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# FINITE VOLUME APPROXIMATION OF THE LINEARIZED SHALLOW WATER EQUATIONS IN HYPERBOLIC MODE

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**Abstract.** In this article, we consider the linearized inviscid shallow water equations in space dimension two in a rectangular domain. We implement a finite volume discretization and prove the numerical stability and convergence of the scheme for three cases that depend on the background flow $\tilde{u}_0$, $\tilde{v}_0$, and $\tilde{\phi}_0$ (sub- or super-critical flow at each part of the boundary). The three cases that we consider are fully hyperbolic modes.

**Key words.** shallow water equations, finite volume method, stability, and convergence.

## 1. Introduction

This article aims to study the finite volume approximation of the initial and boundary value problem for the linearized shallow water (SW) equations in a rectangle. This article builds on two previous articles [15] and [9]. In the theoretical paper [15] the authors determine all the boundary conditions that one can associate to the linearized shallow water equations and find, as explained below, five different situations depending on the respective values of $\tilde{u}_0$, $\tilde{v}_0$, $\tilde{\phi}_0$ corresponding to the (constant) background flow around which the linearization is made. Omitting the non generic cases where one of these constants vanish, we can assume, by a change of variables that $\tilde{u}_0$, $\tilde{v}_0$, $\tilde{\phi}_0$ are $> 0$. The article [15] raises of course the question of the approximation of the SW equations in the rectangle in these different situations. This question was investigated in [9] which considers the approximation of the inviscid linearized shallow water equations in the so-called supercritical (supersonic) case, that is when $\tilde{u}_0^2 + \tilde{v}_0^2 > g\tilde{\phi}_0$ (see below). Four cases remain to be studied and we consider in this article three of them for which the stationary part of the SW equations are fully hyperbolic. We do not discuss in this article the approximation of the fifth case for which the stationary part of the SW equations is partly hyperbolic and partly elliptic as this case necessitates a different approach.

Theoretically, we extended the results in [15] to more general hyperbolic systems in [16] and possibly to more general polygonal-like domains in the fully hyperbolic case (see [16, Remark 2.3]). Hence, we could also study the finite volume approximation in the more general setting. However, in this article, we prefer to consider the shallow water equations in a rectangular domain to stay close from our initial motivation of this work that is the study of the Local Area Models (LAMs) in the atmosphere and oceans sciences, see e.g. [22].

The linearized shallow water equations that we consider read

$$
\begin{cases}
u_t + \tilde{u}_0 u_x + \tilde{v}_0 u_y + g\phi_x = f_u, \\
v_t + \tilde{u}_0 v_x + \tilde{v}_0 v_y + g\phi_y = f_v, \\
\phi_t + \tilde{u}_0 \phi_x + \tilde{v}_0 \phi_y + \tilde{\phi}_0 (u_x + v_y) = f_\phi,
\end{cases}
\tag{1.1}
$$

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where $(x, y) \in \mathcal{M} := (0, L_x) \times (0, L_y)$, $(u, v)$ are the horizontal components of the velocity and $\phi$ is the potential height. The advection velocities $\tilde{u}_0$, $\tilde{v}_0$ and the mean geopotential height $\tilde{\phi}_0$ are constants, $g$ is the gravitational acceleration, $f_u$, $f_v$, and $f_\phi$ are the source terms. As shown in [15], the boundary conditions which can be associated with these equations depend on the relative values of the velocities ($\tilde{u}_0^2, \tilde{v}_0^2 >$ or $< g\tilde{\phi}_0$), that is whether these velocities are sub- or supercritical (sub- or supersonic). The three supersonic cases, when $\tilde{u}_0^2 + \tilde{v}_0^2 - g\tilde{\phi}_0 > 0$, that we consider are called: the mixed hyperbolic case (two sub-cases) and the fully hyperbolic subcritical case. The supercritical case, when $\tilde{u}_0 > \sqrt{g\tilde{\phi}_0}$, $\tilde{v}_0 > \sqrt{g\tilde{\phi}_0}$, has been considered in [9]. In this article we will focus on the other three cases. For the mixed hyperbolic case, we only consider one sub-case, where

$$\tilde{u}_0,\, \tilde{v}_0,\, \tilde{\phi}_0 > 0, \quad \tilde{u}_0 < \sqrt{g\tilde{\phi}_0},\;\; \tilde{v}_0 > \sqrt{g\tilde{\phi}_0}, \tag{1.2}$$

since the other sub-case where

$$\tilde{u}_0,\, \tilde{v}_0,\, \tilde{\phi}_0 > 0, \quad \tilde{u}_0 > \sqrt{g\tilde{\phi}_0},\;\; \tilde{v}_0 < \sqrt{g\tilde{\phi}_0},$$

would be similar. In the fully hyperbolic subcritical case, we assume that

$$\tilde{u}_0,\, \tilde{v}_0,\, \tilde{\phi}_0 > 0, \quad \tilde{u}_0 < \sqrt{g\tilde{\phi}_0},\;\; \tilde{v}_0 < \sqrt{g\tilde{\phi}_0},\;\; \tilde{u}_0^2 + \tilde{v}_0^2 - g\tilde{\phi}_0 > 0. \tag{1.3}$$

We will study the cases (1.2) and (1.3) separately in Section 2 and 3.

As we know, the literature on the shallow water equations is very vast, both on the theoretical and computational aspects, considering the viscous equations or the partly or totally inviscid equations and considering that the height is either always strictly positive or that it can vanish. See e.g. [1,2,4,8,12,21] on the computational side and see e.g. [5,6,10,11,14,17–20] on the theoretical side. Regarding the numerical stability of time discretized finite volume schemes, see e.g. [8], [9], and [12]. The proof of the convergence results follows the same methods as e.g. [3] and [13].

This article is organized as follows. At the end of this introductory section, we present some notations which we will use throughout this article. Section 2 and 3 are devoted to show the stability and convergence results of the finite volume scheme for the linearized SW equations in the mixed hyperbolic case and in the fully hyperbolic subcritical case, respectively.

We now write (1.1) in the compact form

$$\mathbf{u}_t + \mathcal{E}_1 \mathbf{u}_x + \mathcal{E}_2 \mathbf{u}_y = \mathbf{f}, \tag{1.4}$$

where $\mathbf{u} = (u, v, \phi)^T$, $\mathbf{f} = (f_u, f_v, f_\phi)^T$ and

$$\mathcal{E}_1 = \begin{pmatrix} \tilde{u}_0 & 0 & g \\ 0 & \tilde{u}_0 & 0 \\ \tilde{\phi}_0 & 0 & \tilde{u}_0 \end{pmatrix}, \quad \mathcal{E}_2 = \begin{pmatrix} \tilde{v}_0 & 0 & 0 \\ 0 & \tilde{v}_0 & g \\ 0 & \tilde{\phi}_0 & \tilde{v}_0 \end{pmatrix}.$$

Note that $\mathcal{E}_1$, $\mathcal{E}_2$ admit a symmetrizer $S_0 = \operatorname{diag}(1, 1, g/\tilde{\phi}_0)$, which means that $S_0\mathcal{E}_1, S_0\mathcal{E}_2$ are both symmetric (see e.g. [7, Chapter 1]).

Here and in the following, we endow the space $H = L^2(\mathcal{M})^3$ with the Hilbert scalar products and norms, for $\mathbf{u} = (u, v, \phi)^T$, $\mathbf{u}' = (u', v', \phi')^T$:

$$\begin{aligned} &\langle \mathbf{u}, \mathbf{u}' \rangle = (S_0 \mathbf{u}, \mathbf{u}') = (u, u') + (v, v') + \frac{g}{\tilde{\phi}_0}(\phi, \phi'), \quad && |\mathbf{u}| = \{\langle \mathbf{u}, \mathbf{u} \rangle\}^{1/2}, \\ &(\mathbf{u}, \mathbf{u}') = \mathbf{u}'^T \mathbf{u} = (u, u') + (v, v') + (\phi, \phi'), && \|\mathbf{u}\| = \{(\mathbf{u}, \mathbf{u})\}^{1/2}. \end{aligned} \tag{1.5}$$

where $(\cdot,\cdot)$ denotes the standard scalar product on $L^2(\mathcal{M})$. The appearance of the coefficient $g/\tilde{\phi}_0$ in the inner product $\langle\cdot,\cdot\rangle$ is needed for physical (dimensional) reasons.

## 2. The mixed hyperbolic case for the linearized shallow water equations

In this section we consider the mixed hyperbolic case, see (1.2). We associate with (1.1) the initial conditions:

$$\mathbf{u} = (u, v, \phi)^T = (u^0, v^0, \phi^0)^T, \text{ at } t = 0. \tag{2.1}$$

**2.1. Preliminary theoretical results.** The results in this subsection are taken from [15]. From [15, Sections 1 and 3.2], we know that there exists a real non-singular matrix $P$ such that $P^T S_0 \mathcal{E}_1 P$ and $P^T S_0 \mathcal{E}_2 P$ are both diagonal. We set $\tilde{\kappa}_0 = \sqrt{g(\tilde{u}_0^2 + \tilde{v}_0^2 - g\tilde{\phi}_0)/\tilde{\phi}_0}$, and we have:

$$P^{-1} = \begin{pmatrix} \tilde{u}_0 & -\tilde{u}_0 & \tilde{\kappa}_0 \\ \tilde{v}_0 & -\tilde{u}_0 & -\tilde{\kappa}_0 \\ \tilde{u}_0 & \tilde{v}_0 & g \end{pmatrix}.$$

Direct computations show that

$$\begin{aligned} P^T S_0 \mathcal{E}_1 P = D_1 &:= \operatorname{diag}(a_1, a_2, a_3) \\ &:= \operatorname{diag}\Big(\frac{\tilde{u}_0\tilde{\kappa}_0 + g\tilde{v}_0}{2(\tilde{u}_0^2 + \tilde{v}_0^2)\tilde{\kappa}_0}, \frac{\tilde{u}_0\tilde{\kappa}_0 - g\tilde{v}_0}{2(\tilde{u}_0^2 + \tilde{v}_0^2)\tilde{\kappa}_0}, \frac{\tilde{u}_0}{\tilde{u}_0^2 + \tilde{v}_0^2}\Big), \\ P^T S_0 \mathcal{E}_2 P = D_2 &:= \operatorname{diag}(b_1, b_2, b_3) \\ &:= \operatorname{diag}\Big(\frac{\tilde{v}_0\tilde{\kappa}_0 - g\tilde{u}_0}{2(\tilde{u}_0^2 + \tilde{v}_0^2)\tilde{\kappa}_0}, \frac{\tilde{v}_0\tilde{\kappa}_0 + g\tilde{u}_0}{2(\tilde{u}_0^2 + \tilde{v}_0^2)\tilde{\kappa}_0}, \frac{\tilde{v}_0}{\tilde{u}_0^2 + \tilde{v}_0^2}\Big). \end{aligned} \tag{2.2}$$

Under assumption (1.2), we find that

$$a_1,\ a_3,\ b_1,\ b_2,\ b_3 > 0, \quad \text{and} \quad a_2 < 0. \tag{2.3}$$

We introduce the new variables

$$\boldsymbol{\Xi} := \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = P^{-1}\mathbf{u} = \begin{pmatrix} \tilde{v}_0 u - \tilde{u}_0 v + \tilde{\kappa}_0 \phi \\ \tilde{v}_0 u - \tilde{u}_0 v - \tilde{\kappa}_0 \phi \\ \tilde{u}_0 u + \tilde{v}_0 v + g\phi \end{pmatrix}. \tag{2.4}$$

Then using the new variables $\boldsymbol{\Xi}$, we rewrite (1.4) as

$$D_0 \boldsymbol{\Xi}_t + D_1 \boldsymbol{\Xi}_x + D_2 \boldsymbol{\Xi}_y = P^T S_0 \mathbf{f}, \tag{2.5}$$

where, as we noticed, $D_0 = P^T S_0 P$ is symmetric and positive-definite. Since now $D_1$ and $D_2$ are diagonal, we can assign the boundary conditions in the $\boldsymbol{\Xi}$ variables for the parts of $\partial\mathcal{M}$ corresponding to the incoming characteristics. According to the signs of the $a_i$'s and $b_i$'s ($i = 1, 2, 3$), we associate with (2.5) the following boundary conditions in the $\boldsymbol{\Xi}$ variables:

$$\begin{cases} \xi = \zeta = 0, & \text{on } \Gamma_W = \{x = 0\}, \\ \eta = 0, & \text{on } \Gamma_E = \{x = L_x\}, \\ \xi = \eta = \zeta = 0, & \text{on } \Gamma_S = \{y = 0\}. \end{cases} \tag{2.6}$$

Therefore, the boundary conditions we associate with (1.1) read in the original $\mathbf{u}$ variables (see [15, Section 3.2]):

$$\begin{cases} \tilde{v}_0 u - \tilde{u}_0 v + \tilde{\kappa}_0 \phi = \tilde{u}_0 u + \tilde{v}_0 v + g\phi = 0, \text{ on } \Gamma_W = \{x = 0\}, \\ \tilde{v}_0 u - \tilde{u}_0 v - \tilde{\kappa}_0 \phi = 0, \text{ on } \Gamma_E = \{x = L_x\}, \\ u = v = \phi = 0, \text{ on } \Gamma_S = \{y = 0\}. \end{cases} \tag{2.7}$$

We now define two unbounded operators on $H = L^2(\Omega)^3$:

$$\begin{aligned} \mathbf{A}\mathbf{u} &= \mathcal{E}_1 \mathbf{u}_x + \mathcal{E}_2 \mathbf{u}_y, \qquad \forall\, \mathbf{u} \in \mathcal{D}(\mathbf{A}), \\ \tilde{\mathbf{A}}\boldsymbol{\Xi} &= D_1 \boldsymbol{\Xi}_x + D_2 \boldsymbol{\Xi}_y, \qquad \forall\, \boldsymbol{\Xi} \in \mathcal{D}(\tilde{\mathbf{A}}), \end{aligned} \tag{2.8}$$

where the domains $\mathcal{D}(\mathbf{A})$ and $\mathcal{D}(\tilde{\mathbf{A}})$ are

$$\mathcal{D}(\mathbf{A}) = \{\mathbf{u} \in H \ : \mathbf{A}\mathbf{u} \in H \ , \mathbf{u} \text{ satisfies the boundary conditions (2.7)}\},$$
$$\mathcal{D}(\tilde{\mathbf{A}}) = \left\{\boldsymbol{\Xi} \in H \ : \tilde{\mathbf{A}}\boldsymbol{\Xi} \in H \ , \boldsymbol{\Xi} \text{ satisfies the boundary conditions (2.6)}\right\}.$$

Using the relation (2.4) between $\mathbf{u}$ and $\boldsymbol{\Xi}$, we immediately find that

$$\mathbf{A}\mathbf{u} = (P^T S_0)^{-1} \tilde{\mathbf{A}}\boldsymbol{\Xi}. \tag{2.9}$$

**Lemma 2.1.** *Assume that* (1.2) *holds. Then for any sufficiently smooth* $\mathbf{u} \in \mathcal{D}(\mathbf{A})$ *and* $\boldsymbol{\Xi} \in \mathcal{D}(\tilde{\mathbf{A}})$*, there holds*

$$\langle \mathbf{A}\mathbf{u}, \mathbf{u} \rangle = (\tilde{\mathbf{A}}\boldsymbol{\Xi}, \boldsymbol{\Xi}) \geq 0. \tag{2.10}$$

*Proof.* We infer from (2.4) and (2.9) that

$$\langle \mathbf{A}\mathbf{u}, \mathbf{u} \rangle = \langle (P^T S_0)^{-1} \tilde{\mathbf{A}}\boldsymbol{\Xi}, P\boldsymbol{\Xi} \rangle = (S_0 (P^T S_0)^{-1} \tilde{\mathbf{A}}\boldsymbol{\Xi}, P\boldsymbol{\Xi}) = (\tilde{\mathbf{A}}\boldsymbol{\Xi}, \boldsymbol{\Xi}),$$

and the positivity of the operator $\tilde{\mathbf{A}}$ is achieved by integrations by parts:

$$\begin{aligned} (\tilde{\mathbf{A}}\boldsymbol{\Xi}, \boldsymbol{\Xi}) &= (D_1 \boldsymbol{\Xi}_x + D_2 \boldsymbol{\Xi}_y, \Xi) \\ &= \int_0^{L_y} (a_1 \xi^2)\big|_{x=0}^{x=L_x} + (a_2 \eta^2)\big|_{x=0}^{x=L_x} + (a_3 \zeta^2)\big|_{x=0}^{x=L_x} \,\mathrm{d}y \\ &\quad + \int_0^{L_x} (b_1 \xi^2)\big|_{y=0}^{y=L_y} + (b_2 \eta^2)\big|_{y=0}^{y=L_y} + (b_3 \zeta^2)\big|_{y=0}^{y=L_y} \,\mathrm{d}x \\ &\geq 0, \end{aligned}$$

where the last inequality follows from the boundary conditions (2.6). □

**Remark 2.1.** *The fact that the initial and boundary value problem* (1.1), (2.1), (2.7) *is well posed is a recent result proved in [15]. The proof relies on the semigroup theory and necessitates in particular proving (by approximation) that* $\langle \mathbf{A}\mathbf{u}, \mathbf{u} \rangle \geq 0$ *for all* $\mathbf{u} \in \mathcal{D}(\mathbf{A})$*. The fact that* (2.7) *makes sense for such* $\mathbf{u}$*'s results from a trace theorem also proved in [15, Section 2.1].*

**2.2. Finite volume discretization.** We decompose $\mathcal{M} = (0, L_x) \times (0, L_y)$ into $N_x \times N_y$ cells denoted by $(k_{i,j})_{1 \leq i \leq N_x, 1 \leq j \leq N_y}$ of size $\Delta x \times \Delta y$ with $N_x \Delta x = L_x$ and $N_y \Delta y = L_y$.

For $0 \leq i \leq N_x$ and for $0 \leq j \leq N_y$ let

$$x_{i+1/2} = i\Delta x, \ \textit{and} \ y_{j+1/2} = j\Delta y. \tag{2.11}$$

Then the cells $(k_{i,j})$ are, for $1 \leq i \leq N_x, 1 \leq j \leq N_y$,

$$k_{i,j} = (x_{i-1/2}, x_{i+1/2}) \times (y_{j-1/2}, y_{j+1/2}). \tag{2.12}$$

We also define the center $(x_i, y_j)$ of each cell $k_{ij}$,

$$\left\{\begin{array}{l} x_i = \frac{1}{2}(x_{i-1/2} + x_{i+1/2}) = (i-1)\Delta x + \frac{\Delta x}{2},\ 1 \le i \le N_x, \\ y_j = \frac{1}{2}(y_{j-1/2} + y_{j+1/2}) = (j-1)\Delta y + \frac{\Delta y}{2},\ 1 \le j \le N_y. \end{array}\right. \tag{2.13}$$

In view of imposing the boundary conditions, we add fictitious cells on the four sides of the boundary:

$$\begin{aligned} &k_{0,j} = (-\Delta x, 0) \times (y_{j-1/2}, y_{j+1/2}), \\ &\quad \text{centered at } (x_0 = -\frac{\Delta x}{2}, y_j),\ \ 1 \le j \le N_y, \\ &k_{N_x+1,j} = (L_x, L_x + \Delta x) \times (y_{j-1/2}, y_{j+1/2}), \\ &\quad \text{centered at } (x_{N_x+1} = L_x + \frac{\Delta x}{2}, y_j),\ \ 1 \le j \le N_y, \\ &k_{i,0} = (x_{i-1/2}, x_{i+1/2}) \times (-\Delta y, 0), \\ &\quad \text{centered at } (x_i, y_0 = -\frac{\Delta y}{2}),\ \ 1 \le i \le N_x, \\ &k_{i,N_y+1} = (x_{i-1/2}, x_{i+1/2}) \times (L_y, L_y + \Delta y), \\ &\quad \text{centered at } (x_i, y_{N_y+1} = L_y + \frac{\Delta y}{2}),\ \ 1 \le i \le N_x. \end{aligned} \tag{2.14}$$

The finite volume scheme is found by integrating the equations (1.4) or (2.5) over each control volume $(k_{i,j})_{1\le i\le N_x, 1\le j\le N_y}$: for $1 \le i \le N_x$, $1 \le j \le N_y$,

$$\begin{aligned} &\frac{d}{dt}\frac{1}{\Delta x\Delta y}\int_{k_{i,j}} \mathbf{u}\,\mathrm{d}x\mathrm{d}y + \frac{\mathcal{E}_1}{\Delta x\Delta y}\int_{y_{j-1/2}}^{y_{j+1/2}} [\mathbf{u}(x_{i+1/2}, y, t) - \mathbf{u}(x_{i-1/2}, y, t)]\,\mathrm{d}y \\ &\quad + \frac{\mathcal{E}_2}{\Delta x\Delta y}\int_{x_{i-1/2}}^{x_{i+1/2}} [\mathbf{u}(x, y_{j+1/2}, t) - \mathbf{u}(x, y_{j-1/2}, t)]\,\mathrm{d}x \\ &\quad = \frac{1}{\Delta x\Delta y}\int_{k_{i,j}} \mathbf{f}(x, y, t)\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{2.15}$$

Let us denote

$$\begin{aligned} \mathbf{V}_h = \{&\mathbf{u} = (u, v, \phi) \text{ are step functions over } \mathcal{M}\ : \\ &\mathbf{u}|_{k_{i,j}} = \mathbf{u}_{i,j},\ \forall\, 0 \le i \le N_x + 1,\ 0 \le j \le N_y + 1\}. \end{aligned} \tag{2.16}$$

Now we can define the subspaces $\mathcal{V}_h \subset \mathbf{V}_h$ and $P^{-1}\mathcal{V}_h \subset \mathbf{V}_h$, that take into account the boundary conditions (2.7):

$$\begin{aligned} \mathcal{V}_h = \{&\mathbf{u} = (u, v, \phi) \text{ are step functions over } \mathcal{M}\ :\ \mathbf{u}|_{k_{i,j}} = \mathbf{u}_{i,j}, \\ &\forall\, 0 \le i \le N_x + 1, 0 \le j \le N_y + 1, \text{ and } \mathbf{u} \text{ satisfies:} \\ &\tilde{v}_0 u_{0,j} - \tilde{u}_0 v_{0,j} + \tilde{\kappa}_0 \phi_{0,j} = \tilde{u}_0 u_{0,j} + \tilde{v}_0 v_{0,j} + g\phi_{0,j} = 0, \\ &\ \text{for } 1 \le j \le N_y, \\ &\tilde{v}_0 u_{N_x+1,j} - \tilde{u}_0 v_{N_x+1,j} - \tilde{\kappa}_0 \phi_{N_x+1,j} = 0,\ \text{for } 1 \le j \le N_y, \\ &u_{i,0} = v_{i,0} = \phi_{i,0} = 0,\ \text{for } 1 \le i \le N_x.\}, \end{aligned} \tag{2.17}$$

and,

$$
\begin{aligned}
P^{-1}\mathcal{V}_h = \{\boldsymbol{\Xi} = (\xi,\eta,\zeta) \text{ are step functions over } \mathcal{M} : \boldsymbol{\Xi}|_{k_{i,j}} = \boldsymbol{\Xi}_{i,j},\\
&\forall\, 0 \le i \le N_x+1,\\
&0 \le j \le N_y+1, \text{ and } \boldsymbol{\Xi} \text{ satisfies:}\\
&\xi_{0,j} = \zeta_{0,j} = 0, \text{ for } 1 \le j \le N_y,\\
&\eta_{N_x+1,j} = 0, \text{ for } 1 \le j \le N_y,\\
&\xi_{i,0} = \eta_{i,0} = \zeta_{i,0} = 0, \text{ for } 1 \le i \le N_x.\}.
\end{aligned}
\tag{2.18}
$$

We now define the discrete scalar products and norms over $\mathbf{V}_h$.

**Definition 2.1.** *Let $\mathbf{u}_h = (u_h, v_h, \phi_h) \in \mathbf{V}_h$ and $\tilde{\mathbf{u}}_h = (\tilde{u}_h, \tilde{v}_h, \tilde{\phi}_h)_h \in \mathbf{V}_h$; we set*

$$
\begin{aligned}
&(\mathbf{u}_h, \tilde{\mathbf{u}}_h)_h = \Delta x \Delta y \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left(u_{i,j}\tilde{u}_{i,j} + v_{i,j}\tilde{v}_{i,j} + \phi_{i,j}\tilde{\phi}_{i,j}\right),\\
&\langle \mathbf{u}_h, \tilde{\mathbf{u}}_h\rangle_h = (S_0\mathbf{u}_h, \tilde{\mathbf{u}}_h)_h,\\
&\textit{and,}\\
&|\mathbf{u}|_h = \langle \mathbf{u}_h, \mathbf{u}_h\rangle_h = (S_0\mathbf{u}_h, \mathbf{u}_h)_h,\\
&\|\mathbf{u}\|_h = (\mathbf{u}_h, \mathbf{u}_h)_h.
\end{aligned}
\tag{2.19}
$$

Since the boundary conditions are easier to compute in terms of $\boldsymbol{\Xi}$, we will define our spatial scheme through $\boldsymbol{\Xi}$. For the $x$ derivatives, we are using an upwind scheme for $\xi$ and $\zeta$ and a downwind scheme for $\eta$ due to the signs of the $a_i$'s and $b_i$'s in this case. For the $y$ derivatives, we are using an upwind scheme for $\xi$, $\eta$ and $\zeta$. Therefore, the discretized version $\tilde{\mathbf{A}}_{\mathbf{h}}$ of $\tilde{\mathbf{A}}$ is defined as follows: for all $\boldsymbol{\Xi}_h \in P^{-1}\mathcal{V}_h$,

$$
\tilde{\mathbf{A}}_{\mathbf{h}}\boldsymbol{\Xi}_h := \tilde{\mathbf{A}}^x_{\mathbf{h}}\boldsymbol{\Xi}_h + \tilde{\mathbf{A}}^y_{\mathbf{h}}\boldsymbol{\Xi}_h,
\tag{2.20}
$$

where, for $1 \le i \le N_x$ and $1 \le j \le N_y$:

$$
\begin{aligned}
(\tilde{\mathbf{A}}^x_{\mathbf{h}}\boldsymbol{\Xi}_h)_{i,j} &= \left(a_1\frac{\xi_{i,j}-\xi_{i-1,j}}{\Delta x}, a_2\frac{\eta_{i+1,j}-\eta_{i,j}}{\Delta x}, a_3\frac{\zeta_{i,j}-\zeta_{i-1,j}}{\Delta x}\right),\\
(\tilde{\mathbf{A}}^y_{\mathbf{h}}\boldsymbol{\Xi}_h)_{i,j} &= \left(b_1\frac{\xi_{i,j}-\xi_{i,j-1}}{\Delta y}, b_2\frac{\eta_{i,j}-\eta_{i,j-1}}{\Delta y}, b_3\frac{\zeta_{i,j}-\zeta_{i,j-1}}{\Delta y}\right).
\end{aligned}
\tag{2.21}
$$

Similar to the relation (2.9), we define the discretized version $\mathbf{A}_h$ of $\mathbf{A}$ as: for $\mathbf{u}_h \in \mathcal{V}_h$,

$$
\mathbf{A}_h\mathbf{u}_h := (P^TS_0)^{-1}\tilde{\mathbf{A}}_{\mathbf{h}}\boldsymbol{\Xi}_h, \text{ where } \boldsymbol{\Xi}_h = P^{-1}\mathbf{u}_h.
\tag{2.22}
$$

The discretized version $\mathbf{f}_h(t) \in \mathbf{V}_h$ of the forcing term $\mathbf{f}(t)$ is defined by

$$
\mathbf{f}_h(t)\big|_{k_{i,j}} = \mathbf{f}_{i,j}(t), \quad \forall\, 1 \le i \le N_x,\ 1 \le j \le N_y,
\tag{2.23}
$$

where

$$
\mathbf{f}_{i,j}(t) = \frac{1}{\Delta x\Delta y}\int_{k_{i,j}} \mathbf{f}(x,y,t)\mathrm{d}x\mathrm{d}y.
$$

With these definitions, the finite volume spatial approximation of equation (2.15) is

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_h(t) + \mathbf{A}_h\mathbf{u}_h(t) = \mathbf{f}_h(t),\\
&\qquad\qquad \mathbf{u}_h(t) \in \mathcal{V}_h.
\end{aligned}
\tag{2.24}
$$

**Lemma 2.2.** *The operators* $\tilde{\mathbf{A}}_{\mathbf{h}}$ *and* $\mathbf{A}_{\mathbf{h}}$ *defined in* (2.20) *and* (2.22) *are positive over* $\mathcal{P}^{-1}\mathcal{V}_h$ *and* $\mathcal{V}_h$ *, respectively. That is for any* $\mathbf{u}_h \in \mathcal{V}_h$ *and* $\boldsymbol{\Xi}_h = P^{-1}\mathbf{u}_h \in \mathcal{P}^{-1}\mathcal{V}_h$*, we have*

$$\langle \mathbf{A}_h \mathbf{u}_h, \mathbf{u}_h \rangle_h = \left( \tilde{\mathbf{A}}_h \boldsymbol{\Xi}_h, \boldsymbol{\Xi}_h \right)_h \geq 0, \tag{2.25}$$

*where the scalar products are defined in* (2.19)*.*

*Proof.* Using relation (2.22), we find that

$$\begin{aligned} \langle \mathbf{A}_h \mathbf{u}_h, \mathbf{u}_h \rangle_h &= \langle (PS_0)^{-1} \tilde{\mathbf{A}}_h \boldsymbol{\Xi}_h, P\boldsymbol{\Xi}_h \rangle_h \\ &= \left( S_0 (PS_0)^{-1} \tilde{\mathbf{A}}_h \boldsymbol{\Xi}_h, P\boldsymbol{\Xi}_h \right)_h \\ &= \left( \tilde{\mathbf{A}}_h \boldsymbol{\Xi}_h, \boldsymbol{\Xi}_h \right)_h . \end{aligned} \tag{2.26}$$

We are now left to prove that $\left( \tilde{\mathbf{A}}_h \boldsymbol{\Xi}_h, \boldsymbol{\Xi}_h \right)_h \geq 0$. Using the fact that

$$2(a-b)a = a^2 - b^2 + (a-b)^2, \tag{2.27}$$

and taking into account that $\Xi_h \in P^{-1}\mathcal{V}_h$ we have the following equalities

$$\begin{aligned} &\Delta x \Delta y a_1 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \frac{\xi_{i,j} - \xi_{i-1,j}}{\Delta x} \xi_{i,j} \\ &\qquad = \frac{a_1 \Delta y}{2} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left[ |\xi_{i,j}|^2 - |\xi_{i-1,j}|^2 + |\xi_{i,j} - \xi_{i-1,j}|^2 \right] \\ &\qquad = \frac{a_1 \Delta y}{2} \sum_{j=1}^{N_y} \left[ |\xi_{N_x,j}|^2 + \sum_{i=1}^{N_x} |\xi_{i,j} - \xi_{i-1,j}|^2 \right], \\ &\Delta x \Delta y a_3 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \frac{\zeta_{i,j} - \xi_{i-1,j}}{\Delta x} \xi_{i,j} \\ &\qquad = \frac{a_3 \Delta y}{2} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left[ |\zeta_{i,j}|^2 - |\zeta_{i-1,j}|^2 + |\zeta_{i,j} - \zeta_{i-1,j}|^2 \right] \\ &\qquad = \frac{a_3 \Delta y}{2} \sum_{j=1}^{N_y} \left[ |\zeta_{N_x,j}|^2 + \sum_{i=1}^{N_x} |\zeta_{i,j} - \zeta_{i-1,j}|^2 \right], \end{aligned} \tag{2.28}$$

$$
\begin{aligned}
&\Delta x \Delta y b_1 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \frac{\xi_{i,j} - \xi_{i,j-1}}{\Delta y} \xi_{i,j} \\
&\quad = \frac{b_1 \Delta x}{2} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left[ |\xi_{i,j}|^2 - |\xi_{i,j-1}|^2 + |\xi_{i,j} - \xi_{i,j-1}|^2 \right] \\
&\quad = \frac{b_1 \Delta x}{2} \sum_{i=1}^{N_x} \left[ |\xi_{i,N_y}|^2 + \sum_{j=1}^{N_y} |\xi_{i,j} - \xi_{i,j-1}|^2 \right], \\
&\Delta x \Delta y b_2 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \frac{\eta_{i,j} - \eta_{i,j-1}}{\Delta y} \eta_{i,j} \\
&\quad = \frac{b_2 \Delta x}{2} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left[ |\eta_{i,j}|^2 - |\eta_{i,j-1}|^2 + |\eta_{i,j} - \eta_{i,j-1}|^2 \right] \\
&\quad = \frac{b_2 \Delta x}{2} \sum_{i=1}^{N_x} \left[ |\eta_{i,N_y}|^2 + \sum_{j=1}^{N_y} |\eta_{i,j} - \eta_{i,j-1}|^2 \right],
\end{aligned} \tag{2.29}
$$

$$
\begin{aligned}
&\Delta x \Delta y b_3 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \frac{\zeta_{i,j} - \zeta_{i,j-1}}{\Delta y} \zeta_{i,j} \\
&\quad = \frac{b_3 \Delta x}{2} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left[ |\zeta_{i,j}|^2 - |\zeta_{i,j-1}|^2 + |\zeta_{i,j} - \xi_{i,j-1}|^2 \right] \\
&\quad = \frac{b_1 \Delta x}{2} \sum_{i=1}^{N_x} \left[ |\zeta_{i,N_y}|^2 + \sum_{j=1}^{N_y} |\zeta_{i,j} - \zeta_{i,j-1}|^2 \right].
\end{aligned} \tag{2.30}
$$

Now from

$$
2(a-b)b = a^2 - b^2 - (a-b)^2, \tag{2.31}
$$

we deduce that

$$
\begin{aligned}
&\Delta x \Delta y a_2 \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \frac{\eta_{i+1,j} - \eta_{i,j}}{\Delta x} \eta_{i,j} \\
&\quad = \frac{a_2 \Delta y}{2} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left[ |\eta_{i+1,j}|^2 - |\eta_{i,j}|^2 - |\eta_{i+1,j} - \eta_{i,j}|^2 \right] \\
&\quad = -\frac{a_2 \Delta y}{2} \sum_{j=1}^{N_y} \left[ |\eta_{1,j}|^2 + \sum_{i=1}^{N_y} |\eta_{i+1,j} - \eta_{i,j}|^2 \right].
\end{aligned} \tag{2.32}
$$

Since $a_1, a_3, b_1, b_2, b_3 > 0$ and $a_2 < 0$, we infer the positivity of $\left(\tilde{\mathbf{A}}_h \mathbf{\Xi}_h, \mathbf{\Xi}_h\right)_h$ from equations (2.28)-(2.32).

□

**2.3. Time discretization: the Euler implicit scheme.** We now study the time discretization of equations (2.24) and introduce the classical Euler implicit scheme in time. We define a time step $\Delta t$ with $N_t \Delta t = T$ and we set $t_n = n\Delta t$

for $0 \leq n \leq N_t$. We denote by $\{\mathbf{u}_h^n \in \mathcal{V}_h, 0 \leq n \leq N_t\}$ the discrete unknowns. We start with $\mathbf{u}_h^0 \in \mathcal{V}_h$ given by

$$\mathbf{u}_{i,j}^0 = \frac{1}{\Delta x \Delta y} \int_{k_{i,j}} \mathbf{u}^0(x,y)\mathrm{d}x\mathrm{d}y, \text{ for } 1 \leq i \leq N_x,\ 1 \leq j \leq N_y, \tag{2.33}$$

where $\mathbf{u}^0$ is the (given) initial data appearing in (2.1). We notice also using the Cauchy-Schwarz inequality that

$$|\mathbf{u}_h^0| \leq |\mathbf{u}^0|. \tag{2.34}$$

The Euler implicit scheme for (2.24) then reads

$$\begin{cases} \dfrac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t} + \mathbf{A}_h \mathbf{u}_h^{n+1} = \mathbf{f}_h^{n+1}, \\ \mathbf{u}_h^n \in \mathcal{V}_h, \text{ for } 0 \leq n \leq N_t. \end{cases} \tag{2.35}$$

with

$$\mathbf{f}_h^{n+1} = \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} \mathbf{f}_h(t)\mathrm{d}t.$$

We notice that $(I + \Delta t \mathbf{A}_h)$ is invertible because $\mathbf{A}_h$ is positive and therefore $I + \Delta t \mathbf{A}_h$ is positive definite. For this reason the system (2.35) admits a unique solution $\mathbf{u}_h^{n+1}$ when $\mathbf{u}_h^n$ is known.

To obtain an estimate on $|\mathbf{u}_h^{n+1}|_h$, we take the scalar product of (2.35) with $2\Delta t \mathbf{u}_h^{n+1}$ (scalar product $\langle \cdot, \cdot \rangle_h$) and find that

$$|\mathbf{u}_h^{n+1}|_h^2 - |\mathbf{u}_h^n|_h^2 + |\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|_h^2 + 2\Delta t \langle \mathbf{A}_h \mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1} \rangle_h = 2\Delta t \langle \mathbf{f}_h^{n+1}, \mathbf{u}_h^{n+1} \rangle_h. \tag{2.36}$$

Using the Cauchy-Schwarz inequality, the right hand side of (2.36) can be estimated as

$$\begin{aligned} 2\Delta t \langle \mathbf{f}_h^{n+1}, \mathbf{u}_h^{n+1} \rangle_h &= 2\Delta t \langle \mathbf{f}_h^{n+1}, \mathbf{u}_h^{n+1} - \mathbf{u}_h^n \rangle_h + 2\Delta t \langle \mathbf{f}_h^{n+1}, \mathbf{u}_h^n \rangle_h \\ &\leq \Delta t^2 |\mathbf{f}_h^{n+1}|_h^2 + |\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|_h^2 + \Delta t |\mathbf{f}_h^{n+1}|_h^2 + \Delta t |\mathbf{u}_h^n|_h^2, \end{aligned} \tag{2.37}$$

which, together with the positivity of $\mathbf{A}_h$ (see Lemma 2.2), permits us to infer from (2.36) that

$$|\mathbf{u}_h^{n+1}|_h^2 \leq (1 + \Delta t)|\mathbf{u}_h^n|_h^2 + \Delta t(1 + \Delta t)|\mathbf{f}_h^{n+1}|_h^2. \tag{2.38}$$

We now estimate $\Delta t |\mathbf{f}_h^{n+1}|_h^2$ as follows:

(2.39)

$$\begin{aligned} \Delta t |\mathbf{f}_h^{n+1}|_h^2 &\leq \|S_0\|^2 \int_{n\Delta t}^{(n+1)\Delta t} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \int_{k_{i,j}} |\mathbf{f}(x,y,t)|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t, \\ &\leq \|S_0\|^2 \int_{n\Delta t}^{(n+1)\Delta t} \int_{\mathcal{M}} |\mathbf{f}(x,y,t)|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t := \|S_0\|^2 |\mathbf{f}|_{L^2(t_n, t_{n+1}, L^2(\mathcal{M}))}^2. \end{aligned}$$

In view of equation (2.39), equation (2.38) yields

$$|\mathbf{u}_h^{n+1}|_h^2 \leq (1 + \Delta t) \left[ |\mathbf{u}_h^n|_h^2 + \|S_0\|^2 |\mathbf{f}|_{L^2(t_n, t_{n+1}, L^2(\mathcal{M}))}^2 \right], \tag{2.40}$$

and iterating (2.40) and using that $1+\Delta t \leq e^{\Delta t}$, we obtain that

$$
\begin{aligned}
|\mathbf{u}_h^{n+1}|_h^2 &\leq (1+\Delta t)\left[|\mathbf{u}_h^n|_h^2 + \|S_0\|^2 |\mathbf{f}|^2_{L^2(t_n,t_{n+1},L^2(\mathcal{M}))}\right], \\
&\leq (1+\Delta t)\,[(1+\Delta t)(\,|\mathbf{u}_h^{n-1}|_h^2 + \|S_0\|^2 |\mathbf{f}|_{L^2(t_{n-1},t_n,L^2(\mathcal{M}))}\,) \\
&\qquad + \|S_0\|^2 |\mathbf{f}|^2_{L^2(t_n,t_{n+1},L^2(\mathcal{M}))}\Big] \\
&\leq (1+\Delta t)^2\left[|\mathbf{u}_h^{n-1}|_h^2 + \|S_0\|^2 |\mathbf{f}|^2_{L^2(t_{n-1},t_{n+1},L^2(\mathcal{M}))}\right] \\
&\leq \cdots \\
&\leq (1+\Delta t)^{n+1}\left[|\mathbf{u}_h^0|_h^2 + \|S_0\|^2 |\mathbf{f}|^2_{L^2(0,t_{n+1},L^2(\mathcal{M}))}\right], \\
&\leq e^{(n+1)\Delta t}\left[|\mathbf{u}^0|^2 + \|S_0\|^2 |\mathbf{f}|^2_{L^2(0,t_{n+1},L^2(\mathcal{M}))}\right].
\end{aligned}
\tag{2.41}
$$

From equation (2.41) and the fact that $\Delta t N_t = T$, we conclude that

$$
|\mathbf{u}_h^n|_h^2 \leq e^T\left[|\mathbf{u}^0|^2 + \|S_0\|^2 |\mathbf{f}|^2_{L^2(0,T,L^2(\mathcal{M}))}\right], \quad \forall\, n = 0,1,\cdots,N_t. \tag{2.42}
$$

**Theorem 2.1.** *The scheme defined by the equations* (2.35)*,* (2.20)*,* (2.22)*, is stable in* $L^\infty(0,T;L^2(\mathcal{M}))$ *in the sense of* (2.42)*.*

**2.4. Time discretization: the Euler explicit scheme.** For the time discretization, in this section, we use the Euler explicit scheme that we apply to the equations (2.24). We keep the same spatial discretization as in the previous section. For $\mathbf{u}_h^0 \in \mathcal{V}_h$ given by (2.33), the Euler explicit scheme for (2.24) reads

$$
\begin{cases}
\dfrac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t} + \mathbf{A}_h \mathbf{u}_h^n = \mathbf{f}_h^n, \\
\mathbf{u}_h^n \in \mathcal{V}_h, \text{ for } 0 \leq n \leq N_t,
\end{cases}
\tag{2.43}
$$

with

$$
\mathbf{f}_h^n = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} \mathbf{f}_h(t)\mathrm{d}t.
$$

We take the scalar product of $(2.43)_1$ with $2\Delta t \mathbf{u}_h^n$ (scalar product $\langle \cdot,\cdot\rangle_h$) and obtain

$$
|\mathbf{u}_h^{n+1}|_h^2 \leq |\mathbf{u}_h^n|_h^2 + |\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|_h^2 - 2\Delta t\langle \mathbf{A}_h\mathbf{u}_h^n, \mathbf{u}_h^n\rangle_h + 2\Delta t\langle \mathbf{f}_h^n, \mathbf{u}_h^n\rangle_h. \tag{2.44}
$$

Since we have $\langle \mathbf{A}_h\mathbf{u}_h^n, \mathbf{u}_h^n\rangle_h = \left(\tilde{\mathbf{A}}_h \boldsymbol{\Xi}_h^n, \boldsymbol{\Xi}_h^n\right)_h$ by (2.26), we then conclude from (2.28)-(2.32) that

$$
\begin{aligned}
&-\langle \mathbf{A}_h\mathbf{u}_h^n, \mathbf{u}_h^n\rangle_h \leq \\
&\quad -\frac{a_1\Delta y}{2}\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}|\xi_{i,j}^n - \xi_{i-1,j}^n|^2 + \frac{a_2\Delta y}{2}\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}|\eta_{i+1,j}^n - \eta_{i,j}^n|^2 \\
&\quad -\frac{a_3\Delta y}{2}\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}|\zeta_{i,j}^n - \zeta_{i-1,j}^n|^2 - \frac{b_1\Delta x}{2}\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}|\xi_{i,j}^n - \xi_{i,j-1}^n|^2 \\
&\quad -\frac{b_2\Delta x}{2}\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}|\eta_{i,j}^n - \eta_{i,j-1}^n|^2 - \frac{b_3\Delta x}{2}\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}|\zeta_{i,j}^n - \zeta_{i,j-1}^n|^2.
\end{aligned}
\tag{2.45}
$$

From the starting equation $(2.43)_1$ we have

$$
\begin{aligned}
|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|_h^2 &= |\mathbf{f}_h^n - \mathbf{A}_h\mathbf{u}_h^n|_h^2\Delta t^2 \leq 2|\mathbf{A}_h\mathbf{u}_h^n|_h^2\Delta t^2 + +2|\mathbf{f}_h^n|_h^2\Delta t^2 \\
&\leq 2\|P^{-1}S_0^{-1}P^{-T}\|\|\tilde{\mathbf{A}}_h\boldsymbol{\Xi}_h^n\|_h^2\Delta t^2 + 2|\mathbf{f}_h^n|_h^2\Delta t^2.
\end{aligned}
\tag{2.46}
$$

Let us denote by

$$\mu_0 := \|P^{-1}S_0^{-1}P^{-T}\| = \sup_{\mathbf{x}\in\mathbb{R}^3,\ \|\mathbf{x}\|\le 1} \|P^{-1}S_0^{-1}P^{-T}\mathbf{x}\|. \tag{2.47}$$

Therefore, we infer from (2.46) that

$$\begin{aligned}
&|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|_h^2 \le \\
&\qquad 2\mu_0 \frac{\Delta t^2 \Delta y}{\Delta x} \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \big[a_1^2|\xi_{i,j}^n - \xi_{i-1,j}^n|^2 + a_2^2|\eta_{i+1,j}^n - \eta_{i,j}^n|^2 \\
&\qquad\qquad + a_3^2|\zeta_{i,j}^n - \zeta_{i-1,j}^n|^2\big] \\
&\qquad + 2\mu_0 \frac{\Delta t^2 \Delta x}{\Delta y} \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \big[b_1^2|\xi_{i,j}^n - \xi_{i,j-1}^n|^2 + b_2^2|\eta_{i,j}^n - \eta_{i,j-1}^n|^2 \\
&\qquad\qquad + b_3^2|\zeta_{i,j}^n - \zeta_{i,j-1}^n|^2\big] \\
&\qquad + 2|\mathbf{f}_h^n|_h^2 \Delta t^2.
\end{aligned} \tag{2.48}$$

Under the assumptions:

$$\frac{\Delta t}{\Delta x} \le \frac{1}{2\mu_0}\min(\frac{1}{a_1}, -\frac{1}{a_2}, \frac{1}{a_3}), \quad \frac{\Delta t}{\Delta y} \le \frac{1}{2\mu_0}\min(\frac{1}{b_1}, \frac{1}{b_2}, \frac{1}{b_3}), \tag{2.49}$$

we obtain from equations (2.45) and (2.48) that

$$|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|_h^2 - 2\Delta t\langle \mathbf{A}_h \mathbf{u}_h^n, \mathbf{u}_h^n\rangle_h \le 2|\mathbf{f}_h^n|_h^2 \Delta t^2. \tag{2.50}$$

Therefore, under these assumptions, inequality (2.44) yields

$$|\mathbf{u}_h^{n+1}|_h^2 \le |\mathbf{u}_h^n|_h^2 + 2\Delta t\langle \mathbf{f}_h^n, \mathbf{u}_h^n\rangle_h + 2|\mathbf{f}_h^n|_h^2 \Delta t^2, \tag{2.51}$$

which, by using the Cauchy-Schwarz inequality, implies that

$$\begin{aligned}
|\mathbf{u}_h^{n+1}|_h^2 &\le |\mathbf{u}_h^n|_h^2 + \Delta t|\mathbf{u}_h^n|_h^2 + \Delta t|\mathbf{f}_h^n|_h^2 + 2\Delta t^2|\mathbf{f}_h^n|_h^2, \\
&\le (1+\Delta t)|\mathbf{u}_h^n|_h^2 + (2\Delta t + 1)\Delta t|\mathbf{f}_h^n|_h^2.
\end{aligned} \tag{2.52}$$

From equation (2.39) and the fact that $\Delta t \le T$, we deduce that

$$\begin{aligned}
|\mathbf{u}_h^{n+1}|_h^2 &\le (1+\Delta t)|\mathbf{u}_h^n|_h^2 + (1+2T)\|S_0\|^2|\mathbf{f}|^2_{L^2(t_n,t_{n+1},L^2(\mathcal{M}))}] \\
&\le (1+\Delta t)^2|\mathbf{u}_h^{n-1}|_h^2 \\
&\qquad + (1+\Delta t)(1+2T)\|S_0\|^2|\mathbf{f}|^2_{L^2(t_{n-1},t_n,L^2(\mathcal{M}))} \\
&\qquad + (1+2T)\|S_0\|^2|\mathbf{f}|^2_{L^2(t_n,t_{n+1},L^2(\mathcal{M}))} \\
&\le \cdots \\
&\le (1+\Delta t)^n|\mathbf{u}_h^0|_h^2 \\
&\qquad + (1+2T)\sum_{s=1}^{n}(1+\Delta t)^{n-s}\|S_0\|^2|\mathbf{f}|^2_{L^2(t_s,t_{s+1},L^2(\mathcal{M}))},
\end{aligned} \tag{2.53}$$

which, together with $1+x \le e^x$, implies that

$$|\mathbf{u}_h^{n+1}|_h^2 \le e^{n\Delta t}\left[|\mathbf{u}^0|^2 + (1+2T)\|S_0\|^2|\mathbf{f}|^2_{L^2(0,t_{n+1},L^2(\mathcal{M}))}\right]. \tag{2.54}$$

Hence, we have the stability result

$$|\mathbf{u}_h^n|_h^2 \le e^T\left[|\mathbf{u}^0|^2 + (1+2T)\|S_0\|^2|\mathbf{f}|^2_{L^2(0,T,L^2(\mathcal{M}))}\right], \quad \forall\, n = 0, \cdots, N_t. \tag{2.55}$$

**Theorem 2.2.** *Under the following CFL conditions*

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{2\mu_0} \min(\frac{1}{a_1}, -\frac{1}{a_2}, \frac{1}{a_3}), \quad \frac{\Delta t}{\Delta y} \leq \frac{1}{2\mu_0} \min(\frac{1}{b_1}, \frac{1}{b_2}, \frac{1}{b_3}), \tag{2.56}$$

*the scheme defined by the equations* (2.43), (2.20), (2.22), *is stable in* $L^\infty(0,T;L^2(\mathcal{M}))$ *in the sense of* (2.55),

**2.5. Convergence results.** In this section we prove the convergence of functions associated with the $\mathbf{u}_h^n$ for both the Euler explicit and implicit schemes in time. More results on the convergence on finite volume schemes can be found in [3] and [13]. We first remark that we have the uniqueness of the solution for the linearized SWEs (1.1) (see [15, Theorem 9]).

We now define the adjoint operator $\mathbf{A}^*$ of $\mathbf{A}$ as follows (for more details, see [15, Section 3.2.1]):

$$\mathbf{A}^*\mathbf{v} := -\mathcal{E}_1\mathbf{v}_x - \mathcal{E}_2\mathbf{v}_y, \quad \forall\, \mathbf{v} \in \mathcal{D}(\mathbf{A}^*),$$

with

(2.57)
$$\begin{aligned} \mathcal{D}(\mathbf{A}^*) = \{\mathbf{v} = (v^1, v^2, v^3) \in H \ &: \mathbf{A}^*\mathbf{v} \in H \text{ and } \mathbf{v} \text{ satisfies:} \\ &\tilde{v}_0 v^1 - \tilde{u}_0 v^2 + \tilde{\kappa}_0 v^3 = \tilde{u}_0 v^1 + \tilde{v}_0 v^2 + g v^3 = 0, \text{ on } \Gamma_E = \{x = L_x\}, \\ &\tilde{v}_0 v^1 - \tilde{u}_0 v^2 - \tilde{\kappa}_0 v^3 = 0, \text{ on } \Gamma_W = \{x = 0\}, \\ &v^1 = v^2 = v^3 = 0, \text{ on } \Gamma_N = \{y = L_y\}\}. \end{aligned}$$

Of course, the usual relation for adjoint operator holds, i.e.

$$\langle \mathbf{A}\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{A}^*\mathbf{v}\rangle, \quad \forall\, \mathbf{u} \in \mathcal{D}(\mathbf{A}),\ \forall\, \mathbf{v} \in \mathcal{D}(\mathbf{A}^*).$$

Similarly, the adjoint operator $\tilde{\mathbf{A}}^*$ of $\tilde{\mathbf{A}}$ is defined as follows: for $\mathbf{\Psi} = (\Psi^1, \Psi^2, \Psi^3) \in \mathcal{D}(\tilde{\mathbf{A}}^*)$,

$$\tilde{\mathbf{A}}^*\mathbf{\Psi} := -D_1\mathbf{\Psi}_x - D_2\mathbf{\Psi}_y, \tag{2.58}$$

where

$$\begin{aligned} \mathcal{D}(\tilde{\mathbf{A}}^*) = \{\mathbf{\Psi} = (\Psi^1, \Psi^2, \Psi^3) \in H \ &: \tilde{\mathbf{A}}^*\mathbf{\Xi} \in H \text{ and } \mathbf{\Xi} \text{ satisfies:} \\ &\Psi^1 = \Psi^3 = 0, \text{ on } \Gamma_E = \{x = L_x\}, \\ &\Psi^2 = 0, \text{ on } \Gamma_W = \{x = 0\}, \\ &\Psi^1 = \Psi^2 = \Psi^3 = 0, \text{ on } \Gamma_N = \{y = L_y\}\}, \end{aligned} \tag{2.59}$$

and we have the usual relation

$$(\tilde{\mathbf{A}}\mathbf{\Xi}, \mathbf{\Psi}) = (\mathbf{\Xi}, \tilde{\mathbf{A}}^*\mathbf{\Psi}), \quad \forall\, \mathbf{\Xi} \in \mathcal{D}(\tilde{\mathbf{A}}),\ \forall\, \mathbf{\Psi} \in \mathcal{D}(\tilde{\mathbf{A}}^*).$$

We also have a similar relation as in (2.9) for the adjoint operators $\mathbf{A}^*$ and $\tilde{\mathbf{A}}^*$, that is

$$\mathbf{A}^*\mathbf{v} = (P^T S_0)^{-1}\tilde{\mathbf{A}}^*\Psi, \qquad \text{with } \Psi = P^{-1}\mathbf{v}. \tag{2.60}$$

Then we define $\tilde{\mathbf{A}}_h^*$, the discretized version of $\tilde{\mathbf{A}}^*$:

$$\tilde{\mathbf{A}}_h^*\mathbf{\Psi}_h := \tilde{\mathbf{A}}_h^{*,x}\mathbf{\Psi}_h + \tilde{\mathbf{A}}_h^{*,y}\mathbf{\Psi}_h, \quad \forall\, \mathbf{\Psi}_h \in P^{-1}\mathcal{V}_h^*,$$

where, for $1 \le i \le N_x$ and $1 \le j \le N_y$,

(2.61)
$$(\tilde{\mathbf{A}}_{\mathbf{h}}^{*,x}\mathbf{\Psi}_h)_{i,j} = \left(-a_1\frac{\Psi^1_{i+1,j}-\Psi^1_{i,j}}{\Delta x}, -a_2\frac{\Psi^2_{i,j}-\Psi^2_{i-1,j}}{\Delta x}, -a_3\frac{\Psi^3_{i+1,j}-\Psi^3_{i,j}}{\Delta x}\right),$$
$$(\tilde{\mathbf{A}}_{\mathbf{h}}^{*,y}\mathbf{\Psi}_h)_{i,j} = \left(-b_1\frac{\Psi^1_{i,j+1}-\Psi^1_{i,j}}{\Delta y}, -b_2\frac{\Psi^2_{i,j+1}-\Psi^2_{i,j}}{\Delta y}, -b_3\frac{\Psi^3_{i,j+1}-\Psi^3_{i,j}}{\Delta y}\right),$$

and

(2.62)
$$\begin{aligned} P^{-1}\mathcal{V}_h^* = \{&\mathbf{\Psi}_h = (\Psi_h^1, \Psi_h^2, \Psi_h^3) \text{ are step functions over } \mathcal{M} \,: \\ &\mathbf{\Psi}_h\big|_{k_{i,j}} = \mathbf{\Psi}_{i,j},\ \forall\, 0 \le i \le N_x+1,\ 0 \le j \le N_y+1, \\ &\text{ and } \mathbf{\Psi}_h \text{ satisfies:} \\ &\Psi^1_{N_x+1,j} = \Psi^3_{N_x+1,j} = 0,\ \text{for } 1 \le j \le N_y, \\ &\Psi^2_{0,j} = 0,\ \text{for } 1 \le j \le N_y, \\ &\Psi^1_{i,N_y+1} = \Psi^2_{i,N_y+1} = \Psi^3_{i,N_y+1} = 0,\ \text{for } 1 \le i \le N_x.\} \end{aligned}$$

From the equations (2.22)-(2.21), we obtain that, the following adjoint relation,

(2.63)
$$(\tilde{\mathbf{A}}_h\mathbf{\Xi}_h, \mathbf{\Psi}_h)_h = (\mathbf{\Xi}_h, \tilde{\mathbf{A}}_h^*\mathbf{\Psi}_h)_h,$$

holds for all $\mathbf{\Xi} \in P^{-1}\mathcal{V}_h$ and $\mathbf{\Psi}_h \in P^{-1}\mathcal{V}_h^*$.

Finally, we define $\mathbf{A}_h^*$, the discretized version of $\mathbf{A}^*$: for all $\mathbf{v}_h \in \mathcal{V}_h^*$,

(2.64)
$$\mathbf{A}_h^*\mathbf{v}_h = (P^TS_0)^{-1}\tilde{\mathbf{A}}_{\mathbf{h}}^*\mathbf{\Psi}_h, \quad \text{ where } \mathbf{\Psi}_h = P^{-1}\mathbf{v}_h,$$

where

(2.65)
$$\begin{aligned} \mathcal{V}_h^* = \{&\mathbf{v}_h = (v_h^1, v_h^2, v_h^3) \text{ are step functions over } \mathcal{M} \,: \\ &\mathbf{v}_h\big|_{k_{i,j}} = \mathbf{v}_{i,j},\ \forall\, 0 \le i \le N_x+1,\ 0 \le j \le N_y+1,\ \text{and } \mathbf{v}_h \text{ satisfies:} \\ &\tilde{v}_0 v^1_{N_x+1,j} - \tilde{u}_0 v^2_{N_x+1,j} + \tilde{\kappa}_0 v^3_{N_x+1,j} = 0,\ \text{for } 1 \le j \le N_y, \\ &\tilde{u}_0 v^1_{N_x+1,j} + \tilde{v}_0 v^2_{N_x+1,j} + g v^3_{N_x+1,j} = 0,\ \text{for } 1 \le j \le N_y, \\ &\tilde{v}_0 v^1_{0,j} - \tilde{u}_0 v^2_{0,j} - \tilde{\kappa}_0 v^3_{0,j} = 0,\ \text{for } 1 \le j \le N_y, \\ &v^1_{i,N_y+1} = v^2_{i,N_y+1} = v^3_{i,N_y+1} = 0,\ \text{for } 1 \le i \le N_x\}. \end{aligned}$$

Similarly, we also have

(2.66)
$$\langle \mathbf{A}_h\mathbf{u}_h, \mathbf{v}_h\rangle_h = \langle \mathbf{u}_h, \mathbf{A}_h^*\mathbf{v}_h\rangle_h, \quad \forall\, \mathbf{u}_h \in \mathcal{V}_h,\ \mathbf{v}_h \in \mathcal{V}_h^*.$$

Let us define $\tilde{\mathbf{r}}_h : (\mathcal{C}^\infty(\overline{\mathcal{M}}))^3 \cap \mathcal{D}(\tilde{\mathbf{A}}^*) \longrightarrow P^{-1}\mathcal{V}_h^*$, for $\mathbf{\Psi} = (\Psi^1, \Psi^2, \Psi^3) \in (\mathcal{C}^\infty(\overline{\mathcal{M}}))^3 \cap \mathcal{D}(\tilde{\mathbf{A}}^*)$

(2.67)
$$\tilde{\mathbf{r}}_h\mathbf{\Psi} := (\tilde{r}_h\Psi^1, \tilde{r}_h\Psi^1, \tilde{r}_h\Psi^1),$$

where for $\Psi^\sharp = \Psi^1, \Psi^2, \Psi^3$;

(2.68)
$$\begin{aligned} &(\tilde{r}_h\Psi^\sharp)_{i,j} = \frac{1}{\Delta x\Delta y}\int_{k_{i,j}} \Psi^\sharp(x,y)\mathrm{d}x\mathrm{d}y,\ \text{for } 1 \le i \le N_x,\ 1 \le j \le N_y, \\ &\text{and,} \\ &(\tilde{r}_h\Psi^1)_{N_x+1,j} = (\tilde{r}_h\Psi^3)_{N_x+1,j} = 0,\ \text{for } 1 \le j \le N_y, \\ &(\tilde{r}_h\Psi^2)_{0,j} = 0,\ \text{for } 1 \le j \le N_y, \\ &(\tilde{r}_h\Psi^1)_{i,N_y+1} = (\tilde{r}_h\Psi^2)_{i,N_y+1} = (\tilde{r}_h\Psi^3)_{i,N_y+1} = 0,\ \text{for } 1 \le i \le N_x. \end{aligned}$$

**Lemma 2.3.** *For all* $\mathbf{\Psi} \in (\mathcal{C}^\infty(\overline{\mathcal{M}}))^3 \cap \mathcal{D}(\tilde{\mathbf{A}}^*)$*, we have that as* $(\Delta x, \Delta y) \to 0$*,*

$$\begin{aligned} \tilde{\mathbf{r}}_h \mathbf{\Psi} &\longrightarrow \mathbf{\Psi}, \ \textit{strongly in } L^2(\mathcal{M}), \\ \tilde{\mathbf{A}}^*_{\mathbf{h}} \tilde{r}_h \mathbf{\Psi} &\longrightarrow \tilde{\mathbf{A}}^* \mathbf{\Psi}, \ \textit{strongly in } L^2(\mathcal{M}). \end{aligned} \tag{2.69}$$

The proof of Lemma 2.3 is an application of the Taylor's formula and we refer the readers to [13, Section 3.1] for a detailed proof.

We also define $\mathbf{r}_h : (\mathcal{C}^\infty(\overline{\mathcal{M}}))^3 \cap \mathcal{D}(\mathbf{A}^*) \longrightarrow \mathcal{V}_h^*$, for $\mathbf{v} = (v^1, v^2, v^3) \in (\mathcal{C}^\infty(\overline{\mathcal{M}}))^3 \cap \mathcal{D}(\mathbf{A}^*)$ by

$$\mathbf{r}_h \mathbf{v} = P \tilde{\mathbf{r}}_h (P^{-1} \mathbf{v}). \tag{2.70}$$

As an immediate consequence of Lemma 2.3 and identities (2.60) and (2.64), we have the following result.

**Lemma 2.4.** *For all* $\mathbf{v} \in (\mathcal{C}^\infty(\overline{\mathcal{M}}))^3 \cap \mathcal{D}(\mathbf{A}^*)$*, we have that as* $(\Delta x, \Delta y) \to 0$*,*

$$\begin{aligned} \mathbf{r}_h \mathbf{v} &\longrightarrow \mathbf{v} \ \textit{strongly in } L^2(\mathcal{M}), \\ \mathbf{A}^*_{\mathbf{h}} \mathbf{r}_h \mathbf{v} &\longrightarrow \mathbf{A}^* \mathbf{v}, \ \textit{strongly in } L^2(\mathcal{M}). \end{aligned} \tag{2.71}$$

In order to prove the convergence result, we first consider the Euler implicit scheme (2.35) and then briefly study the Euler explicit scheme (2.43).

**Euler implicit scheme.** We first introduce the approximated solution denoted by $\bar{\mathbf{u}}_{\mathbf{h}}$: for each $t \in I_n := (n\Delta t, (n+1)\Delta t]$ and $n = 0, \ldots, N_T - 1$, we set

$$\bar{\mathbf{u}}_{\mathbf{h}}(t) = \mathbf{u}_h^{n+1}, \quad \forall\, t \in I_n, \tag{2.72}$$

that is, $\bar{\mathbf{u}}_{\mathbf{h}}$ is the step function on the interval $(0, T)$ with values taken from the right of each interval $I_n$. We also define $\bar{\mathbf{f}}_{\mathbf{h}}$ in the same fashion, i.e. $\bar{\mathbf{f}}_{\mathbf{h}} = \mathbf{f}_h^{n+1}$ on $I_n$ for $n = 0, \cdots, N_T - 1$.

**Lemma 2.5.** *For* $\mathbf{u}_h^n$ *solution of* (2.35)*, there exists a subsequence* $h'$ *such that*

$$\bar{\mathbf{u}}_{\mathbf{h}'} \rightharpoonup \mathbf{u} \ \textit{weak-star in } L^\infty(0, T; (L^2(\mathcal{M}))^3). \tag{2.73}$$

*Proof.* The result directly follows from the uniform estimate (2.42). □

Let $\varphi \in \mathcal{C}^\infty([0, T])$ with $\varphi(T) = 0$, $\mathbf{v} \in \mathcal{C}^\infty(\overline{\mathcal{M}}) \cap D(\mathbf{A}^*)$, and define $\bar{\varphi}$ and $\tilde{\varphi}$ by setting

$$\begin{aligned} &\bar{\varphi}(t) = \varphi(t_n), \quad \forall\, t \in I_n, \\ &\tilde{\varphi}(t) = \frac{\varphi(t_{n+1}) - \varphi(t_n)}{\Delta t}(t - t_n) + \varphi(t_n), \quad \forall\, t \in I_n, \end{aligned} \tag{2.74}$$

where $t_n = n\Delta t$, for all $n = 0, \cdots, N_T$. By the Mean-Value Theorem, we obtain

$$\bar{\varphi},\ \tilde{\varphi},\ \tilde{\varphi}_t \longrightarrow \varphi,\ \varphi,\ \varphi_t, \ \text{in } \mathcal{C}([0, T]), \ \text{respectively, as } \Delta t \to 0. \tag{2.75}$$

We now take the $L^2(\mathcal{M})$ inner product of $(2.35)_1$ with $\varphi(t_n)\mathbf{r}_h\mathbf{v}$ and then sum for $n = 0, \ldots, N_T - 1$; we arrive at

$$\begin{aligned} \sum_{n=0}^{N_T-1} \langle \frac{\mathbf{u}_{h'}^{n+1} - \mathbf{u}_{h'}^n}{\Delta t}, \varphi(t_n)\mathbf{r}_{h'}\mathbf{v} \rangle + \sum_{n=0}^{N_T-1} \langle \mathbf{A}_h \mathbf{u}_{h'}^{n+1}, \varphi(t_n)\mathbf{r}_{h'}\mathbf{v} \rangle \\ = \sum_{n=0}^{N_T-1} \langle \mathbf{f}_{h'}^{n+1}, \varphi(t_n)\mathbf{r}_{h'}\mathbf{v} \rangle. \end{aligned} \tag{2.76}$$

Using the fact that $\varphi(T)=0$ and equation (2.66), the left-hand side of (2.76) can be written as follows

$$-\sum_{n=0}^{N_T-1}\langle \mathbf{u}_{h'}^{n+1}, \frac{\varphi(t_{n+1})-\varphi(t_n)}{\Delta t}\mathbf{r}_{h'}\mathbf{v}\rangle - \frac{\varphi(0)}{\Delta t}\langle \mathbf{u}_{h'}^0, \mathbf{r}_{h'}\mathbf{v}\rangle + \sum_{n=0}^{N_T-1}\langle \mathbf{u}_{h'}^{n+1}, \varphi(t_n)A_h^*\mathbf{r}_{h'}\mathbf{v}\rangle.$$

Therefore, equation (2.76) reads

$$\begin{aligned} -\int_0^T \langle \bar{\mathbf{u}}_{\mathbf{h}'}, \tilde{\varphi}_t\mathbf{r}_{h'}\mathbf{v}\rangle \mathrm{d}t + \int_0^T \langle \bar{\mathbf{u}}_{\mathbf{h}'}, \bar{\varphi}\mathbf{A}_h^*\mathbf{r}_{h'}\mathbf{v}\rangle \mathrm{d}t \\ = \langle \mathbf{u}_{h'}^0, \mathbf{r}_{h'}\mathbf{v}\rangle\varphi(0) + \int_0^T \langle \bar{\mathbf{f}}_{\mathbf{h}'}, \bar{\varphi}\mathbf{r}_{h'}\mathbf{v}\rangle \mathrm{d}t. \end{aligned} \tag{2.77}$$

Now passing to the limit and using Lemma 2.4, 2.5 and the convergence result (2.75), we obtain

$$-\int_0^T \langle \mathbf{u}, \varphi_t\mathbf{v}\rangle \mathrm{d}t + \int_0^T \langle \mathbf{u}, \varphi\mathbf{A}^*\mathbf{v}\rangle \mathrm{d}t = \langle \mathbf{u}^0, \mathbf{v}\rangle\varphi(0) + \int_0^T \langle \mathbf{f}(t), \varphi\mathbf{v}\rangle \mathrm{d}t, \tag{2.78}$$

where we also implicitly used the convergences of $\mathbf{u}_{h'}^0$ to $\mathbf{u}^0$ and $\bar{\mathbf{f}}_{\mathbf{h}'}$ to $\mathbf{f}$ in $L^2$, which can be obtained by the application of the Taylor's formula (see Lemma 2.3 or [13, Section 3.1]) since $\mathcal{C}_c^\infty(\mathcal{M})$ (resp. $\mathcal{C}_c^\infty([0,T]\times\mathcal{M})$) is dense in $L^2(\mathcal{M})$ (resp. $L^2((0,T)\times\mathcal{M})$).

By taking $\varphi$ compactly supported on $(0,T)$ in (2.78), we conclude that

$$\frac{d}{\mathrm{d}t}\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{A}^*\mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle. \tag{2.79}$$

Now, (2.79) is valid for all $\mathbf{v}$ smooth, and we deduce that (2.78) is also valid for all $\mathbf{v}\in\mathcal{D}(\mathbf{A}^*)$ since $\mathcal{C}^\infty(\overline{\mathcal{M}})\cap D(\mathbf{A}^*)$ is dense in $D(\mathbf{A}^*)$ (see [15]).

Multiplying (2.79) by $\varphi\in C([0,T])$ with $\varphi(T)=0$ and then integrating by part we find

$$-\int_0^T \langle \mathbf{u}, \varphi_t\mathbf{v}\rangle \mathrm{d}t + \int_0^T \langle \mathbf{u}, \varphi\mathbf{A}^*\mathbf{v}\rangle \mathrm{d}t = \langle \mathbf{u}(0), \mathbf{v}\rangle\varphi(0) + \int_0^T \langle \mathbf{f}(t), \varphi\mathbf{v}\rangle \mathrm{d}t, \tag{2.80}$$

for any $\mathbf{v}\in\mathcal{D}(\mathbf{A}^*)$. Comparing (2.78) and (2.80) and taking $\varphi(0)\neq 0$, we find that

$$\langle \mathbf{u}(0), \mathbf{v}\rangle = \langle \mathbf{u}^0, \mathbf{v}\rangle, \ \ \forall \mathbf{v}\in\mathcal{D}(\mathbf{A}^*). \tag{2.81}$$

Because $\mathcal{D}(\mathbf{A}^*)$ is dense in $(L^2(\mathcal{M}))^3$ (see [15, Section 3.2.1]), we have that

$$\mathbf{u}(0) = \mathbf{u}^0 \quad \text{(as elements of } L^2(\mathcal{M})^3\text{)}. \tag{2.82}$$

Note that since we have uniqueness of the solution $\mathbf{u}$ for the linearized SWEs (1.1) with initial condition (2.82) (see [15]), we can conclude that every subsequence $\mathbf{u}_{h'}$ converge to the same limit.

Hence, we proved the following convergence theorem for the Euler implicit scheme.

**Theorem 2.3.** *For $\bar{\mathbf{u}}_h$, defined as in equation* (2.72)*, with $\mathbf{u}_h^{n+1}$ the solution of* (2.35) *we have that*

$$\bar{\mathbf{u}}_h \textit{ converges to } \mathbf{u} \textit{ in } L^\infty(0,T;(L^2(\mathcal{M}))^3) \textit{ weak-star}, \tag{2.83}$$

*as $\Delta t$, $h\to 0$ and $\mathbf{u}$ satisfies the following equation equivalent to* (1.1), (2.82)*:*

$$\begin{cases} \dfrac{d}{dt}\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{A}^*\mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle, \ \forall \mathbf{v}\in\mathcal{D}(\mathbf{A}^*), \\ \mathbf{u}(x,y,t=0) = \mathbf{u}^0(x,y). \end{cases} \tag{2.84}$$

**Euler explicit scheme.** We redefine $\bar{\mathbf{u}}_{\mathbf{h}}$, $\bar{\mathbf{f}}_{\mathbf{h}}$ and $\bar{\varphi}$ by

$$\begin{aligned}\bar{\mathbf{u}}_{\mathbf{h}}(t) &= \mathbf{u}_h^n, \quad \forall\, t \in I_n,\\ \bar{\mathbf{f}}_{\mathbf{h}}(t) &= \mathbf{f}_h^n, \quad \forall\, t \in I_n,\\ \bar{\varphi}(t) &= \varphi(t_{n+1}), \quad \forall\, t \in I_n,\end{aligned} \tag{2.85}$$

and $\tilde{\varphi}$ is the same as in (2.74). We still have the convergence (2.75) and the following result, which is an immediate consequence of the uniform estimate (2.55) and the uniqueness of the solution $\mathbf{u}$ for the linearized SWEs (1.1).

**Lemma 2.6.** *For $\bar{\mathbf{u}}_h$, defined in equation* $(2.85)_1$, *the solution of* (2.43) *we have that*

$$\bar{\mathbf{u}}_{\mathbf{h}} \rightharpoonup \mathbf{u}, \quad \textit{weak-star in } L^\infty(0,T;(L^2(\mathcal{M}))^3). \tag{2.86}$$

Taking the inner product of (2.43) with $\varphi(t_{n+1})\mathbf{r}_h\mathbf{v}$ in $L^2(\mathcal{M})$ and summing for $n = 0, \ldots, N_T - 1$, we arrive at

$$\begin{aligned}\sum_{n=0}^{N_T-1} \langle \frac{\mathbf{u}_{h'}^{n+1} - \mathbf{u}_{h'}^n}{\Delta t}, \varphi(t_{n+1})\mathbf{r}_{h'}\mathbf{v}\rangle &+ \sum_{n=0}^{N_T-1} \langle \mathbf{A}_h\mathbf{u}_{h'}^n, \varphi(t_{n+1})\mathbf{r}_{h'}\mathbf{v}\rangle\\ &= \sum_{n=0}^{N_T-1} \langle \mathbf{f}_{h'}^n, \varphi(t_{n+1})\mathbf{r}_{h'}\mathbf{v}\rangle.\end{aligned} \tag{2.87}$$

Using $\varphi(T) = 0$ and (2.66), we rewrite the left-hand side of (2.87) as

$$-\sum_{n=0}^{N_T-1} \langle \mathbf{u}_{h'}^n, \frac{\varphi(t_{n+1}) - \varphi(t_n)}{\Delta t}\mathbf{r}_{h'}\mathbf{v}\rangle - \frac{\varphi(0)}{\Delta t}\langle \mathbf{u}_{h'}^0, \mathbf{r}_{h'}\mathbf{v}\rangle + \sum_{n=0}^{N_T-1} \langle \mathbf{u}_{h'}^n, \varphi(t_{n+1})A_h^*\mathbf{r}_{h'}\mathbf{v}\rangle.$$

Therefore, equation (2.87) reads exactly the same as (2.77), with $\bar{\mathbf{u}}_{\mathbf{h}}$, $\bar{\mathbf{f}}_{\mathbf{h}}$ and $\bar{\varphi}$ defined by (2.85),

$$\begin{aligned}-\int_0^T \langle \bar{\mathbf{u}}_{\mathbf{h}'}, \tilde{\varphi}_t\mathbf{r}_{h'}\mathbf{v}\rangle \mathrm{d}t &+ \int_0^T \langle \bar{\mathbf{u}}_{\mathbf{h}'}, \bar{\varphi}\mathbf{A}_h^*\mathbf{r}_{h'}\mathbf{v}\rangle \mathrm{d}t\\ &= \langle \mathbf{u}_{h'}^0, \mathbf{r}_{h'}\mathbf{v}\rangle\varphi(0) + \int_0^T \langle \bar{\mathbf{f}}_{\mathbf{h}'}, \bar{\varphi}\mathbf{r}_{h'}\mathbf{v}\rangle \mathrm{d}t.\end{aligned} \tag{2.88}$$

Then passing to the limit using Lemma 2.4 and 2.6, we find

$$-\int_0^T \langle \mathbf{u}, \varphi_t\mathbf{v}\rangle \mathrm{d}t + \int_0^T \langle \mathbf{u}, \varphi\mathbf{A}^*\mathbf{v}\rangle \mathrm{d}t = \langle \mathbf{u}^0, \mathbf{v}\rangle\varphi(0) + \int_0^T \langle \mathbf{f}(t), \varphi\mathbf{v}\rangle \mathrm{d}t. \tag{2.89}$$

We can also obtain that $\mathbf{u}(0) = \mathbf{u}^0$ (as elements of $L^2(\mathcal{M})^3$). by following the same arguments used for the Euler implicit scheme in (2.79)-(2.82).

Since we have uniqueness of the solution $u$, we can conclude that every subsequence $\mathbf{u}_h'$ converge to the same limit. Therefore, we proved the following theorem.

**Theorem 2.4.** *For $\bar{\mathbf{u}}_h$, defined as in equation* (2.72), *with $\mathbf{u}_h^{n+1}$ the solution of* (2.43) *we have that*

$$\bar{\mathbf{u}}_h \textit{ converges to } \mathbf{u} \textit{ in } L^\infty(0,T;(L^2(\mathcal{M}))^3) \textit{ weak-star}, \tag{2.90}$$

*as $\Delta t$, $h \to 0$ and $\mathbf{u}$ satisfies the following equations equivalent to* (1.1), (2.82)*:*

$$\begin{cases} \dfrac{d}{dt}\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{A}^*\mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle, \quad \forall\, \mathbf{v} \in \mathcal{D}(\mathbf{A}^*),\\ \mathbf{u}(x,y,t=0) = \mathbf{u}^0(x,y).\end{cases} \tag{2.91}$$

## 3. The fully hyperbolic subcritical case for the linearized shallow water equations

In this section we consider the fully hyperbolic subcritical case, see (1.3). We associate with (1.1) the initial conditions:

$$\mathbf{u} = (u, v, \phi)^T = (u^0, v^0, \phi^0)^T, \text{ at } t = 0. \tag{3.1}$$

**3.1. Preliminary theoretical results.** In the fully hyperbolic subcritical case, we still have the diagonalization (2.2), but under the assumption (1.3), we find that

$$a_1,\, a_3,\, b_2,\, b_3 > 0, \quad \text{and} \quad a_2,\, b_1 < 0. \tag{3.2}$$

Using the same variables $\boldsymbol{\Xi}$ introduced in (2.4), we still have (2.5).

According to the signs of $a_i$'s and $b_i$'s ($i = 1, 2, 3$), we associate with (2.5) the following boundary conditions in the $\boldsymbol{\Xi}$ variables:

$$\begin{cases} \xi = \zeta = 0, & \text{on } \Gamma_W = \{x = 0\}, \\ \eta = 0, & \text{on } \Gamma_E = \{x = L_x\}, \\ \eta = \zeta = 0, & \text{on } \Gamma_S = \{y = 0\}, \\ \xi = 0, & \text{on } \Gamma_N = \{y = L_y\}. \end{cases} \tag{3.3}$$

Therefore, we associate with (1.1) the boundary conditions in the $\mathbf{u}$ variables (see [15, Section 3.3]):

$$\begin{cases} \tilde{v}_0 u - \tilde{u}_0 v + \tilde{\kappa}_0 \phi = \tilde{u}_0 u + \tilde{v}_0 v + g\phi = 0, \text{ on } \Gamma_W = \{x = 0\}, \\ \tilde{v}_0 u - \tilde{u}_0 v - \tilde{\kappa}_0 \phi = 0, \text{ on } \Gamma_E = \{x = L_x\}, \\ \tilde{v}_0 u - \tilde{u}_0 v - \tilde{\kappa}_0 \phi = \tilde{u}_0 u + \tilde{v}_0 v + g\phi = 0, \text{ on } \Gamma_S = \{y = 0\}, \\ \tilde{v}_0 u - \tilde{u}_0 v + \tilde{\kappa}_0 \phi = 0, \text{ on } \Gamma_N = \{y = L_y\}. \end{cases} \tag{3.4}$$

As in Subsection 2.1, we define the unbounded operators $\mathbf{A}$ and $\tilde{\mathbf{A}}$ on $H = L^2(\Omega)^3$ by (2.8) but the domains are now:

$$\mathcal{D}(\mathbf{A}) = \{\mathbf{u} \in H \ : \mathbf{A}\mathbf{u} \in H \ , \mathbf{u} \text{ satisfies the boundary conditions (3.4)}\},$$
$$\mathcal{D}(\tilde{\mathbf{A}}) = \left\{\boldsymbol{\Xi} \in H \ : \tilde{\mathbf{A}}\boldsymbol{\Xi} \in H \ , \boldsymbol{\Xi} \text{ satisfies the boundary conditions (3.3)}\right\}.$$

We still have the relation (2.9) and the positivity results for $\mathbf{A}$ and $\tilde{\mathbf{A}}$ as in Lemma 2.1:

**Lemma 3.1.** *Assume that* (1.3) *holds. Then for every sufficiently smooth* $\mathbf{u} \in \mathcal{D}(\mathbf{A})$ *and* $\boldsymbol{\Xi} \in \mathcal{D}(\tilde{\mathbf{A}})$*, there holds*

$$\langle \mathbf{A}\mathbf{u}, \mathbf{u} \rangle = (\tilde{\mathbf{A}}\boldsymbol{\Xi}, \boldsymbol{\Xi}) \geq 0. \tag{3.5}$$

The proof of Lemma 3.1 is exactly the same as the proof of Lemma 2.1 by taking into account the boundary conditions (3.3). Also, as observed in Remark 2.1, (3.5) is extended in [15] to any $\mathbf{u}$ in $\mathcal{D}(\mathbf{A})$.

**3.2. Finite volume discretization.** In this section we define the finite volume spaces and discrete operator $\mathbf{A}_h$ for the fully hyperbolic subcritical case. The finite volume spaces are similar to the ones in the previous section but with different boundary conditions. In this case, the spatial discretization is the same as in Section 2.2, see (2.11) -(2.16), we define two new subspaces $\mathcal{W}_h \subset \mathbf{V}_h$ and $P^{-1}\mathcal{W}_h \subset \mathbf{V}_h$, that take into account the boundary conditions (3.4):

$$
\begin{aligned}
\mathcal{W}_h = \{\mathbf{u}_h = (u_h, v_h, \phi_h) &\text{ are step functions over } \mathcal{M} : \mathbf{u}_h|_{k_{i,j}} = \mathbf{u}_{i,j}, \\
&\forall\, 0 \le i \le N_x + 1,\ 0 \le j \le N_y + 1, \text{ and } \mathbf{u}_h \text{ satisfies:} \\
&\tilde{v}_0 u_{0,j} - \tilde{u}_0 v_{0,j} + \tilde{\kappa}_0 \phi_{0,j} = \tilde{u}_0 u_{0,j} + \tilde{v}_0 v_{0,j} + g\phi_{0,j} = 0, \\
&\ \text{for } 1 \le j \le N_y, \\
&\tilde{v}_0 u_{N_x+1,j} - \tilde{u}_0 v_{N_x+1,j} - \tilde{\kappa}_0 \phi_{N_x+1,j} = 0, \text{ for } 1 \le j \le N_y, \\
&\tilde{v}_0 u_{i,0} - \tilde{u}_0 v_{i,0} - \tilde{\kappa}_0 \phi_{i,0} = \tilde{u}_0 u_{i,0} + \tilde{v}_0 v_{i,0} + g\phi_{i,0} = 0, \\
&\ \text{for } 1 \le i \le N_x, \\
&\tilde{v}_0 u_{i,N_y+1} - \tilde{u}_0 v_{i,N_y+1} + \tilde{\kappa}_0 \phi_{i,N_y+1} = 0, \text{ for } 1 \le i \le N_x.\},
\end{aligned} \tag{3.6}
$$

and,

$$
\begin{aligned}
P^{-1}\mathcal{W}_h = \{\mathbf{\Xi} = (\xi, \eta, \zeta) &\text{ are step functions over } \mathcal{M} : \mathbf{\Xi}_h|_{k_{i,j}} = \mathbf{\Xi}_{i,j}, \\
&\forall 0 \le i \le N_x + 1,\ 0 \le j \le N_y + 1, \text{ and } \mathbf{\Xi}_h \text{ satisfies:} \\
&\xi_{0,j} = \zeta_{0,j} = 0, \text{ for } 1 \le j \le N_y, \\
&\eta_{N_x+1,j} = 0, \text{ for } 1 \le j \le N_y, \\
&\eta_{i,0} = \zeta_{i,0} = 0, \text{ for } 1 \le i \le N_x, \\
&\xi_{i,N_y+1} = 0, \text{ for } 1 \le i \le N_x.\}.
\end{aligned} \tag{3.7}
$$

We still adopt Definition 2.1 of the inner products and norms for the space $\mathbf{V}_h$. However, we use a different scheme rather than the one in Section 2.2: for the $x$ derivatives, we use an upwind scheme for $\xi$, $\zeta$ and a downwind scheme for $\eta$; for the $y$ derivatives, we use an upwind scheme for $\eta$ and $\zeta$, and a downwind scheme for $\xi$. Therefore, the discretized version $\tilde{\mathbf{A}}_{\mathbf{h}}$ of $\tilde{\mathbf{A}}$ is defined as follows: for $\mathbf{\Xi}_h \in P^{-1}\mathcal{W}_h$,

$$
\tilde{\mathbf{A}}_{\mathbf{h}}\mathbf{\Xi}_h := \tilde{\mathbf{A}}^x_{\mathbf{h}}\mathbf{\Xi}_h + \tilde{\mathbf{A}}^y_{\mathbf{h}}\mathbf{\Xi}_h, \tag{3.8}
$$

where, for $1 \le i \le N_x$ and $1 \le j \le N_y$:

$$
\begin{aligned}
(\tilde{\mathbf{A}}^x_{\mathbf{h}}\mathbf{\Xi}_h)_{i,j} &= \left( a_1 \frac{\xi_{i,j} - \xi_{i-1,j}}{\Delta x}, a_2 \frac{\eta_{i+1,j} - \eta_{i,j}}{\Delta x}, a_3 \frac{\zeta_{i,j} - \zeta_{i-1,j}}{\Delta x} \right), \\
(\tilde{\mathbf{A}}^y_{\mathbf{h}}\mathbf{\Xi}_h)_{i,j} &= \left( b_1 \frac{\xi_{i,j+1} - \xi_{i,j}}{\Delta y}, b_2 \frac{\eta_{i,j} - \eta_{i,j-1}}{\Delta y}, b_3 \frac{\zeta_{i,j} - \zeta_{i,j-1}}{\Delta y} \right).
\end{aligned} \tag{3.9}
$$

Similarly as in Section 2.2, the discretized version $\mathbf{A}_h$ of $\mathbf{A}$ is defined by

$$
\mathbf{A}_h\mathbf{u}_h := (P^T S_0)^{-1}\tilde{\mathbf{A}}_{\mathbf{h}}\mathbf{\Xi}_h, \text{ where } \mathbf{\Xi}_h = P^{-1}\mathbf{u}_h, \quad \forall\, \mathbf{u}_h \in \mathcal{W}_h. \tag{3.10}
$$

With the definitions in (3.8)-(3.10), the spatial approximation of equation (2.15) is written as

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_h(t) + \mathbf{A}_h\mathbf{u}_h(t) &= \mathbf{f}_h(t), \\
\mathbf{u}_h(t) &\in \mathcal{W}_h,
\end{aligned} \tag{3.11}
$$

where $\mathbf{f}_h(t)$ is defined as in (2.23).

**Lemma 3.2.** *The operators $\tilde{\mathbf{A}}_{\mathbf{h}}$ and $\mathbf{A}_{\mathbf{h}}$ defined in* (3.8)-(3.10) *is positive over $P^{-1}\mathcal{W}_h$ and $\mathcal{W}_h$ , respectivly. That is for any $\mathbf{u}_h \in \mathcal{W}_h$ and $\mathbf{\Xi}_h = P^{-1}\mathbf{u}_h \in P^{-1}\mathcal{W}_h$ , we have*

$$
\langle \mathbf{A}_h\mathbf{u}_h, \mathbf{u}_h \rangle_h = \left( \tilde{\mathbf{A}}_h\mathbf{\Xi}_h, \mathbf{\Xi}_h \right)_h \ge 0. \tag{3.12}
$$

*Proof.* The proof is the same as for Lemma 2.2 except for the first coordinate of $(\tilde{\mathbf{A}}^y_{\mathbf{h}}\Xi_h)_{i,j}$. Using again (2.31) we deduce that

$$\Delta x \Delta y b_1 \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \frac{\xi_{i,j+1}-\xi_{i,j}}{\Delta y}\xi_{i,j} = \frac{b_1 \Delta x}{2}\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left[|\xi_{i,j+1}|^2 - |\xi_{i,j}|^2 - |\xi_{i,j+1}-\xi_{i,j}|^2\right]$$

$$= -\frac{b_1\Delta x}{2}\sum_{i=1}^{N_x}\left[|\xi_{i,0}|^2 + \sum_{j=1}^{N_y}|\xi_{i,j+1}-\xi_{i,j}|^2\right].$$

Since $b_1 < 0$, the positivity of $\left(\tilde{\mathbf{A}}_h\Xi_h, \Xi_h\right)_h$ is proven.

□

**3.3. Time discretization: the Euler implicit scheme.** In this section we apply the Euler implicit scheme in time to the equations (3.11). We define a time step $\Delta t$ with $N_t\Delta t = T$ and we set $t_n = n\Delta t$ for $0 \le n \le N_t$. We denote by $\{\mathbf{u}_h^n \in \mathcal{W}_h, 0 \le n \le N_t\}$ the discrete unknowns. For $\mathbf{u}_h^0 \in \mathcal{W}_h$ given by (2.33), the Euler implicit scheme for (3.11) reads

$$\begin{cases} \dfrac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t} + \mathbf{A}_h\mathbf{u}_h^{n+1} = \mathbf{f}_h^{n+1}, \\ \mathbf{u}_h^n \in \mathcal{W}_h, \text{ for } 0 \le n \le N_t. \end{cases} \tag{3.13}$$

with

$$\mathbf{f}_h^{n+1} = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} \mathbf{f}_h(t)\mathrm{d}t.$$

We first remark that $(I+\Delta t\mathbf{A}_h)$ is invertible because $\mathbf{A}_h$ is positive and therefore $I+\Delta t\mathbf{A}_h$ is positive definite. For this reason the system (3.13) admits a unique solution $\mathbf{u}_h^{n+1}$ when $\mathbf{u}_h^n$ is known.

To obtain an estimate of $|\mathbf{u}_h^{n+1}|_h$, we take the scalar product $(\langle\cdot,\cdot\rangle_h)$ of equation $(3.13)_1$ with $2\Delta t\mathbf{u}_h^{n+1}$, and using the same arguments as in Section 2.3 we obtain

$$|\mathbf{u}_h^{n+1}|_h^2 \le (1+\Delta t)\left[|\mathbf{u}_h^n|_h^2 + \|S_0\|^2|\mathbf{f}|^2_{L^2(t_n,t_{n+1},L^2(\mathcal{M}))}\right], \tag{3.14}$$

and iterating (3.14), we have

$$\begin{aligned} |\mathbf{u}_h^{n+1}|_h^2 &\le (1+\Delta t)\left[|\mathbf{u}_h^n|_h^2 + \|S_0\|^2|\mathbf{f}|^2_{L^2(t_n,t_{n+1},L^2(\mathcal{M}))}\right], \\ &\le (1+\Delta t)^2\left[|\mathbf{u}_h^{n-1}|_h^2 + \|S_0\|^2|\mathbf{f}|^2_{L^2(t_{n-1},t_{n+1},L^2(\mathcal{M}))}\right], \\ &\le (1+\Delta t)^{n+1}\left[|\mathbf{u}_h^0|^2 + \|S_0\|^2|\mathbf{f}|^2_{L^2(0,t_{n+1},L^2(\mathcal{M}))}\right]. \end{aligned} \tag{3.15}$$

From equation (3.15) and since $\Delta t N_t = T$, we conclude that

$$|\mathbf{u}_h^n|_h^2 \le e^T\left[|\mathbf{u}^0|^2 + \|S_0\|^2\int_0^T\int_{\mathcal{M}}|\mathbf{f}|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\right], \quad \forall\, n = 0,1,\cdots,N_t. \tag{3.16}$$

**Theorem 3.1.** *The scheme defined by the equations* (3.13), (3.8)-(3.10)*, is stable in* $L^\infty(0,T;L^2(\mathcal{M}))$ *in the sense of* (3.16).

**3.4. Time discretization: the Euler explicit scheme.** In this section we apply the Euler explicit scheme in time to the equations (3.11). With the same spatial discretization as in the previous section and For $\mathbf{u}_h^0 \in \mathcal{W}_h$ given by (2.33), the Euler explicit scheme for (3.11) reads

$$\begin{cases} \dfrac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t} + \mathbf{A}_h \mathbf{u}_h^n = \mathbf{f}_h^n, \\ \mathbf{u}_h^n \in \mathcal{W}_h, \text{ for } 0 \le n \le N_t, \end{cases} \tag{3.17}$$

with

$$\mathbf{f}_h^n = \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} \mathbf{f}_h(t) \mathrm{d}t.$$

We take the scalar product ($\langle \cdot, \cdot \rangle_h$) of equation $(3.17)_1$ with $2\Delta t \mathbf{u}_h^n$ to obtain

$$|\mathbf{u}_h^{n+1}|_h^2 \le |\mathbf{u}_h^n|_h^2 + |\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|_h^2 - 2\Delta t \langle \mathbf{A}_h \mathbf{u}_h^n, \mathbf{u}_h^n \rangle_h + 2\Delta t \langle \mathbf{f}_h^n, \mathbf{u}_h^n \rangle_h. \tag{3.18}$$

Since $\langle \mathbf{A}_h \mathbf{u}_h^n, \mathbf{u}_h^n \rangle_h = \left( \tilde{\mathbf{A}}_h \mathbf{\Xi}_h^n, \mathbf{\Xi}_h^n \right)_h$, we have

$$\begin{aligned} - \langle \mathbf{A}_h \mathbf{u}_h^n, \mathbf{u}_h^n \rangle_h \le& \\ & - \frac{a_1 \Delta y}{2} \sum_j \sum_i |\xi_{i,j}^n - \xi_{i-1,j}^n|^2 + \frac{a_2 \Delta y}{2} \sum_j \sum_i |\eta_{i+1,j}^n - \eta_{i,j}^n|^2 \\ & - \frac{a_3 \Delta y}{2} \sum_j \sum_i |\zeta_{i,j}^n - \zeta_{i-1,j}^n|^2 - \frac{b_1 \Delta x}{2} \sum_j \sum_i |\xi_{i,j+1}^n - \xi_{i,j}^n|^2 \\ & - \frac{b_2 \Delta x}{2} \sum_j \sum_i |\eta_{i,j}^n - \eta_{i,j-1}^n|^2 - \frac{b_3 \Delta x}{2} \sum_j \sum_i |\zeta_{i,j}^n - \zeta_{i,j-1}^n|^2. \end{aligned} \tag{3.19}$$

Similar to (2.46), we deduce from the equation (3.17) that

$$|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|_h^2 \le |\mathbf{f}_h^n - \mathbf{A}_h \mathbf{u}_h^n|_h^2 \Delta t^2 \le 2\mu_0 \Delta t^2 \|\tilde{\mathbf{A}}_h \mathbf{\Xi}_h^n\|_h^2 + 2\Delta t^2 |\mathbf{f}_h^n|_h^2, \tag{3.20}$$

where $\mu_0$ is defined in (2.47). Therefore

$$\begin{aligned} |\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|_h^2 \le& \\ & 2\mu_0 \frac{\Delta t^2 \Delta y}{\Delta x} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \big[ a_1^2 |\xi_{i,j}^n - \xi_{i-1,j}^n|^2 + a_2^2 |\eta_{i+1,j}^n - \eta_{i,j}^n|^2 \\ & + a_3^2 |\zeta_{i,j}^n - \zeta_{i-1,j}^n|^2 \big] \\ & + 2\mu_0 \frac{\Delta t^2 \Delta x}{\Delta y} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \big[ b_1^2 |\xi_{i,j+1}^n - \xi_{i,j}^n|^2 + b_2^2 |\eta_{i,j}^n - \eta_{i,j-1}^n|^2 \\ & + b_3^2 |\zeta_{i,j}^n - \zeta_{i,j-1}^n|^2 \big] \\ & + 2\Delta t^2 |\mathbf{f}_h^n|^2. \end{aligned} \tag{3.21}$$

Under the assumptions:

$$\frac{\Delta t}{\Delta x} \le \frac{1}{2\mu_0} \min(\frac{1}{a_1}, -\frac{1}{a_2}, \frac{1}{a_3}), \quad \frac{\Delta t}{\Delta y} \le \frac{1}{2\mu_0} \min(-\frac{1}{b_1}, \frac{1}{b_2}, \frac{1}{b_3}), \tag{3.22}$$

we find from equations (3.19) and (3.21) that

$$|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|_h^2 - 2\Delta t \langle \mathbf{A}_h \mathbf{u}_h^n, \mathbf{u}_h^n \rangle_h \le 2\Delta t^2 |\mathbf{f}_h^n|_h^2. \tag{3.23}$$

Therefore, inequality (3.18) yields

$$
\begin{aligned}
|\mathbf{u}_h^{n+1}|_h^2 &\leq |\mathbf{u}_h^n|_h^2 + 2\Delta t \langle \mathbf{f}_h^n, \mathbf{u}_h^n \rangle_h + 2\Delta t^2 |\mathbf{f}_h^n|_h^2 \\
&\leq (1+\Delta t)|\mathbf{u}_h^n|_h^2 + (2\Delta t + 1)\Delta t |\mathbf{f}_h^n|_h^2.
\end{aligned} \tag{3.24}
$$

From equation (3.24) and the fact that $\Delta t \leq T$, we deduce that

$$
\begin{aligned}
|\mathbf{u}_h^{n+1}|_h^2 &\leq (1+\Delta t)|\mathbf{u}_h^n|_h^2 + (1+2T)\|S_0\|^2 |\mathbf{f}|^2_{L^2(t_n,t_{n+1},L^2(\mathcal{M}))}] \\
&\leq (1+\Delta t)^2 |\mathbf{u}_h^{n-1}|_h^2 + (1+\Delta t)(1+2T)\|S_0\|^2 |\mathbf{f}|^2_{L^2(t_{n-1},t_n,L^2(\mathcal{M}))} \\
&\qquad + (1+2T)\|S_0\|^2 |\mathbf{f}|^2_{L^2(t_n,t_{n+1},L^2(\mathcal{M}))} \\
&\leq \dots \\
&\leq (1+\Delta t)^n |\mathbf{u}_h^0|_h^2 \\
&\qquad + (1+2T)\sum_{s=1}^{n} (1+\Delta t)^{n-s} \|S_0\|^2 |\mathbf{f}|^2_{L^2(t_s,t_{s+1},L^2(\mathcal{M}))}, \\
&\leq (1+\Delta t)^n |\mathbf{u}^0|^2 \\
&\qquad + (1+2T)\sum_{s=1}^{n} (1+\Delta t)^{n-s} \|S_0\|^2 |\mathbf{f}|^2_{L^2(t_s,t_{s+1},L^2(\mathcal{M}))},
\end{aligned} \tag{3.25}
$$

which, together with $1 + x \leq e^x$, implies that

$$
|\mathbf{u}_h^{n+1}|_h^2 \leq e^{n\Delta t} \left[ |\mathbf{u}^0|^2 + (1+2T)\|S_0\|^2 |\mathbf{f}|^2_{L^2(0,t_{n+1},L^2(\mathcal{M}))} \right]. \tag{3.26}
$$

Hence, we have the stability result

$$
|\mathbf{u}_h^n|_h^2 \leq e^T \left[ |\mathbf{u}^0|^2 + (1+2T)\|S_0\|^2 |\mathbf{f}|^2_{L^2(0,T,L^2(\mathcal{M}))} \right], \quad \forall\, n = 0, \cdots, N_t. \tag{3.27}
$$

**Theorem 3.2.** *Under the following CFL conditions*

$$
\frac{\Delta t}{\Delta x} \leq \frac{1}{2\mu_0} \min\left(\frac{1}{a_1}, -\frac{1}{a_2}, \frac{1}{a_3}\right), \quad \frac{\Delta t}{\Delta y} \leq \frac{1}{2\mu_0} \min\left(-\frac{1}{b_1}, \frac{1}{b_2}, \frac{1}{b_3}\right), \tag{3.28}
$$

*the scheme defined by the equations* (3.17), (3.8)-(3.10), *is stable in* $L^\infty(0,T;L^2(\mathcal{M}))$ *in the sense of* (3.27),

**3.5. Convergence results.** In this section we prove the convergence of the $\mathbf{u}_h^n$ for both the Euler explicit and implicit scheme in time for the fully hyperbolic subcritical case. We also remark that we have the uniqueness of the solution for the linearized SWEs (1.1) (see [15, Theorem 9]).

We define the adjoint operator $\mathbf{A}^*$ of $\mathbf{A}$ as follows (for more details, see [15, Section 3.3.1]):

$$
\mathbf{A}^*\mathbf{v} := -\mathcal{E}_1 \mathbf{v}_x - \mathcal{E}_2 \mathbf{v}_y, \quad \forall\, \mathbf{v} \in \mathcal{D}(\mathbf{A}^*),
$$

with

$$
\begin{aligned}
\mathcal{D}(\mathbf{A}^*) = \{\mathbf{v} = (v^1, v^2, v^3) \in H \ :\ & \mathbf{A}^*\mathbf{v} \in H \text{ and } \mathbf{v} \text{ satisfies:} \\
& \tilde{v}_0 v^1 - \tilde{u}_0 v^2 + \tilde{\kappa}_0 v^3 = \tilde{u}_0 v^1 + \tilde{v}_0 v^2 + g v^3 = 0, \\
& \qquad \text{on } \Gamma_E = \{x = L_x\}, \\
& \tilde{v}_0 v^1 - \tilde{u}_0 v^2 - \tilde{\kappa}_0 v^3 = 0, \text{ on } \Gamma_W = \{x = 0\}, \\
& \tilde{v}_0 v^1 - \tilde{u}_0 v^2 - \tilde{\kappa}_0 v^3 = \tilde{u}_0 v^1 + \tilde{v}_0 v^2 + g v^3 = 0, \\
& \qquad \text{on } \Gamma_E = \{y = L_x\}, \\
& \tilde{v}_0 v^1 - \tilde{u}_0 v^2 + \tilde{\kappa}_0 v^3 = 0, \text{ on } \Gamma_W = \{y = 0\}\}.
\end{aligned} \tag{3.29}
$$

Of course, the usual relation for adjoint operator holds, i.e.

$$\langle \mathbf{A}\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{A}^*\mathbf{v}\rangle, \quad \forall\, \mathbf{u} \in \mathcal{D}(\mathbf{A}),\ \forall\, \mathbf{v} \in \mathcal{D}(\mathbf{A}^*).$$

Similarly, the adjoint operator $\tilde{\mathbf{A}}^*$ of $\tilde{\mathbf{A}}$ is defined as follows: for $\boldsymbol{\Psi} = (\Psi^1, \Psi^2, \Psi^3) \in \mathcal{D}(\tilde{\mathbf{A}}^*)$,

$$\tilde{\mathbf{A}}^*\boldsymbol{\Psi} := -D_1\boldsymbol{\Psi}_x - D_2\boldsymbol{\Psi}_y, \tag{3.30}$$

where

$$\begin{aligned} \mathcal{D}(\tilde{\mathbf{A}}^*) = \{\boldsymbol{\Psi} = (\Psi^1, \Psi^2, \Psi^3) \in H\ :\ & \tilde{\mathbf{A}}^*\boldsymbol{\Xi} \in H \text{ and } \boldsymbol{\Xi} \text{ satisfies:} \\ & \Psi^1 = \Psi^3 = 0, \text{ on } \Gamma_E = \{x = L_x\}, \\ & \Psi^2 = 0, \text{ on } \Gamma_W = \{x = 0\}, \\ & \Psi^2 = \Psi^3 = 0, \text{ on } \Gamma_E = \{y = L_y\}, \\ & \Psi^1 = 0, \text{ on } \Gamma_W = \{y = 0\}\}, \end{aligned} \tag{3.31}$$

and we have the usual relation

$$(\tilde{\mathbf{A}}\boldsymbol{\Xi}, \boldsymbol{\Psi}) = (\boldsymbol{\Xi}, \tilde{\mathbf{A}}^*\boldsymbol{\Psi}), \quad \forall\, \boldsymbol{\Xi} \in \mathcal{D}(\tilde{\mathbf{A}}),\ \forall\, \boldsymbol{\Psi} \in \mathcal{D}(\tilde{\mathbf{A}}^*).$$

We also have a similar relation as in (2.9) for the adjoint operators $\mathbf{A}^*$ and $\tilde{\mathbf{A}}^*$, that is

$$\mathbf{A}^*\mathbf{v} = (P^T S_0)^{-1}\tilde{\mathbf{A}}^*\Psi, \qquad \text{with } \Psi = P^{-1}\mathbf{v}.$$

Then we define $\tilde{\mathbf{A}}_h^*$, the discretized version of $\tilde{\mathbf{A}}^*$:

$$\tilde{\mathbf{A}}_h^*\boldsymbol{\Psi}_h := \tilde{\mathbf{A}}_h^{*,x}\boldsymbol{\Psi}_h + \tilde{\mathbf{A}}_h^{*,y}\boldsymbol{\Psi}_h, \quad \forall\, \boldsymbol{\Psi}_h \in P^{-1}\mathcal{W}_h^*,$$

where

$$\begin{aligned} (\tilde{\mathbf{A}}_{\mathbf{h}}^{*,x}\boldsymbol{\Psi}_h)_{i,j} &= \left(-a_1\frac{\Psi^1_{i+1,j} - \Psi^1_{i,j}}{\Delta x}, -a_2\frac{\Psi^2_{i,j} - \Psi^2_{i-1,j}}{\Delta x}, -a_3\frac{\Psi^3_{i+1,j} - \Psi^3_{i,j}}{\Delta x}\right), \\ (\tilde{\mathbf{A}}_{\mathbf{h}}^{*,y}\boldsymbol{\Psi}_h)_{i,j} &= \left(-b_1\frac{\Psi^1_{i,j} - \Psi^1_{i,j-1}}{\Delta y}, -b_2\frac{\Psi^2_{i,j+1} - \Psi^2_{i,j}}{\Delta y}, -b_3\frac{\Psi^3_{i,j+1} - \Psi^3_{i,j}}{\Delta y}\right), \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} P^{-1}\mathcal{W}_h^* = \{\boldsymbol{\Psi}_h = (\Psi_h^1, \Psi_h^2, \Psi_h^3) & \text{ are step functions over } \mathcal{M}\ : \\ & \boldsymbol{\Psi}_h\big|_{k_{i,j}} = \boldsymbol{\Psi}_{i,j},\ \forall\, 0 \le i \le N_x + 1, 0 \le j \le N_y + 1, \\ & \text{ and } \boldsymbol{\Psi}_h \text{ satisfies:} \\ & \Psi^1_{N_x+1,j} = \Psi^3_{N_x+1,j} = 0, \text{ for } 1 \le j \le N_y, \\ & \Psi^2_{0,j} = 0, \text{ for } 1 \le j \le N_y, \\ & \Psi^2_{i,N_y+1} = \Psi^3_{i,N_y+1} = 0, \text{ for } 1 \le i \le N_x, \\ & \Psi^1_{i,0} = 0, \text{ for } 1 \le i \le N_y\}. \end{aligned} \tag{3.33}$$

From the equations (3.9)-(3.10), we obtain the following adjoint relation:

$$(\tilde{\mathbf{A}}_h\boldsymbol{\Xi}_h, \boldsymbol{\Psi}_h) = (\boldsymbol{\Xi}_h, \tilde{\mathbf{A}}_h^*\boldsymbol{\Psi}_h) \tag{3.34}$$

holds for all $\boldsymbol{\Xi} \in P^{-1}\mathcal{W}_h$ and $\boldsymbol{\Psi}_h \in P^{-1}\mathcal{W}_h^*$.

Finally, we define $\mathbf{A}_h^*$, the discretized version of $\mathbf{A}^*$: for all $\mathbf{v}_h \in \mathcal{W}_h^*$,

$$\mathbf{A}_h^*\mathbf{v}_h = (P^T S_0)^{-1}\tilde{\mathbf{A}}_{\mathbf{h}}^*\boldsymbol{\Psi}_h, \quad \text{where } \boldsymbol{\Psi}_h = P^{-1}\mathbf{v}_h, \tag{3.35}$$

$$
\begin{aligned}
\mathcal{W}_h^* = \{\mathbf{v}_h = (v_h^1, v_h^2, v_h^3) &\text{ are step functions over } \mathcal{M} \ : \\
&\mathbf{v}_h\Big|_{k_{i,j}} = \mathbf{v}_{i,j}, \, \forall\, 0 \le i \le N_x+1, \, 0 \le j \le N_y+1, \text{ and } \mathbf{v}_h \text{ satisfies:} \\
&\tilde{v}_0 v^1_{N_x+1,j} - \tilde{u}_0 v^2_{N_x+1,j} + \tilde{\kappa}_0 v^3_{N_x+1,j} = 0, \text{ for } 1 \le j \le N_y, \\
&\tilde{u}_0 v^1_{N_x+1,j} + \tilde{v}_0 v^2_{N_x+1,j} + g v^3_{N_x+1,j} = 0, \text{ for } 1 \le j \le N_y, \\
&\tilde{v}_0 v^1_{0,j} - \tilde{u}_0 v^2_{0,j} - \tilde{\kappa}_0 v^3_{0,j} = 0, \text{ for } 1 \le j \le N_y, \\
&\tilde{v}_0 v^1_{i,N_y+1} - \tilde{u}_0 v^2_{i,N_y+1} - \tilde{\kappa}_0 v^3_{i,N_y+1} = 0, \text{ for } 1 \le i \le N_x, \\
&\tilde{u}_0 v^1_{i,N_y+1} + \tilde{v}_0 v^2_{i,N_y+1} + g v^3_{i,N_y+1} = 0, \text{ for } 1 \le i \le N_x, \\
&\tilde{v}_0 v^1_{i,0} - \tilde{u}_0 v^2_{i,0} + \tilde{\kappa}_0 v^3_{i,0} = 0, \text{ for } 1 \le i \le N_x\}.
\end{aligned} \tag{3.36}
$$

Similarly, we also have

$$
\langle \mathbf{A}_h \mathbf{u}_h, \mathbf{v}_h \rangle = \langle \mathbf{u}_h, \mathbf{A}_h^* \mathbf{v}_h \rangle, \quad \forall\, \mathbf{u}_h \in \mathcal{W}_h, \ \mathbf{v}_h \in \mathcal{W}_h^*. \tag{3.37}
$$

We have the same Lemmas and Theorems as in Section 2.5, the only differences are the definition of $\tilde{\mathbf{A}}^*$ and $\mathbf{A}^*$, therefore we will not prove the two theorems below as the proof is the same as in Section 2.5.

**Euler implicit scheme.** With the Euler implicit time discretization we have the following convergence result:

**Theorem 3.3.** *For $\bar{\mathbf{u}}_h$, defined as in equation* (2.72)*, with $\mathbf{u}_h^{n+1}$ the solution of* (3.13) *we have that*

$$
\bar{\mathbf{u}}_h \text{ converges to } \mathbf{u} \text{ in } L^\infty(0,T;(L^2(\mathcal{M}))^3) \text{ weak-star,} \tag{3.38}
$$

*as $\Delta t$, $h \to 0$ and $\mathbf{u}$ satisfies the following equation equivalent to* (1.1), (2.82)*:*

$$
\begin{cases}
\dfrac{d}{dt}\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{A}^*\mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle, \ \forall \mathbf{v} \in \mathcal{D}(\mathbf{A}^*), \\
\mathbf{u}(x,y,t=0) = \mathbf{u}^0(x,y).
\end{cases} \tag{3.39}
$$

**Euler explicit scheme.** With the Euler explicit time discretization we have the following convergence result:

**Theorem 3.4.** *For $\bar{\mathbf{u}}_h$, defined as in equation* $(2.85)_1$*, with $\mathbf{u}_h^{n+1}$ the solution of* (3.17) *we have that*

$$
\bar{\mathbf{u}}_h \text{ converges to } \mathbf{u} \text{ in } L^\infty(0,T;(L^2(\mathcal{M}))^3) \text{ weak-star,} \tag{3.40}
$$

*as $\Delta t$, $h \to 0$ and $\mathbf{u}$ satisfies the following equation equivalent to* (1.1), (2.82)*:*

$$
\begin{cases}
\dfrac{d}{dt}\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{A}^*\mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle, \ \forall \mathbf{v} \in \mathcal{D}(\mathbf{A}^*), \\
\mathbf{u}(x,y,t=0) = \mathbf{u}^0(x,y).
\end{cases} \tag{3.41}
$$

## 4. Concluding remarks

In this article, we proposed a unified way to implement finite volume discretization for the linearized 2D inviscid SWEs in a rectangular domain with the boundary conditions proposed in [15], where the well-posedness result for the linearized 2D inviscid SWEs is established, and we believe that the finite volume scheme we proposed here can be also applied to more general linear hyperbolic initial and boundary value problems in a rectangle (see [16] for theoretic results). We also proved the numerical stability and convergence result for the scheme in the fully hyperbolic case, that is when $\tilde{u}_0^2 + \tilde{v}_0^2 > g\tilde{\phi}_0$ (see (1.2)-(1.3)). As we known in [15], there is another important case called the elliptic-hyperbolic case remaining to be investigated, which is left for future work.

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