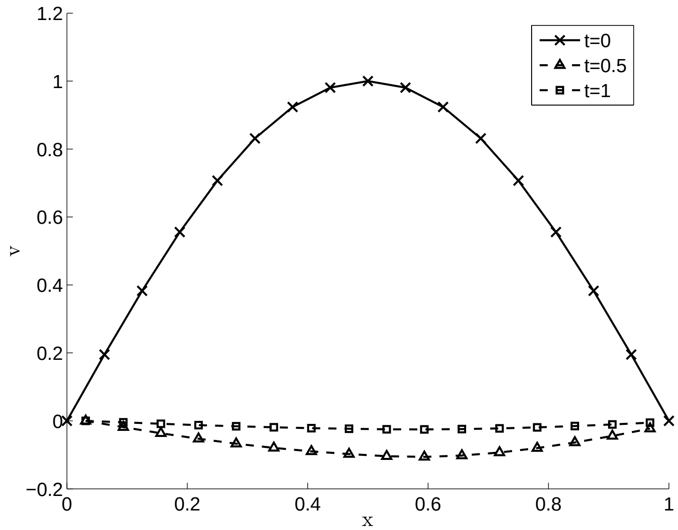

FIGURE 2. Numerical results at different time moments - velocity (v). The initial values at discretization points are denoted with ×.

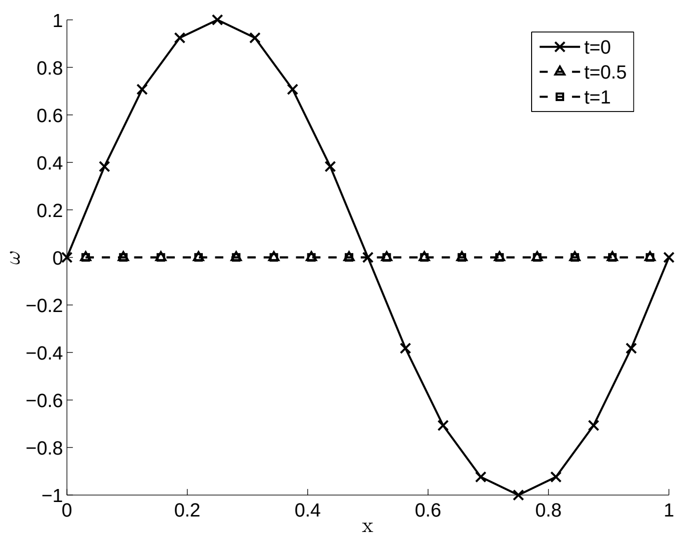

FIGURE 3. Numerical results at different time moments - microrotation velocity (ω). The initial values at discretization points are denoted with ×.

We consider here the numerical solutions of problem (2.1)-(2.7) defined with the following initial functions:

$$\rho_0(x) = \left|x^2 - \frac{1}{4}\right| + 1, \tag{10.2}$$

$$v_0(x) = \sin(\pi x), \tag{10.3}$$

$$\omega_0(x) = \sin(2\pi x), \tag{10.4}$$

$$\theta_0(x) = 2 + \cos(\pi x), \tag{10.5}$$

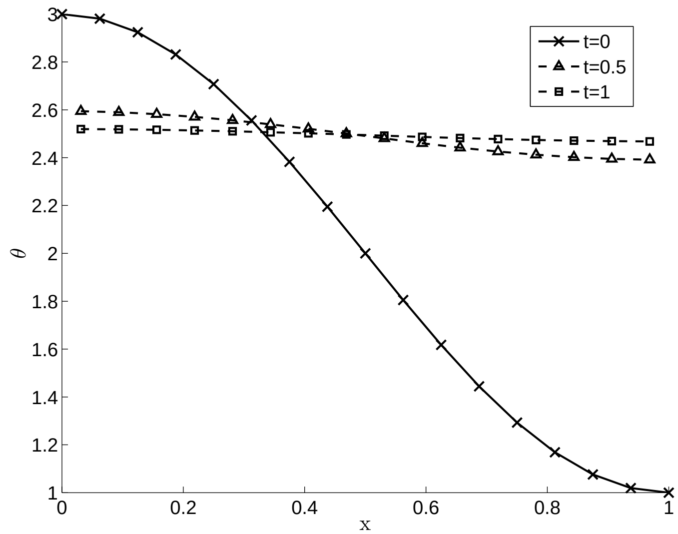

FIGURE 4. Numerical results at different time moments - temperature (θ). The initial values at discretization points are denoted with x.

TABLE 1. L^∞ norm for the differences between numerical solution at $t = 20$ and stationary solution.

N	$\|\rho^N(\cdot, 20) - \rho^S\|_\infty$	$\|\omega^N(\cdot, 20) - \omega^S\|_\infty$	$\|v^N(\cdot, 20) - v^S\|_\infty$	$\|\theta^N(\cdot, 20) - \theta^S\|_\infty$
8	1.11×10^{-3}	1.54×10^{-14}	5.90×10^{-13}	2.34×10^{-2}
16	2.79×10^{-4}	3.14×10^{-14}	6.76×10^{-14}	5.98×10^{-3}
32	6.99×10^{-5}	9.30×10^{-14}	3.93×10^{-14}	1.50×10^{-3}
64	1.75×10^{-5}	4.68×10^{-13}	3.44×10^{-14}	3.76×10^{-4}

and parameters $A = K = D = 1$.

In Figures 1-4 we present the numerical solution of the considered problem at different time moments. We used $N = 16$ points in the spatial discretization. The calculations were carried out on a sufficiently fine grid in time, which eliminates the error of approximation in time in comparison with the approximation error in space.

Since the exact analytical solution of the considered problem is unknown, we can not compare numerical solutions with the exact ones. However, we can use the fact that that the solution $(\rho, v, \omega, \theta)$ of the system (2.1)-(2.7) converges, as $t \to \infty$, to the stationary constant solution $(\rho^S, v^S, \omega^S, \theta^S) = (\alpha^{-1}, 0, 0, E_1)$ as explained in Section 2 (see [14]), so that the obtained numerical solution at some large time moment can be compared with this stationary solution. In the considered example we have $\alpha^{-1} = 1.226285790315$ and $E_1 = 2.5$. The difference of the numerical solution obtained with finite difference method from this stationary solution is taken at $t = 20$. The numerical results are presented in Table 1.

11. Conclusion

In this paper the finite difference scheme for the nonstationary 1D flow of the compressible viscous and heat-conducting micropolar fluid, which is in the thermodynamical sense perfect and polytropic, with the homogeneous boundary conditions for velocity, microrotation and heat flux, is defined and analyzed. The sequence of the approximate solution is constructed as a solution of the finite difference approximate equations system, which is derived by using the appropriate finite difference spatial discretization. The properties of these approximate solutions are analyzed and their convergence to the strong solution of our problem globally in time is proved. In this way the global existence of the solution is verified. The numerical properties of the proposed scheme are presented on the chosen test example.

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