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A PARALLEL VARIATIONAL MULTISCALE METHOD FOR INCOMPRESSIBLE FLOWS BASED ON THE PARTITION OF UNITY

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Abstract. A parallel variational multiscale method based on the partition of unity is proposed for incompressible flows in this paper. Based on two-grid method, this algorithm localizes the global residual problem of variational multiscale method into a series of local linearized residual problems. To decrease the undesirable effect of the artificial homogeneous Dirichlet boundary condition of local sub-problems, an oversampling technique is also introduced. The globally continuous finite element solutions are constructed by assembling all local solutions together using the partition of unity functions. Numerical simulations demonstrate the high efficiency and flexility of the new algorithm.

Key words. Incompressible flows, variational multiscale method, local and parallel, partition of unity, oversampling.

1. Introduction

The variational multiscale method was proposed to solve multiscale problems by Hughes and co-workers in [1, 2]. A projection of the large scales in Large Eddy Simulation method into appropriate subspaces was introduced. Since then much attention has been paid in this field. For example, John and Kaya [3] gave the finite element analysis of a variational multiscale method for the Navier-Stokes equations. Gravemeier et al. [4] also presented the three-level variational multiscale method. Zheng et al. improved the finite element variational multiscale method by introducing two Gauss integration method [5] and adaptive technique [6]. Zhang et al. [7], Yu et al. [8], Shan et al. [9] et al. presented subgrid model, projection basis and modular type to improve the variational multiscale methods, respectively.

Based on the observation that in numerical simulations low frequency components can be approximated well by the relative coarse grid and high frequency components can be computed on a fine grid by some local and parallel procedure, the parallel finite element computations have been widely used [10, 11, 12, 13]. Combining the partition of unity method [14, 15] and the parallel adaptive algorithm from [11], Holst [16, 17] constructed the parallel partition of unity method (PPUM). Zheng et al. [19, 20] developed some local and parallel finite element algorithms based on the partition of unity. Song et al. [18] presented an adaptive local postprocessing technique based on the partition of unity method for the Navier-Stokes equations. There are also some papers improving the variational multiscale methods by combining with two-grid method or local and parallel techniques [21, 22].

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It is natural to consider to add the local parallel method to the variational multiscale method in order to retain the best features of both methods and overcome many of their defects. In particular, we use the variational multiscle method based on two local Gauss integrations [5] since it avoids constructing the projection operator, keeps the same efficiency and does not need extra storage compared with common VMS method. Comparing with the parallel method in [22], we add an artificial stabilization term in the local and parallel procedure by considering the residual as a subgrid value, which keeps the sub-problems stable. Then, an oversampling technique is introduced in order to overcome the undesirable effect of the artificial homogeneous Dirichlet boundary conditions of local sub-problems. The interesting points in this algorithm lie in: firstly, a class of partition of unity is derived by a given triangulation, which guides the domain decomposition; secondly, the series of local linearized residual problems are implemented in parallel, and they require less communication between each other; finally, the globally continuous finite element solution is obtained by assembling all local solutions together via the partition of unity functions.

The outline of the paper is as follows. We introduce the Navier-Stokes equations, the notations and some well-known results for the finite element methods in section 2. In section 3, we first propose the parallel variational multiscale method based on the partition of unity and then derive the error estimates. In section 4, the implementation and some numerical simulations are presented to illustrate the efficiency of our method. And finally a short conclusion is presented in section 5.

2. The Navier-Stokes Equations

We consider the following incompressible flows

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \ \ \texttt{in}\ \Omega, \\ \nabla\cdot\mathbf{u} &= 0 \ \ \texttt{in}\ \Omega, \\ \mathbf{u} &= 0 \ \ \texttt{on}\ \partial\Omega, \end{aligned} \tag{1}$$

where Ω represents a polyhedral domain in R^d (d =2, 3) with boundary $\partial\Omega$, $\mathbf{u}$, p, $\mathbf{f}$ and $\nu > 0$ represent the velocity vector, pressure, prescribed body force, kinematic viscosity respectively. And ν is inversely proportional to the Reynolds number Re.

For a bounded domain $\Omega \subset R^d$, we use the standard notations for Sobolev spaces $W^{s,k}(\Omega)$ and their associated norms [23, 24]. Especially when $k = 2$, $H^s(\Omega) = W^{s,2}(\Omega)$ denotes the usual Soblev space, $\|\cdot\|_{s,\Omega} = \|\cdot\|_{s,2,\Omega}$ denotes standard Soblev norm, $(\cdot,\cdot)_s$ denotes the inner product in $L^2(\Omega)$ or its vector value version. The space $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$ is equipped with the usual norm $\|\nabla\cdot\|_{0,\Omega}$ or its equivalent norm $\|\cdot\|_{1,\Omega}$ due to the Poincare's inequality. $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. In the following we will denote the spaces consisting of vector-valued functions in boldface.

For sub-domains $D \subset G \subset \Omega$, $D \subset\subset G$ means that $\mathrm{dist}(\partial D\backslash\partial\Omega, \partial G\backslash\partial\Omega) > 0$. Throughout the paper we use C to denote a generic positive constant whose value may change from place to place but remains independent of the mesh parameter h.

The standard variational formulation of (1) is given by: find $(\mathbf{u}, p) \in (\mathbf{X}, M)$ satisfying

$$\nu a(\mathbf{u},\mathbf{v}) + b(\mathbf{u},\mathbf{u},\mathbf{v}) - d(\mathbf{v},p) + d(\mathbf{u},q) = (\mathbf{f},\mathbf{v}), \quad \forall(\mathbf{v},q) \in (\mathbf{X}, M), \tag{2}$$

where

$$\mathbf{X} = \mathbf{H}_0^1(\Omega), \quad M = L_0^2(\Omega) = \{q \in L^2(\Omega); \int_\Omega q\mathbf{dx} = 0\},$$

and

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &= (\nabla\mathbf{u},\nabla\mathbf{v}), \quad d(\mathbf{v},p) = (\nabla\cdot\mathbf{v},p), \\ b(\mathbf{u},\mathbf{w},\mathbf{v}) &= ((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v}) + \frac{1}{2}((\mathrm{div}\mathbf{u})\mathbf{w},\mathbf{v}) \\ &= \frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v}) - \frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}). \end{aligned}$$

They have the following properties:

$$\begin{aligned} &a(\mathbf{u},\mathbf{v}) \le \|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}, \quad \forall \mathbf{u},\mathbf{v}\in\mathbf{X}, \\ &d(\mathbf{v},q)| \le \|\mathbf{v}\|_{1,\Omega}\|q\|_{0,\Omega}, \quad \forall(\mathbf{v},q)\in(\mathbf{X},M), \\ &b(\mathbf{u},\mathbf{v},\mathbf{w}) = b(\mathbf{u},\mathbf{w},\mathbf{v}), \quad \forall\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}, \\ &|b(\mathbf{u},\mathbf{v},\mathbf{w})| \le C\|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}\|\mathbf{w}\|_{1,\Omega}, \quad \forall\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}. \end{aligned} \tag{3}$$

It is well-known [25, 26] that, if ν and $\mathbf{f}$ satisfy the following conditions

$$\frac{N_b||f||_{-1,\Omega}}{\nu^2} < 1, \quad N_b = \sup_{\mathbf{u},\mathbf{w},\mathbf{v}\in H_0^1(\Omega),\mathbf{u},\mathbf{w},\mathbf{v}\neq 0} \frac{b(\mathbf{u},\mathbf{w},\mathbf{v})}{||\mathbf{u}||_{1,\Omega}||\mathbf{w}||_{1,\Omega}||\mathbf{v}||_{1,\Omega}},$$

then, the problem (2) has a unique solution.

Let τ_h be a regular triangulation of the domain Ω, and h denote the maximum diameter of the elements in τ_h. We use $P_2 - P_1$ elements in this paper, which means that $\mathbf{X}^h$ and M^h contain piecewise polynomials of degree 2 and 1 respectively. $(\mathbf{X}^H, M^H)$ is defined in the same way but on τ_H with coarser mesh size H where $H > h$. Set $(\mathbf{X}_0^h, M_0^h) = (\mathbf{X}^h, M^h)\bigcap(\mathbf{X}, M)$.

It is known that the standard Galerkin finite element discretization of (2) is unstable in the case of high Reynolds number (or smaller viscosity). Therefore, we consider the finite element variational multiscale method [5]: find $(\mathbf{u}_h, p_h) \in (\mathbf{X}_0^h, M_0^h)$ satisfying

(4)
$$\nu a(\mathbf{u}_h,\mathbf{v}) + b(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}) - d(\mathbf{v},p_h) + d(\mathbf{u}_h,q) + G(\mathbf{u}_h,\mathbf{v}) = (\mathbf{f},\mathbf{v}) \quad \forall(\mathbf{v},q)\in(\mathbf{X}_0^h, M_0^h),$$

where $G(\mathbf{u}_h,\mathbf{v}) = \alpha((I-\Pi_h)\nabla\mathbf{u}_h, (I-\Pi_h)\nabla\mathbf{v})$. Let $\mathbb{L} = L^2(\Omega)^{d\times d}$ and $\Pi_h : \mathbb{L} \to \mathbb{L}^h$ be the orthogonal projection operator with the following properties:

$$((I-\Pi_h)\mathbf{r}, \mathbf{g}_h) = 0, \quad \forall\mathbf{r}\in\mathbb{L}, \ \mathbf{g}_h\in\mathbb{L}^h, \tag{5}$$

$$\|\Pi\mathbf{r}\|_0 \le C\|\mathbf{r}\|_0, \quad \forall\mathbf{r}\in\mathbb{L}, \tag{6}$$

$$\|(I-\Pi_h)\mathbf{r}\|_0 \le Ch\|\mathbf{r}\|_1, \quad \forall\mathbf{r}\in\mathbb{L}\cap H^1(\Omega)^{d\times d}, \tag{7}$$

where I is the identify operator.

According to [5], we can use the equivalent formulation of G based on two local Gauss integrations as follows,

$$G(\mathbf{u}_h,\mathbf{v}) = \alpha\sum_{\Omega_e\in\tau_h}\{\int_{\Omega_e,s}\nabla\mathbf{u}_h\nabla\mathbf{v}\mathbf{d}x - \int_{\Omega_e,1}\nabla\mathbf{u}_h\nabla\mathbf{v}\mathbf{d}x\} \quad \forall\mathbf{u}_h, \ \mathbf{v}\in\mathbf{X}^h,$$

where $\int_{\Omega_e,i} g(x)\mathbf{d}x$ denotes an appropriate Gauss integral over Ω_e which is exact for polynomials of degree i, $i = 1, 2$. For all test functions $\mathbf{v}\in\mathbf{X}^h$, $\nabla\mathbf{u}_h$ must be piecewise constant when $i = 1$. And set $\alpha = O(h^2)$ in order to keep the rates of convergence.

In [22] Shang proved the following theorem.

Theorem 1. *Assume that* $(\mathbf{u}, p)$ *is a nonsingular solution to the Navier-Stokes equations satisfying* $(\mathbf{u}, p) \in (H^3(\Omega)^n \bigcap H_0^1(\Omega)^n) \times (H^2(\Omega)\bigcap L_0^2(\Omega))$, *and* α *tends* *to zero as h tends to zero. Then the solution $(\mathbf{u}_h, p_h)$ computed by the numerical scheme (4) satisfies*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq ch^2 + c\alpha, \tag{8}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} + \|p - p_h\|_{-1,\Omega} \leq ch(h^2 + \alpha) + c\alpha^2. \tag{9}$$

3. The Parallel Variational Multiscale Method based on the Partition of Unity

In this section, we will derive a partition of unity based on a given triangulation, and propose a framework for domain decomposition.

First, choose a regular conforming triangulation τ_{H_p} for Ω. For each node $x_i \in \tau_{H_p}, \imath = 1, 2, \cdots, N$, (here N is the number of nodes on τ_{H_p}), define corresponding continuous linear Lagrange basis function φ_i, such that $\varphi_i(x_m) = \delta_{i,m}$. Let $\omega^i = supp\varphi_i \cap \Omega,\ i = 1, 2, \cdots, N$ denote the local subdomain.

Then, we denote $\omega^{i,0} = \omega^i$, which means the local domain without oversampling. To enlarge this domain we introduce one layer oversampling $\omega^{i,1}$, which is the union of the supports of φ_i and one layer of its neighbors, and also multiple layers oversampling $\omega^{i,s}$:

$$\omega^{i,1} = \bigcup_{x_m \in \omega^{i,0}} \omega^m, \quad \omega^{i,s} = \bigcup_{x^m \in \omega^{i,s-1}} \omega^m.$$

We will use Fig1 below to demonstrate the definition of oversampling.

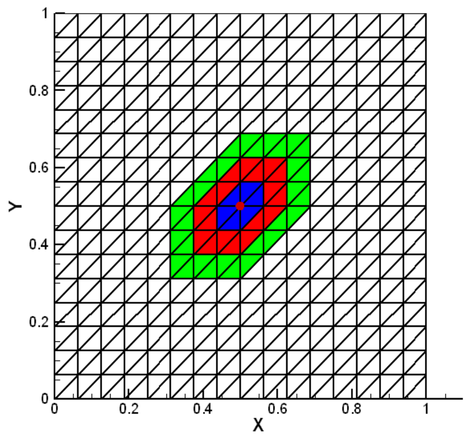


FIGURE 1. Local domain with oversampling. $\omega^{i,0}$=blue region, $\omega^{i,1}$=blue and red regions, $\omega^{i,2}$=blue, red and green regions.

It's easy to check that, for any given s, $\{\omega^{i,s}\}_1^N$ is an open cover of Ω and $\{\varphi_i\}_1^N$ is a partition of unity subordinate to the cover $\{\omega^{i,s}\}_1^N$ which satisfying

$$\texttt{supp}\varphi_i \subset \overline{\omega^{i,s}},\ \forall i. \tag{10}$$

$$\sum_i \varphi_i \equiv 1 \texttt{ on } \Omega. \tag{11}$$

$$||\varphi_i||_{L^\infty(R^n)} \leq C_\infty. \tag{12}$$

$$||\nabla\varphi_i||_{L^\infty(R^n)} \leq \frac{C_G}{H_p}. \tag{13}$$

where $C_\infty,\ C_G$ are two constants.

Based on above special partition of unity, we develop a new local and parallel variational mutiscale method as follows.

ALGORITHM PVMS-PU:

Step 1. Use the variational multiscale method to find a globally coarse grid solution $(\mathbf{u}_H, p_H) \in (\mathbf{X}_0^H, M_0^H)$ such that

$$\begin{aligned} \nu a(\mathbf{u}_H, \mathbf{v}) + b(\mathbf{u}_H, \mathbf{u}_H, \mathbf{v}) - d(\mathbf{v}, p_H) + d(\mathbf{u}_H, q) \quad &+G(\mathbf{u}_H, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \\ &\forall(\mathbf{v}, q) \in (\mathbf{X}_0^H, M_0^H). \end{aligned} \tag{14}$$

Step 2. For a given τ_{H_p}, fix $s \geq 1$, correct the residue $(\mathbf{e}^i, \epsilon^i)$ on a fine grid of each overlapping subdomain $\omega^{i,s}$ of τ_{H_p} in parallel, $(\mathbf{e}^i, \epsilon^i) \in (\mathbf{X}_0^h(\omega^{i,s}), M_0^h(\omega^{i,s})),\ i = 1, 2, \cdots, N$, such that

$$\begin{aligned} &\nu a(\mathbf{e}^i, \mathbf{v}) + b(\mathbf{e}^i, \mathbf{u}_H, \mathbf{v}) + b(\mathbf{u}_H, \mathbf{e}^i, \mathbf{v}) - d(\mathbf{v}, \epsilon^i) + d(\mathbf{e}^i, q) + \beta(\nabla \mathbf{e}^i, \nabla \mathbf{v}) \\ &\quad = (\mathcal{R}(\mathbf{u}_H, p_H), \mathbf{v}), \quad \forall(\mathbf{v}, q) \in (\mathbf{X}_0^h(\omega^{i,s}), M_0^h(\omega^{i,s})), \end{aligned} \tag{15}$$

where $(\mathcal{R}(\mathbf{u}_H, p_H), \mathbf{v}) = (\mathbf{f}, \mathbf{v}) - \nu(\mathbf{u}_H, \mathbf{v}) - b(\mathbf{u}_H, \mathbf{u}_H, \mathbf{v}) + d(\mathbf{v}, \mathbf{p}_H) - d(\mathbf{u}_H, q)$. Here

$$\begin{aligned} \mathbf{X}_0^h(\omega^{i,s}) \quad &:= \quad \{\mathbf{v} \in \mathbf{X}^h(\Omega) : \ \text{supp } \mathbf{v} \subset\subset \omega^{i,s}\}, \\ M_0^h(\omega^{i,s}) \quad &:= \quad \{q \in M^h(\Omega) : \ \text{supp } q \subset\subset \omega^{i,s} \text{ and } \int_{\omega^{i,s}} q \mathrm{d}x = 0\}. \end{aligned}$$

Step 3. Update: $(\mathbf{u}^i, p^i) = (\mathbf{u}_H, p_H) + (\mathbf{e}^i, \epsilon^i)$ in $\omega^{i,s}$.

Step 4. Obtain the finite element solution $\mathbf{u}^h = \sum_{i=1}^{N} \varphi_i \mathbf{u}^i,\ p^h = \sum_{i=1}^{N} \varphi_i p^i$.

In order to get the error estimate of this algorithm we first introduce a lemma and regularity property from [12], which are listed as two lemmas here.

Lemma 1. *Suppose that* $\mathbf{g} \in H^{-1}(\Omega)^n, 0 < H \leq \bar{h}_0$ *and* $S \subset\subset \Omega_0 \subset \Omega$. *Then* $(\mathbf{w}, r) \in X_h(\Omega) \times M_h(\Omega)$ *defined by*

$$\begin{aligned} &\nu a(\mathbf{w}, v) + (\mathbf{u}_h, \mathbf{w}, \mathbf{v}) + b(\mathbf{w}, \mathbf{u}_H, \mathbf{v} - d(\mathbf{v}, r) + d(\mathbf{w}, q) = (\mathbf{g}, \mathbf{v}), \\ &\quad \forall(\mathbf{v}, q) \in X_0^h(\Omega_0) \times M_0^h(\Omega_0) \end{aligned} \tag{16}$$

satisfies

$$\|\mathbf{w}\|_{1,D} + \|r\|_{0,D} \leq C(\|\mathbf{w}\|_{0,\Omega_0} + \|r\|_{-1,\Omega_0} + \|g\|_{-1,\Omega_0}). \tag{17}$$

Lemma 2. *There exists a unique* $(\Psi_h, \Psi_h) \in X_h^0(\Omega) \times M_h^0(\Omega)$ *satisfying the dual problem:*

(18)

$$\begin{aligned} &(\nu + \beta)a(\mathbf{v}, \Phi_h) + b(\mathbf{u}_h, \mathbf{v}, \Phi_h) + b(\mathbf{v}, \mathbf{u}_h, \Phi_h) + d(\mathbf{v}, \Psi_h) - d(\Phi_h, q) = (\psi, \mathbf{v}) + (\phi, q), \\ &\quad \forall(\mathbf{v}, q) \in H_0^1(\Omega) \times L_0^2(\Omega) \end{aligned}$$

and has the following estimates

$$\|\Phi - \Phi_h\|_{1,\Omega} + \|\Psi - \Psi_h\|_{0,\Omega} \leq Ch(\|\phi\|_{0,\Omega} + \|\psi\|_{1,\Omega}),$$

$$\|\Phi_h\|_{1,\Omega} + \|\Psi_h\|_{0,\Omega} \leq C(\|\phi\|_{0,\Omega} + \|\psi\|_{1,\Omega}).$$

We also need the following lemma which can be proved easily so that we don't show the details here.

Lemma 3. *Let $C_0 > 0$ be a constant, and $\{\varphi_i\}_1^N$ be the partition of unity based on τ_{H_p} with $H_p \geq C_0$. Then there exist constant C_1, C_2, C_3, C_4 independent of N satisfying the following inequalities which will be used in the proof of next theorem.*

$$\|\sum_{i=1}^{N}\varphi_i\mathbf{v}\|_{1,\Omega} \leq C_1\sum_{i=1}^{N}\|\varphi_i\mathbf{v}\|_{1,\Omega}, \quad \forall{\in}H^1(\Omega), \tag{19}$$

$$\|\sum_{i=1}^{N}\varphi_i q\|_{0,\Omega} \leq C_2\sum_{i=1}^{N}\|\varphi_i q\|_{0,\Omega}, \quad \forall q \in L^2(\Omega), \tag{20}$$

$$\|\mathbf{v}\|_{1,\Omega}^2 + \|q\|_{0,\Omega}^2 \leq C_3(\sum_{i=1}^{N}(\|\varphi_i\mathbf{v}\|_{1,\Omega}^2 + \|\varphi_i q\|_{0,\Omega}^2)), \tag{21}$$

$$\|\varphi_i\mathbf{v}\|_{1,\Omega}^2 + \|\varphi_i q\|_{1,\Omega}^2 \leq C_4(\|\mathbf{v}\|_{1,\omega^i}^2 + \|q\|_{1,\omega^i}^2). \tag{22}$$

Then we can prove the following theorem.

Theorem 2. *Assume that the conditions of Theorem 1 hold, $0 < H \leq h$, for a given $H_p \geq C_0$ and $s \geq 1$, the solution $(\mathbf{u}^h, p^h)$ defined by Algorithm PVMS-PU satisfies*

$$\|\mathbf{u}_h - \mathbf{u}^h\|_{1,\Omega} + \|p_h - p^h\|_{0,\Omega} \leq C(H^3 + H\alpha_H + \alpha_H^2 + \beta(H^2 + \alpha_H)). \tag{23}$$

Proof. **Step1**. In order to get the final result, let $D = \omega^i$, $\Omega_0 = \Omega^{i,s}$, then, first estimate $\|\mathbf{e}^i\|_{0,\Omega_0} + \|\epsilon^i\|_{-1,\Omega_0}$. From (4) and (14) we can easily get the equality:

$$\begin{aligned}&\nu a(\mathbf{u}_h - \mathbf{u}_H, \Phi_H) + b(\mathbf{u}_h - \mathbf{u}_H, \mathbf{u}_H, \Phi_H) + b(\mathbf{u}_H, \mathbf{u}_h - \mathbf{u}_H, \Phi_H)\\ &+b(\mathbf{u}_h - \mathbf{u}_H, \mathbf{u}_h - \mathbf{u}_H, \Phi_H) - d(\Phi_H, p_h - p_H) + d(\mathbf{u}_h - \mathbf{u}_H, \Psi_H)\\ &+G(\mathbf{u}_h,, \Phi_H) - G(\mathbf{u}_H, \Phi_H) = 0.\end{aligned} \tag{24}$$

It follows from (4) and (15) that

$$\begin{aligned}&(\nu+\beta)a(\mathbf{e}^i, \mathbf{v}) + b(\mathbf{e}^i, \mathbf{u}_H, \mathbf{v}) + b(\mathbf{u}_H, \mathbf{e}^i, \mathbf{v}) - d(\mathbf{v}, \epsilon^i) + d(\mathbf{e}^i, q)\\ &= \nu a(\mathbf{u}_h - \mathbf{u}_H, \mathbf{v}) + b(\mathbf{u}_h - \mathbf{u}_H, \mathbf{u}_H, \mathbf{v}) + b(\mathbf{u}_H, \mathbf{u}_h - \mathbf{u}_H, \mathbf{v})\\ &\quad +b(\mathbf{u}_h - \mathbf{u}_H, \mathbf{u}_h - \mathbf{u}_H, \mathbf{v}) - d(\mathbf{v}, p_h - p_H) + d(\mathbf{u}_h - \mathbf{u}_H, q) + G(\mathbf{u}_h, \mathbf{v}).\end{aligned} \tag{25}$$

It's also easy to see that the dual problem is given below:

$$\begin{aligned}&(\phi, \mathbf{e}^i) + (\psi, \epsilon^i)\\ &= (\nu+\beta)a(\mathbf{e}^i, \Phi_h) + b(\mathbf{u}_h, \mathbf{e}^i, \Phi_h) + b(\mathbf{e}^i, \mathbf{u}_H, \Phi_h) + d(\mathbf{e}^i, \Psi_h) - d(\Phi_h, \epsilon^i)\\ &= \nu a(\mathbf{u}_h - \mathbf{u}_H, \Phi_h) + b(\mathbf{u}_h - \mathbf{u}_H, \mathbf{u}_H, \Phi_h) + b(\mathbf{u}_H, \mathbf{u}_h - \mathbf{u}_H, \Phi_h)\\ &\quad +b(\mathbf{u}_h - \mathbf{u}_H, \mathbf{u}_h - \mathbf{u}_H, \Phi_h) - d(\boldsymbol{\Phi}_h, p_h - p_H) + d(\mathbf{u}_h - \mathbf{u}_H, \Psi_h) + G(\mathbf{u}_h, \Phi_h)\\ &= \nu a(\mathbf{u}_h - \mathbf{u}_H, \Phi_h - \Phi_H) + b(\mathbf{u}_h - \mathbf{u}_H, \mathbf{u}_H, \Phi_h - \Phi_H)\\ &\quad +b(\mathbf{u}_H, \mathbf{u}_h - \mathbf{u}_H, \Phi_h - \Phi_H) + b(\mathbf{u}_h - \mathbf{u}_H, \mathbf{u}_h - \mathbf{u}_H, \Phi_h - \Phi_H)\\ &\quad -d(\Phi_h - \Phi_H, p_h - p_H) + d(\mathbf{u}_h - \mathbf{u}_H, \Psi_h - \Psi_H)\\ &\quad +G(\mathbf{u}_h, \Phi_h - \Phi_H) + G(\mathbf{u}_H, \Phi_H).\end{aligned} \tag{26}$$

Clearly we need to estimate each term on the right side. Using (3) and lemma 2, we obtain that:

$$\begin{aligned}&\nu a(\mathbf{u}_h - \mathbf{u}_H, \Phi_h - \Phi_H) \leq CH\|\mathbf{u}_h - \mathbf{u}_H\|_{1,\Omega_0}(\|\phi\|_{0,\Omega_0} + \|\psi\|_{1,\Omega_0}),\\ &b(\mathbf{u}_h - \mathbf{u}_H, \mathbf{u}_H, \Phi_h - \Phi_H) + b(\mathbf{u}_H, \mathbf{u}_h - \mathbf{u}_H, \Phi_h - \Phi_H)\\ &\leq CH\|\mathbf{u}_h - \mathbf{u}_H\|_{1,\Omega_0}(\|\phi\|_{0,\Omega_0} + \|\psi\|_{1,\Omega_0}),\end{aligned}$$

$$
\begin{aligned}
&b(\mathbf{u}_h-\mathbf{u}_H,\mathbf{u}_h-\mathbf{u}_H,\Phi_h-\Phi_H)\le CH\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\Omega_0}^2(\|\phi\|_{0,\Omega_0}+\|\psi\|_{1,\Omega_0}),\\
&d(\Phi_h-\Phi_H,p_h-p_H)\le CH\|p_h-p_H\|_{0,\Omega_0}(\|\phi\|_{0,\Omega_0}+\|\psi\|_{1,\Omega_0}),\\
&d(\mathbf{u}_h-\mathbf{u}_H,\Psi_h-\Psi_H)\le CH\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\Omega_0}(\|\phi\|_{0,\Omega_0}+\|\psi\|_{1,\Omega_0}),\\
&G(\mathbf{u}_h,\Phi_h-\Phi_H)\le C\alpha_h(\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega_0}+h\|\mathbf{u}\|_{2,\Omega_0})(\|\phi\|_{0,\Omega_0}+\|\psi\|_{1,\Omega_0}),\\
&G(\mathbf{u}_H,\Phi_H)\le C\alpha_H(\|\mathbf{u}-\mathbf{u}_H\|_{1,\Omega_0}+H\|\mathbf{u}\|_{2,\Omega_0})(\|\phi\|_{0,\Omega_0}+\|\psi\|_{1,\Omega_0}).
\end{aligned}
$$

It follows from the above inequalities that

(27)
$$
\begin{aligned}
&(\phi,\mathbf{e}^i)+(\psi,\epsilon^i)\\
\le\ & C(H\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\Omega_0}+H\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\Omega_0}^2+H\|p_h-p_H\|_{0,\Omega_0}+\alpha_h\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega_0}\\
&+\alpha_H\|\mathbf{u}-\mathbf{u}_H\|_{1,\Omega_0}+(\alpha_h h+\alpha_H H)\|\mathbf{u}\|_{2,\Omega_0})(\|\phi\|_{0,\Omega_0}+\|\psi\|_{1,\Omega_0}),
\end{aligned}
$$

which yields:

$$
\begin{aligned}
&\|\mathbf{e}^i\|_{0,\Omega_0}+\|\epsilon^i\|_{-1,\Omega_0}\\
&\le C(H\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\Omega_0}+H\|p_h-p_H\|_{0,\Omega_0}+H^3\|\mathbf{u}\|_{2,\Omega_0}).
\end{aligned}\tag{28}
$$

Step2. We have the following equality from (4) and the Algorithm:

$$
\begin{aligned}
&(\nu+\beta)a(\mathbf{u}_h-\mathbf{u}^h,\mathbf{v})+b(\mathbf{u}_h-\mathbf{u}^h,\mathbf{u}_H,\mathbf{v})+b(\mathbf{u}_H,\mathbf{u}_h-\mathbf{u}^h,\mathbf{v})-d(\mathbf{v},p_h-p^h)\\
&+d(\mathbf{u}_h-\mathbf{u}^h,q)=-G(\mathbf{u}_h,\mathbf{v})+\beta a(\mathbf{u}_h-\mathbf{u}_H,\mathbf{v})-b(\mathbf{u}_h-\mathbf{u}_H,\mathbf{u}_h-\mathbf{u}_H,\mathbf{v}).
\end{aligned}
$$

Then using Lemma 1 and (28) we easily obtain:

(29)
$$
\begin{aligned}
&\|\mathbf{u}_h-\mathbf{u}^i\|_{1,D}+\|p_h-p^i\|_{0,D}\\
\le\ & C(\|\mathbf{u}_h-\mathbf{u}^i\|_{0,\Omega_0}+\|p_h-p^i\|_{-1,\Omega_0}+\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\Omega_0}^2+\beta\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\Omega_0}\\
&+\alpha_h\|\mathbf{u}_h\|_{1,\Omega_0})\\
\le\ & C(\|\mathbf{u}_h-\mathbf{u}_H\|_{0,\Omega_0}+\|p_h-p_H\|_{-1,\Omega_0}+\|\mathbf{e}^i\|_{0,\Omega_0}+\|\epsilon^i\|_{-1,\Omega_0}+\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\Omega_0}^2\\
&+\beta\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\Omega_0}+\alpha_h\|\mathbf{u}_h\|_{1,\Omega_0})\\
\le\ & C(\|\mathbf{u}_h-\mathbf{u}_H\|_{0,\Omega_0}+\|p_h-p_H\|_{-1,\Omega_0}+\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\Omega_0}^2+H\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\Omega_0}\\
&+H\|p_h-p_H\|_{0,\Omega_0}+H^3\|\mathbf{u}\|_{2,\Omega_0}+\beta\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\Omega_0}+\alpha_h\|\mathbf{u}_h\|_{1,\Omega_0}).
\end{aligned}
$$

Thus, the inequality (29) is valid for every $D=\omega^i,\Omega_0=\omega^{i,s},\ i=1,2,\cdots,N$.
We then get the global error estimate by using Theorem 1 and Lemma 2:

$$
\begin{aligned}
&\|\mathbf{u}_h-\mathbf{u}^h\|_{1,\Omega}+\|p_h-p^h\|_{0,\Omega}\\
&=\|\sum_{i=1}^N\varphi_i(\mathbf{u}_h-\mathbf{u}^i)\|_{1,\Omega}+\|\sum_{i=1}^N\varphi_i(p_h-p^i)\|_{0,\Omega}\\
&\le C(\sum_{i=1}^N\|\varphi_i(\mathbf{u}_h-\mathbf{u}^i)\|_{1,\Omega}^2+\sum_{i=1}^N\|\varphi_i(p_h-p^i)\|_{0,\Omega}^2)^{\frac12}\\
&\le C(\sum_{i=1}^N\|\varphi_i(\mathbf{u}_h-\mathbf{u}^i)\|_{1,\omega^i}^2+\sum_{i=1}^N\|\varphi_i(p_h-p^i)\|_{0,\omega^i}^2)^{\frac12}\\
&\le C(\sum_{i=1}^N\|\mathbf{u}_h-\mathbf{u}^i\|_{1,\omega^i}^2+\sum_{i=1}^N\|p_h-p^i\|_{0,\omega^i}^2)^{\frac12}\\
&\le C(\sum_{i=1}^N(\|\mathbf{u}_h-\mathbf{u}_H\|_{0,\omega^{i,s}}^2+\|p_h-p_H\|_{-1,\omega^{i,s}}^2+\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\omega^{i,s}}^4\\
&\quad+H^2\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\omega^{i,s}}^2+H^2\|p_h-p_H\|_{0,\omega^{i,s}}^2+H^6\|\mathbf{u}\|_{2,\omega^{i,s}}^2\\
&\quad+\beta^2\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\omega^{i,s}}^2+\alpha_h^2\|\mathbf{u}_h\|_{1,\omega^{i,s}}^2))^{\frac12}
\end{aligned}\tag{30}
$$

$$
\begin{aligned}
&\le C(\sum_{i=1}^{N}\sum_{E_i\in\omega^{i,s}}(\|\mathbf{u}_h-\mathbf{u}_H\|_{0,E_i}^2+\|p_h-p_H\|_{-1,E_i}^2+\|\mathbf{u}_h-\mathbf{u}_H\|_{1,E_i}^4\\
&\quad +H^2\|\mathbf{u}_h-\mathbf{u}_H\|_{1,E_i}^2+H^2\|p_h-p_H\|_{0,E_i}^2+H^6\|\mathbf{u}\|_{2,E_i}^2\\
&\quad +\beta^2\|\mathbf{u}_h-\mathbf{u}_H\|_{1,E_i}^2+\alpha_h^2\|\mathbf{u}_h\|_{1,E_i}^2))^{\frac{1}{2}}\\
&\le CC_{ov}(\|\mathbf{u}_h-\mathbf{u}_H\|_{0,\Omega}^2+\|p_h-p_H\|_{-1,\Omega}^2+\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\Omega}^4+H^2\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\Omega}^2\\
&\quad +H^2\|p_h-p_H\|_{0,\Omega}^2+H^6\|\mathbf{u}\|_{2,\Omega}^2+\beta^2\|\mathbf{u}_h-\mathbf{u}_H\|_{1,\Omega}^2+\alpha_h^2\|\mathbf{u}_h\|_{1,\Omega}^2)^{\frac{1}{2}}\\
&\le C(H^3+H\alpha_H+\alpha_H^2+\beta(H^2+\alpha_H)).
\end{aligned}
$$

Here, C_{ov} is a finite integer defined as the maximal number of elements E_j contained in each subdomain $\omega^{i,s}$. It is determined by the layer index s, the minimum angle of the regular triangulation τ_{H_p}, and is independent of N. □

Remark 1. *It has been mentioned in* [5] *that α should been chosen as $O(h^2)$ in the computation. Thus we can make a conclusion that β only needs to be $O(H)$ to keep the rate of convergence.*

Using the triangle inequality we can get the following theorem directly from theorem 1 and theorem 2.

Theorem 3. *Assume that the conditions of Theorem 1 hold,* $0 < H \le h$, *for a given* $H_p \ge C_0$ *and* $s \ge 1$, *choose* $\alpha_H = O(H^2)$, $\beta = O(H)$, *then, the solution* $(\mathbf{u}^h, p^h)$ *defined by Algorithm PVMS-PU satisfies*

$$
\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}+\|p-p^h\|_{0,\Omega}\le C(h^2+H^3). \tag{31}
$$

4. Numerical Tests

The algorithm in all experiments is implemented by the public finite element software Freefem++ [27]. All simulations were performed on a dawning parallel cluster composed of 32 nodes, each with eight-core 2.0 GHz CPU, 2 GB × 8 DRAM, and connected together by 20Gbps InfiniBand. The message-passing is supported by MPICH.

4.1. Implementation. To verify the analysis results, we consider 2D numerical examples. Dividing Ω into sub-squares with equal sizes h (or H, H_p), and drawing the diagonal in each sub-square, we obtain the regular triangulation τ_h (or τ_H, τ_{H_p}).

For convenience of presentation, we introduce the following notations:

SFEM means the standard finite element method. Namely, the nonlinear systems are solved by Newton iteration.

GVMS means the finite element variational multiscale method based on two local Gauss integrations (4).

PVMS-PU means `ALGORITHM PVMS-PU`.

4.2. Rates of convergence study. The first test problem is a smooth problem in $\Omega = [0,1]\times[0,1]$, where the exact solution of the stationary Navier-Stokes equations (1) is given by ($\mathbf{u} = (u_1, u_2), p$):

$$
\begin{aligned}
&u_1 = 10x^2(x-1)^2y(y-1)(2y-1),\\
&u_2 = -10x(x-1)(2x-1)y^2(y-1)^2,\\
&p = 10(2x-1)(2y-1),
\end{aligned}
$$

here, $\nu = 1.0$ for simplicity, $\mathbf{f}$ and the boundary conditions are set by ($\mathbf{u} = (u_1, u_2), p$).

To get the optimal orders for H^1-norm of velocity and L^2-norm of pressure, we should choose H and h such that $h \sim H^{\frac{3}{2}}$. In this example, we compute the finite element solutions by PVMS-PU with coarse mesh sizes $H = \frac{1}{12n}$ (n=1, 2, 3, 4) and the corresponding fine mesh sizes $h = H/m$ (m=4, 5, 6, 7). Besides, according to Theorem 1and 3, we choose $\alpha_H = 0.1H^2$, $\beta = 0.1H$. The corresponding linear algebraic system is solved by LU factorization. Convergence of the Newton iteration is achieved when the relative H^1-error of successive iterative velocities is within a fixed tolerance of 10^{-6}, i.e., the following condition is satisfied:

$$\frac{||\mathbf{u}_\mu^{n+1} - \mathbf{u}_\mu^n||_{1,\Omega}}{||\mathbf{u}_\mu^{n+1}||_{1,\Omega}} \leq 10^{-6},$$

where $\mathbf{u}_\mu^n$ (μ could be h, H) is the nth Newton iterative solution.

For PVMS-PU, we fix P_1-PU on τ_{H_p}, $H_p = 1/12$, $s = 1$, thus, $N = 169$, all simulations are implemented with 32 processors.

Table 1. The errors of GVMS

h	$\|\|\mathbf{u} - \bar{\mathbf{u}}_h\|\|_{1,\Omega}$	Order	$\|\|p - \bar{p}_h\|\|_{0,\Omega}$	Order	CPU
1/48	0.000365636	-	0.00112071	-	3.62
1/120	5.89085e-05	1.9943	0.00017977	1.99814	33.46
1/216	1.92045e-05	1.99225	5.68389e-05	1.98735	153.99
1/336	-	-	-	-	-

Table 2. The errors of PVMS-PU

H	h	$\|\|\mathbf{u} - \mathbf{u}^h\|\|_{1,\Omega}$	Order	$\|\|p - p^h\|\|_{0,\Omega}$	Order	Wall time
1/12	1/48	0.00102771	-	0.00118434	-	2.85
1/24	1/120	0.000126349	2.05336	0.00018409	1.99975	12.17
1/36	1/216	3.57543e-05	2.11658	5.64822e-05	2.01571	50.62
1/48	1/336	1.47535e-05	2.05336	2.35785e-05	1.99975	182.75

To further test our PVMS-PU, we also consider another smooth problem (referred as Solution 2) with exact solution $\mathbf{u} = (u_1, u_2)$

$$\begin{aligned} u_1 &= \sin(\pi x)^2 \sin(2\pi y), \\ u_2 &== -\sin(2\pi x)\sin(\pi y)^2, \\ p &= \cos(\pi x)\cos(\pi y). \end{aligned}$$

In present computations, the same parameters h, H and s for PVMS-PU are chosen as Solution 1. The results are tabulated in Tables 3 and 4. From these two tables, we can observe similar phenomena and draw same conclusion as found from Tables 1 and 2.

Table 3. Solution 2, the errors of GVMS

h	$\|\|\mathbf{u} - \bar{\mathbf{u}}_h\|\|_{1,\Omega}$	Order	$\|\|p - \bar{p}_h\|\|_{0,\Omega}$	Order	Wall time
1/48	0.00566745	-	0.000179236	-	3.32
1/120	0.00090770	1.99891	2.86259e-05	2.00198	29.82
1/216	0.000280193	1.99977	9.00352e-06	1.96788	153.65
1/336	-	-	-	-	-

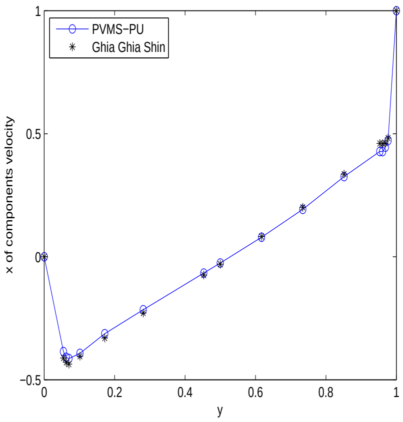
TABLE 4. Solution 2, the errors of PVMS-PU

H	h	$\|\|\mathbf{u}-\mathbf{u}^h\|\|_{1,\Omega}$	Order	$\|\|p-p^h\|\|_{0,\Omega}$	Order	Wall time
1/12	1/48	0.00809542	-	0.00283574	-	5.82
1/24	1/120	0.000937267	2.35306	0.00019754	2.9075	13.33
1/36	1/216	0.000284110	2.03068	4.44115e-05	2.53909	51.48
1/48	1/336	0.000116765	2.01252	1.56627e-05	2.35885	193.94

4.3. The driven cavity flow. A popular benchmark problem for testing numerical schemes is the 'fluid driven cavity'. This problem is chosen because some benchmark data is available for comparison. In this problem, computations are carried out in the domain $\Omega = [0,1]\times[0,1]$. Flow is driven by the tangential velocity field applied to the top boundary in the absence of other body forces. On the top side $\{(x,1) : 0 < x < 1\}$, the velocity is equal to $\mathbf{u} = (1,0)$, and zero Dirichlet conditions are imposed on the rest of the boundary.

Based on PVMS-PU with P_1-PU on τ_{H_p}, $H_p = 1/12$, $s = 1$, we compute for Reynolds numbers $Re = 5000$ with fixed $H = 1/48$, $h = 96$ and $Re = 10000$ with fixed $H = 1/60$, $h = 120$, and $\alpha_H = 0.1H$, $\beta = 0.1H$. the computational results are shown in Figures 2 and 3, comparing with the results of Ghia, Ghia, and Shin [28]. Ghia et al.'s algorithm is based on the time dependent stream function using the coupled implicit and multigrid methods.

For different Reynolds numbers, the x component of velocity along the vertical centerline and y component of velocity along the horizontal centerlines by PUPVMS are drawn in Fig 2 and 3. The accuracy of the computed solutions by PUPVMS has good agreements with the benchmark data of Ghia et al. [28].



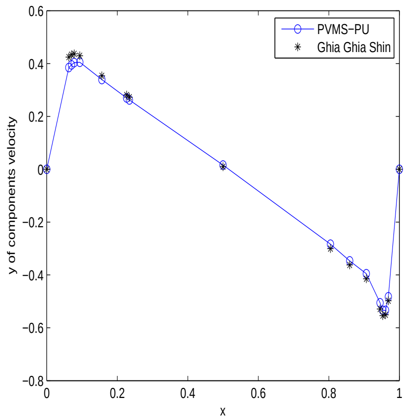
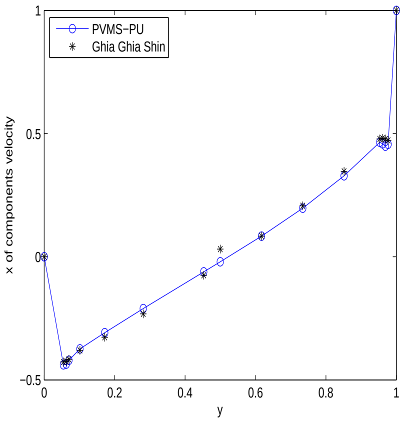


FIGURE 2. PUPVMS for $Re = 5000$. Left, x component of velocity along the vertical centerline; right, y component of velocity along the horizontal centerline.

5. Conclusion

In this paper we proposed the parallel variational multiscale method for the incompressible flows based on the partition of unity and derived the global error estimates. The implementation and some numerical simulations were presented to



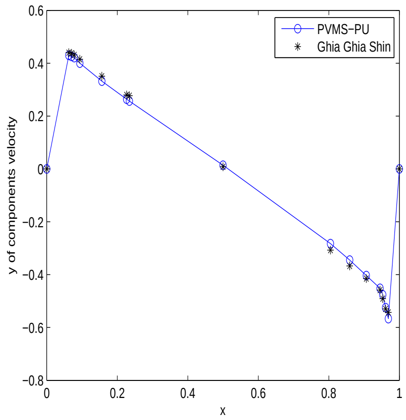


FIGURE 3. PUPVMS for $Re = 10000$. Left, x component of velocity along the vertical centerline; right, y component of velocity along the horizontal centerline.

illustrate the high efficiency and flexility of the new algorithm. Our next aim is to develop some adaptive strategies for oversampling and refinements.

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