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FINITE ELEMENT APPROXIMATION OF OPTIMAL CONTROL FOR SYSTEM GOVERNED BY IMMISCIBLE DISPLACEMENT IN POROUS MEDIA

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Abstract. In this work, we study the finite element approximation of a model optimal control problem governed by the system describing the two-phase incompressible flow in porous media, with the aim to maximize production of oil from petroleum reservoirs. We first give the proof for the existence of the solutions of the control problem. The optimality conditions are then obtained and the existence of the solution of the adjoint equations is shown. After that we consider its finite element approximation. We have obtained the a priori error estimates with the optimal orders and minimum regularity requirements. Finally, we carry out some numerical tests.

Key words. Finite element approximation, optimal control problem, immiscible displacement.

1. Motivation

The field of petroleum engineering is concerned with the search for ways to extract more oil and gas from the earths subsurface. In a world in which an increase in production of tenths of a percentage may result into a growth in profit of millions of dollars, no stone is left unturned.

A common technique in oil recovery, known as "water flooding", makes use of two types of wells: injection and production wells. The production wells are used to transport liquid and gas from the reservoir to the subsurface. The injection wells inject water into the oil reservoir with the aim to push the oil towards the production wells and keep up the pressure difference. The oil-water front progresses toward the production wells until water breaks through into the production stream. An increasing amount of water is used, while the oil production rate diminishes, until at some time the recovery is no longer profitable and production is brought to an end. Using water flooding, up to about 35 percent of the oil can be recovered economically. Due to the strongly heterogeneous nature of oil reservoirs, the oil-water front does not travel uniformly towards the production wells, but is usually irregularly shaped. As a result, large amounts of oil may be still trapped within the reservoir as water breakthrough occurs and production is brought to an end.

Recent advances in petroleum engineering allow for advanced well downhole measurement and control devices, which expand the possibilities to manipulate and control fluid flow paths through the oil reservoir. The ability to manipulate the progression of the oil-water front provides the possibility to search for a control strategy that will result in maximization of oil recovery. A straightforward approach to find such a control strategy is to use the optimal control technique to increase recovery by delaying water breakthrough and increasing sweep, based on a predictive reservoir model. Obviously, this problem can be described as an optimal control problem of PDEs where the goal is to find a control q over a time interval

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$[0, T]$ that maximizes a certain performance measure $\mathcal{J}(q)$. Meantime, one realistic way to control the flow paths through the reservoir is to manipulate the quantity of water injected by the control valve settings.

Reservoir simulators use conservation of mass and momentum equations to describe the flow of oil, water or gas through the reservoir rock. Although oil consists of a large number of chemical components with varying properties, in many reservoir modeling cases the Black Oil Model is adopted for simplicity reasons. This model distinguishes between three phases: water, oil and gas. For further simplification, in the oil reservoirs models used within this work no gas is assumed present, hence reducing the number of phases to two.

In this study we carry out some initial investigations on the finite element approximation of this kind of optimal control problem, which to our best knowledge is not much studied in the literature. In our first model we assume that the reservoir is isolated so that with the water being injected, the remaining oil at any time can be estimated by the integral of the concentration over the reservoir at that time. This is of course a much simplified model, but the essential mathematical difficulties to be dealt in this kind of control problems are clearly displayed in it. Thus hopefully this will pave the way to the study of more complex and realistic situations. Although our initial objective is to minimize the remaining oil by adjusting the amount of what injected, the water injected needs to be purified and is expensive. Therefore the cost of water injection needs to be considered as well. We then extend our objective functional into weighted sum of the total remaining oil and the total water injected. Of course it is natural to consider a linear functional to express the cost of oil remained and water injected. However in order to effectively compute such a control problem, still it needs to be reglazed by adding quadratic terms. Thus in this work we will directly consider a quadratic model. Assume the water injection period is between [0,T]. Then in the case of 2-d, our model is governed by a nonlinear coupled system of equations for the movement of two-phase incompressible and completely immiscible fluids in a reservoir $\Omega \subset R^2$ of unit thickness:

$$\min J(q) = \min_{q\in K} \frac{1}{2}\int_\Omega \tilde{\omega} c^2(T) + \frac{\alpha_0}{2}\int_0^T \int_\Omega \sum_{i=1}^{N_w} \delta_{w,i} q_i^2 \tag{1}$$

subject to

$$\begin{cases} \phi(x)\dfrac{\partial c}{\partial t} + b(c)u\cdot\nabla c - \nabla\cdot(d(c)\nabla c) = f(c)\displaystyle\sum_{i=1}^{N_w}\delta_{w,i}q_{w,i}, \\ \nabla\cdot u = \displaystyle\sum_{i=1}^{N_w}\delta_{w,i}q_{w,i} - \sum_{j=1}^{N_o}\delta_{o,j}q_{o,j}, \\ u = -a(c)\nabla p, \\ q = \displaystyle\sum_{i=1}^{N_w} q_{w,i} = \sum_{j=1}^{N_o} q_{o,j}, \end{cases} \tag{2}$$

where N_w, N_o are the total numbers of the injective wells and the production wells respectively, $\delta_{w,i}$, $\delta_{o,j}$ are the Dirac functions located at the $i-$th injection well and the $j-$th production well respectively, $K = \{q \in L^\infty[0, T] : \quad 0 \le q \le \hat{q}\}$ and $\hat{q}$ is a positive constant. This system is subject to the following boundary conditions:

$$\begin{cases} \mathbf{u}\cdot\mathbf{n} = 0, & \mathbf{x}\in\partial\Omega, \quad t\in J, \\ d(c)\nabla c\cdot\mathbf{n} = 0, & \mathbf{x}\in\partial\Omega, \quad t\in J \end{cases} \tag{3}$$

and an initial condition

$$c(\mathbf{x},0) = c_0(\mathbf{x}), \quad \mathbf{x}\in\Omega, \tag{4}$$

as well as a compatible condition

$$\int_\Omega p = 0. \tag{5}$$

Here $a(c) = k(\mathbf{x})\varrho(c), b(c) = \varrho_o'(c), d(c) = d(\mathbf{x},c) = k(\mathbf{x})\varrho(c)\varrho_o(c)\varrho_w(c)p_c'(c), f(c) = -\varrho_o(c)$,

$c(\mathbf{x},t)$	the saturation of oil in the two-phase fluid,
$\phi(\mathbf{x})$	the porosity of the rock,
$k(\mathbf{x})$	the permeability of the porous rock,
$\varrho(c)$	the total mobolity of the two-phase fluid,
$\varrho_o(c)$	the relative mobolity of the oil,
$\varrho_w(c)$	the relative mobolity of the water,
$u(\mathbf{x},t)$	the Darcy velocity of the mixture,
$q(t)$	the flow rate,
$p_c(c)$	the capillary pressure,
$\tilde{\omega}$	the price of oil,
α_0	the price of water.

Moreover, we assume that $0 < a_* \le a(c) \le a^*$, $\Phi_* < \phi(\mathbf{x}) < \Phi^*$ and $0 < d_* \le d(c) \le d^*$.

To further simplify our analysis, we here first study the case of one injection well and one production well:

$$(\mathcal{QP}): \quad \min J(q) = \min_{q\in K}\left\{\frac{1}{2}\int_\Omega \tilde{\omega}c^2(T) + \frac{\alpha_0}{2}\int_0^T\int_\Omega \delta_0 q^2\right\} \tag{6}$$

subject to

$$\begin{cases} \phi(\mathbf{x})\dfrac{\partial c}{\partial t} + b(c)\mathbf{u}\cdot\nabla c - \nabla\cdot(d(c)\nabla c) = f(c)\delta_0 q, \\ \nabla\cdot\mathbf{u} = (\delta_0-\delta_1)q, \\ \mathbf{u} = -a(c)\nabla p, \end{cases} \tag{7}$$

where $t\in J = [0,T]$, $\Omega = [0,1]\times[0,1]$. We assume the location of injection well is at the point $\mathbf{x}_0 = (0,0)$ and production well is at the point $\mathbf{x}_1 = (1,1)$. So, δ_0 and δ_1 are Dirac functions at the wells respectively. Again we emphasize this further simplified model reflects the key mathematical difficulties to be handled in our analysis. Based on the model this work chooses the quantity of water injection in water flooding as control in our optimal control setting.

In order to handle Dirac functions δ_0, δ_1 and point-wise value numerically, we approximate the objective functional with some averages. Let $\Omega_0,\Omega_1\subset\Omega$, $\mathbf{x}_0\in\Omega_0$ and $\mathbf{x}_1\in\Omega_1$. Moreover, $\Omega_0\cap\Omega_1 = \phi$ and $|\Omega_0| = |\Omega_1| = \sigma$ where $0<\sigma<<1$. Then, we define

$$r_i = \begin{cases} 1/\sigma, & \mathbf{x}\in\Omega_i, \\ 0, & \mathbf{x}\in\Omega\backslash\Omega_i, \end{cases} \qquad i = 0,1, \tag{8}$$

and

$$\omega(\mathbf{x},t)=\begin{cases}\tilde{\omega}/\varepsilon, & (\mathbf{x},t)\in\Omega\times[T-\varepsilon,T],\\ 0, & (\mathbf{x},t)\in\Omega\times[0,T-\varepsilon).\end{cases} \tag{9}$$

Then the optimal control problem $(\mathcal{QP})$ can be rewritten as:

$$(\mathcal{P}):\quad \min J(q)=\min_{q\in K}\Big\{\frac{1}{2}\int_0^T\int_\Omega \omega(\mathbf{x},t)c^2+\frac{\alpha_0}{2}\int_0^T q^2\Big\} \tag{10}$$

subject to

$$\begin{cases}\phi(\mathbf{x})\dfrac{\partial c}{\partial t}+b(c)\mathbf{u}\cdot\nabla c-\nabla\cdot(d(c)\nabla c)=f(c)r_0q,\\ \nabla\cdot\mathbf{u}=(r_0-r_1)q,\\ \mathbf{u}=-a(c)\nabla p.\end{cases} \tag{11}$$

The finite element approximation of optimal control problems has been extensively studied in the literature. There have been extensive studies in convergence of the standard finite element approximation of optimal control problems, see, some examples in [2], [3], [13], [15], [16] and [23]-[26], although it is impossible to give even a very brief review here. For optimal control problems governed by linear state equations, a prior error estimates of the finite element approximation were established long ago; see, for example [13] and [15]. But it is much more difficult to obtain such error estimates for nonlinear control problems. For some classes of nonlinear optimal control problems, a priori error estimates were established in [4], [18], [21],and [22]. However to our best knowledge there is little work on this optimal control in the literature, where the state equations are some coupled complex nonlinear convection-diffusion parabolic and elliptic equations. Furthermore, there does not seem to exist systematical studies in the literature on its finite element approximation and analysis, although there exists an extensive body of reference in the state system and its finite element approximation, see for example, in [8], [9], [10] and [11].

The aim of this work is to systematically investigate this simplified control problem and its finite element approximation. Even so, as to be seen below, this involves much complex mathematical analysis and substantial computational work. One of our main contributions to the existing literature is that we have obtained the a priori error estimates with the optimal orders and minimum regularity requirements for its finite element approximation, which involve substantial novel mathematical analysis.

This paper is organized as follows. Weak form and existence of the optimal control problem are presented in Section 2. In Section 3, we study the optimality condition of the optimal control; in Sections 4,5 we present the finite element approximation of this optimal control and derive a priori error estimate of the approximation. In Section 6, some numerical tests are presented.

2. Weak form and existence of the solution

Throughout the paper, we adopt the standard notation $W^{m,s}(\Omega)$ for Sobolev space on Ω as follows ([1]):

$$W^{m,s}(\Omega)=\big\{u\in L^s(\Omega):\partial^\alpha u\in L^s(\Omega),|\alpha|\le m\big\},\quad m\ge 0,\ 1\le s\le\infty.$$

The norm in $W^{m,s}(\Omega)$ is denoted by $\|\cdot\|_{m,s,\Omega}$ and defined by

$$\|u\|_{m,s,\Omega}=\Big(\sum_{|\alpha|\le m}\int_\Omega|\partial^\alpha u|^s\Big)^{1/s},\quad 1\le s<\infty$$

and

$$\|u\|_{m,\infty,\Omega} = \max_{|\alpha|\le m}\left\{ess\sup_{\mathbf{x}\in\Omega}|\partial^\alpha u|\right\}.$$

The Sobolev space $W_0^{m,s}(\Omega)$ is the closure of the space $C_0^\infty(\Omega)$ in the norm of $W^{m,s}(\Omega)$. And we denote

$$W^{m,2}(\Omega) = H^m(\Omega),\ W_0^{m,2}(\Omega) = H_0^m(\Omega),$$
$$\|\cdot\|_{m,2,\Omega} = \|\cdot\|_{m,\Omega}.$$

Introduce function spaces

$$V = \big\{\mathbf{v}\in H(\mathrm{div};\Omega)|\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\big\}$$

endowed with the norm:

$$\|\mathbf{v}\|_V = \|\mathbf{v}\|_{H(\mathrm{div};\Omega)} = \left(\|\mathbf{v}\|_{0,\Omega}^2 + \|\mathrm{div}\mathbf{v}\|_{0,\Omega}^2\right)^{1/2},$$

and

$$W = \big\{w\in L^2(\Omega),\quad (w,1) = 0\big\}.$$

We also require the following spaces which incorporate time dependence and norms. Let $[a,b]\subset J$ and X be a Sobolev space. For $f(\mathbf{x},t)$ defined on $\Omega\times[a,b]$, we set

$$H^m(a,b;X) = \Big\{f:\int_a^b\|\frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t)\|_X^2<\infty,\ |\alpha|\le m\Big\},$$
$$\|f\|_{H^m(a,b;X)} = \Big[\sum_{|\alpha|\le m}\int_a^b\|\frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t)\|_X^2\Big]^{\frac{1}{2}},\quad m\ge 0,$$
$$W^{m,\infty}(a,b;X) = \Big\{f: ess\sup_{[a,b]}\|\frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t)\|_X<\infty,\ |\alpha|\le m\Big\},$$
$$\|f\|_{W^{m,\infty}(a,b;X)} = \max_{|\alpha|\le m} ess\sup_{[a,b]}\|\frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t)\|_X,\quad m\ge 0,$$
$$L^2(a,b;X) = H^0(a,b;X),\quad L^\infty(a,b;X) = W^{0,\infty}(a,b;X).$$

We also adopt the space([20])

$$W(0,T) = \big\{f:\ f\in L^2(0,T;H^1(\Omega)),\ \frac{df}{dt}\in L^2(0,T;H^{-1}(\Omega))\big\},$$

endowed with the norm

$$\|f\|_{W(0,T)} = \Big(\int_0^T\|f(t)\|_{H^1(\Omega)}^2dt + \int_0^T\|\frac{df}{dt}\|_{H^{-1}(\Omega)}^2dt\Big)^{\frac{1}{2}}.$$

Then following the standard references on the weak formula of the state system (see [12]), we can recast the weak form of this control such that:

$$\min J(q) = \min_{q\in K}\Big\{\frac{1}{2}\int_0^T\int_\Omega\omega(\mathbf{x},t)c^2 + \frac{\alpha_0}{2}\int_0^T q^2\Big\}, \tag{12}$$

where $c\in W(0,T),\ u\in L^2(0,T;V),\ q(t)\in K, p\in L^2(0,T;W)$ subject to

$$\begin{cases}(\phi\frac{\partial c}{\partial t},z) + (b(c)\mathbf{u}\cdot\nabla c,z) + (d(c)\nabla c,\nabla z) = (f(c)r_0q,z), & \forall\ z\in H^1(\Omega),\\ (\nabla\cdot\mathbf{u},w) = ((r_0-r_1)q,w), & \forall\ w\in W,\\ (\alpha(c)\mathbf{u},\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) = 0, & \forall\ \mathbf{v}\in V,\end{cases} \tag{13}$$

associated with an initial condition $c|_{t=0} = c_0$, where $\alpha(c) = \dfrac{1}{a(c)}$.

Following the standard assumptions on studying this class of nonlinear equations (see [10], [12]), we suppose that there exists a uniform constant M_0 such that

$$(A):\begin{cases}\|f(c)\|_{L^\infty(0,T;L^\infty(\Omega))}\le M_0, \quad \|a(c)\|_{L^\infty(0,T;L^\infty(\Omega))}\le M_0,\\ \|b(c)\|_{L^\infty(0,T;L^\infty(\Omega))}\le M_0, \quad \|d(c)\|_{L^\infty(0,T;L^\infty(\Omega))}\le M_0,\\ a(c), b(c), d(c), f(c) \text{ are Lipschitz continuous functions about } c.\end{cases}$$

Throughout this paper, we denote M the generic constant independent of mesh size h defined in the following parts.

Theorem 2.1. *Supposing that assumption (A) is satisfied, then problem $(\mathcal{P})$ admits at least a solution.*

Proof. The proof is divided into six parts.

(1) Let q_n be a minimization sequence. Then it is clear that the set $\{\int_0^T q_n^2\}_{n=1}^\infty$ is bounded and $\{q_n\}_{n=1}^\infty \subset L^2(0,T)$. Thus there is a subsequence, still denoted by q_n, such that

$$q_n \to \bar{q}, \quad \text{weakly in } L^2(0,T). \tag{14}$$

(2) By the theory of partial differential equation of [7, 17] and the elliptic equation $-\nabla\cdot(a(c_n)\nabla p_n) = (r_0 - r_1)q_n$, we have the regular property:

$$\int_0^T \|p_n(t)\|_{H^1}^2 \le M\int_0^T \hat{q}^2\|r_0 - r_1\|_{L^2}^2 \tag{15}$$

and

$$\int_0^T \|\mathbf{u}_n(t)\|_{L^2}^2 = \int_0^T \|a(c_n)\nabla p_n(t)\|_{L^2}^2 \le M\int_0^T \hat{q}^2\|r_0 - r_1\|_{L^2}^2. \tag{16}$$

Thus $\{p_n\}_{n=1}^\infty \subset L^2(0,T;\ H^1(\Omega))$ and there is a subsequence, still denoted by p_n such that p_n converges weakly to $\bar{p}$ in $L^2(0,T;H^1(\Omega))$. That is to say that there is a subsequence, still denoted by $\mathbf{u}_n$ such that $\mathbf{u}_n$ converges weakly to $\bar{\mathbf{u}}$ in $L^2(0,T;(L^2(\Omega))^2)$.

(3) Prove the estimate

$$\|c_n\|_{L^2(0,T;\ H^1(\Omega))} + \|c_n\|_{L^\infty(0,T;\ L^2(\Omega))} \le M. \tag{17}$$

Noticing that if taking $z = c_n$ in (13), we get

$$(\phi\frac{\partial c_n}{\partial t}, c_n) + (b(c_n)\mathbf{u}_n\cdot\nabla c_n, c_n) + (d(c_n)\nabla c_n, \nabla c_n) = (f(c_n)r_0 q_n, c_n), \tag{18}$$

i.e

$$(\phi\frac{\partial c_n}{\partial t}, c_n) + (d(c_n)\nabla c_n, \nabla c_n) = -(b(c_n)\mathbf{u}_n\cdot\nabla c_n, c_n) + (f(c_n)r_0 q_n, c_n). \tag{19}$$

It is obvious that

$$-(b(c_n)\mathbf{u}_n\cdot\nabla c_n, c_n) = -(\mathbf{u}_n\cdot\nabla\varrho_o(c_n), c_n) = ((r_0-r_1)q_n\varrho_o(c_n), c_n) + (\mathbf{u}_n\cdot\nabla c_n, \varrho_o(c_n)).$$

By (15), we have

$$\frac{d}{dt}\|\sqrt{\phi}c_n\|_{L^2}^2 + d_*\|\nabla c_n\|_{L^2}^2 \le Mq_n\|c_n\|_{H^1}. \tag{20}$$

Integrating (20) about time from $t = 0$ to $t = T$ and using the Gronwall lemma, we can obtain

$$\max_{0\le t\le T}\|c_n(t)\|_{L^2}^2 + \int_0^T \|\nabla c_n(t)\|_{L^2}^2 dt \le M\int_0^T q_n^2 dt. \tag{21}$$

Thus (17) holds.

(4) Prove the estimate

$$\|\frac{\partial c_n}{\partial t}\|_{L^2(0,T;H^{-1}(\Omega))} \leq M. \tag{22}$$

By taking any $z \in L^2(0,T;H^1(\Omega))$, we get

$$\begin{aligned} \int_0^T (\phi\frac{\partial c_n}{\partial t}, z) = -\int_0^T (d(c_n)\nabla c_n, \nabla z) + \int_0^T ((r_0 - r_1)q_n\varrho_o(c_n), z) \\ + \int_0^T (\mathbf{u}_n \cdot \nabla z, \varrho_o(c_n)) + \int_0^T (f(c_n)r_0q_n, z). \end{aligned} \tag{23}$$

From the embedding theorems, it is obvious that

$$\begin{aligned} \Big|\int_0^T (\phi\frac{\partial c_n}{\partial t}, z)\Big| \leq C\Big\{\|c_n\|_{L^2(0,T;H^1(\Omega))}\|z\|_{L^2(0,T;H^1(\Omega))} \\ + \|\mathbf{u}_n\|_{L^2(0,T;L^2(\Omega))}\|z\|_{L^2(0,T;H^1(\Omega))} + \|q_n\|_{L^2(0,T)}\|z\|_{L^2(0,T;L^2(\Omega))}\Big\}, \end{aligned} \tag{24}$$

such that (22) holds.

Because

$$\|c_n\|_{L^2(0,T;\ H^1(\Omega))} \leq M \tag{25}$$

and

$$\|\frac{\partial c_n}{\partial t}\|_{L^2(0,T;\ H^{-1}(\Omega))} \leq M, \tag{26}$$

then we can extract a subsequence, again denoted by c_n, such that

$$\begin{cases} c_n \to \bar{c} \ \ strongly\ in\ \ L^2(0,T;L^2(\Omega)), \\ \dfrac{\partial c_n}{\partial t} \to \dfrac{\partial \bar{c}}{\partial t} \ \ weakly\ in\ \ L^2(0,T;\ H^{-1}(\Omega)), \\ c_n \to \bar{c} \ \ weakly\ in\ \ L^2(0,T;\ H^1(\Omega)). \end{cases} \tag{27}$$

(5) For each smooth enough w, we note that

$$\begin{aligned} &|\int_0^T \big(a(c_n)\nabla p_n, \nabla w\big) - \int_0^T (a(\bar{c})\nabla\bar{p}, \nabla w)| \\ &\leq |\int_0^T \big((a(c_n) - a(\bar{c}))\nabla p_n, \nabla w\big)| + |\int_0^T \big(a(\bar{c})\nabla(p_n - \bar{p}), \nabla w\big)| \\ &\leq \|a(c_n) - a(\bar{c})\|_{L^2(0,T;L^2(\Omega))}\|\nabla p_n\|_{L^2(0,T;L^2(\Omega))}\|\nabla w\|_{L^\infty(0,T;L^\infty(\Omega))} \\ &\quad + |\int_0^T \big(a(\bar{c})\nabla(p_n - \bar{p}), \nabla w\big)|. \end{aligned} \tag{28}$$

Since $a(c)$ is a Lipschitz continuous function about c by assumption (A), $\{p_n\}_{n=1}^\infty$ are bounded in $L^2(0,T;H^1(\Omega))$, $(a(\bar{c})\nabla p, \nabla w)$ is a continuous functional about p and $p_n \to \bar{p}$ weakly in $L^2(0,T;\ H^1(\Omega))$, from (28) we know that

$$\int_0^T \big(a(c_n)\nabla p_n, \nabla w\big) \to \int_0^T (a(\bar{c})\nabla\bar{p}, \nabla w). \tag{29}$$

Similarly we get

$$\begin{aligned}
&\int_0^T (d(c_n)\nabla c_n, \nabla z) \to \int_0^T (d(\bar{c})\nabla \bar{c}, \nabla z),\\
&\int_0^T (f(c_n) r_0 q_n, z) \to \int_0^T (f(\bar{c}) r_0 \bar{q}, z),\\
&\int_0^T ((r_0 - r_1) q_n \varrho_o(c_n), z) \to \int_0^T ((r_0 - r_1)\bar{q}\varrho_o(\bar{c}), z),\\
&\int_0^T (\mathbf{u}_n \varrho_o(c_n), \nabla z) \to \int_0^T (\bar{\mathbf{u}}\varrho_o(\bar{c}), \nabla z).
\end{aligned} \tag{30}$$

At the meantime, since

$$\int_0^T (b(c_n)\mathbf{u}_n \cdot \nabla c_n, z) = -\int_0^T ((r_0 - r_1) q_n \varrho_o(c_n), z) - \int_0^T (\mathbf{u}_n \varrho_o(c_n), \nabla z), \tag{31}$$

we know that

$$\int_0^T (b(c_n)\mathbf{u}_n \cdot \nabla c_n, z) \to \int_0^T (b(\bar{c})\bar{\mathbf{u}} \cdot \nabla \bar{c}, z). \tag{32}$$

(6) By choosing z, w arbitrarily and using the above convergence results, we have

$$\begin{cases}
(\phi \dfrac{\partial \bar{c}}{\partial t}, z) + (b(\bar{c})\bar{\mathbf{u}} \cdot \nabla \bar{c}, z) + (d(\bar{c})\nabla \bar{c}, \nabla z) = (f(\bar{c}) r_0 \bar{q}, z), & \forall\ z \in H^1(\Omega),\\
\\
(a(\bar{c})\nabla \bar{p}, \nabla w) = ((r_0 - r_1)\bar{q}, w), & \forall\ w \in H^1(\Omega),
\end{cases} \tag{33}$$

and by continuity and convexity of the objective founctional

$$\frac{1}{2}\int_0^T \int_\Omega \omega \bar{c}^2\, dt + \frac{\alpha_0}{2}\int_0^T \bar{q}^2\, dt \leq \varliminf_{n\to\infty} \Big\{ \frac{1}{2}\int_0^T \int_\Omega \omega c_n^2 + \frac{\alpha_0}{2}\int_0^T q_n^2 \Big\}. \tag{34}$$

This implies that $(\bar{c}, \bar{p}, \bar{\mathbf{u}}, \bar{q})$ is a solution of the control problem. □

3. Optimality conditions

3.1. The first-order optimality conditions. In this section, we derive the co-state equations and the first-order optimality conditions. Besides above assumptions, we assume that there exists a uniform constant M_1 such that

$$(B):\begin{cases}
\|u\|_{L^\infty([0,T];L^\infty)} \leqslant M_1, \quad \|\nabla c\|_{L^\infty([0,T];L^\infty)} \leqslant M_1,\\
\|\alpha'(c)\|_{L^\infty([0,T];L^\infty)} \leqslant M_1, \quad \|d'(c)\|_{L^\infty([0,T];L^\infty)} \leqslant M_1.
\end{cases}$$

This assumption is again usually used in the study of this state system in the literature, see [10] and [12].

Theorem 3.1. *Suppose that Assumptions* (A) *and* (B) *hold. If* $(c, q, \mathbf{u}, p) \in W(0,T) \times K \times L^2(0,T;V) \times L^2(0,T;W)$ *is a solution of the control problem* (10) *then there exists a co-state* $(c^*, \mathbf{u}^*, p^*) \in W(0,T) \times L^2(0,T;V) \times L^2(0,T;W)$ *such* *that* $(c, c^*, q, \mathbf{u}, \mathbf{u}^*, p, p^*)$ *satisfies the following system:*

$$(35)\quad \begin{cases} (\phi\dfrac{\partial c}{\partial t}, z) + (b(c)\mathbf{u}\cdot\nabla c, z) + (d(c)\nabla c, \nabla z) = (f(c)r_0 q, z), & \forall\ z \in H^1(\Omega), \\ (\nabla\cdot\mathbf{u}, w) = ((r_0 - r_1)q, w), & \forall\ w \in W, \\ (\alpha(c)\mathbf{u}, \mathbf{v}) - (p, \nabla\cdot\mathbf{v}) = 0, & \forall\ \mathbf{v} \in V, \\ (-\phi\dfrac{\partial c^*}{\partial t}, z) - ((b(c)\mathbf{u} - d'(c)\nabla c)\cdot\nabla c^*, z) + (d(c)\nabla c^*, \nabla z) & \\ \quad + (\alpha'(c)\mathbf{u}^*\cdot\mathbf{u}, z) + (r_1 q b(c) c^*, z) = (\omega c, z), & \forall\ z \in H^1(\Omega), \\ (c^* b(c)\nabla c + \alpha(c)\mathbf{u}^*, \mathbf{v}) - (p^*, \nabla\cdot\mathbf{v}) = 0, & \forall\ \mathbf{v} \in V, \\ (\nabla\cdot\mathbf{u}^*, w) = 0, & \forall\ w \in W, \\ \displaystyle\int_0^T \int_\Omega (f(c)r_0 c^* - (r_0 - r_1)p^* + \alpha_0 q)(\tilde{q} - q) \geq 0, & \forall\ \tilde{q} \in K, \end{cases}$$

associated with the initial conditions:

$$(36)\qquad c|_{t=0} = c_0, \quad c^*|_{t=T} = 0.$$

Proof. Firstly, it is clear that the direction derivative of the objective functional reads

$$(37)\qquad J'(q)(\delta q) = \int_0^T \int_\Omega \omega(\mathbf{x}, t) c\delta c + \alpha_0 \int_0^T q\delta q,$$

where $\tilde{q} \in K,\ \delta q = \tilde{q} - q$ denoting the direction, and $\delta c = c'(q)(\tilde{q} - q)$.

We differentiate the equations of the weak formula at q in a direction of δq to obtain:

$$(38)\quad \begin{cases} \displaystyle\int_0^T [(\phi\frac{\partial \delta c}{\partial t}, z) + (b(c)\delta\mathbf{u}\cdot\nabla c, z) + (b(c)\mathbf{u}\cdot\nabla\delta c, z) + (b'(c)\delta c\mathbf{u}\cdot\nabla c, z) \\ \qquad + (d(c)\nabla\delta c, \nabla z) - (d'(c)\delta c\nabla c, \nabla z)] = \displaystyle\int_0^T [(f(c)r_0\delta q, z) + (f'(c)r_0 q\delta c, z)], \\ \displaystyle\int_0^T (\nabla\cdot\delta\mathbf{u}, w) = \int_0^T ((r_0 - r_1)\delta q, w), \\ \displaystyle\int_0^T [(\alpha(c)\delta\mathbf{u}, \mathbf{v}) + (\alpha'(c)\delta c\mathbf{u}, \mathbf{v}) - (\delta p, \nabla\cdot\mathbf{v})] = 0, \end{cases}$$

where $\delta\mathbf{u} = u'(q)(\tilde{q} - q),\ \delta p = p'(q)(\tilde{q} - q)$.

Taking $z = c^*, \mathbf{v} = \mathbf{u}^*, w = p^*$, letting $c^*(\mathbf{x}, T) = 0,\ d(c)\nabla c^*\cdot\mathbf{n}|_{\partial\Omega} = 0$ and integrating by parts, it follows

$$(39)\quad \begin{cases} \displaystyle\int_0^T [-(\phi\frac{\partial c^*}{\partial t}, \delta c) + (c^* b(c)\nabla c, \delta\mathbf{u}) + (c^* b'(c)\mathbf{u}\cdot\nabla c, \delta c) - (b(c)\mathbf{u}\cdot\nabla c^*, \delta c) \\ \qquad - (\nabla\cdot(b(c)\mathbf{u})c^*, \delta c) - (\nabla\cdot(d(c)\nabla c^*), \delta c) + (d'(c)\nabla c\cdot\nabla c^*, \delta c)] \\ \qquad = \displaystyle\int_0^T [(f(c)r_0 c^*, \delta q) + (f'(c)r_0 q c^*, \delta c)], \\ \displaystyle\int_0^T (p^*, \nabla\cdot\delta\mathbf{u}) = \int_0^T ((r_0 - r_1)p^*, \delta q), \\ \displaystyle\int_0^T [(\alpha(c)\mathbf{u}^*, \delta\mathbf{u}) + (\alpha'(c)\mathbf{u}^*\cdot\mathbf{u}, \delta c) - (\nabla\cdot\mathbf{u}^*, \delta p)] = 0. \end{cases}$$

Summing above all, we can have:

$$\begin{aligned}\int_0^T &[(-\phi\frac{\partial c^*}{\partial t}-(b(c)\mathbf{u}-d'(c)\nabla c)\cdot\nabla c^*-\nabla\cdot(d(c)\nabla c^*)+(b'(c)\mathbf{u}\cdot\nabla c\\ &\quad -\nabla\cdot(b(c)\mathbf{u})-f'(c)r_0q)c^*+\alpha'(c)\mathbf{u}^*\cdot u,\delta c)\\ &\quad +(c^*b(c)\nabla c+\alpha(c)\mathbf{u}^*,\delta\mathbf{u})-(p^*,\nabla\cdot\delta\mathbf{u})-(\nabla\cdot\mathbf{u}^*,\delta p)]\\ &=\int_0^T(f(c)r_0c^*-(r_0-r_1)p^*,\delta q).\end{aligned}\tag{40}$$

Let us define $(c^*,\mathbf{u}^*,p^*)$ to be the co-state which satisfies the equations following:
(41)

$$\begin{cases}(-\phi\dfrac{\partial c^*}{\partial t}-(b(c)\mathbf{u}-d'(c)\nabla c)\cdot\nabla c^*-\nabla\cdot(d(c)\nabla c^*)\\ \quad +(b'(c)\mathbf{u}\cdot\nabla c-\nabla\cdot(b(c)\mathbf{u})-f'(c)r_0q)c^*+\alpha'(c)\mathbf{u}^*\cdot\mathbf{u},z)=(\omega c,z), & \forall\ z\in H^1(\Omega),\\ (c^*b(c)\nabla c+\alpha(c)\mathbf{u}^*,\mathbf{v})-(p^*,\nabla\cdot\mathbf{v})=0, & \forall\ \mathbf{v}\in V,\\ (\nabla\cdot\mathbf{u}^*,w)=0, & \forall\ w\in W.\end{cases}$$

Now, if take $z=\delta c, w=\delta p, \mathbf{v}=\delta\mathbf{u}$ in (41), we obtain:
(42)

$$\begin{cases}(-\phi\dfrac{\partial c^*}{\partial t}-(b(c)\mathbf{u}-d'(c)\nabla c)\cdot\nabla c^*-\nabla\cdot(d(c)\nabla c^*)\\ \quad +(b'(c)\mathbf{u}\cdot\nabla c-\nabla\cdot(b(c)\mathbf{u})-f'(c)r_0q)c^*+\alpha'(c)\mathbf{u}^*\cdot\mathbf{u},\delta c)=(\omega(\mathbf{x},t)c,\delta c),\\ (c^*b(c)\nabla c+\alpha(c)\mathbf{u}^*,\delta\mathbf{u})-(p^*,\nabla\cdot\delta\mathbf{u})=0,\\ (\nabla\cdot\mathbf{u}^*,\delta p)=0.\end{cases}$$

By (40) it is obviously to see that $(\omega(\mathbf{x},t)c,\delta c)=(f(c)r_0c^*-(r_0-r_1)p^*,\delta q)$.

Furthermore, the optimality condition reads

$$\int_0^T\int_\Omega(f(c)r_0c^*-(r_0-r_1)p^*+\alpha_0q)(\tilde{q}-q)\geq 0.\tag{43}$$

It is well-known that the solution of this inequality reads:

$$q(t)=\min\Big\{\max\{0,-\frac{1}{\alpha_0}\int_\Omega(f(c)r_0c^*-(r_0-r_1)p^*)\},\hat{q}\Big\}.\tag{44}$$

From the definition of coefficients, we know that $b'(c)\mathbf{u}\cdot\nabla c-\nabla\cdot(b(c)\mathbf{u})-f'(c)r_0q=r_1qb(c)$, and if we let $\mathbf{u}^*\cdot n|_{\partial\Omega}=0$, the adjoint system can be the following equivalent form:
(45)

$$\begin{cases}-\phi\dfrac{\partial c^*}{\partial t}-(b(c)\mathbf{u}-d'(c)\nabla c)\cdot\nabla c^*-\nabla\cdot(d(c)\nabla c^*)+\alpha'(c)\mathbf{u}^*\cdot u+r_1qb(c)c^*=\omega c,\\ c^*b(c)\nabla c+\nabla p^*+\alpha(c)\mathbf{u}^*=0,\\ \nabla\cdot\mathbf{u}^*=0.\end{cases}$$

The existence of the solution of adjoint system will be presented below. Now, the proof completes. □

3.2. Existence of the solution of adjoint problem.

Theorem 3.2. *Suppose that assumptions* (A) *and* (B) *are satisfied, then adjoint equations* (45) *admit at least a solution.*

Proof. First of all, we present the weak formula for the system (45):

(46)
$$\begin{cases} -(\phi\dfrac{\partial c^*}{\partial t}, z) - ((b(c)\mathbf{u} - d'(c)\nabla c)\cdot\nabla c^*, z) + (d(c)\nabla c^*, \nabla z) \\ \qquad + (\alpha'(c)\mathbf{u}^*\cdot\mathbf{u}, z) + (r_1 q b(c) c^*, z) = (\omega c, z), & \forall\ z \in H^1(\Omega), \\ (\alpha(c)\mathbf{u}^*, \mathbf{v}) - (p^*, \nabla\cdot\mathbf{v}) = -(c^* b(c)\nabla c, \mathbf{v}), & \forall\ \mathbf{v}\in V, \\ (\nabla\cdot\mathbf{u}^*, w) = 0, & \forall\ w \in W. \end{cases}$$

Then the last two equations of (46) are equivalent to the time-parametrized saddle-point problem of finding a map $(\mathbf{u}^*, p^*) : (0,T] \to V \times W$, such that

$$\begin{cases} A(c;\mathbf{u}^*,\mathbf{v}) - B(\mathbf{v}, p^*) = -(c^* b(c)\nabla c, \mathbf{v}), & \forall\ \mathbf{v}\in V, \\ B(\mathbf{u}^*, w) = 0, & \forall\ w\in W, \end{cases} \tag{47}$$

where

$$\begin{cases} A(c;\mathbf{u},\mathbf{v}) = (\alpha(c)\mathbf{u},\mathbf{v}), & \forall\ c\ \in H^1(\Omega),\ u\ \in V,\ v\ \in V, \\ B(\mathbf{v}, p) = (p, \nabla\cdot\mathbf{v}), & \forall\ v\ \in V, p\ \in W. \end{cases}$$

Let $\tilde{V}\subset V$, $\tilde{W}\subset W$ be Raviart-Thomas mixed finite element spaces. Furthermore let

$$Z = \{\mathbf{v}\in V:\ B(\mathbf{v},\phi) = 0, \phi\in W\}$$

and

$$\tilde{Z} = \{\mathbf{v}\in\tilde{V} : B(\mathbf{v},\phi) = 0, \phi\in\tilde{W}\}.$$

Note that the boundary condition $v\cdot\mathbf{n} = 0$ for $v\in\tilde{V}$ implies $div\cdot\tilde{V}\subset\tilde{W}$. Hence $v\in\tilde{Z}$ implies $div\cdot\mathbf{v} = 0$, and since $Z = H(div;\Omega)\cap\{div\cdot v = 0\ \ in\ \Omega\ and\ v\cdot\mathbf{n} = 0\ on\ \partial\Omega\}$, it follows that $\tilde{Z}\subset Z$. Thus

$$\|v\|_V = \|v\|_{L^2(\Omega)}, \quad v\in\tilde{Z}.$$

So, if $\mathbf{v}\in\tilde{Z}$

$$A(c;\mathbf{v},\mathbf{v}) = \sum_{i=1}^{2}(\frac{1}{a(c)}v_i, v_i) \geqslant \frac{1}{a^*}\|\mathbf{v}\|_V^2.$$

Next from the standard properties of Raviart-Thomas elements [27], we have

$$\sup_{\mathbf{v}\in\tilde{V}/\{0\}}\frac{B(\mathbf{v},\phi)}{\|v\|_V} \geqslant \theta\|\phi\|_W, \quad \forall\ \phi\in\tilde{W}.$$

Hence, it follows from the standard results of the mixed element approximation ([27, 28]), for any fixed C^*, there exists a generic constant M independent of h such that any solution (U^*, P^*) of (48) below satisfies:

$$\|U^*\|_V + \|P^*\|_W \quad\leq\quad M\|C^*\|_{L^2(\Omega)}.$$

where the map $\{U^*, P^*\} : (0,T] \to \tilde{V}\times\tilde{W}$ satisfies:

$$\begin{cases} A(c, U^*, v) - B(v, P^*) = -(C^* b(c)\nabla c, v), & \forall\ v\in\tilde{V}, \\ B(U^*, w) = 0, & \forall\ w\in\tilde{W}. \end{cases} \tag{48}$$

Let $\tilde{M} \subset H^1(\Omega)$ be a standard Galerkin finite element space. Define $C^* : (0;T] \to \tilde{M}$ and $\{U^*, P^*\} : (0;T] \to \tilde{V} \times \tilde{W}$ to satisfy:

$$(49)\quad \begin{cases} -(\phi\dfrac{\partial C^*}{\partial t}, z) - ((b(c)u - d'(c)\nabla c) \cdot \nabla C^*, z) + (d(c)\nabla C^*, \nabla z) \\ \qquad + (\alpha'(c)U^* \cdot u, z) + (r_1 q b(c)C^*, z) = (wc, z), & \forall z \in \tilde{M}, \\ (\alpha(c)U^*, v) - (P^*, \nabla \cdot v) = -(C^* b(c)\nabla c, v), & \forall v \in \tilde{V}, \\ (\nabla \cdot U^*, w) = 0, & \forall w \in \tilde{W}. \end{cases}$$

We first prove that for homogeneous problem (i.e. $wc = 0$), the above finite element system has and only has trivial zero solution. The equation for C^* reads

$$(50)\quad \begin{aligned} &-(\phi\frac{\partial C^*}{\partial t}, z) - ((b(c)u - d'(c)\nabla c) \cdot \nabla C^*, z) + (d(c)\nabla C^*, \nabla z) \\ &\quad +(\alpha'(c)U^* \cdot u, z) + (r_1 q b(c)C^*, z) = 0, \quad \forall z \in \tilde{M}. \end{aligned}$$

Obviously $C^* = 0$, $U^* = 0$, $P^* = 0$ are solutions of this homogeneous problem, so we only need to prove the uniqueness.

Suppose there exists another solution $C^* \neq 0$ which satisfies (50). It follows from (48) that

$$(51)\quad \|U^*\|_V + \|P^*\|_W \leq M\|C^*\|_{L^2(\Omega)}.$$

Choosing $z = C^*$ in (50), we have

$$(52)\quad \begin{aligned} &-(\phi\frac{\partial C^*}{\partial t}, C^*) - ((b(c)u - d'(c)\nabla c) \cdot \nabla C^*, C^*) + (d(c)\nabla C^*, \nabla C^*) \\ &\quad +(\alpha'(c)U^* \cdot u, C^*) + (r_1 q b(c)C^*, C^*) = 0. \end{aligned}$$

It follows from (51) that

$$(53)\quad -\phi^* \frac{\|C^*\|^2_{L^2(\Omega)}}{dt} + \frac{d_*}{2}\|\nabla C^*\|^2_{L^2(\Omega)} \leq M\|C^*\|^2_{L^2(\Omega)}.$$

Integrating time from T to t, noting that $C^*(T) = 0$ and using Gronwall inequality, we can derive

$$(54)\quad \phi^*\|C^*\|^2_{L^2(\Omega)} + \frac{d_*}{2}\int_t^T \|\nabla C^*\|^2_{L^2(\Omega)} \leq 0.$$

It can be seen from (54) that $C^* = 0$, which conflicts with our former supposition. So the uniqueness of the zero-solutions have been validated.

Since the finite element problem (49) is linear and full rank, we obtain the existence of its solution directly from the uniqueness of the zero-solution of its corresponding homogeneous problem.

Now let $\{C^*, P^*, U^*\}$ satisfy (49). We still have (51) holds. Taking $z = C^*$ in the first equation of (49), and using the same techniques as above, we can obtain

$$(55)\quad \phi^*\|C^*\|^2_{L^2(\Omega)} + \frac{d_*}{2}\int_t^T \|\nabla C^*\|^2_{L^2(\Omega)} \leq M\|wc\|^2_{L^2(\Omega)}.$$

Therefore,

(56)

$$\|C^*\|_{L^2(0,T;H^1(\Omega))} \leq M, \quad \|C^*\|_{L^\infty(0,T;L^2(\Omega))} \leq M, \quad \|U^*\|_V \leq M, \quad \|P^*\|_W \leq M.$$

Similarly, we have

$$\|\frac{\partial C^*}{\partial t}\|_{L^2(0,T;H^{-1}(\Omega))} \leq M.$$

Hence we may extract a subsequence denoted by $\{C_n^*, P_n^*, U_n^*\}$ such that

$$\begin{cases} C_n^* \to c^* & \text{strongly in } L^2(0,T;L^2(\Omega)), \\ U_n^* \to u^* & \text{weakly in } V, \\ P_n^* \to p^* & \text{weakly in } W. \end{cases} \tag{57}$$

Then, it holds

(58)

$$\begin{cases} -(\phi\frac{\partial c^*}{\partial t}, z) - ((b(c)\mathbf{u} - d'(c)\nabla c)\cdot\nabla c^*, z) + (d(c)\nabla c^*, \nabla z) \\ \quad + (\alpha'(c)\mathbf{u}^*\cdot u, z) + (r_1 q b(c) c^*, z) = (\omega c, z), & \forall\ z \in H^1(\Omega), \\ (\alpha(c)\mathbf{u}^*, v) - (p^*, \nabla\cdot\mathbf{v}) + (c^* b(c)\nabla c, v) = 0, & \forall\ v \in V, \\ (\nabla\cdot\mathbf{u}^*, w) = 0, & \forall\ w \in W. \end{cases}$$

Now the proof completes. □

4. Finite element approximation

We are now able to introduce a finite-element based approximation of the optimal control problem. Let us note that the finite element approximation of the state and the co-state system is widely studied in the literature, see e.g., [10], [12], [20] and [21], on which our approximation is based. To this end, we consider a family of triangulations $(\mathcal{T}_h)$, $h > 0$, of $\bar{\Omega}$. With each element $\tau \in \mathcal{T}_h$, we associate two parameters $\rho(\tau)$ and $\sigma(\tau)$, where $\rho(\tau)$ denotes the diameter of the set τ and $\sigma(\tau)$ is the diameter of the largest ball contained in τ. The mesh size of the grid is defined by $h = \max_{\tau\in T_h} \rho(\tau)$. We suppose that triangulations $(\mathcal{T}_h)$ satisfy the following regularity assumption:

(H_1) *There exist two positive constants ρ and σ such that*

$$\frac{\rho(\tau)}{\sigma(\tau)} \le \sigma, \quad \frac{\sigma(\tau)}{\rho(\tau)} \le \rho$$

hold for all $\tau \in \mathcal{T}_h$ and all $0 < h \le 1$. Now we can see $\Omega = \Omega_h = \bigcup_{\tau\in\mathcal{T}_h}$.

Introduce finite element spaces as follows:

$$Z_h = \{y_h \in H^1(\Omega) : \quad y_h|_\tau \in P_l(\tau), \quad \tau \in \mathcal{T}_h\},$$

where $l \ge 1$ and denotes by P_l function space of polynomials of degree less or equal than l.

Next we introduce the k-order R-T mixed finite element spaces: $V_h \times W_h \subset V \times W$ such that for a positive constant β_0, the following *inf-sup* condition satisfies ([27, 28]):

$$\inf_{0\ne q_h\in W_h} \sup_{\mathbf{0}\ne\mathbf{v}_h\in V_h} \frac{(\nabla\cdot\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_V \|q_h\|_{L^2}} \ge \beta_0. \tag{59}$$

Then the possible semi-discrete finite dimensional approximation of the optimal control problem is to seek $(c_h, q_h, \mathbf{u}_h, p_h) \in \Big(H^1(0,T;Z_h) \cap L^2(0,T;Z_h)\Big) \times K \times L^2(0,T;V_h) \times L^2(0,T;W_h)$ such that

$$\min\ J_h(q_h) = \min_{q_h\in K} \left\{\frac{1}{2}\int_0^T \int_\Omega \omega c_h^2 + \frac{\alpha_0}{2}\int_0^T q_h^2\right\} \tag{60}$$

subject to

$$\begin{cases} (\phi\dfrac{\partial c_h}{\partial t}, z_h) + (b(c_h)\mathbf{u}_h \cdot \nabla c_h, z_h) + (d(c_h)\nabla c_h, \nabla z_h) = (f(c_h)r_0 q_h, z_h), & \forall\ z_h \in Z_h, \\ (\nabla \cdot \mathbf{u}_h, w_h) = ((r_0 - r_1)q_h, w_h), & \forall\ w_h \in W_h, \\ (\alpha(c_h)\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = 0, & \forall\ \mathbf{v}_h \in V_h, \end{cases}$$

associated with the initial condition:

$$c_h|_{t=0} = c_{0,h}. \tag{61}$$

where $c_{0,h} \in Z_h$ is an approximation of c_0.

Similarly, we know $(c_h, q_h, \mathbf{u}_h, p_h)$ is the solution of (60) if there is a co-state $(c_h^*, \mathbf{u}_h^*, p_h^*)$, such that

$$\begin{cases} (\phi\dfrac{\partial c_h}{\partial t}, z_h) + (b(c_h)\mathbf{u}_h \cdot \nabla c_h, z_h) + (d(c_h)\nabla c_h, \nabla z_h) = (f(c_h)r_0 q_h, z_h), & \forall\ z_h \in Z_h, \\ (\nabla \cdot \mathbf{u}_h, w_h) = ((r_0 - r_1)q_h, w_h), & \forall\ w_h \in W_h, \\ (\alpha(c_h)\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = 0, & \forall\ \mathbf{v}_h \in V_h; \\ (-\phi\dfrac{\partial c_h^*}{\partial t}, z_h) - ((b(c_h)\mathbf{u}_h - d'(c_h)\nabla c_h) \cdot \nabla c_h^*, z_h) + (d(c_h)\nabla c_h^*, \nabla z_h) & \\ \qquad + (\alpha'(c_h)\mathbf{u}_h^* \cdot \mathbf{u}_h, z_h) + (r_1 q_h b(c_h) c_h^*, z_h) = (\omega c_h, z_h), & \forall\ z_h \in Z_h, \\ (c_h^* b(c_h)\nabla c_h, \mathbf{v}_h) + (\alpha(c_h)\mathbf{u}_h^*, \mathbf{v}_h) - (p_h^*, \nabla \cdot \mathbf{v}_h) = 0, & \forall\ \mathbf{v}_h \in V_h, \\ (\nabla \cdot \mathbf{u}_h^*, w_h) = 0, & \forall\ w_h \in W_h, \\ \displaystyle\int_0^T \int_\Omega (f(c_h)c_h^* - p_h^* + \alpha_0 q_h)(\tilde{q} - q_h) \geq 0, & \forall\ \tilde{q} \in K, \end{cases} \tag{62}$$

associated with the initial conditions:

$$c_h|_{t=0} = c_{0,h}, \quad c_h^*|_{t=T} = 0. \tag{63}$$

It can also be shown the solution of the variational inequality reads:

$$q_h = \min\Big\{\max\big\{0, -\frac{1}{\alpha_0}\int_\Omega (f(c_h)r_0 c_h^* - (r_0 - r_1)p_h^*)\big\}, \hat{q}\Big\}. \tag{64}$$

For easy of presentation, we only consider a priori error estimates for the semi-discrete finite dimensional approximation of the optimal control problem in the following section.

5. A priori error estimate

In this section, we will give the a priori error estimates about the solution $(c_h, q_h, \mathbf{u}_h, p_h, c_h^*, \mathbf{u}_h^*, p_h^*)$ of scheme (62). Moreover, in this part, we need assume

$l \geq 1, k \geq 1$, and that there exists a uniform constant M_2 such that:

(65)

$$(C):\begin{cases} \|\alpha'(c)\|_{L^\infty(0,T;L^\infty)} + \|b'(c)\|_{L^\infty(0,T;L^\infty)} + \|d'(c)\|_{L^\infty(0,T;L^\infty)} \leq M_2; \\ \|\alpha''(c)\|_{L^\infty(0,T;L^\infty)} + \|b''(c)\|_{L^\infty(0,T;L^\infty)} + \|d''(c)\|_{L^\infty(0,T;L^\infty)} \leq M_2; \\ \|\mathbf{u}\|_{L^\infty(0,T;L^\infty)} + \|\frac{\partial \mathbf{u}}{\partial t}\|_{L^\infty(0,T;L^\infty)} \leq M_2; \\ \|p\|_{L^\infty(0,T;H^{k+3})} \leq M_2; \\ \|c\|_{L^\infty(0,T;H^{l+1})} + \|\frac{\partial c}{\partial t}\|_{L^\infty(0,T;H^{l+1})} \leq M_2; \\ \|c\|_{L^\infty(0,T;W^{1,\infty})} + \|\frac{\partial c}{\partial t}\|_{L^\infty(0,T;W^{1,\infty})} \leq M_2; \\ \|\mathbf{u}^*\|_{L^\infty(0,T;L^\infty)} + \|\frac{\partial \mathbf{u}^*}{\partial t}\|_{L^\infty(0,T;L^\infty)} \leq M_2; \\ \|p^*\|_{L^\infty(0,T;H^{k+3})} \leq M_2; \\ \|c^*\|_{L^\infty(0,T;H^{l+1})} + \|\frac{\partial c^*}{\partial t}\|_{L^\infty(0,T;H^{l+1})} \leq M_2. \end{cases}$$

Now, let us define the intermediate functions $(\hat{c}_h, \hat{\mathbf{u}}_h, \hat{p}_h, \hat{c}_h^*, \hat{\mathbf{u}}_h^*, \hat{p}_h^*)$ satisfying the following system:

(66)

$$\begin{cases} (\phi\frac{\partial \hat{c}_h}{\partial t}, z_h) + (b(\hat{c}_h)\hat{\mathbf{u}}_h \cdot \nabla \hat{c}_h, z_h) + (d(\hat{c}_h)\nabla \hat{c}_h, \nabla z_h) = (f(\hat{c}_h)r_0 q, z_h), & \forall\ z_h \in Z_h, \\ (\nabla \cdot \hat{\mathbf{u}}_h, w_h) = ((r_0 - r_1)q, w_h), & \forall\ w_h \in W_h, \\ (\alpha(\hat{c}_h)\hat{\mathbf{u}}_h, \mathbf{v}_h) - (\hat{p}_h, \nabla \cdot \mathbf{v}_h) = 0, & \forall\ \mathbf{v}_h \in V_h, \\ (-\phi\frac{\partial \hat{c}_h^*}{\partial t}, z_h) - ((b(\hat{c}_h)\hat{\mathbf{u}}_h - d'(\hat{c}_h)\nabla \hat{c}_h) \cdot \nabla \hat{c}_h^*, z_h) + (d(\hat{c}_h)\nabla \hat{c}_h^*, \nabla z_h) & \\ \qquad + (\alpha'(\hat{c}_h)\hat{\mathbf{u}}_h^* \cdot \hat{\mathbf{u}}, z_h) + (r_1 q b(\hat{c}_h)\hat{c}_h^*, z_h) = (\omega \hat{c}_h, z_h), & \forall\ z_h \in Z_h, \\ (\hat{c}_h^* b(\hat{c}_h)\nabla \hat{c}_h, \mathbf{v}_h) - (\hat{p}_h^*, \nabla \cdot \mathbf{v}_h) + (\alpha(\hat{c}_h)\hat{\mathbf{u}}_h^*, \mathbf{v}_h) = 0, & \forall\ \mathbf{v}_h \in V_h, \\ (\nabla \cdot \hat{\mathbf{u}}_h^*, w_h) = 0, & \forall\ w_h \in W_h, \end{cases}$$

associated with the initial conditions:

(67) $$\hat{c}_h|_{t=0} = c_{0,h}, \quad \hat{c}_h^*|_{t=T} = 0.$$

Assume that the initial approximation satisfies the error estimate:

(68) $$\|c_0 - c_{0,h}\|_{L^2} \leq Mh^{l+1}\|c_0\|_{H^{l+1}}.$$

We firstly derive the estimate of $\|q - q_h\|_{L^2(0,T)}$. To this end, we need the following convexity assumption:

(69) $$c_0\|q - q_h\|^2_{L^2(0,T)} \leq (J_h'(q) - J_h'(q_h), q - q_h),$$

where c_0 is a positive constant, and h is small enough. Although it is no-trivial to prove such an assumption for nonlinear system (see [29, 30] for the relevant work on the flow control governed by the Navies-Stokes equations), it is widely assumed in error analysis of the finite element approximation of nonlinear optimal control. So, we assume (69) in this initial stage.

Lemma 5.1. *If Assumptions $(A), (B)$ and (C) mentioned above hold, there holds the estimate:*

(70)

$$\|q - q_h\|_{L^2(0,T)} \leq M\Big\{\|\hat{c}_h - c\|_{L^2(0,T;L^2)} + \|\hat{c}_h^* - c^*\|_{L^2(0,T;L^2)} + \|\hat{p}_h^* - p^*\|_{L^2(0,T;W)}\Big\}.$$

Proof. It follows from the convexity assumption that

$$
\begin{aligned}
& c_0\|q-q_h\|^2_{L^2(0,T)} \\
& \le (J_h'(q)-J_h'(q_h), q-q_h) \\
& = \int_0^T\int_\Omega (r_0 f(\hat{c}_h)\hat{c}_h^* + (r_0-r_1)\hat{p}_h^* + \alpha_0 q)(q-q_h) \\
& \quad - \int_0^T\int_\Omega (r_0 f(c_h)c_h^* + (r_0-r_1)p_h^* + \alpha_0 q_h)(q-q_h) \\
& \le \int_0^T\int_\Omega (r_0 f(\hat{c}_h)\hat{c}_h^* - r_0 f(c)c^* + (r_0-r_1)(\hat{p}_h^* - p^*))(q-q_h) \\
& \le M\Big\{\|\hat{c}_h - c\|_{L^2(0,T;L^2)} + \|\hat{c}_h^* - c^*\|_{L^2(0,T;L^2)} + \|\hat{p}_h^* - p^*\|_{L^2(0,T;W)}\Big\}\|q-q_h\|_{L^2(0,T)}.
\end{aligned}
$$

Then (70) is derived. The proof of Lemma 5.1 is completed. □

Let us show two useful lemmas about the error estimate of intermediate states variables.

Lemma 5.2. *Suppose the regularity assumption* (C) *holds. Let* $(c, \mathbf{u}, p)$ *and* $(\hat{c}_h, \hat{\mathbf{u}}_h, \hat{p}_h)$ *be the solutions of* (35) *and* (66) *respectively. There hold the a priori error estimates*

$$
(71)\qquad
\begin{aligned}
&(a)\quad \|c-\hat{c}_h\|_{L^\infty(0,T;L^2)} \le M\Big\{h^{l+1}+h^{k+1}\Big\}, \\
&(b)\quad \|\mathbf{u}-\hat{\mathbf{u}}_h\|_{L^\infty(0,T;V)} + \|p-\hat{p}_h\|_{L^\infty(0,T;W)} \le M\Big\{h^{l+1}+h^{k+1}\Big\},
\end{aligned}
$$

where M *is independent of* h.

Proof. To obtain optimal L^2-norm error estimate, define the auxiliary function $(\tilde{\mathbf{u}}_h, \tilde{p}_h)$ such that

$$
(72)\qquad
\begin{cases}
(\alpha(c)\tilde{\mathbf{u}}_h, \mathbf{v}_h) - (\tilde{p}_h, \nabla\cdot\mathbf{v}_h) = 0, & \forall\, \mathbf{v}_h \in V_h, \\
(\nabla\cdot\tilde{\mathbf{u}}_h, w) = ((r_0-r_1)q, w_h), & \forall\, w_h \in W_h.
\end{cases}
$$

It is clear that

$$
(73)\qquad
\begin{cases}
(\alpha(c)(\mathbf{u}-\tilde{\mathbf{u}}_h), \mathbf{v}_h) - (p-\tilde{p}_h, \nabla\cdot\mathbf{v}_h) = 0, & \forall\, \mathbf{v}_h \in V_h, \\
(\nabla\cdot(\mathbf{u}-\tilde{\mathbf{u}}_h), w_h) = 0, & \forall\, w_h \in W_h
\end{cases}
$$

and

$$
(74)\qquad
\begin{cases}
(\alpha(\hat{c}_h)(\hat{\mathbf{u}}_h-\tilde{\mathbf{u}}_h), \mathbf{v}_h) - (\hat{p}_h-\tilde{p}_h, \nabla\cdot\mathbf{v}_h) = ((\alpha(c)-\alpha(\hat{c}_h))\tilde{\mathbf{u}}_h, \mathbf{v}_h), & \forall\, \mathbf{v}_h \in V_h, \\
(\nabla\cdot(\hat{\mathbf{u}}_h-\tilde{\mathbf{u}}_h), w_h) = 0, & \forall\, w_h \in W_h.
\end{cases}
$$

It follows from the results of [9] that

$$
(75)\qquad \|\mathbf{u}-\tilde{\mathbf{u}}_h\|_V + \|p-\tilde{p}_h\|_W \le Mh^{k+1}\|p\|_{H^{k+2}}
$$

and

$$
(76)\qquad \|\hat{\mathbf{u}}_h-\tilde{\mathbf{u}}_h\|_V + \|\hat{p}_h-\tilde{p}_h\|_W \le M\|c-\hat{c}_h\|_{L^2}.
$$

Next, let $\tilde{c}_h$ be the projection of c given by

$$
(77)\qquad (d(c)\nabla(c-\tilde{c}_h), \nabla z_h)) + (b(c)\mathbf{u}\cdot\nabla(c-\tilde{c}_h), z_h) + \lambda(c-\tilde{c}_h, z_h) = 0, \quad \forall\, z_h \in Z_h,
$$

where the $\lambda > 0$ is chosen to assure the coercivity of the bilinear form on the left-hand side of the above equality. Standard arguments show that, for any $t \in [0, T]$,

$$\begin{cases} \|c - \tilde{c}_h\|_{L^2} \leq Mh^{l+1}\|c\|_{H^{l+1}}, \\ \|\frac{\partial}{\partial t}(c - \tilde{c}_h)\|_{L^2} \leq Mh^{l+1}\Big\{\|c\|_{H^{l+1}} + \|\frac{\partial c}{\partial t}\|_{H^{l+1}}\Big\}, \end{cases} \tag{78}$$

where M is a generic constant depending on the L^∞-norm of $\dfrac{\partial \mathbf{u}}{\partial t}$ and $\mathbf{u}$.

We bound $c - \hat{c}_h$. Let $\xi = \hat{c}_h - \tilde{c}_h$ and $\eta = c - \tilde{c}_h$. It follows from (10) and (60) that

$$\begin{aligned} &(\phi\frac{\partial \xi}{\partial t}, z_h) + (b(\hat{c}_h)\mathbf{u} \cdot \nabla\xi, z_h) + (d(\hat{c}_h)\nabla\xi, \nabla z_h) \\ &\quad = (\phi\frac{\partial \eta}{\partial t}, z_h) - \lambda(\eta, z_h) - (b(\hat{c}_h)(\hat{\mathbf{u}}_h - \mathbf{u}) \cdot \nabla\hat{c}_h + (b(\hat{c}_h) - b(c))\mathbf{u} \cdot \nabla\tilde{c}_h, z_h) \\ &\quad - ((d(\hat{c}_h) - d(c))\nabla\tilde{c}_h, \nabla z_h) + ((f(\hat{c}_h) - f(c))r_0 q, z_h). \end{aligned} \tag{79}$$

Letting $z_h = \xi$, we estimate (79) term by term.

$$|(b(\hat{c}_h)\mathbf{u} \cdot \nabla\xi, \xi)| \leq \|b(\hat{c}_h)\mathbf{u}\|_{L^\infty}\|\nabla\xi\|_{L^2}\|\xi\|_{L^2}. \tag{80}$$

In order to estimate the next nonlinear items, we follow the standard induction technique that is widely used in the error analysis of the mixed finite element approximation of the state equations, see [9, 10]. Firstly make the induction assumption that there exits a constant h_0 such that:

$$h^{-1}\|\xi\|_{L^\infty(0,T;L^2)} \leq 1, \quad 0 < h \leq h_0. \tag{81}$$

Then using the inverse inequality ([7], [31]), we see

$$\begin{aligned} &|((b(\hat{c}_h)\hat{\mathbf{u}}_h - b(\hat{c}_h)\mathbf{u}) \cdot \nabla\hat{c}_h, \xi)| \\ &\leq |(b(\hat{c}_h)(\hat{\mathbf{u}}_h - \mathbf{u}) \cdot \nabla(\hat{c}_h - \tilde{c}_h), \xi)| + |(b(\hat{c}_h)(\hat{\mathbf{u}}_h - \mathbf{u}) \cdot \nabla\tilde{c}_h, \xi)| \\ &\leq M\|b(\hat{c}_h)\|_{L^\infty}\|\nabla\xi\|_{L^2}\|\mathbf{u} - \hat{\mathbf{u}}_h\|_{L^2}\|\xi\|_{L^\infty} + M\|b(\hat{c}_h)\|_{L^\infty}\|\nabla\tilde{c}_h\|_{L^\infty}\|\mathbf{u} - \hat{\mathbf{u}}_h\|_{L^2}\|\xi\|_{L^2} \\ &\leq M\|\nabla\xi\|_{L^2}\|\mathbf{u} - \hat{\mathbf{u}}_h\|_V h^{-1}\|\xi\|_{L^2} + M\|\nabla c\|_{L^\infty}\|\mathbf{u} - \hat{\mathbf{u}}_h\|_V\|\xi\|_{L^2}, \end{aligned} \tag{82}$$

Further, we have

$$|((b(\hat{c}_h) - b(c))\mathbf{u} \cdot \nabla\tilde{c}_h, \xi)| \leq M\|b'(c)\mathbf{u}\|_{L^\infty}\|\nabla\tilde{c}_h\|_{L^\infty}\|c - \hat{c}_h\|_{L^2}\|\xi\|_{L^2}, \tag{83}$$

$$|((d(c) - d(\hat{c}_h))\nabla\tilde{c}_h, \nabla\xi)| \leq M\|d'(c)\nabla\tilde{c}_h\|_{L^\infty}\|c - \hat{c}_h\|_{L^2}\|\nabla\xi\|_{L^2} \tag{84}$$

and

$$|((f(c) - f(\hat{c}_h))r_0 q, \xi)| \leq M\|q\|_{L^\infty}\|c - \hat{c}_h\|_{L^2}\|\xi\|_{L^2}. \tag{85}$$

By using ε-inequality and Sobelov inequality, we have the inequality

$$\frac{d}{dt}(\phi\xi, \xi) + \|\nabla\xi\|_{L^2}^2 \leq M\Big\{\|\xi\|_{L^2}^2 + \|\mathbf{u} - \hat{\mathbf{u}}\|_V^2 + \|\eta\|_{L^2}^2 + \|\frac{\partial \eta}{\partial t}\|_{L^2}^2\Big\}. \tag{86}$$

Using the Gronwall lemma gives

$$\|\xi\|_{L^\infty(0,T;L^2)} \leq M\Big\{h^{l+1} + h^{k+1}\Big\}. \tag{87}$$

Now we turn to proving the induction hypothesis (81) for $0 < h \leq h_0$, where h_0 is the sufficiently small so that $M(h_0^k + h_0^l) < 1$. Noting that $\xi(0) = 0$, we have $\|\xi(0)\|_{L^2} = 0$. Thus, there exist some $0 < t^* \leq T$, such that

$$h^{-1}\|\xi\|_{L^\infty(0,t^*;L^2)} \leq 1.$$

Let $T^* = \max\{t^*; h^{-1}\|\xi\|_{L^\infty(0,t^*;L^2)} \le 1\}$. We will prove $T^* = T$, which implies the induction hypothesis (81) to be true. Assume $T^* < T$. It follows from the proof of (87) that

$$\|\xi\|_{L^\infty(0,T^*;L^2)} \le M\Big\{h^{l+1} + h^{k+1}\Big\}$$

such that

$$h^{-1}\|\xi\|_{L^\infty(0,T^*;L^2)} \le M\Big\{h^l + h^k\Big\} \le M\Big\{h_0^l + h_0^k\Big\} < 1, \quad 0 < h \le h_0.$$

Particularly, $h^{-1}\|\xi(T^*)\|_{L^2} < 1$. Thus there exist some $T^* < t^* \le T$ such that

$$h^{-1}\|\xi\|_{L^\infty(0,t^*;L^2)} \le 1, \quad 0 < h \le h_0,$$

which conflicts with the definition of T^*. So the assumption $T^* < T$ is false. □

Lemma 5.3. *Let $(c^*, \mathbf{u}^*, p^*)$ and $(\hat{c}_h^*, \hat{\mathbf{u}}_h^*, \hat{p}_h^*)$ be the solutions of* (35) *and* (66) *respectively. There exits a constant $h_0 > 0$ such that for $0 < h \le h_0$,*

$$\begin{aligned} &(a)\ \ \|c^* - \hat{c}_h^*\|_{L^\infty(0,T;L^2)} \le M\Big\{h^{l+1} + h^{k+1}\Big\}, \\ &(b)\ \ \|\mathbf{u}^* - \hat{\mathbf{u}}_h^*\|_{L^\infty(0,T;V)} + \|p^* - \hat{p}_h^*\|_{L^\infty(0,T;W)} \le M\Big\{h^{l+1} + h^{k+1}\Big\}, \end{aligned} \tag{88}$$

where M is independent of h.

Proof. Introduce $(\tilde{\mathbf{u}}_h^*, \tilde{p}_h^*)$ such that

$$\begin{cases} (\alpha(c)\tilde{\mathbf{u}}_h^*, \mathbf{v}_h) - (\tilde{p}_h^*, \nabla\cdot\mathbf{v}_h) = -(c^* b(c)\nabla c, \mathbf{v}_h), & \forall\ \mathbf{v}_h \in V_h, \\ (\nabla\cdot\tilde{\mathbf{u}}_h^*, w_h) = 0, & \forall\ w_h \in W_h. \end{cases} \tag{89}$$

It follows from the results of [9] that

$$\|\tilde{\mathbf{u}}_h^* - \mathbf{u}^*\|_V + \|\tilde{p}_h^* - p^*\|_W \le M\Big\{\inf_{\mathbf{v}_h\in V_h}\|\mathbf{u}^* - \mathbf{v}_h\|_V + \inf_{w_h\in W_h}\|p^* - w_h\|_W\Big\}. \tag{90}$$

From (89), we see

$$\begin{cases} ((\tilde{\mathbf{u}}_h^* - \hat{\mathbf{u}}_h^*)\alpha(\hat{c}_h), \mathbf{v}_h) - (\tilde{p}_h^* - \hat{p}_h^*, \nabla\cdot\mathbf{v}_h) = (\hat{c}_h^* b(\hat{c}_h)\nabla\hat{c}_h - c^* b(c)\nabla c, \mathbf{v}_h) \\ \qquad + ((\alpha(\hat{c}_h) - \alpha(c))\tilde{\mathbf{u}}_h^*, \mathbf{v}_h), \\ (\nabla\cdot(\tilde{\mathbf{u}}_h^* - \hat{\mathbf{u}}_h^*), w_h) = 0. \end{cases} \tag{91}$$

Noting that

$$(\hat{c}_h^* b(\hat{c}_h)\nabla(\hat{c}_h - c), \mathbf{v}_h) = -((\hat{c}_h - c)\nabla(\hat{c}_h^* b(\hat{c}_h)), \mathbf{v}_h) - (\hat{c}_h^* b(\hat{c}_h)(\hat{c}_h - c), \nabla\cdot\mathbf{v}_h)$$

such that

$$\begin{aligned} |(\hat{c}_h^* b(\hat{c}_h)\nabla\hat{c}_h - c^* b(c)\nabla c, \mathbf{v}_h)| &\le |(\hat{c}_h^* b(\hat{c}_h)\nabla(\hat{c}_h - c), \mathbf{v}_h)| \\ &\quad + |(\hat{c}_h^*(b(\hat{c}_h) - b(c))\nabla c, \mathbf{v}_h)| + |((\hat{c}_h^* - c^*)b(c)\nabla c, \mathbf{v}_h)| \\ &\le M\big(\|b'(\hat{c}_h)\nabla\hat{c}_h\|_{L^\infty}\|\hat{c}_h^*\|_{L^\infty} + \|b(\hat{c}_h)\|_{L^\infty}\|\hat{c}_h^*\|_{W^{1,\infty}}\big)\|\hat{c}_h - c\|_{L^2}\|\mathbf{v}_h\|_V \\ &\quad + M\|\hat{c}_h^*\|_{H^1}\|b'(c)\nabla c\|_{L^\infty}\|c - \hat{c}_h\|_{L^2}\|\mathbf{v}_h\|_{L^2} \\ &\quad + M\|b(c)\nabla c\|_{L^\infty}\|c^* - \hat{c}_h^*\|_{L^2}\|\mathbf{v}_h\|_{L^2}, \end{aligned} \tag{92}$$

and from the result of [9], we have

$$\|\tilde{\mathbf{u}}_h^* - \hat{\mathbf{u}}_h^*\|_V + \|\tilde{p}_h - \hat{p}_h^*\|_W \le M\Big\{\|c - \hat{c}_h\|_{L^2} + \|c^* - \hat{c}_h^*\|_{L^2}\Big\}. \tag{93}$$

Now, let us turn to estimating $\hat{c}_h^* - c^*$. Here we recast the adjoint system:

(94)
$$\begin{cases} -(\phi\frac{\partial \hat{c}_h^*}{\partial t}, z_h) - ((b(\hat{c}_h)\hat{\mathbf{u}}_h - d'(\hat{c}_h)\nabla\hat{c}_h)\cdot\nabla\hat{c}_h^*, z_h) + (d(\hat{c}_h)\nabla\hat{c}_h^*, \nabla z_h) \\ \quad + (\alpha'(\hat{c}_h)\hat{\mathbf{u}}_h^*\cdot\hat{\mathbf{u}}, z_h) + (r_1 q b(\hat{c}_h)c^*, z_h) = (\omega\hat{c}_h, z_h), & \forall\, z_h \in Z_h, \\ (\hat{c}_h^* b(\hat{c}_h)\nabla\hat{c}_h, \mathbf{v}_h) - (\hat{p}*, \nabla\cdot\mathbf{v}_h) + (\alpha(\hat{c}_h)\hat{\mathbf{u}}_h^*, \mathbf{v}_h) = 0, & \forall\, \mathbf{v}_h \in V_h, \\ (\nabla\cdot\hat{\mathbf{u}}_h^*, w_h) = 0, & \forall\, w_h \in W_h. \end{cases}$$

Similar to the proof in Lemma 5.2, we also define $\tilde{c}_h^*$ which satisfies

(95)
$$-((b(c)\mathbf{u} - d'(c)\nabla c)\cdot(\nabla c^* - \nabla\tilde{c}_h^*), z_h) + (d(c)(\nabla c^* - \nabla\tilde{c}_h^*), \nabla z_h) + \lambda(c^* - \tilde{c}_h^*, z_h) = 0,$$

where the $\lambda > 0$ is chosen to ensure the coercivity too. Then, we can also obtain

$$\begin{cases} \|c^* - \tilde{c}_h^*\|_{L^2} \le Mh^{l+1}\|c^*\|_{H^{l+1}}, \\ \|\frac{\partial}{\partial t}(c^* - \tilde{c}_h^*)\|_{L^2} \le Mh^{l+1}\Big\{\|c^*\|_{H^{l+1}} + \|\frac{\partial c^*}{\partial t}\|_{H^{l+1}}\Big\}. \end{cases} \tag{96}$$

Furthermore, letting $\xi = \hat{c}_h^* - \tilde{c}_h^*$ and $\eta = c^* - \tilde{c}_h^*$, we derive that

(97)
$$\begin{aligned} &-(\phi\frac{\partial \xi}{\partial t}, z_h) - ((b(\hat{c}_h)\mathbf{u} - d'(\hat{c})\nabla c)\cdot\nabla\xi, z_h) + (d(\hat{c}_h)\nabla\xi, \nabla z_h) + (r_0 q b(\hat{c}_h)\xi, z_h) \\ &\quad = -(\phi\frac{\partial \eta}{\partial t}, z_h) - (\lambda\eta, z_h) + ((b(\hat{c}_h)\hat{\mathbf{u}}_h - b(\hat{c}_h)\mathbf{u})\cdot\nabla\hat{c}_h^*, z_h) \\ &\qquad + ((b(\hat{c}_h)\mathbf{u} - b(c)\mathbf{u})\cdot\nabla\tilde{c}_h^*, z_h) - ((d'(\hat{c}_h)\nabla\hat{c}_h - d'(\hat{c}_h)\nabla c)\cdot\nabla\hat{c}_h^*, z_h) \\ &\qquad - ((d'(\hat{c}_h)\nabla c - d'(c)\nabla c)\cdot\nabla\tilde{c}_h^*, z_h) - ((d(\hat{c}_h) - d(c))\nabla\tilde{c}_h^*, \nabla z_h) \\ &\qquad + (\omega(\hat{c}_h - c), z_h) + (\alpha'(c)\mathbf{u}^*\cdot\mathbf{u}, z_h) - (\alpha'(\hat{c}_h)\hat{\mathbf{u}}_h^*\cdot\hat{\mathbf{u}}_h, z_h) \\ &\qquad + (r_0 q(b(\hat{c}_h) - b(c))\tilde{c}_h^*, z_h), \quad \forall\, z_h \in Z_h. \end{aligned}$$

Choosing $z_h = \xi$, we can carry out the estimates term by term. Assuming that there exists a constant $h_0 > 0$ such that

$$h^{-1}\|\xi\|_{L^\infty(0,T;L^2)} \le 1, \quad 0 < h \le h_0. \tag{98}$$

Similarly as the proofs of (80)-(85), we obtain

(99)
$$\begin{aligned} &|((b(\hat{c}_h)\mathbf{u} - d'(\hat{c})\nabla c)\cdot\nabla\xi, \xi)| \le \|(b(\hat{c}_h)\mathbf{u} - d'(\hat{c})\nabla c)\|_{L^\infty}\|\nabla\xi\|_{L^2}\|\xi\|_{L^2}, \\ &|((b(\hat{c}_h)\hat{\mathbf{u}}_h - b(\hat{c}_h)\mathbf{u})\cdot\nabla\hat{c}_h^*, \xi)| \\ &\quad \le M\|\nabla\xi\|_{L^2}\|\mathbf{u} - \hat{\mathbf{u}}_h\|_V h^{-1}\|\xi\|_{L^2} + M\|\nabla c^*\|_{L^\infty}\|\mathbf{u} - \hat{\mathbf{u}}_h\|_V\|\xi\|_{L^2}, \\ &|((b(\hat{c}_h)\mathbf{u} - b(c)\mathbf{u})\cdot\nabla\tilde{c}_h^*, \xi)| \le M\|b'(c)\mathbf{u}\|_{L^\infty}\|\nabla c^*\|_{L^2}\|c - \hat{c}_h\|_{H^1}\|\xi\|_{H^1}, \\ &|((d'(\hat{c}_h)\nabla c - d'(c)\nabla c)\cdot\nabla\tilde{c}_h^*, \xi)| \le M\|d''(c)\nabla c\|_{L^\infty}\|\nabla c^*\|_{L^2}\|c - \hat{c}_h\|_{H^1}\|\xi\|_{H^1} \end{aligned}$$

and

(100)
$$\begin{aligned} &|((d'(\hat{c}_h)\nabla\hat{c}_h - d'(\hat{c}_h)\nabla c)\cdot\nabla\hat{c}_h^*, \xi)| \\ &\le |(\nabla(d(\hat{c}_h) - d(c))\cdot\nabla\hat{c}_h^*, \xi)| + |((d'(\hat{c}_h) - d'(c))\nabla c\cdot\nabla\hat{c}_h^*, \xi)| \\ &\le |((d(\hat{c}_h) - d(c))\Delta\hat{c}_h^*, \xi)| + |((d(\hat{c}_h) - d(c))\nabla\hat{c}_h^*, \nabla\xi)| + |((d'(\hat{c}_h) - d'(c))\nabla c\cdot\nabla\hat{c}_h^*, \xi)| \\ &\le M\|c - \hat{c}_h\|_{L^2}\|\xi\|_{H^1}. \end{aligned}$$

Moreover, we get

$$
\begin{aligned}
&|(\alpha'(c)\mathbf{u}^*\cdot\mathbf{u},\xi)-(\alpha'(\hat{c}_h)\hat{\mathbf{u}}_h^*\cdot\hat{\mathbf{u}}_h,\xi)| \\
&\le |((\alpha'(c)-\alpha'(\hat{c}_h))\mathbf{u}^*\cdot\mathbf{u},\xi)|+|(\alpha'(\hat{c}_h)(\mathbf{u}^*-\hat{\mathbf{u}}_h^*)\cdot\mathbf{u},\xi)| \\
&\quad +|(\alpha'(\hat{c}_h)\hat{\mathbf{u}}_h^*\cdot(\mathbf{u}-\hat{\mathbf{u}}_h),\xi)| \\
&\le M\Big\{\|\mathbf{u}\|_V\|\mathbf{u}^*\|_V\|\hat{c}_h-c\|_{L^2}\|\xi\|_{H^1}+\|\mathbf{u}\|_V\|\hat{\mathbf{u}}_h^*-\mathbf{u}^*\|_V\|\xi\|_{L^2} \\
&\quad +\|\hat{\mathbf{u}}_h-\mathbf{u}\|_V\|\xi\|_{H^1}\|\hat{\mathbf{u}}_h^*\|_{L^2}\Big\}.
\end{aligned}
\tag{101}
$$

Treating other terms as in the proofs in Lemma 5.2, then we also obtain that

$$
\begin{aligned}
&-\frac{d}{dt}(\phi\xi,\xi)+(d_*\nabla\xi,\nabla\xi) \\
&\le M\Big\{\|\xi\|_{L^\infty}^2\|c-\hat{c}_h\|_{L^2}^2+\|\xi\|_{L^2}^2+\|\mathbf{u}^*-\hat{\mathbf{u}}_h^*\|_V^2+\|\eta\|_{L^2}^2+\|\frac{\partial\eta}{\partial t}\|_{L^2}^2\Big\} \\
&\le M\Big\{(1+h^{-2}\|\xi\|_{L^2}^2)\|c-\hat{c}_h\|_{L^2}^2+\|\xi\|_{L^2}^2+h^{2l+2}+h^{2k+2}\Big\}.
\end{aligned}
\tag{102}
$$

It follows from (102) and the Gronwall lemma that

$$
\|\xi\|_{L^\infty(0,T;L^2)}\le e^{2MT}\Big\{M(h^{2l+2}+h^{2k+2})+\|\xi(T)\|_{L^2}^2\Big\},
\tag{103}
$$

and thus

$$
\|\hat{c}_h^*-c^*\|_{L^\infty(0,T;L^2)}\le M\Big\{h^{l+1}+h^{k+1}\Big\}.
\tag{104}
$$

Then, it leads to

$$
\|\hat{\mathbf{u}}_h^*-\mathbf{u}^*\|_{L^2(0,T;V)}\le M\Big\{h^{l+1}+h^{k+1}\Big\}.
\tag{105}
$$

The proof of the induction hypothesis (98) is similar to that in Lemma 5.2. □

Applying Lemmas 5.1 - 5.3, we have the following convergence results.

Theorem 5.1. *Assume that all the assumptions in Lemmas* 5.1 − 5.3 *hold for sufficient small* h*. There holds the a priori error estimate*

$$
\|q-q_h\|_{L^2(0,T)}\le M\Big\{h^{l+1}+h^{k+1}\Big\}.
\tag{106}
$$

Theorem 5.2. *Let* $(c,q,\mathbf{u},p,c^*,\mathbf{u}^*,p^*)$ *and* $(c_h,q_h,\mathbf{u}_h,p_h,c_h^*,\mathbf{u}_h^*,p_h^*)$ *be the solution of* (35) *and* (62) *respectively. Suppose that all the assumptions in Lemmas* 5.1 − 5.3 *hold. There exists a constant* $h_0>0$ *such that for* $0<h\le h_0$*, there hold the following estimates*

$$
\|c-c_h\|_{L^\infty(0,T;L^2)}+\|\mathbf{u}-\mathbf{u}_h\|_{L^\infty(0,T;V)}+\|p-p_h\|_{L^\infty(0,T;W)}\le M\Big\{h^{l+1}+h^{k+1}\Big\}
\tag{107}
$$

and

$$
\|c^*-c_h^*\|_{L^\infty(0,T;L^2)}+\|\mathbf{u}^*-\mathbf{u}_h^*\|_{L^\infty(0,T;V)}+\|p^*-p_h^*\|_{L^\infty(0,T;W)}\le M\Big\{h^{l+1}+h^{k+1}\Big\}.
\tag{108}
$$

Proof. Standard arguments show the following estimates

$$
\|c-c_h\|_{L^\infty(0,T;L^2)}\le\|c-\tilde{c}_h\|_{L^\infty(0,T;L^2)}+\|\tilde{c}_h-c_h\|_{L^\infty(0,T;L^2)}
$$

and

$$
\|c^*-c_h^*\|_{L^\infty(0,T;L^2)}\le\|c^*-\tilde{c}_h^*\|_{L^\infty(0,T;L^2)}+\|\tilde{c}_h^*-c_h^*\|_{L^\infty(0,T;L^2)}.
$$

As to u and p, the similar estimates also hold, so we omit the details to avoid repeating.

Now, based on the results above, we only need to estimate $\|\tilde{c}_h - c_h\|_{L^\infty(0,T;L^2)}$ and $\|\tilde{c}_h^* - c_h^*\|_{L^\infty(0,T;L^2)}$. Define $\xi = c_h - \tilde{c}_h$, $\eta = c - \tilde{c}_h$. It follows from (10), (60) and (77) that we have

(109)
$$\begin{aligned}&(\phi\frac{\partial \xi}{\partial t}, z_h) + (b(c_h)\mathbf{u}\cdot\nabla\xi, z_h) + (d(c_h)\nabla\xi, \nabla z_h)\\ &= (\phi\frac{\partial \eta}{\partial t}, z_h) - (\eta\lambda, z_h) - ((b(c_h)\mathbf{u}_h - b(c_h)\mathbf{u})\cdot\nabla c_h + (b(c_h)\mathbf{u} - b(c)\mathbf{u})\cdot\nabla\tilde{c}_h, z_h)\\ &\quad -((d(c_h) - d(c))\nabla\tilde{c}_h, \nabla z_h) + ((f(c_h) - f(c))r_0 q, z_h) + (f(c_h)r_0(q_h - q), z_h).\end{aligned}$$

Similarly as (93), we can obtain the following estimates:

$$\|\mathbf{u} - \mathbf{u}_h\|_V + \|p - p_h\|_W \le M\Big\{\|c - c_h\|_{L^2} + |q - q_h|\Big\}. \tag{110}$$

Following the proof of Lemma 5.2, we have

(111)
$$\frac{d}{dt}(\phi\xi, \xi) + (d_*\nabla\xi, \nabla\xi) \le M\Big\{\|\xi\|_{L^2}^2 + \|\mathbf{u} - \mathbf{u}_h\|_V^2 + \|\eta\|_{L^2}^2 + \|\frac{\partial\eta}{\partial t}\|_{L^2}^2 + |q - q_h|^2\Big\},$$

where we used the induction hypothesis

$$h^{-1}\|\xi\|_{L^\infty(0,T;L^2)} \le 1, \tag{112}$$

whose proof is similar to that in Lemma 5.2.

Combining (111) with the result of Theorem 5.1 and using the Gronwall Lemma, we have

$$\|\xi\|_{L^\infty(0,T;L^2)} \le M\Big\{h^{l+1} + h^{k+1}\Big\}. \tag{113}$$

Then (107) can be proved.

Similarly, we can obtain the second estimate (108) of Theorem 5.2. □

6. Numerical Simulations

In this section, we present some numerical experiments to have some initial understanding on the optimal water injection plans. The numerical experiments simulate immiscible displacement within a horizontal reservoir of one unit thickness.

We first recall the object functional as below:

$$J(q) = \frac{1}{2}\int_0^T\int_\Omega \omega(x,t)c^2 + \frac{\alpha}{2}\int_0^T q^2. \tag{114}$$

Without losing generality, we suppose the domain is $\Omega = [0,1]\times[0,1]$, the injection well is located at the upper right corner (1,1), and the production well is located at the lower left corner (0, 0). In our computations we assume the porosity of the rock is a constant, which means the porous medium is homogeneous. The parameters are re-scaled (see [32] and [33]) from a real problem to have similar characterizations compared with real situations. For example these parameters show the convection dominated property of the concentration equation. In summary we set the parameters as follows:

the initial concentration of the oil is $c_0(x) = 0.6$,
the permeability of the porous rock $k(x) = 0.5$,
the porosity of the rock is supposed to be $\phi(x) = 0.3$,
the viscosity of the oil and the water in the reservoir are 16 and 0.5 respectively,
the derivative of the capillary pressure is 0.1,
the water price index is taken as $\alpha = 0.01$,
let $\tilde{\omega}/\varepsilon = w$: the oil price index changes from $w = 4$ to $w = 4.0e+03$.

It is clear that a (relatively) lower price of water will cause a larger amount of water injection, which makes the flow very convection-dominated and thus causes much computational difficulty.

Example 1. In this example, we try to find out the regular pattern of the optimal water injection curves with different price fraction w/α when the injection period T is fixed ($T = 1, 3, 5, 8, 10$). The optimal injection curves are shown in the following figures (x-axis indicates time, and y-axis presents water injection rate) .

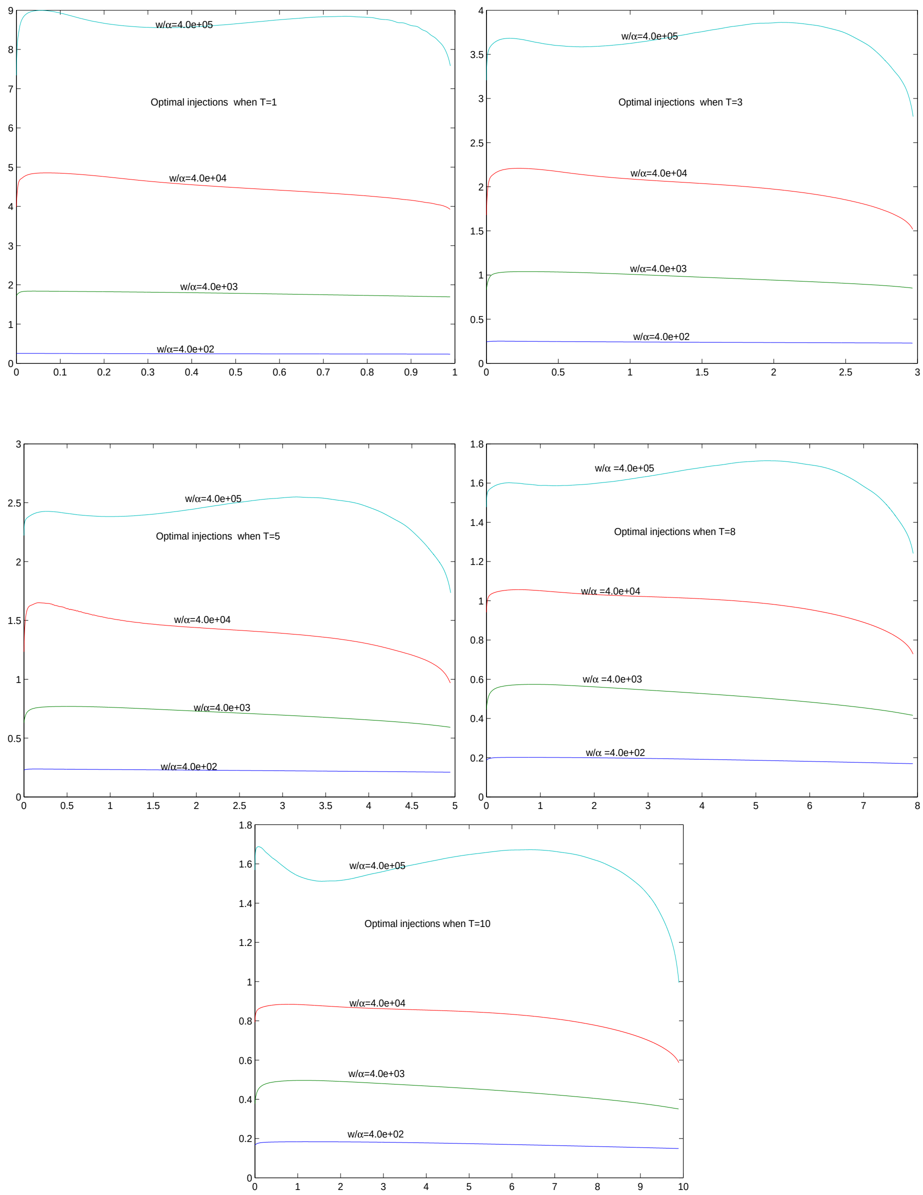


Figure 1. The optimal injection curves with fixed injection period for $T = 1, 3, 5, 8, 10$

From Figure 1 it is interesting to observe that when the price fraction is low, the optimal injection curves are very flat like a quasi-constant water injection plan that is widely used in practice. This may explain its popularity. However when the fraction is higher, the curves fluctuated more.

For a fixed injection period T, the higher the price fraction, the higher the optimal injection curves go. This indeed agrees with the rules used in real practice. This is clear sensible when water is more expensive than oil there is no need to inject any water. In practical, this case does exist, such as in the period of the global financial crises in 2009, the international oil price greatly dropped, many wells in China stopped injecting water from economic benefit consideration.

From Figure 1 we also observe that during a very short period at the beginning of injection, the injection rate of the optimal injection plan rapidly increases from a lower value to a high one. The possible explanation is that the initial concentration of oil is high, and the relative permeability of water is low, so that the water flow has a poor translation. As some water is injected into the injection well, the relative permeability of water near the injection well goes up rapidly, and consequently more water is allowed to be injected.

There is another interesting phenomenon in Figure 1. As we can see when the price fraction $w/\alpha = 4.0e+05$, the pattern of the optimal injection curve changes compared with the other cases the curve slowly rises after a slow falling. After comparing the concentration's changes at different time, we found the slowly rising started when the water injected in the injection well flowed out from the production well.

From Figure 2, we know as the decrease of the oil's concentration, the convection diffusion becomes definite.

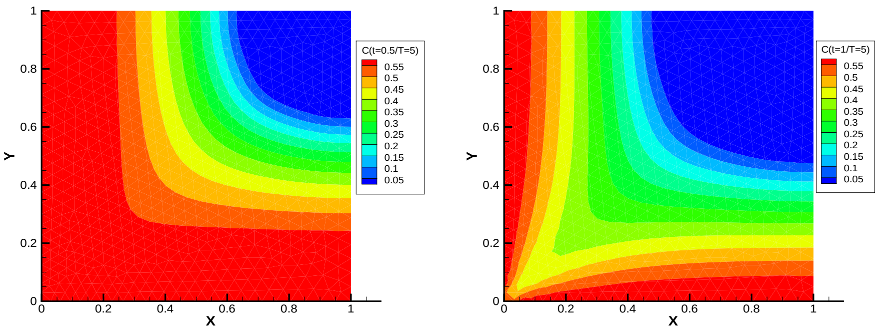
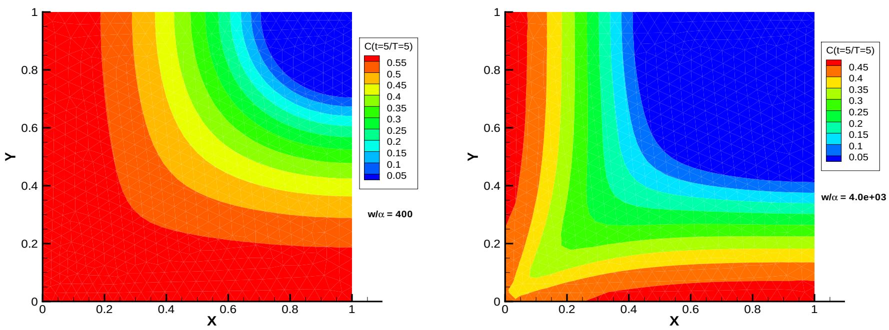


Figure 2. The oil concentration distribution at an early time and an later time.

Below we also present some corresponding concentration distributions for T=5,10 as shown in the following figures.



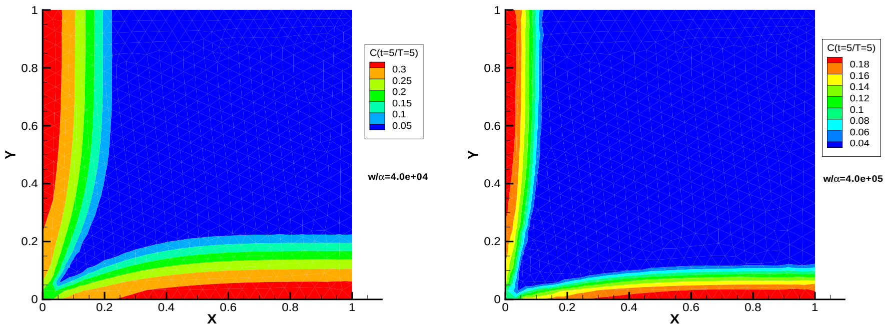


Figure 3. The final concentration distributions with different price fractions when T=5

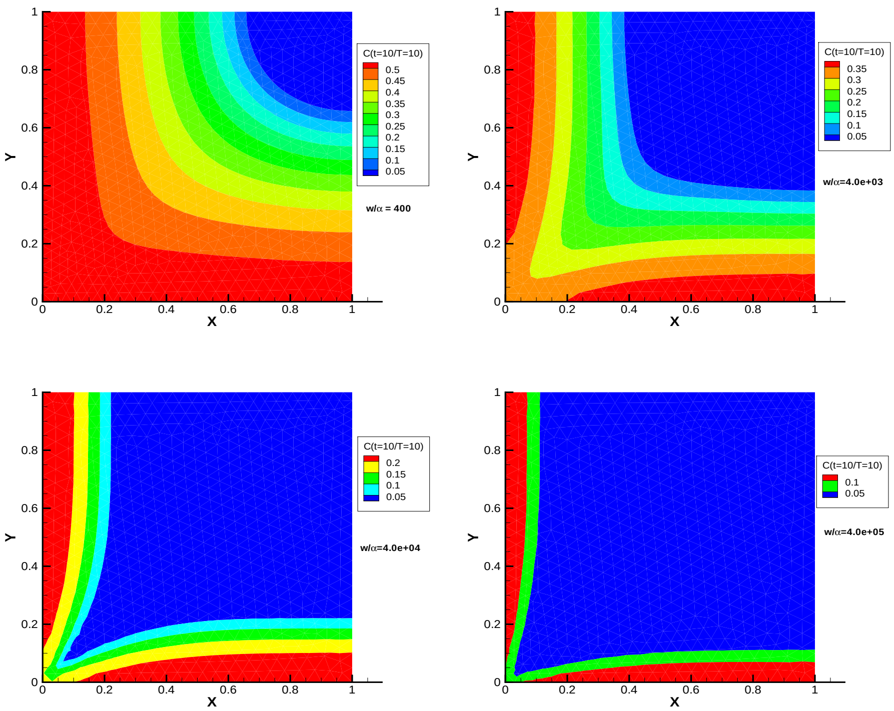


Figure 4. The final concentration distributions with different price fractions when T=10

Example 2. In this numerical experiment, we compare the optimal water injection plans we obtained with other water injection plans. Since in practice often a quasi-constant rate injection plan is used, here we just compare the objective functional values of the optimal injection plans, and the constant injection plans. The results are shown in the following graphs for a fix the price fraction w/α:

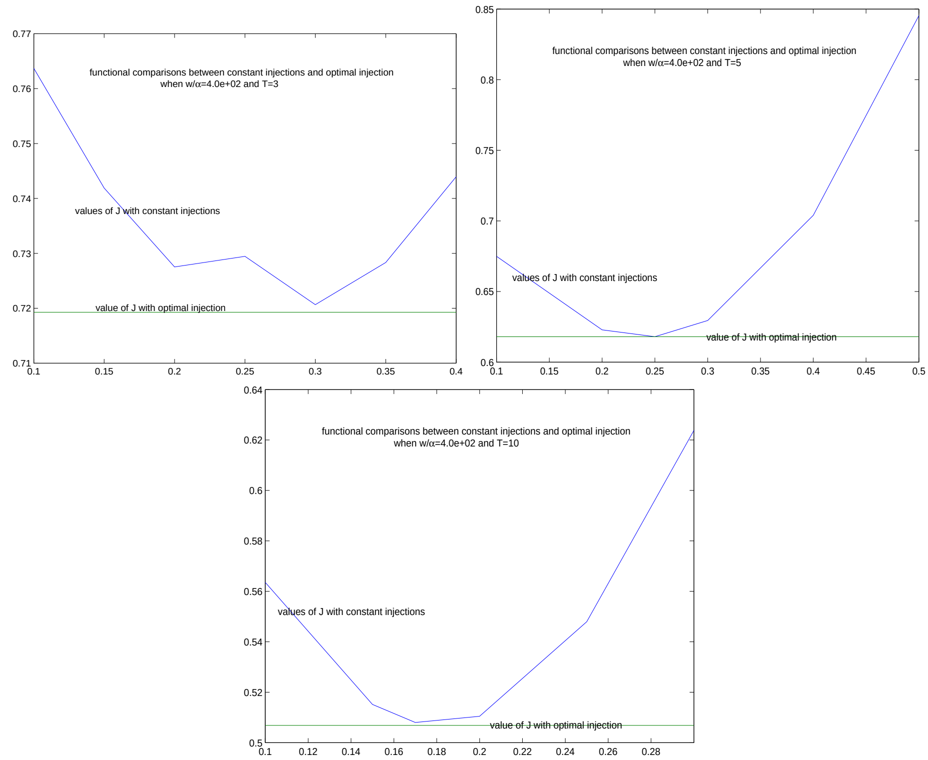


Figure 5. The functional comparisons for $T = 3,\ 5,\ 10$ when $w/\alpha = 4.0e + 02$

In these graphs the horizontal line indicates the optimal value of the objective functional for the given the price fraction w/α, while the curves above the line indicate the functional values for a constant injection plan at the given water injection rate on the x-axis. It is clear that a suitable constant water injection plan could achieve a very good approximation to the optimal value for some cases. That may explain the wide use of quasi-constant water injection plans, besides its simple operation benefit. However the non-global-convex nature of the underlying optimal control does appear from time to time, when there are multi-minimizers, and gaps between the objective values of the optimal and a constant water injection plans. Let us note that for such applications, even very marginal improvement over the minimum functional values, could bring huge economic benefit.

From the results shown in Figure 5, it is clear that when we fix the price fraction w/α, the longer the injection period T is, the lower value of the functional J is. The possible reason is that when we have a longer T, obviously we can inject the water slowly to sweep the oil out of the oil reservoir more completely. Otherwise, we may need to keep a large water injection for the case of shorter T, then water channeling would happen, and that will greatly reduce effectiveness of the water injection.

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