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AN ERROR ESTIMATE OF THE COUPLED FINITE-INFINITE ELEMENT METHOD FOR SCATTERING FROM AN ARC

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Abstract. The scattering problem from time-harmonic waves by a Neumann type crack in $\mathbb{R}^2$ is considered. A PML technique is used for solving the problem with a bounded domain instead of the infinite domain. A coupled finite-infinite element method is employed in the computation. Because of the singularity of the solution, the infinite element method is used near the crack tip. An error analysis is presented for the numerical approximation. The convergence order of the method is higher than FEM's.

Key words. error estimate, coupled infinite-finite element, Helmholtz equation, crack.

1. Introduction

The scattering problem for an arc has attracted more and more attention in the past ten years not only because of pure mathematical interest but also of considerable interest for crack-detecting problems in material sciences. The problem can be governed by the Helmholtz equation with boundary conditions on both sides of the arc and the radiation condition at infinity for scattered wave. The difficulty of the scattering problem by an arc as compared to the case of closed smooth boundary is the presence of the tips of crack. Krutitskii [21,22] reported that the solution has a square root singularity at the end of the arc in Dirichlet and Neumann cases. The solution does not belong to $H^{\frac{3}{2}}(\mathbb{R}^2\backslash\Gamma)$ since the solution has a singularity of the form $r^{\frac{1}{2}}\phi(\theta)$, where (r,θ) are the polar coordinates centered at the crack tip.

Up to now, the main method to solve the problem is the application of integral equations. In [16], Kress used it to solve the Dirichlet problem by using cosine transformation and its numerical solution via fully discrete collocation methods. Mönch [19] converted the unbounded Neumann problem into a boundary integral equation. In [17], Kress and Lee extended the method to the case of the impedance boundary condition. Liu [18] considered the scattering problem by a crack in $\mathbb{R}^2$ with different impedance type boundary conditions on different sides. In his paper, the solution is represented in the form of the combined angular potential and single-layer potential.

We will consider the numerical computation of the scattering problem for an arc in this paper. The first difficulty of the problem is infinity of the domain. We use a PML technique for solving the problem with a bounded domain instead of the infinite domain (to limit the computational region). From the first paper [4] about PML technique, various constructions of PML absorbing layers have been proposed and studied in the literature (Chen [9,10], Collino and Monk [11]). The method developed in the present paper is based on [9].

The second difficulty of the problem is the singularity of the solution. Infinite element is considered for such reason. The infinite element was first investigated by Bettess [5]. The method is to extend the element towards infinity in one direction. Thus, shape functions are non polynomial but integrable over the infinite

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element, such as mapped infinite element developed by Bettess and Zienkiewicz [6], and original wave envelope element by Astley [2]. The issue of formulating a mathematically variational statement has been addressed by [7] and [15]. Demkowicz and Gerdes [14] presented a convergence analysis for the infinite element method for the Helmholtz equation in exterior domains. In [13], Demkowicz and Ihlenburg analyzed the error of the coupled finite-infinite element method (FIEM). The error involves the radius of the computational domain covered with finite elements and the number N of the radial terms of infinite elements. Zheng [24] estimated the error of FIEM for the exterior Poisson equation . The result involves not only the size h but also the order p of the quasiuniform finite element approximation without any assumptions. The number N of radial terms of infinite elements has an algebraic convergence.

In this paper, we present a novel error estimate of the coupled finite-infinite element for scattering from an arc. The method is better than the usual FEM. The number of meshes is less than the usual FEM. The method reflects the singularity near the tip accurately. Thus the convergence order is higher. The error depends on N exponentially.

The organization of this paper is as follows. In section 2, we estimate the solution formulation in the neighborhood of the vertexes of the arc and its approximation. In section 3, we introduce the PML problem of the arc and the uniqueness and the convergence of the PML problem. A finite-infinite element subspace is constructed in section 4. The uniqueness of the discrete problem and the error analysis is given.

Let us consider the following model problem. For a given arc $\Gamma \subset \mathbb{R}^2$, we denote the end points of the crack with P, Q, respectively. Denote the left-hand and right-hand side of the crack by Γ^+, Γ^-. The outward normal to Γ^+ is written by n^+ and the opposite direction is written by n^-. For the given incident plane wave $u^i(x) = u^i(r,\theta) = e^{ikx\cdot d}$ with wave number $k \in \mathbb{R}^+$ and incident direction d, consider the following scattering problem for total wave $v(x) = u^i(x) + u^s(x)$,

$$\Delta v + k^2 v = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \Gamma, \tag{1}$$

$$\frac{\partial v}{\partial n^\pm} = 0 \quad \text{on} \quad \Gamma^\pm, \tag{2}$$

$$\sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) \to 0 \quad \text{as} \quad r = |x| \to \infty. \tag{3}$$

In (2),

$$\frac{\partial v(x)}{\partial n^\pm} = \lim_{h\to 0^+} \pm n(x)\cdot \bigtriangledown v(x \pm hn(x)), \tag{4}$$

where the boundary condition at the ends of arc Γ is not required.

Noticing that u^i is an entire function, then u^s satisfies the following system:

$$\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \Gamma, \tag{5}$$

$$\frac{\partial u^s}{\partial n^\pm} = -\frac{\partial u^i}{\partial n^\pm} = g^\pm \quad \text{on} \quad \Gamma^\pm, \tag{6}$$

$$\sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) \to 0 \quad \text{as} \quad r = |x| \to \infty. \tag{7}$$

Throughout the paper, for a given domain Ω, we denote by $|u|_{1,\Omega}, \|u\|_{1,\Omega}$ the standard semi-norm and norm of the function u in the space $H^1(\Omega)$.

2. Formulation of solution near endpoints of the arc and its approximation

Define

$$B_\rho^P = \{x \in \mathbb{R}^2, |x-P| \le \rho\ \}, \quad B_\rho^Q = \{x \in \mathbb{R}^2, |x-Q| \le \rho\ \}. \tag{8}$$

and $S_\rho^P = \{x \in \mathbb{R}^2, |x-P| = \rho\}, S_\rho^Q = \{x \in \mathbb{R}^2, |x-Q| = \rho\}$. Let R_0 be a real number such that the section of Γ in $B_{R_0}^i$ can approximate as a line segment, where $i = P, Q$, and $R_0 < \frac{\pi}{3k}$. Let $R < min(1, R_0)$. Let us associate each vertex i of the arc Γ with polar coordinate (r_i, θ_i) (such that the line $\theta_i = 0$ coincides with the section of Γ in $B_{R_0}^i$).

Proposition 2.1 $\{\cos \frac{n}{2}\theta\}_{n=0}^\infty$ and $\{\sin \frac{n}{2}\theta\}_{n=0}^\infty$ are both sets of orthogonal bases of $L^2(0, 2\pi)$, respectively.

The total wave $v(x)$ in $B_{R_0}^i \backslash \Gamma$ has the following expansion from separation of variables,

$$v|_{B_{R_0}^i \backslash \Gamma} = \frac{a_0}{2} J_0(kr_i) + \sum_{n=1}^\infty a_n J_{\frac{n}{2}}(kr_i) \cos \frac{n}{2}\theta_i, \tag{9}$$

where $J_\gamma(z)$ are Bessel functions of the first kind of order γ. If (2) is the homogenous Dirichlet boundary condition instead, the expansion of total wave is

$$v|_{B_{R_0}^i \backslash \Gamma} = \frac{a_0}{2} J_0(kr_i) + \sum_{n=1}^\infty b_n J_{\frac{n}{2}}(kr_i) \sin \frac{n}{2}\theta_i.$$

Proposition 2.2

$$\left|\frac{J_\gamma(kr)}{J_\gamma(kR_0)}\right| \le M\left(\frac{r}{R_0}\right)^\gamma \le M, \tag{10}$$

where M is a constant independent of γ, if $\gamma > 0$, $R_0 < \frac{\pi}{3k}$, $0 < r < R_0$.

Proof. From [20], we know the Possion integral representation of Bessel functions

$$J_\gamma(kr) = \frac{1}{\sqrt{\pi}\ \Gamma(\gamma + \frac{1}{2})}\left(\frac{kr}{2}\right)^\gamma \int_0^\pi \cos(kr\cos\theta) \sin^{2\gamma}\theta d\theta.$$

Thus

$$\left|\frac{J_\gamma(kr)}{J_\gamma(kR_0)}\right| = \left(\frac{r}{R_0}\right)^\gamma \left|\frac{\int_0^\pi \cos(kr\cos\theta)\sin^{2\gamma}\theta d\theta}{\int_0^\pi \cos(kR_0\cos\theta)\sin^{2\gamma}\theta d\theta}\right|, \tag{11}$$

since $0 < r < R_0 < \frac{\pi}{3k}$, $\frac{1}{2} < \cos(kr\cos\theta) < 1$.

We have

$$\left|\frac{J_\gamma(kr)}{J_\gamma(kR_0)}\right| \le M\left(\frac{r}{R_0}\right)^\gamma \le M.$$

The proof is completed.

Proposition 2.3[14] Consider a series $\sum_{n=0}^\infty p_n$ with $p_n \ge 0$. Then

$$\sum_{n=N+1}^\infty p_n \le Ce^{-\kappa N},$$

holds for some $C > 0$, $\kappa > 0$, if and only if there exists some $D > 0$ such that

$$p_n \le De^{-\kappa n}, \quad \forall n$$

for the same $\kappa > 0$.

Proposition 2.4

$$\sum_{n=0}^{\infty} a_n b_n \leq (\sum_{n=0}^{\infty} a_n\ )(\sum_{n=0}^{\infty} b_n),$$

if $a_n, b_n \geq 0$.

Proposition 2.5 Under the condition of Proposition 2.2 ,

$$|\frac{\{J_{\frac{n}{2}}(kr)\}'}{J_{\frac{n}{2}}(kR_0)}| \leq K\frac{n}{2}\frac{1}{r}(\frac{r}{R_0})^{\frac{n}{2}}, \tag{12}$$

where K is a constant independent of n, if $r < R_0$.

Proof. From [20], we have

$$\begin{aligned} |\frac{kJ'_{\frac{n}{2}}(kr)}{J_{\frac{n}{2}}(kR_0)}| &= |\frac{k[J_{\frac{n}{2}+1}(kr) - \frac{n}{2kr}J_{\frac{n}{2}}(kr)]}{J_{\frac{n}{2}}(kR_0)}| \\ &\leq |\frac{kJ_{\frac{n}{2}+1}(kr)}{J_{\frac{n}{2}}(kR_0)}| + |\frac{n}{2r}\frac{J_{\frac{n}{2}}(kr)}{J_{\frac{n}{2}}(kR_0)}|. \end{aligned} \tag{13}$$

According to Proposition 2.2,

$$\frac{n}{2r}|\frac{J_{\frac{n}{2}}(kr)}{J_{\frac{n}{2}}(kR_0)}| \leq M\frac{n}{2}\frac{1}{r}(\frac{r}{R_0})^{\frac{n}{2}}, \tag{14}$$

so

$$\begin{aligned} |\frac{kJ_{\frac{n}{2}+1}(kr)}{J_{\frac{n}{2}}(kR_0)}| &= |k\frac{J_{\frac{n}{2}+1}(kr)}{J_{\frac{n}{2}}(kr)}||\frac{J_{\frac{n}{2}}(kr)}{J_{\frac{n}{2}}(kR_0)}| \\ &\leq |k\ \frac{\Gamma(\frac{n}{2}+\frac{1}{2})}{\Gamma(\frac{n}{2}+\frac{3}{2})}\frac{\int_0^\pi \cos(kr\cos\theta)\sin^{n+2}\theta d\theta}{\int_0^\pi \cos(kr\cos\theta)\sin^{n}\theta d\theta}(\frac{r^{\frac{n}{2}+1}}{R_0^{\frac{n}{2}}})| \\ &\leq Mk\frac{2}{n+1}r\{\frac{r}{R_0}\}^{\frac{n}{2}} \\ &< K\frac{n}{2}\frac{1}{r}(\frac{r}{R_0})^{\frac{n}{2}}, \end{aligned} \tag{15}$$

since $r < R$. This completes the proof of the proposition.

Lemma 2.1 Under the condition of Proposition 2.2, if $u(R_0, \cdot) \in H^\nu(0, 2\pi)$, where

$$u = u(r,\theta) = \frac{a_0}{2}\frac{J_0(kr)}{J_0(kR_0)} + \sum_{n=1}^{\infty}\frac{J_{\frac{n}{2}}(kr)}{J_{\frac{n}{2}}(kR_0)}a_n\cos\frac{n}{2}\theta, \quad r < R,$$

and

$$u_N = u_N(r,\theta) = \frac{a_0}{2}\frac{J_0(kr)}{J_0(kR_0)} + \sum_{n=1}^{N}\frac{J_{\frac{n}{2}}(kr)}{J_{\frac{n}{2}}(kR_0)}a_n\cos\frac{n}{2}\theta,$$

we have

$$|u - u_N|_{1,B_R\backslash\Gamma} \leq Ce^{-\kappa\frac{N}{2}}|u(R_0,\cdot)|_{\nu,\ (0,2\pi)}, \tag{16}$$

$$\|(u-u_N)(R,\cdot)\|_{1,(0,2\pi)} \leq Ce^{-\kappa\frac{N}{2}}|u(R_0,\cdot)|_{\nu,\ (0,2\pi)}, \tag{17}$$

where $N \geq 1$, C is a constant independent of ν and κ is a positive constant only dependent on R and R_0.

Proof.

$$|u-u_N|^2_{1,B_R\backslash\Gamma} = |\frac{\partial}{\partial r}(u-u_N)|^2_{0,B_R\backslash\Gamma} + |\frac{1}{r}\frac{\partial}{\partial\theta}(u-u_N)|^2_{0,B_R\backslash\Gamma}. \tag{18}$$

According to Proposition 2.2,

$$
\begin{aligned}
&|\frac{1}{r}\frac{\partial}{\partial\theta}(u-u_N)|^2_{0,B_R\setminus\Gamma} \\
\le& M^2\sum_{N+1}^{\infty}(\frac{n}{2})^2\int_0^R\frac{r}{r^2}(\frac{r}{R_0})^n|a_n|^2dr \\
\le& M^2\sum_{N+1}^{\infty}(\frac{n}{2})^2\frac{1}{n}(\frac{R}{R_0})^n|a_n|^2.
\end{aligned} \tag{19}
$$

From Proposition 2.5,

$$|\frac{\partial}{\partial r}(u-u_N)|^2_{0,B_R\setminus\Gamma} \tag{20}$$

$$=\sum_{N+1}^{\infty}\int_0^R r\{\frac{kJ'_{\frac{n}{2}}(kr)}{J_{\frac{n}{2}}(kR_0)}\}^2|a_n|^2dr \tag{21}$$

$$\le\sum_{N+1}^{\infty}|a_n|^2K^2\int_0^R(\frac{n}{2})^2\frac{r}{r^2}(\frac{r}{R_0})^ndr \tag{22}$$

$$=K^2\sum_{N+1}^{\infty}(\frac{n}{2})^2\frac{1}{n}(\frac{R}{R_0})^n|a_n|^2. \tag{23}$$

We have

$$
\begin{aligned}
|u-u_N|^2_{1,B_R\setminus\Gamma}\le& C^2\sum_{N+1}^{\infty}(\frac{n}{2})^2\frac{1}{n}(\frac{R}{R_0})^n|a_n|^2 \\
\le& C^2\ \{\sum_{N+1}^{\infty}(\frac{2}{n})^{2\nu-1}(\frac{R}{R_0})^n\}\{\sum_{N+1}^{\infty}\ (\frac{n}{2})^{2\nu}|a_n|^2\},
\end{aligned} \tag{24}
$$

where $C=\max\{M,K\}$. Since $N\ge 1$,

$$(\frac{2}{n})^{2\nu-1}(\frac{R}{R_0})^n<(\frac{R}{R_0})^n=e^{-\kappa n}, \tag{25}$$

where $\kappa=\ln\frac{R_0}{R}>0$. According to Proposition 2.3, we get (16)

$$|u-u_N|_{1,B_R\setminus\Gamma}\le C\ e^{-\kappa\frac{N}{2}}|u(R_0,\cdot)|_{\nu,\ (0,2\pi)}.$$

It is easy to see

$$\|(u-u_N)(R,\cdot)\|^2_{0,(0.2\pi)}\le M^2\sum_{N+1}^{\infty}|a_n|^2(\frac{R}{R_0})^n, \tag{26}$$

$$|(u-u_N)(R,\cdot)|^2_{1,\ (0,2\pi)}\le M^2\sum_{N+1}^{\infty}(\frac{n}{2})^2\{\frac{R}{R_0}\}^n|a_n|^2, \tag{27}$$

So

$$
\begin{aligned}
\|(u-u_N)(R,\cdot)\|^2_{1,(0,2\pi)}\le& M^2\sum_{N+1}^{\infty}(\frac{n}{2})^2\{\frac{R}{R_0}\}^n|a_n|^2 \\
\le& M^2\{\sum_{N+1}^{\infty}(\frac{2}{n})^{2\nu-2}(\frac{R}{R_0})^n\}\{\sum_{N+1}^{\infty}(\frac{n}{2})^{2\nu}|a_n|^2\} \\
\le& C^2\ e^{-\kappa N}|u(R_0,\cdot)|^2_{\nu,\ (0,2\pi)}.
\end{aligned} \tag{28}
$$

3. The PML formulation of the problem

In this section, we consider the existence, uniqueness and the convergence of the solution of PML problem. Let Γ be contained in the interior of the disc $B_{\tilde{R}} = \{x \in \mathbb{R}^2 : |x| < \tilde{R}\}$. Denote $\Omega_{\tilde{R}} = B_{\tilde{R}} \backslash \Gamma$. We surround the domain $\Omega_{\tilde{R}}$ with a PML layer $\Omega^{PML} = \{x \in \mathbb{R}^2 : \tilde{R} < |x| < \rho\}$. Let $\Gamma_\rho = \{x \in \mathbb{R}^2 : |x| = \rho\}$, $\Omega_\rho = B_\rho \backslash \Gamma$, $\Omega_\rho^e = \Omega_\rho \backslash (B_R^P \cup B_R^Q)$.

Let $\alpha(r) = 1 + i\sigma(r)$ be the model medium property which satisfies

$$\sigma \in C(\mathbb{R}), \quad \sigma \geq 0, \quad \text{and} \quad \sigma = 0 \;\; \text{for} \;\; r \leq \tilde{R}.$$

Denote by $\tilde{r}$ the complex radius defined by

$$\tilde{r} = \tilde{r}(r) = \begin{cases} r & r \leq \tilde{R}, \\ \int_0^r \alpha(t)dt = r\beta(r) & r \geq \tilde{R}. \end{cases} \tag{29}$$

From [9], the PML solution $\hat{u}$ in $\Omega_\rho = B_\rho \backslash \Gamma$ is defined as the solution of the following system

$$\begin{aligned} &\nabla \cdot (A \nabla \hat{u}) + \alpha\beta k^2 \hat{u} = 0 \quad \text{in} \;\; \Omega_\rho, \\ &\tfrac{\partial \hat{u}}{\partial n^\pm} = g^\pm \quad \text{on} \;\; \Gamma^\pm, \quad \hat{u} = 0 \quad \text{on} \;\; \Gamma_\rho, \end{aligned} \tag{30}$$

where $A = A(x)$ is a matrix which satisfies, in polar coordinates,

$$\nabla \cdot (A \nabla) = \frac{1}{r}\frac{\partial}{\partial r}(\frac{\beta r}{\alpha}\frac{\partial}{\partial r}) + \frac{\alpha}{\beta r^2}\frac{\partial^2}{\partial \theta^2}.$$

The problem (30) is shown as an approximation of problem (5)-(7).

Let $a : H^1(\Omega_\rho) \times H^1(\Omega_\rho) \to \mathbb{C}$ be a sesquilinear form

$$a(\hat{u}, \psi) = \int_{\Omega_\rho} (\frac{\beta}{\alpha} r \frac{\partial \hat{u}}{\partial r}\frac{\partial \overline{\psi}}{\partial r} + \frac{\alpha}{\beta r}\frac{\partial \hat{u}}{\partial \theta}\frac{\partial \overline{\psi}}{\partial \theta} - \alpha\beta k^2 r \hat{u}\overline{\psi})dx, \tag{31}$$

Then the weak formulation for (30) is: Find $\hat{u} \in H_E^1(\Omega_\rho)$ such that

$$a(\hat{u}, \psi) = < g^+, \psi >_{\Gamma^+} + < g^-, \psi >_{\Gamma^-} \quad \forall \psi \in H_E^1(\Omega_\rho), \tag{32}$$

where $H_E^1(\Omega_\rho) = \{u : u \in H^1(\Omega_\rho), u|_{\Gamma_\rho} = 0\}$, and $< \cdot, \cdot >_\Gamma$ stands for the inner product on $L^2(\Gamma)$.

We make the following assumption

$$\sigma = \sigma_0 \{\frac{r - \tilde{R}}{\rho - \tilde{R}}\}^m \tag{33}$$

for some constant $\sigma_0 > 0$ and some integer $m > 1$. Let $\beta(r) = 1 + i\hat{\sigma}(r)$, $\alpha_0 = 1 + i\sigma_0$. It is easily to see that $0 < \hat{\sigma} \leq \sigma \leq \sigma_0$. Thus

$$Re \; a(u, u) = \int_{\Omega_\rho} [\frac{1 + \sigma\hat{\sigma}}{1 + \sigma^2} r |\frac{\partial u}{\partial r}|^2 + \frac{1 + \sigma\hat{\sigma}}{1 + \hat{\sigma}^2}\frac{1}{r}|\frac{\partial u}{\partial \theta}|^2 + (\sigma\hat{\sigma} - 1)k^2 r |u|^2] dr d\theta, \tag{34}$$

where

$$\frac{1 + \sigma\hat{\sigma}}{1 + \sigma^2}, \frac{1 + \sigma\hat{\sigma}}{1 + \hat{\sigma}^2} \geq \frac{1}{\sigma_0^2}.$$

Next, we introduce another PML problem in order to represent the singularity of P, Q, because it satisfies the homogeneous boundary condition on Γ. Let $u = \hat{u} + u^i$. The problem (30) is equivalent to the following system

$$\nabla \cdot (A\nabla u) + \alpha\beta k^2 u = f \quad \text{in} \ \ \Omega_\rho, \tag{35}$$

$$\frac{\partial u}{\partial n^\pm} = 0, \quad \text{on} \ \ \Gamma^\pm, \tag{36}$$

$$u = u^i, \quad \text{on} \ \ \Gamma_\rho, \tag{37}$$

where $f = \nabla \cdot (A\nabla u^i) + \alpha\beta k^2 u^i$. It is easy to see $f = 0$ in $B_{\tilde{R}}$.

We set $U = u - u^i\phi$, where $\phi \in \mathcal{C}^\infty(\bar{\Omega}_\rho)$ satisfies

$$\phi = 1 \quad \text{on} \ \ \Gamma_\rho, \quad \phi = 0 \quad \text{in} \ \ B_{\tilde{R}}.$$

It is easy to see that U is a solution of the following system

$$\nabla \cdot (A\nabla U) + \alpha\beta k^2 U = f + g \quad \text{in} \ \ \Omega_\rho, \tag{38}$$

$$\frac{\partial U}{\partial n^\pm} = 0, \quad \text{on} \ \ \Gamma^\pm, \tag{39}$$

$$U = 0 \quad \text{on} \ \ \Gamma_\rho, \tag{40}$$

where $g = -\nabla \cdot [A\nabla(u^i\phi)] - \alpha\beta k^2 u^i\phi$. Clearly, $g = 0$ in $B_{\tilde{R}}$.

The variational formulation of the problem (38)-(40) is to find $U \in H^1_E(\Omega_\rho)$ such that

$$(\nabla AU, v) - k^2(\alpha\beta U, v) = (f + g, v) \quad \forall v \in H^1_E(\Omega_\rho).$$

Similar to the arguments in [11], we define two operators $\mathcal{M} : H^1_E(\Omega_\rho) \to H^1(\Omega_\rho)$ and $\mathcal{N} : H^1_E(\Omega_\rho) \to H^1(\Omega_\rho)$ by Riez representation as follows

$$(\nabla\mathcal{M}u, \nabla v) + (\mathcal{M}u, v) = (A\nabla u, \nabla v) + (u, v), \tag{41}$$

$$(\nabla\mathcal{N}u, \nabla v) + (\mathcal{N}u, v) = ((\alpha\beta k^2 + 1)u, v), \tag{42}$$

for all $u, v \in H^1_E(\Omega_\rho)$. Under the condition of (33), $\mathcal{M}^{-1}$ exists and is bounded. The next step is to prove $\mathcal{N}$ is compact. If $\omega = \mathcal{N}u$, $\omega \in H^1(\Omega_\rho)$ satisfies the following boundary problem:

$$-\Delta\omega + \omega = (\alpha\beta k^2 + 1)u \quad \text{in} \ \ \Omega_\rho, \tag{43}$$

$$\frac{\partial \omega}{\partial r} = 0 \quad \text{on} \ \ \Gamma_\rho, \tag{44}$$

$$\frac{\partial \omega}{\partial n^\pm} = 0 \quad \text{on} \ \ \Gamma^\pm. \tag{45}$$

From the variational formulation and standard elliptic regularity theory, the arguments of [8] imply that $\|\omega\|_{\varepsilon,\Omega_\rho} \le C\|u\|_{1,\Omega_\rho}$, where $1 < \varepsilon < \frac{3}{2}$. Since $H^\varepsilon(\Omega_\rho)$ is compactly embedded in $H^1(\Omega_\rho)$, $\mathcal{N}$ is compact.

Now let $G \in H^1(\Omega_\rho)$ be defined by

$$(\nabla G, \nabla v) + (G, v) = (f + g, v) \quad \forall v \in H^1(\Omega_\rho).$$

Then the problem (38)-(40) is equivalent to finding $U \in H^1_E(\Omega_\rho)$ such that

$$U - \mathcal{M}^{-1}\mathcal{N}U = \mathcal{M}^{-1}G. \tag{46}$$

Since $U = u - u^i\phi$, u satisfies

$$u - \mathcal{M}^{-1}\mathcal{N}u = \mathcal{M}^{-1}F, \tag{47}$$

where F is some function in $H^1(\Omega_\rho)$. Clearly, the Fredholm alternative theorem is applicable.

Remark. In general, $I : H^\varepsilon(\Omega_\rho) \mapsto H^1(\Omega_\rho)$ is compact under the condition that Ω_ρ is Lipschitz. But [1] presents that I is still compact if Ω_ρ satisfies the cone condition.

Proposition 3.1 The problem (35)-(37) has a unique solution $u \in H^1(\Omega_\rho)$ except for a discrete set of values of k.

The arguments of Chen [9] imply that the PML solution converges to the solution of original scattering problem (5)-(7) in $\Omega_{\tilde{R}}$ for a large σ_0 as the thickness of the PML layer tends to infinity.

Let $u^e = u|_{\Omega_\rho^e}, u_j = u|_{B_R^j}$, γ_j is the trace operator of S_R^j in $H^1(\Omega_\rho^e)$. Define

$$V := \{u|u|_{\Omega_\rho^e} \in H^1(\Omega_\rho^e), u|_{B_R^j} = \sum_{n=0}^{\infty} b_{n,j} \frac{J_{\frac{n}{2}}(kr)}{J_{\frac{n}{2}}(kR)} \cos\frac{n}{2}\theta_j, \gamma_j u^e = u_j|_{S_R^j}, j = P, Q, \},$$

where

$$b_{n,j} = \frac{1}{2\pi}\int_0^{2\pi} \gamma_j u^e \cos\frac{n}{2}\theta d\theta \in \mathbb{R}.$$

Therefore, $V \subset H^1(\Omega_\rho)$. And

$$\|u\|_V = \|u\|_{1,\Omega_\rho}.$$

Obviously, $u \in V$ can be decided by $u|_{\Omega_\rho^e}$ uniquely.

Theorem 3.1 V is a closed subspace of $H^1(\Omega_\rho)$.

Proof. Define

$$V_0 = \{\sum_{n=0}^{\infty} b_n \frac{J_{\frac{n}{2}}(kr)}{J_{\frac{n}{2}}(kR)} \cos\frac{n}{2}\theta, \quad \forall\ b_n \in \mathbb{R}, \quad 0 \le r \le R\ \}.$$

Let functions $\{a_m\} \in V_0 \subset H^1(B_R)$ converge to $a \in H^1(B_R)$, i.e. $\lim\limits_{m\to\infty} \|a_m - a\|_1 = 0$. So functions $\{a_m(R, \cdot)\} \subset H^{\frac{1}{2}}(0, 2\pi)$ is convergent according to trace theorem. Assume that the limit is

$$a_m(R, \cdot) \to b' = \sum_{n=0}^{\infty} b_n \cos\frac{n}{2}\theta. \quad (m \to \infty)$$

Let

$$b = \sum_{n=0}^{\infty} b_n \frac{J_{\frac{n}{2}}(kr)}{J_{\frac{n}{2}}(kR)} \cos\frac{n}{2}\theta \in V_0.$$

Obviously, $b' = b(R, \cdot)$. From Lemma 2.1,

$$\|a_m - b\|_1 \le C\|(a_m - b)(R, \cdot\ )\|_{\frac{1}{2},(0,2\pi)} \to 0. \tag{48}$$

So $a = b$. V_0 is closed. Thus V is a closed subspace. The proof is completed.

Define

$$\begin{aligned} \Pi\ :\ H^1(\Omega_\rho)\ &\to\ V \\ u\ &\to\ v \end{aligned}$$

$$\text{s.t}\ \ \Pi u|_{\Omega_\rho^e} = u\ |_{\Omega_\rho^e}.$$

Since v can only be decided by $u\,|_{\Omega_\rho^e}$, the operator Π is linear, injective, but not one-to-one. Still Π is bounded.

$$
\begin{aligned}
|\Pi u|^2_{1,\Omega_\rho} &\le |u|^2_{1,\Omega_\rho^e} + \sum_{j=P,Q} |\sum_{n=0}^{\infty} b_{n,j}\frac{J_n(kr_j)}{J_n(kR)}\cos\frac{n}{2}\theta_j|^2_{1,B_R^j} \\
&\le |u|^2_{1,\Omega_\rho^e} + \sum_{j=P,Q} |u(R,\theta_j)|^2_{\frac{1}{2},(0,2\pi)} \\
&\le C|u|^2_{1,\Omega_\rho}.
\end{aligned} \tag{49}
$$

Since (9), the solution of (47) in $H^1(\Omega_\rho)$ must be in V. Let (47) multiple Π.

$$
u - \Pi\mathcal{M}^{-1}\mathcal{N}u = \Pi\mathcal{M}^{-1}F. \tag{50}
$$

$\Pi\mathcal{M}^{-1}\mathcal{N}$ is compact because Π is bounded. Fredholm alternative theorem is applicable. Moreover, the solution of (47) must be the solution of (50). Thus (47) and (50) has the same unique solution except for a discrete set of k.

4. Discrete problem and estimation of error

Let $\mathcal{F}^h = \{K_i^h\}$ be a regular triangulation of the domain Ω_ρ^e, where $h = \text{maxdiam}\{K_i^h\}$. If $K_i^h \cap (B_R^P \cup B_R^Q) = \emptyset$, K_i^h are triangles. If $K_i^h \cap (B_R^P \cup B_R^Q) \neq \emptyset$, K_i^h are curved triangles which have one curved edge align with S_R^Q and S_R^P. Let G_i^h be a one-to-one sufficiently smooth mapping, which maps K_i^h onto the standard triangle,

$$
T = \{(x_1, x_2)| -1 < x_1 < 1, -1 < x_2 < x_1\}.
$$

Let $X_{R,p}^h(\Omega_\rho^e) \subset H^1(\Omega_\rho^e)$ denote the set of functions u such that $u|_{K_i^h} \circ (G_i^h)^{-1}$ is a polynomial of degree $\le p$. The space we have used is the same as [24].

The mesh $\mathcal{F}^h$ induces two partitions $\mathcal{F}_R^{h,i} = \{K_{R,j}^{h,i}\}_{j=1}^{m_i}$ of S_R^i ,where $i = P, Q$. Denote by $\{N_j^{h,i}\}_{j=1}^{m_i}$ the nodal points of $\mathcal{F}_R^{h,i}$, which induces two partitions $\mathcal{F}^{\Delta\theta,i} = \{K_j^{\Delta\theta,i}\}_{j=1}^{m_i}$ of $(0, 2\pi)$ with θ. Let $X_p^{\Delta\theta,i}(0, 2\pi) \subset H^1(0, 2\pi)$ be the set of functions u such that $u|_{K_j^{\Delta\theta,i}}$ is a polynomial of degree $\le p$ with respect to the polar angle variable. We define an interpolation operator Π_{θ_i} such that $\Pi_{\theta_i}(u|_{S_R^i}) \in X_p^{\Delta\theta,i}(0, 2\pi)$.

For $v \in H^\nu(0, 2\pi)$, according to [23] , we have

$$
\|v - \Pi_\theta v\|_{t,(0,2\pi)} \le C\frac{h^{\min(\nu,p+1)-t}}{p^{\nu-t}}\|v\|_{\nu,(0,2\pi)}, \tag{51}
$$

where $\nu > 1, 0 \le t \le 1, p \ge 1$.

Let

$$
Y_{N,R}^{h,p}(B_R^i) = \{\ \sum_{n=0}^{N}\frac{J_{\frac{n}{2}}(kr_i)}{J_{\frac{n}{2}}(kR)}\omega_n(\theta_i),\ \omega_n(\theta_i) \in X_p^{\Delta\theta,i}(0, 2\pi)\ \},
$$

where $i = P$ or Q.

Moreover,

(52)

$$
X_{N,R}^{h,p} = \{u \in H^1(\Omega_\rho) : u|_{\Omega_\rho^e} \in X_{R,p}^h(\Omega_\rho^e), u|_{B_R^p} \in Y_{N,R}^{h,p}(B_R^p), u|_{B_R^Q} \in Y_{N,R}^{h,p}(B_R^Q)\}.
$$

We define

$$
u_{dis}^i \in X_{R,p}^h(\Gamma_\rho) = \{u|_{\Gamma_\rho}\big|u \in X_{R,p}^h(\Omega_\rho^e)\}, \quad u_{dis}^i(N_{\rho,j}^h) = u^i(N_{\rho,j}^h), \tag{53}
$$

where $N_{\rho,j}^h$ is the nodal points of $\mathcal{F}^h$ in Γ_ρ.

Define

$$
\tilde{X}_{N,R}^{h,p} = \{u \in X_{N,R}^{h,p} : u|_{\Gamma_\rho} = u_{dis}^i\}, \quad \tilde{X}_{N,R,0}^{h,p} = \{u \in X_{N,R}^{h,p} : u|_{\Gamma_\rho} = 0\}. \tag{54}
$$

Now we get the following discrete version of (35)-(37):

Find $u_{h,N,R} \in \tilde{X}^{h,p}_{N,R}$, such that

(55)
$$\int_{\Omega_\rho} (\frac{\beta}{\alpha} r \frac{\partial u_{h,N,R}}{\partial r}\frac{\partial \overline{v}}{\partial r} + \frac{\alpha}{\beta r}\frac{\partial u_{h,N,R}}{\partial \theta}\frac{\partial \overline{v}}{\partial \theta} - \alpha\beta k^2 r u_{h,N,R}\ \overline{v}) = < f, v >, \quad \forall v \in \tilde{X}^{h,p}_{N,R,0}.$$

It is easy to see that $X^{h,p}_{N,R}$ is ultimately dense in V. In addition, problem (30) is equivalent to (50). Thus problem (55) has a unique solution expect for a discrete set of values of k.

Theorem 4.1 Let $u \in H^1(\Omega_\rho)$ be the solution of problem (35)-(37). Then if $u|_{\Omega^e_\rho} \in H^m(\Omega^e_\rho), u(R_{0,j},\cdot) \in H^{\nu+\frac{1}{2}}(0,2\pi)$, we have

(56)
$$\inf_{w_{p,N}\in X^{h,p}_{N,R}} |u - w_{p,N}|_{1,\Omega_\rho} \le C \sum_{j=P,Q} \{ \frac{h^{\min(\nu,p)}}{p^\nu}|u(R_{0,j}\ ,\cdot)|_{\nu+\frac{1}{2},(0,2\pi)} + e^{-\kappa\frac{N}{2}}|u(R_{0,j}\ ,\cdot)|_{\nu,(0,2\pi)} + \frac{h^{\min(p,m-1)}}{p^{m-1}}|u|_{m,\Omega^e_\rho}\}.$$

Proof. Assume that $\varphi'_{R,h} \in X^h_{R,p}(\Omega^e_\rho)$ is the finite element approximation of the following system

$$\nabla\cdot(A\nabla\varphi) + \alpha\beta k^2\varphi = f \quad \text{in } \Omega^e_\rho, \tag{57}$$
$$\varphi = u^i \quad \text{on } \Gamma_\rho, \tag{58}$$
$$\varphi = u|_{S^P_R} \quad \text{on } S^P_R, \quad \varphi = u|_{S^Q_R} \quad \text{on } S^Q_R. \tag{59}$$

where u is the solution of (35)-(37).

From the standard theories of finite element method,

$$|\varphi - \varphi'_{R,h}|_{1,\Omega^e_\rho} \le C\frac{h^{\min(p,m-1)}}{p^{m-1}}|\varphi|_{m,\Omega^e_\rho}. \tag{60}$$

Now let

$$\varphi^h_{N,R} = \begin{cases} \varphi'_{R,h}, & x \in \Omega^e_\rho, \\ \dfrac{J_{\frac{1}{2}}(kr)}{J_{\frac{1}{2}}(kR)}\Pi_{\theta_P}(u-u_N)(R_j,\cdot) + \Pi_{\theta_P}u_N, & x \in B^P_R, \\ \dfrac{J_{\frac{1}{2}}(kr)}{J_{\frac{1}{2}}(kR)}\Pi_{\theta_Q}(u-u_N)(R_j,\cdot) + \Pi_{\theta_Q}u_N, & x \in B^Q_R. \end{cases} \tag{61}$$

It is easy to verify that $\varphi^h_{N,R} \in \tilde{X}^{h,p}_{N,R}$. We have

(62)
$$|u - \varphi^h_{N,R}|^2_{1,\Omega_\rho} \le |u - \varphi'_{R,h}|^2_{1,\Omega^e_\rho} + \sum_{i=P,Q}\{|u - \Pi_{\theta_i}u_N|^2_{1,B^i_R\backslash\Gamma} + |\frac{J_{\frac{1}{2}}(kr_i)}{J_{\frac{1}{2}}(kR)}\Pi_{\theta_i}(u-u_N)(R,\cdot)|^2_{1,B^i_R\backslash\Gamma}\}.$$

We get

(63)

$$
\begin{aligned}
&|\frac{J_{\frac{1}{2}}(kr)}{J_{\frac{1}{2}}(kR)}\Pi_\theta(u-u_N)(R,\cdot)|^2_{1,B_R\setminus\Gamma}\\
=&|\frac{\partial}{\partial r}(\frac{J_{\frac{1}{2}}(kr)}{J_{\frac{1}{2}}(kR)})\Pi_\theta(u-u_N)(R,\cdot)\big|^2_{0,B_R\setminus\Gamma}+|\frac{1}{r}\frac{J_{\frac{1}{2}}(kr)}{J_{\frac{1}{2}}(kR)}\frac{\partial}{\partial\theta}\Pi_\theta(u-u_N)(R,\cdot)\big|^2_{0,B_R\setminus\Gamma}\\
=&I_1+I_2,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1=&\int_0^R r\big|\frac{\partial}{\partial r}(\frac{J_{\frac{1}{2}}(kr)}{J_{\frac{1}{2}}(kR)})\big|^2dr\int_0^{2\pi}|\Pi_\theta(u-u_N)(R,\cdot)|^2d\theta\\
\le&K^2\int_0^R(\frac{1}{2})^2\frac{r}{r^2}\frac{r}{R}dr\int_0^{2\pi}|\Pi_\theta(u-u_N)(R,\cdot)|^2d\theta \qquad (64)\\
\le&C\int_0^{2\pi}|\Pi_\theta(u-u_N)(R,\cdot)|^2d\theta,
\end{aligned}
$$

$$
\begin{aligned}
I_2=&\int_0^R\frac{r}{r^2}\big|\frac{J_{\frac{1}{2}}(kr)}{J_{\frac{1}{2}}(kR)}\big|^2dr\int_0^{2\pi}|\frac{\partial}{\partial\theta}\Pi_\theta(u-u_N)(R,\cdot)|^2d\theta\\
\le&M^2\int_0^R\frac{1}{r}\frac{r}{R}dr\int_0^{2\pi}|\frac{\partial}{\partial\theta}\Pi_\theta(u-u_N)(R,\cdot)|^2d\theta \qquad (65)\\
\le&C\int_0^{2\pi}|\frac{\partial}{\partial\theta}\Pi_\theta(u-u_N)(R,\cdot)|^2d\theta.
\end{aligned}
$$

So

$$
\begin{aligned}
&|\frac{J_{\frac{1}{2}}(kr)}{J_{\frac{1}{2}}(kR)}\Pi_\theta(u-u_N)(R,\cdot)|^2_{1,B_R\setminus\Gamma}\\
\le&C\|\Pi_\theta(u-u_N)(R,\cdot)\|_{1,(0,2\pi)} \qquad (66)\\
\le&C\ \|(u-u_N)(R,\cdot)\|_{1,(0,2\pi)}\le Ce^{-\kappa\frac{N}{2}}|u(R_0,\cdot)|_{\nu,(0,2\pi)}.
\end{aligned}
$$

In addition,

$$
|u_N-\Pi_\theta u_N|^2_{1,B_R\setminus\Gamma}=|\frac{\partial}{\partial r}(u_N-\Pi_\theta u_N)|^2_{0,B_R\setminus\Gamma}+|\frac{1}{r}\frac{\partial}{\partial\theta}(u_N-\Pi_\theta u_N)|^2_{0,B_R\setminus\Gamma}\equiv I_3+I_4,
$$

and

$$
\begin{aligned}
I_3=&\int_0^R|\frac{\partial}{\partial r}(u_N-\Pi_\theta u_N)|^2_{0,(0,2\pi)}rdr\le C\frac{h^{2\min(\nu,p+1)}}{p^{2\nu}}\int_0^R|\frac{\partial}{\partial r}u_N|^2_{\nu,(0,2\pi)}rdr\\
=&C\frac{h^{2\min(\nu,p+1)}}{p^{2\nu}}\sum_{n=0}^N\int_0^R\pi(\frac{n}{2})^{2\nu}|a_n|^2\{\frac{kJ'_{\frac{n}{2}}(kr)}{J_{\frac{n}{2}}(kR_0)}\}^2rdr\\
\le&C\frac{h^{2\min(\nu,p+1)}}{p^{2\nu}}\sum_{n=0}^N(\frac{n}{2})^{2\nu}|a_n|^2\int_0^R(\frac{n}{2})^2\frac{1}{r}(\frac{r}{R_0})^ndr \qquad (67)\\
\le&C\frac{h^{2\min(\nu,p+1)}}{p^{2\nu}}\sum_{n=0}^N(\frac{n}{2})^{2\nu+1}|a_n|^2\\
\le&C\frac{h^{2\min(\nu,p)}}{p^{2\nu}}|u(R_0,\cdot)|^2_{\nu+\frac{1}{2},(0,2\pi)},
\end{aligned}
$$

and

$$I_4 = \int_0^R |u_N - \Pi_\theta u_N|^2_{1,(0,2\pi)} \frac{dr}{r} \le C\frac{h^{2\min(\nu+1,p+1)-2}}{p^{2\nu}} \int_0^R |u_N|^2_{\nu+1,(0,2\pi)} \frac{dr}{r} \tag{68}$$

$$= C\frac{h^{2\min(\nu+1,p+1)-2}}{p^{2\nu}} \sum_{n=0}^{N} (\frac{n}{2})^{2\nu+2}|a_n|^2 \int_0^R \{\frac{J_{\frac{n}{2}}(kr)}{J_{\frac{n}{2}}(kR_0)}\}\frac{dr}{r} \tag{69}$$

$$\le C\frac{h^{2\min(\nu,p)}}{p^{2\nu}}|u(R_0,\cdot)|^2_{\nu+\frac{1}{2},(0,2\pi)}. \tag{70}$$

So

$$|u - \Pi_\theta u_N|_{1,B_R\setminus\Gamma} \le C\frac{h^{\min(\nu,p)}}{p^{\nu}}|u(R_0,\cdot)|_{\nu+\frac{1}{2},(0,2\pi)}. \tag{71}$$

From Lemma 2.1, we get the conclusion.

Theorem 4.2 Problem (55) has a unique solution $u_{h,N,R}$ except for a discrete set of values of k, if N, p are large enough, especially, under the condition of Theorem 4.1,

$$\begin{aligned}|u - u_{h,N,R}|_{1,\Omega_\rho} \le C \sum_{j=P,Q} \{ & \frac{h^{\min(\nu,p)}}{p^{\nu}}|u(R_{0,j}\ ,\cdot)|_{\nu+\frac{1}{2},\ (0,2\pi)} + \\ & e^{-\kappa\frac{N}{2}}|u(R_{0,j}\ ,\cdot)|_{\nu,\ (0,2\pi)} + \frac{h^{\min(p,m-1)}}{p^{m-1}}|u|_{m,\Omega_\rho^e}\}.\end{aligned} \tag{72}$$

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