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CONVERGENCE ANALYSIS OF YEE SCHEMES FOR MAXWELL'S EQUATIONS IN DEBYE AND LORENTZ DISPERSIVE MEDIA

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Abstract. We present discrete energy decay results for the Yee scheme applied to Maxwell's equations in Debye and Lorentz dispersive media. These estimates provide stability conditions for the Yee scheme in the corresponding media. In particular, we show that the stability conditions are the same as those for the Yee scheme in a nondispersive dielectric. However, energy decay for the Maxwell-Debye and Maxwell-Lorentz models indicate that the Yee schemes are dissipative. The energy decay results are then used to prove the convergence of the Yee schemes for the dispersive models. We also show that the Yee schemes preserve the Gauss divergence laws on its discrete mesh. Numerical simulations are provided to illustrate the theoretical results.

Key words. Maxwell's equations, Debye, Lorentz dispersive materials, Yee, FDTD method, energy decay, convergence analysis.

1. Introduction

The Yee scheme is a finite difference time domain (FDTD) numerical technique for the discretization of Maxwell's equations in a non-dispersive medium such as free space. It was first presented in [35]. The Yee scheme was extended to discretize Maxwell's equations in linear dispersive media and analyzed in a series of papers [4, 9, 13, 18, 19, 20, 31] involving dispersive media models such as the Debye [14, 20], Lorentz [18, 30], cold plasma [13, 36] and Cole-Cole [9, 12] models among others. Fourier analysis of the Yee scheme in such dispersive media (see for e.g. [4, 31]) indicate that the Yee scheme is stable under the same stability condition as that in a corresponding (having the same relative permittivity) non-dispersive dielectric. However, the Yee scheme in dispersive media is dissipative, unlike its counterpart in a non-dispersive, non-conductive medium, and in addition is more dispersive [5, 32]. The time step in the Yee scheme needs to be chosen to resolve all the time scales associated with a particular dispersive medium such as relaxation times, resonance times, and incident wave periods [32]. Maxwell's equations in such media have been shown to constitute a stiff problem and the time step needed to resolve waves in the numerical grid can be extremely small [32]. Research on the construction and analysis of Yee type finite difference time domain methods for Maxwell's equations in dispersive media is an area of active interest. We refer the reader to the book [33] and the numerous references therein for an introduction to the Yee scheme and its properties.

In this paper we present for the first time an analysis of the Yee scheme in Debye (Maxwell-Debye) and Lorentz (Maxwell-Lorentz) media by deriving energy decay results that indicate the conditional stability and dissipative nature of the schemes. We also present a full convergence analysis of the Yee schemes for the Maxwell-Debye and Maxwell-Lorentz models using the derived energy decay results. Energy methods based on variational techniques for analyzing stability

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and convergence properties of the Yee scheme in a lossy non-dispersive medium and operator splitting FDTD techniques have recently been published in the literature, see for example [7, 10, 15]. Finite element methods (FEM) and discontinuous Galerkin (DG) methods for Maxwell's equations in various dispersive media have also recently been published, for example see [1, 16, 21, 22, 23, 24, 25, 26, 34] and references therein.

We construct exact solutions based on numerical dispersion relations for the Maxwell-Debye and Maxwell-Lorentz models which are useful in understanding the decay of discrete energies in numerical methods for these models. We use these exact solutions to illustrate our stability and convergence analyses in our numerical simulations of the Yee schemes.

The outline of the paper is as follows. In Section 2 we present two dispersive media models and construct the Maxwell-Debye model and Maxwell-Lorentz model in two dimensions. We recall energy decay results for these models from the literature [23]. In Section 3 we outline the discrete meshes and spaces that the electric, magnetic and polarization fields are discretized on and establish discrete curl operators and their properties. In Sections 4 and 5 we recall the Yee schemes for the Maxwell-Debye and the Maxwell-Lorentz models, respectively. For both models we show that the corresponding Yee schemes are second-order accurate in time, establish discrete energy decay results and prove the conditional convergence of the corresponding Yee schemes. In addition, we show that these schemes satisfy the Gauss divergence laws on the discrete Yee mesh. Numerical simulations based on exact solutions are presented in Sections 6 and 7 that illustrate the stability and convergence analyses. Finally, conclusions are made in Section 8.

2. Maxwell's Equations in Dispersive Dielectrics

We consider Maxwell's equations which govern the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ in a domain $\Omega \subset \mathbb{R}^3$ from time 0 to T given as

$$\frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} = \mathbf{0} \text{ in } \Omega \times (0,T), \tag{2.1a}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0} \text{ in } \Omega \times (0,T), \tag{2.1b}$$

$$\nabla \cdot \mathbf{D} = 0 = \nabla \cdot \mathbf{B} \text{ in } \Omega \times (0,T), \tag{2.1c}$$

$$\mathbf{n} \times \mathbf{E} = \mathbf{0} \text{ on } \partial\Omega \times (0,T), \tag{2.1d}$$

$$\mathbf{E}(0,\mathbf{x}) = \mathbf{E}_0;\ \mathbf{H}(0,\mathbf{x}) = \mathbf{H}_0 \text{ in } \Omega. \tag{2.1e}$$

The fields $\mathbf{D}, \mathbf{B}$ are the electric and magnetic flux densities respectively. On the boundary, $\partial\Omega$, we impose a perfect conducting (PEC) boundary condition (2.1d), where the vector $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$. Lastly, we add initial conditions (2.1e) to the system.

Within the dielectric medium we have constitutive relations that relate the flux densities $\mathbf{D}, \mathbf{B}$ to the electric and magnetic fields, respectively, as

$$\mathbf{D} = \epsilon_0 \epsilon_\infty \mathbf{E} + \mathbf{P}, \tag{2.2a}$$

$$\mathbf{B} = \mu_0 \mathbf{H}, \tag{2.2b}$$

where the constants ϵ_0 and μ_0 are the permittivity and permeability of free space, and are connected to the speed of light in vacuum, c_0, by $c_0 = 1/\sqrt{\epsilon_0 \mu_0}$. The quantity $\mathbf{P}$ is called the electric (macroscopic) relaxation polarization, and the coefficient ϵ_∞ is called the infinite frequency relative permittivity.

The constitutive law (2.2a) incorporates the effects of electric polarization, which is defined as the electric field induced disturbance of the charge distribution in a region [2]. This polarization may have an instantaneous component as well as ones that do not occur instantaneously. The relaxation polarization $\mathbf{P}$ is the non-instantaneous part of the electric polarization and usually has associated time constants [2]. The presence of instantaneous polarization is accounted for by the coefficient ϵ_∞ in the constitutive relation (2.2a). We neglect any additional magnetic effects and assume that the magnetic constitutive relation (2.2b) for free space is also valid in the dispersive medium.

To describe the behavior of the media's macroscopic electric polarization $\mathbf{P}$, a general integral equation model is employed in which the polarization explicitly depends on the past history of the electric field [2]. The resulting constitutive law can be given in terms of a convolution involving a displacement susceptibility kernel g as

$$\mathbf{P}(t,\mathbf{x}) = \int_0^t g(t-s,\mathbf{x})\mathbf{E}(s,\mathbf{x})ds, \tag{2.3}$$

inside the dielectric. Here, we consider polarization mechanisms for which, in the time domain, the convolution (2.3) describing the polarization can be converted to an ordinary differential equation (ODE) or systems of ODEs governing the evolution of the relaxation polarization driven by the electric field [2]. In particular, we consider two popular models: the Debye model [14] for orientational polarization and the Lorentz model [30] for electronic polarization.

2.1. Debye Media: Model and Energy estimates. To model wave propagation in polar materials, like water, we use the single-pole Debye model in which the susceptibility kernel in (2.3) is

$$g(t,\mathbf{x}) = \frac{\epsilon_0(\epsilon_s - \epsilon_\infty)}{\tau}\mathrm{e}^{-t/\tau}. \tag{2.4}$$

This gives a model for orientational polarization [2, 14]. Using this form of the susceptibility kernel, equation (2.3) can be re-written as an ODE in time forced by the electric field

$$\tau\frac{\partial \mathbf{P}}{\partial t} + \mathbf{P} = \epsilon_0\epsilon_\infty(\epsilon_q - 1)\mathbf{E}, \text{ in } \Omega \times (0,T). \tag{2.5}$$

In equation (2.5) the parameter ϵ_s is called the static relative permittivity. The ratio of static to infinite permittivities is denoted as $\epsilon_q := \frac{\epsilon_s}{\epsilon_\infty}$. The parameter τ is the *relaxation time* associated with the polarization mechanism [2]. In general τ, ϵ_∞, and ϵ_s can be functions of space, but we assume here that all parameters are constant within the medium, $\epsilon_s > \epsilon_\infty$, i.e. $\epsilon_q > 1$ and $\tau > 0$.

To construct a model for electromagnetic wave propagation in a polar material in two dimensions, we make the assumption that no fields exhibit variation in the z direction, i.e. all partial derivatives with respect to z are zero. The electric field and polarization then have two components each, $\mathbf{E} = (E_x, E_y)^T, \mathbf{P} = (P_x, P_y)^T$ and the magnetic field has one component $H_z = H$. Combining (2.5) with the constitutive relations (2.2a) and (2.2b), and substituting in the Maxwell curl equations (2.1a) and (2.1b) we get the following system of partial differential equations which we call the **2D TE Maxwell-Debye** model :

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \text{ curl } \mathbf{E}, \tag{2.6a}$$

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\epsilon_0\epsilon_\infty}\mathbf{curl}\ H - \frac{(\epsilon_q - 1)}{\tau}\mathbf{E} + \frac{1}{\epsilon_0\epsilon_\infty\tau}\mathbf{P}, \tag{2.6b}$$

$$\frac{\partial \mathbf{P}}{\partial t} = \frac{\epsilon_0\epsilon_\infty(\epsilon_q - 1)}{\tau}\mathbf{E} - \frac{1}{\tau}\mathbf{P}, \tag{2.6c}$$

where for a vector field, $\mathbf{U} = (U_x, U_y)^T$, the scalar curl operator is curl $\mathbf{U} := \frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}$, and for a scalar field, V, the vector curl operator is $\mathbf{curl}\ V := \left(\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial x}\right)^T$ [28]. All the fields in (2.6) are functions of position $\mathbf{x} = (x, y)^T$ and time t.

We first show that system (2.6) along with the PEC boundary conditions (2.1d) and initial conditions $\mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0(\mathbf{x})$, $\mathbf{P}(\mathbf{x}, 0) = \mathbf{P}_0(\mathbf{x})$ and $H(\mathbf{x}, 0) = H_0(\mathbf{x})$ for $\mathbf{x} \in \Omega \subset \mathbb{R}^2$ is well-posed. To this end, we define the following two function spaces:

$$H(\text{curl}, \Omega) = \{\mathbf{u} \in \left(L^2(\Omega)\right)^2 \ \mid \text{curl } \mathbf{u} \in L^2(\Omega)\}, \tag{2.7}$$

$$H_0(\text{curl}, \Omega) = \{\mathbf{u} \in H(\text{curl}, \Omega) \mid \mathbf{n} \times \mathbf{u} = 0 \text{ on } \partial\Omega\}. \tag{2.8}$$

Let $(\cdot, \cdot)$ denote the L^2 inner product and $|| \cdot ||_2$ the corresponding norm. Multiplying (2.6a) by $\mu_0 v \in L^2(\Omega)$, (2.6b) by $\epsilon_0\epsilon_\infty\mathbf{u} \in H_0(\text{curl}, \Omega)$, and (2.6c) by $(\epsilon_0\epsilon_\infty(\epsilon_q - 1))^{-1}\mathbf{w} \in \left(L^2(\Omega)\right)^2$, integrating over the domain $\Omega \subset \mathbb{R}^2$ and applying Green's formula for the **curl** operator

$$(\mathbf{curl}\ H, \mathbf{u}) = (H, \text{curl } \mathbf{u}),\ \forall \mathbf{u} \in H_0(\text{curl}, \Omega), \tag{2.9}$$

we obtain the weak formulation for the 2D Maxwell-Debye system of equations (2.6) as follows

$$\left(\mu_0 \frac{\partial H}{\partial t}, v\right) = (-\text{curl } \mathbf{E}, v)\,, \forall\ v \in L^2(\Omega), \tag{2.10a}$$

$$\left(\epsilon_0\epsilon_\infty \frac{\partial \mathbf{E}}{\partial t}, \mathbf{u}\right) = (H, \text{curl } \mathbf{u}) - \left(\frac{\epsilon_0\epsilon_\infty(\epsilon_q - 1)}{\tau}\mathbf{E}, \mathbf{u}\right) + \left(\frac{1}{\tau}\mathbf{P}, \mathbf{u}\right), \quad \forall\ \mathbf{u} \in H_0(\text{curl}, \Omega), \tag{2.10b}$$

$$\left(\frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q - 1)}\frac{\partial \mathbf{P}}{\partial t}, \mathbf{w}\right) = \left(\frac{1}{\tau}\mathbf{E}, \mathbf{w}\right) - \left(\frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q - 1)\tau}\mathbf{P}, \mathbf{w}\right), \quad \forall\ \mathbf{w} \in \left(L^2(\Omega)\right)^2. \tag{2.10c}$$

The following theorem shows the stability of the 2D Maxwell-Debye model (2.6) by showing that the model exhibits energy decay.

Theorem 2.1 (Maxwell-Debye Energy Decay)**.** *Let $\Omega \subset \mathbb{R}^2$ and suppose that the solutions of the weak formulation* (2.10) *for the 2D Maxwell-Debye system of equations* (2.6) *satisfy the regularity conditions $\mathbf{P} \in C^1(0, T; \left(L^2(\Omega)\right)^2)$, $\mathbf{E} \in C(0, T; H_0(\text{curl}, \Omega)) \cap C^1(0, T; (L^2(\Omega))^2)$, and $H \in C^1(0, T; L^2(\Omega))$. Then the system exhibits energy decay,*

$$\mathcal{E}_D(t) \le \mathcal{E}_D(0), \quad \forall t \ge 0, \tag{2.11}$$

where the energy $\mathcal{E}_D(t)$ is defined as

$$\mathcal{E}_D(t) = \left(\left\|\sqrt{\mu_0}\ H(t)\right\|_2^2 + \left\|\sqrt{\epsilon_0\epsilon_\infty}\ \mathbf{E}(t)\right\|_2^2 + \left\|\frac{1}{\sqrt{\epsilon_0\epsilon_\infty(\epsilon_q - 1)}}\ \mathbf{P}(t)\right\|_2^2\right)^{\frac{1}{2}}. \tag{2.12}$$

Proof. See [6, 21, 23]. □

In [23], it is also shown that the Gauss laws are satisfied by the Maxwell-Debye system if the initial fields are divergence free.

The **2D Maxwell-Debye TE scalar equations** derived from (2.6) are given as

$$\frac{\partial H}{\partial t} = \frac{1}{\mu_0}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right), \tag{2.13a}$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon_0\epsilon_\infty}\frac{\partial H}{\partial y} - \frac{(\epsilon_q - 1)}{\tau}E_x + \frac{1}{\epsilon_0\epsilon_\infty\tau}P_x, \tag{2.13b}$$

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\epsilon_0\epsilon_\infty}\frac{\partial H}{\partial x} - \frac{(\epsilon_q - 1)}{\tau}E_y + \frac{1}{\epsilon_0\epsilon_\infty\tau}P_y, \tag{2.13c}$$

$$\frac{\partial P_x}{\partial t} = \frac{\epsilon_0\epsilon_\infty(\epsilon_q - 1)}{\tau}E_x - \frac{1}{\tau}P_x, \tag{2.13d}$$

$$\frac{\partial P_y}{\partial t} = \frac{\epsilon_0\epsilon_\infty(\epsilon_q - 1)}{\tau}E_y - \frac{1}{\tau}P_y. \tag{2.13e}$$

2.2. Lorentz Media: Model and Energy estimates. For Lorentz Media, the choice of the kernel function in equation (2.3) is

$$g(t, \mathbf{x}) = \frac{\epsilon_0\omega_p^2}{\nu_0}\mathrm{e}^{-t/2\tau}\sin(\nu_0 t), \tag{2.14}$$

where $\omega_p := \omega_0\sqrt{\epsilon_s - \epsilon_\infty}$ is the plasma frequency, ω_0 is the resonant frequency of the medium, $\lambda := \dfrac{1}{2\tau}$ is a damping constant, and $\nu_0 := \sqrt{\omega_0^2 - \lambda^2}$. We assume that the parameters $\epsilon_s, \epsilon_\infty, \omega_0$ and τ (hence also ω_p and λ) are constants. The Lorentz model for electronic polarization in differential form is represented with the second order ODE forced by the electric field given as

$$\frac{\partial^2\mathbf{P}}{\partial t^2} + \frac{1}{\tau}\frac{\partial\mathbf{P}}{\partial t} + \omega_0^2\mathbf{P} = \epsilon_0\omega_p^2\mathbf{E}. \tag{2.15}$$

Rewriting the above second order ODE as a system of two first order ODE's by introducing a new variable $\mathbf{J}_P = \frac{\partial\mathbf{P}}{\partial t}$, the **2D TE Maxwell-Lorentz** model is

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0}\ \mathrm{curl}\ \mathbf{E}, \tag{2.16a}$$

$$\frac{\partial\mathbf{E}}{\partial t} = \frac{1}{\epsilon_0\epsilon_\infty}\mathbf{curl}\ H - \frac{1}{\epsilon_0\epsilon_\infty}\mathbf{J}_P, \tag{2.16b}$$

$$\frac{\partial\mathbf{J}_P}{\partial t} = -\frac{1}{\tau}\mathbf{J}_P - \omega_0^2\mathbf{P} + \epsilon_0\omega_p^2\mathbf{E}, \tag{2.16c}$$

$$\frac{\partial\mathbf{P}}{\partial t} = \mathbf{J}_P. \tag{2.16d}$$

All the fields in (2.16) are functions of position $\mathbf{x} = (x, y)^T$ and time t. For the Maxwell-Lorentz system (2.16) we obtain the weak formulation

$$\left(\mu_0\frac{\partial H}{\partial t}, v\right) = -\left(\mathrm{curl}\ \mathbf{E}, v\right),\ \forall v \in L^2(\Omega), \tag{2.17a}$$

$$\left(\epsilon_0\epsilon_\infty\frac{\partial\mathbf{E}}{\partial t}, \mathbf{u}\right) = \left(H, \mathrm{curl}\ \mathbf{u}\right) - \left(\mathbf{J}_P, \mathbf{u}\right),\ \forall\mathbf{u} \in H_0(\mathrm{curl}, \Omega), \tag{2.17b}$$

$$\left(\frac{1}{\epsilon_0\omega_p^2}\frac{\partial \mathbf{J}_p}{\partial t}, \mathbf{w}\right) = -\left(\frac{1}{\epsilon_0\omega_p^2\tau}\mathbf{J}_p, \mathbf{w}\right) - \left(\frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)}\mathbf{P}, \mathbf{w}\right) + (\mathbf{E}, \mathbf{w}), \ \forall \mathbf{w} \in \left(L^2(\Omega)\right)^2, \tag{2.17c}$$

$$\left(\frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)}\frac{\partial \mathbf{P}}{\partial t}, \mathbf{q}\right) = \left(\frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)}\mathbf{J}_p, \mathbf{q}\right), \ \forall \mathbf{q} \in \left(L^2(\Omega)\right)^2. \tag{2.17d}$$

The following theorem shows the stability of the 2D Maxwell-Lorentz model (2.16) by showing that the model exhibits energy decay (also see [5, 23]).

Theorem 2.2 (Maxwell-Lorentz Energy Decay)**.** *Let* $\Omega \subset \mathbb{R}^2$ *and suppose that the solutions of the weak formulation* (2.17) *for the Maxwell-Lorentz system of equations* (2.16) *satisfy the regularity conditions* $\mathbf{E} \in C(0, T; H_0(\mathrm{curl}, \Omega)) \cap C^1(0, T; (L^2(\Omega))^2)$, $\mathbf{P}, \mathbf{J}_P \in C^1(0, T; \left(L^2(\Omega)\right)^2)$, *and* $H(t) \in C^1(0, T; L^2(\Omega))$. *Then the system exhibits energy decay,*

$$\mathcal{E}_L(t) \le \mathcal{E}_L(0), \quad \forall\ t \ge 0, \tag{2.18}$$

where the energy $\mathcal{E}_L(t)$ *is defined as*

$$\mathcal{E}_L(t) = \left(\left\|\sqrt{\mu_0}\ H(t)\right\|_2^2 + \left\|\sqrt{\epsilon_0\epsilon_\infty}\ \mathbf{E}(t)\right\|_2^2 + \left\|\frac{1}{\sqrt{\epsilon_0\epsilon_\infty(\epsilon_q-1)}}\ \mathbf{P}(t)\right\|_2^2 + \left\|\frac{1}{\sqrt{\epsilon_0}\omega_p}\ \mathbf{J}_P(t)\right\|_2^2\right)^{\frac{1}{2}}. \tag{2.19}$$

Proof. See [6, 23]. □

In [23], it is also shown that the Gauss laws are satisfied by the Maxwell-Lorentz system if the initial fields are divergence free.

The **2D Maxwell-Lorentz TE scalar equations** derived from system (2.16) on which the Yee scheme is based are:

$$\frac{\partial H}{\partial t} = \frac{1}{\mu_0}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right), \tag{2.20a}$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon_0\epsilon_\infty}\left(\frac{\partial H}{\partial y} - J_{P_x}\right), \tag{2.20b}$$

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\epsilon_0\epsilon_\infty}\left(\frac{\partial H}{\partial x} + J_{P_y}\right), \tag{2.20c}$$

$$\frac{\partial J_{P_x}}{\partial t} = \epsilon_0\omega_p^2 E_x - \frac{1}{\tau}J_{P_x} - \omega_0^2 P_x, \tag{2.20d}$$

$$\frac{\partial J_{P_y}}{\partial t} = \epsilon_0\omega_p^2 E_y - \frac{1}{\tau}J_{P_y} - \omega_0^2 P_y, \tag{2.20e}$$

$$\frac{\partial P_x}{\partial t} = J_{P_x}, \tag{2.20f}$$

$$\frac{\partial P_y}{\partial t} = J_{P_y}. \tag{2.20g}$$

3. The Yee scheme: Discretization in Space and Time

In this section we consider the finite difference time domain (FDTD) Yee scheme for discretizing the 2D Maxwell-Debye model (2.6) or (2.13) and 2D Maxwell-Lorentz models (2.16) or (2.20).

Consider the spatial domain $\Omega = [0,a] \times [0,b] \subset \mathbb{R}^2$ and time interval $[0,T]$ with a, b, $T > 0$, and spatial step sizes $\Delta x > 0$ and $\Delta y > 0$ and time step $\Delta t > 0$. The discretization of the intervals $[0,a]$, $[0,b]$, and $[0,T]$ is performed as follows [7]. Define $L = a/\Delta x$, $J = b/\Delta y$ and $N = T/\Delta t$. For $\ell, j, n \in \mathbb{N}$ we consider the discretizations

$$0 = x_0 \le x_1 \le \cdots \le x_\ell \le \cdots \le x_L = a, \tag{3.1}$$

$$0 = y_0 \le y_1 \le \cdots \le y_j \le \cdots \le y_J = b, \tag{3.2}$$

$$0 = t^0 \le t^1 \le \cdots \le t^n \le \cdots \le t^N = T, \tag{3.3}$$

where $x_\ell = \ell\Delta x$, $y_j = j\Delta y$, and $t^n = n\Delta t$ for $0 \le \ell \le L$, $0 \le j \le J$, and $0 \le n \le N$. Define $(x_\alpha, y_\beta, t^\gamma) = (\alpha\Delta x, \beta\Delta y, \gamma\Delta t)$ where α is either ℓ or $\ell + \frac{1}{2}$, β is either j or $j + \frac{1}{2}$, and γ is either n or $n + \frac{1}{2}$ with $\ell, j, n \in \mathbb{N}$. The Yee scheme staggers the electric and magnetic fields in space and time. Fields E_x, E_y, and H are staggered in the x and y directions. We define the discrete meshes

$$\tau_h^{E_x} := \left\{ \left(x_{\ell+\frac{1}{2}}, y_j \right) \mid 0 \le \ell \le L-1, 0 \le j \le J \right\}, \tag{3.4}$$

$$\tau_h^{E_y} := \left\{ \left(x_\ell, y_{j+\frac{1}{2}} \right) \mid 0 \le \ell \le L, 0 \le j \le J-1 \right\}, \tag{3.5}$$

$$\tau_h^{H} := \left\{ \left(x_{\ell+\frac{1}{2}}, y_{j+\frac{1}{2}} \right) \mid 0 \le \ell \le L-1, 0 \le j \le J-1 \right\}, \tag{3.6}$$

to be the sets of spatial grid points on which the E_x, E_y, and H fields, respectively, will be discretized. The components P_x, and J_{P_x} are discretized at the same spatial locations as the field E_x, while the components P_y and J_{P_y} are discretized at the same spatial locations as the field E_y. For the time discretization, the components E_x, E_y, P_x, P_y, J_{P_x} and J_{P_y} are all discretized at integer time steps t^n for $0 \le n \le N$. In the Yee scheme, the magnetic field, H, is staggered in time with respect to E_x and E_y and discretized at time $t^{n+\frac{1}{2}}$ for $0 \le n \le N-1$.

Let U be one of the field variables H, E_x, E_y, P_x, P_y, J_{P_x} or J_{P_y}, let $(x_\alpha, y_\beta) \in \tau_h^H, \tau_h^{E_x}$ or $\tau_h^{E_y}$, and γ be either n or $n+\frac{1}{2}$ with $n \in \mathbb{N}$. We define the *grid functions* or the numerical approximations

$$U_{\alpha,\beta}^\gamma \approx U(x_\alpha, y_\beta, t^\gamma).$$

We will also use the notation $U(t^\gamma)$ to denote the continuous solution on the domain Ω at time t^γ, and the notation U^γ to denote the corresponding grid function on its discrete spatial mesh at time t^γ.

We define in a standard way (see for e.g. [3, 10]) the centered temporal difference operator and a discrete .time averaging operation, respectively, as

$$\delta_t U_{\alpha,\beta}^\gamma := \frac{U_{\alpha,\beta}^{\gamma+\frac{1}{2}} - U_{\alpha,\beta}^{\gamma-\frac{1}{2}}}{\Delta t}, \tag{3.7}$$

$$\overline{U}_{\alpha,\beta}^\gamma := \frac{U_{\alpha,\beta}^{\gamma+\frac{1}{2}} + U_{\alpha,\beta}^{\gamma-\frac{1}{2}}}{2}, \tag{3.8}$$

and the centered spatial difference operators in the x and y direction, respectively, as

$$\delta_x U_{\alpha,\beta}^\gamma := \frac{U_{\alpha+\frac{1}{2},\beta}^{\gamma} - U_{\alpha-\frac{1}{2},\beta}^{\gamma}}{\Delta x}, \tag{3.9}$$

$$\delta_y U_{\alpha,\beta}^\gamma := \frac{U_{\alpha,\beta+\frac{1}{2}}^{\gamma} - U_{\alpha,\beta-\frac{1}{2}}^{\gamma}}{\Delta y}. \tag{3.10}$$

Next, we define the following staggered discrete l^2 normed spaces (see also [11])

(3.11)
$$\mathbb{V}_H := \left\{ U = (U_{l+\frac{1}{2},j+\frac{1}{2}}), (x_{\ell+\frac{1}{2}}, y_{j+\frac{1}{2}}) \in \tau_h^H,\ \|U\|_H < \infty \right\},$$

(3.12)
$$\mathbb{V}_E := \left\{ \mathbf{F} = (F_{x_{l+\frac{1}{2},j}}, F_{y_{l,j+\frac{1}{2}}})^T, (x_{\ell+\frac{1}{2}}, y_j) \times (x_\ell, y_{j+\frac{1}{2}}) \in \tau_h^{E_x} \times \tau_h^{E_y},\ \|\mathbf{F}\|_E < \infty \right\},$$

(3.13)
$$\mathbb{V}_{E,0} := \Big\{ \mathbf{F} \in \mathbb{V}_E \mid F_{x_{\ell+\frac{1}{2},0}} = F_{x_{\ell+\frac{1}{2},J}} = F_{y_{0,j+\frac{1}{2}}} = F_{x_{L,j+\frac{1}{2}}} = 0, \\ 0 \le \ell \le L, 0 \le j \le J \Big\},$$

where the discrete grid norms are defined as

$$\|\mathbf{F}\|_E^2 = \Delta x \Delta y \sum_{\ell=0}^{L-1} \sum_{j=0}^{J-1} \Big(|F_{x_{\ell+\frac{1}{2},j}}|^2 + |F_{y_{\ell,j+\frac{1}{2}}}|^2 \Big), \forall\ \mathbf{F} \in \mathbb{V}_E, \tag{3.14}$$

$$\|U\|_H^2 = \Delta x \Delta y \sum_{\ell=0}^{L-1} \sum_{j=0}^{J-1} |U_{\ell+\frac{1}{2},j+\frac{1}{2}}|^2, \forall\ U \in \mathbb{V}_H, \tag{3.15}$$

with corresponding inner products

(3.16)
$$(\mathbf{F}, \mathbf{G})_E = \Delta x \Delta y \sum_{\ell=0}^{L-1} \sum_{j=0}^{J-1} \Big(F_{x_{\ell+\frac{1}{2},j}} G_{x_{\ell+\frac{1}{2},j}} + F_{y_{\ell,j+\frac{1}{2}}} G_{y_{\ell,j+\frac{1}{2}}} \Big), \forall\ \mathbf{F}, \mathbf{G} \in \mathbb{V}_E,$$

(3.17)
$$(U, V)_H = \Delta x \Delta y \sum_{\ell=0}^{L-1} \sum_{j=0}^{J-1} U_{\ell+\frac{1}{2},j+\frac{1}{2}} V_{\ell+\frac{1}{2},j+\frac{1}{2}}, \forall\ U, V \in \mathbb{V}_H.$$

Finally, we define discrete curl operators on the staggered l^2 normed spaces as

$$\mathrm{curl_h} : \mathbb{V}_{E,0} \longrightarrow \mathbb{V}_H,\ \mathrm{curl_h}\, \mathbf{F} := \delta_x F_y - \delta_y F_x, \tag{3.18}$$

and

$$\mathbf{curl}_\mathrm{h} : \mathbb{V}_H \longrightarrow \mathbb{V}_{E,0},\ \mathbf{curl}_\mathrm{h}\, U := (\delta_y U, -\delta_x U)^T. \tag{3.19}$$

The discrete differential operators mimic properties that are satisfied by their continuous counterparts. In particular, if the PEC conditions (2.1d) are satisfied on the discrete Yee mesh,

$$F_{x_{\ell+\frac{1}{2},0}} = F_{x_{\ell+\frac{1}{2},J}} = F_{y_{0,j+\frac{1}{2}}} = F_{x_{L,j+\frac{1}{2}}} = 0,\ 0 \le \ell \le L,\ 0 \le j \le J, \tag{3.20}$$

i.e. $\forall\ \mathbf{F} \in \mathbb{V}_{E,0}$, discrete integration by parts (also see [3, 10]) yields,

$$(\mathrm{curl_h}\, \mathbf{E}, H)_H = (\mathbf{E}, \mathbf{curl}_\mathrm{h}\, H)_E. \tag{3.21}$$

Thus, the discrete versions of the curl operators remain adjoint to each other, which is essential for obtaining discrete energy estimates [3].

In the rest of the paper we assume a uniform mesh, i.e. $\Delta x = \Delta y = h$. In Sections 4 and 5 we prove discrete energy estimates for the Yee scheme applied to the Maxwell Debye model (2.13) and the Maxwell-Lorentz model (2.20), respectively. In addition, we show that the Yee schemes for these media retain the second order accuracy in space and time that the scheme enjoys in a non-dispersive medium. However, our energy decay results indicate the dissipative nature of the Yee schemes in Debye and Lorentz media, as opposed to the non-dissipative nature of the Yee scheme in a non-dispersive dielectric (also see [4, 31]). Our energy analysis shows that the Yee schemes for the Maxwell-Debye and Maxwell-Lorentz models are conditionally stable with the stability condition.

$$\frac{2c_\infty^2\Delta t^2}{h^2} < 1, \text{ or } 2\nu^2 - 1 < 0, \tag{3.22}$$

where the *Courant number* $\nu := \frac{c_\infty \Delta t}{h}$. Then the stability condition (3.22) implies that $\nu < \frac{1}{\sqrt{2}}$, and we prove convergence of the Yee schemes under this criteria.

4. Yee Scheme for the Maxwell-Debye System

4.1. Discretization. To discretize the 2D TE Maxwell-Debye system (2.13), in addition to staggering electric and magnetic components in space and time, the lower order terms are discretized using averaging. Using the operators defined in (3.7), (3.8), (3.9), and (3.10), the Yee scheme for the 2D TE Maxwell-Debye system (2.13) consists of the following discrete equations:

$$\delta_t H^n_{\ell+\frac12,j+\frac12} = \frac{1}{\mu_0}\Big(\delta_y E^n_{x_{\ell+\frac12,j+\frac12}} - \delta_x E^n_{y_{\ell+\frac12,j+\frac12}}\Big), \tag{4.1a}$$

$$\delta_t E^{n+\frac12}_{x_{\ell+\frac12,j}} = \frac{1}{\epsilon_0\epsilon_\infty}\delta_y H^{n+\frac12}_{\ell+\frac12,j} - \frac{\epsilon_q-1}{\tau}\overline{E}^{n+\frac12}_{x_{\ell+\frac12,j}} + \frac{1}{\tau\epsilon_0\epsilon_\infty}\overline{P}^{n+\frac12}_{x_{\ell+\frac12,j}}, \tag{4.1b}$$

$$\delta_t E^{n+\frac12}_{y_{\ell,j+\frac12}} = -\frac{1}{\epsilon_0\epsilon_\infty}\delta_x H^{n+\frac12}_{\ell,j+\frac12} - \frac{\epsilon_q-1}{\tau}\overline{E}^{n+\frac12}_{y_{\ell,j+\frac12}} + \frac{1}{\tau\epsilon_0\epsilon_\infty}\overline{P}^{n+\frac12}_{y_{\ell,j+\frac12}}, \tag{4.1c}$$

$$\delta_t P^{n+\frac12}_{x_{\ell+\frac12,j}} = \frac{\epsilon_0\epsilon_\infty(\epsilon_q-1)}{\tau}\overline{E}^{n+\frac12}_{x_{\ell+\frac12,j}} - \frac{1}{\tau}\overline{P}^{n+\frac12}_{x_{\ell+\frac12,j}}, \tag{4.1d}$$

$$\delta_t P^{n+\frac12}_{y_{\ell,j+\frac12}} = \frac{\epsilon_0\epsilon_\infty(\epsilon_q-1)}{\tau}\overline{E}^{n+\frac12}_{y_{\ell,j+\frac12}} - \frac{1}{\tau}\overline{P}^{n+\frac12}_{y_{\ell,j+\frac12}}. \tag{4.1e}$$

Re-writing system (4.1) in vector form we consider the problem of solving the discrete Maxwell-Debye system given by the Yee scheme as
Given $\mathbf{E}^0 \in \mathbb{V}_{E,0}$, $\mathbf{P}^0 \in \mathbb{V}_E$ and $H^{-\frac12} \in \mathbb{V}_H$, find $\mathbf{E}^{n+1} \in \mathbb{V}_{E,0}$, $\mathbf{P}^{n+1} \in \mathbb{V}_E$ and $H^{n+\frac12} \in \mathbb{V}_H$ that satisfy

$$\delta_t H^n + \frac{1}{\mu_0}\left(\mathrm{curl}_h \mathbf{E}\right)^n = 0, \tag{4.2a}$$

$$\delta_t \mathbf{E}^{n+\frac12} = \frac{1}{\epsilon_0\epsilon_\infty}\left(\mathbf{curl}_h H\right)^{n+\frac12} - \frac{\epsilon_q-1}{\tau}\overline{\mathbf{E}}^{n+\frac12} + \frac{1}{\tau\epsilon_0\epsilon_\infty}\overline{\mathbf{P}}^{n+\frac12}, \tag{4.2b}$$

$$\delta_t \mathbf{P}^{n+\frac12} = \frac{\epsilon_0\epsilon_\infty(\epsilon_q-1)}{\tau}\overline{\mathbf{E}}^{n+\frac12} - \frac{1}{\tau}\overline{\mathbf{P}}^{n+\frac12}. \tag{4.2c}$$

4.2. Accuracy: Truncation Error Analysis. Similar to the Yee scheme in free space, the Yee scheme for the Maxwell-Debye system is also second-order accurate in both time and space.

Lemma 4.1 (Yee Scheme Truncation Errors for Maxwell-Debye)**.** *Suppose that the solutions to the two-dimensional Maxwell-Debye equations* (2.6) *or* (2.13) *satisfy the regularity conditions* $\mathbf{E} \in C^3([0,T];[C^3(\overline{\Omega})]^2)$, $\mathbf{P} \in C^3([0,T];[C(\overline{\Omega})]^2)$ *and* $H \in C^3\left([0,T];[C^3(\overline{\Omega})]\right)$. *Let* ξ^n_H , $\xi^{n+\frac12}_{E_x}, \xi^{n+\frac12}_{E_y}, \xi^{n+\frac12}_{P_x}, \xi^{n+\frac12}_{P_y}$ *be the truncation errors for the Yee scheme for the Maxwell-Debye model* (4.1)*. Then*

$$max\left\{\left|\xi^n_H\right|, \left|\xi^{n+\frac12}_{E_x}\right|, \left|\xi^{n+\frac12}_{E_y}\right|, \left|\xi^{n+\frac12}_{P_x}\right|, \left|\xi^{n+\frac12}_{P_y}\right|\right\} \le C_D\left(\Delta x^2 + \Delta y^2 + \Delta t^2\right), \tag{4.3}$$

where $C_D = C_D(\epsilon_0, \mu_0, \epsilon_\infty, \epsilon_q, \tau)$ *does not depend on the mesh sizes* Δx, Δy, *and* Δt.

Proof. We perform Taylor expansions and substitute the exact solution to obtain the truncation errors for (4.1a) - (4.1e). We have

(4.4)
$$\begin{aligned}(\xi_H)^n_{\ell+\frac{1}{2},j+\frac{1}{2}} &= \frac{\Delta t^2}{24}\frac{\partial^3 H}{\partial t^3}(x_{\ell+\frac{1}{2}}, y_{j+\frac{1}{2}}, t_{11}) + \frac{\Delta y^2}{24\mu_0}\frac{\partial^3 E_x}{\partial y^3}(x_{\ell+\frac{1}{2}}, y_{11}, t^n)\\ &- \frac{\Delta x^2}{24\mu_0}\frac{\partial^3 E_y}{\partial x^3}(x_{11}, y_{j+\frac{1}{2}}, t^n),\end{aligned}$$

(4.5)
$$\begin{aligned}(\xi_{E_x})^{n+\frac{1}{2}}_{\ell+\frac{1}{2},j} &= \frac{\Delta t^2}{24}\frac{\partial^3 E_x}{\partial t^3}(x_{\ell+\frac{1}{2}}, y_j, t_{21}) + \frac{\Delta t^2}{24\epsilon_0\epsilon_\infty}\frac{\partial^3 P_x}{\partial t^3}(x_{\ell+\frac{1}{2}}, y_j, t_{22})\\ &- \frac{\Delta y^2}{24\epsilon_0\epsilon_\infty}\frac{\partial^3 H}{\partial y^3}(x_{\ell+\frac{1}{2}}, y_{21}, t^{n+\frac{1}{2}}),\end{aligned}$$

(4.6)
$$\begin{aligned}(\xi_{E_y})^{n+\frac{1}{2}}_{\ell,j+\frac{1}{2}} &= \frac{\Delta t^2}{24}\frac{\partial^3 E_y}{\partial t^3}(x_\ell, y_{j+\frac{1}{2}}, t_{31}) + \frac{\Delta t^2}{24\epsilon_0\epsilon_\infty}\frac{\partial^3 P_y}{\partial t^3}(x_\ell, y_{j+\frac{1}{2}}, t_{32})\\ &+ \frac{\Delta x^2}{24\epsilon_0\epsilon_\infty}\frac{\partial^3 H}{\partial x^3}(x_{31}, y_{j+\frac{1}{2}}, t^{n+\frac{1}{2}}),\end{aligned}$$

(4.7)
$$\begin{aligned}(\xi_{P_x})^{n+\frac{1}{2}}_{\ell+\frac{1}{2},j} &= \frac{\Delta t^2}{24}\frac{\partial^3 P_x}{\partial t^3}(x_{\ell+\frac{1}{2}}, y_j, t_{41}) - \frac{\Delta t^2\epsilon_0\epsilon_\infty(\epsilon_q-1)}{8\tau}\frac{\partial^2 E_x}{\partial t^2}(x_{\ell+\frac{1}{2}}, y_j, t_{42})\\ &+ \frac{\Delta t^2}{8\tau}\frac{\partial^2 P_x}{\partial t^2}(x_{\ell+\frac{1}{2}}, y_j, t_{43}),\end{aligned}$$

(4.8)
$$\begin{aligned}(\xi_{P_y})^{n+\frac{1}{2}}_{\ell,j+\frac{1}{2}} &= \frac{\Delta t^2}{24}\frac{\partial^3 P_y}{\partial t^3}(x_\ell, y_{j+\frac{1}{2}}, t_{51}) - \frac{\Delta t^2\epsilon_0\epsilon_\infty(\epsilon_q-1)}{8\tau}\frac{\partial^2 E_y}{\partial t^2}(x_\ell, y_{j+\frac{1}{2}}, t_{52})\\ &+ \frac{\Delta t^2}{8\tau}\frac{\partial^2 P_y}{\partial t^2}(x_\ell, y_{j+\frac{1}{2}}, t_{53}),\end{aligned}$$

where $x_\ell \leq x_{11} \leq x_{\ell+1}$, $y_j \leq y_{11} \leq y_{j+1}$, $t^{n-\frac{1}{2}} \leq t_{11} \leq t^{n+\frac{1}{2}}$, $x_{\ell-\frac{1}{2}} \leq x_{31} \leq x_{\ell+\frac{1}{2}}$, $y_{j-\frac{1}{2}} \leq y_{21} \leq y_{j+\frac{1}{2}}$, $t^n \leq t_{2i}, t_{3i} \leq t^{n+1}$ for $i = 1, 2$, and $t^n \leq t_{4i}, t_{5i} \leq t^{n+1}$ for $i = 1, 2, 3$. $\square$

4.3. Discrete Energy Estimates for Debye media. In this section we prove a discrete version of the energy decay property given in Theorem 2.1 for the 2D Maxwell-Debye model (2.6). Theorem 4.1 proves the conditional stability of the 2D Yee scheme for discretizing the Maxwell-Debye model by showing the decay of a discrete energy in time. To prove the decay of a discrete energy we will need the following lemma.

Lemma 4.2. *The operator $\mathcal{A}_h : \mathbb{V}_{E,0} \to \mathbb{V}_{E,0}$ defined as*

$$\mathcal{A}_h\mathbf{F} = \left(\mathcal{I} - \frac{c_\infty^2\Delta t^2}{4}\,\mathbf{curl}_\mathrm{h}\,\mathrm{curl}_\mathrm{h}\right)\mathbf{F},\ \forall\ \mathbf{F} \in \mathbb{V}_{E,0}. \tag{4.9}$$

satisfies the inequality

$$(\mathcal{A}_h\mathbf{F}, \mathbf{F})_E \geq (1 - 2\nu^2)||\mathbf{F}||_E^2, \forall\ \mathbf{F} \in \mathbb{V}_{E,0}. \tag{4.10}$$

where $\nu := \frac{c_\infty\Delta t}{h}$ is the Courant number, and h is the (uniform) mesh step size.

Proof. In two dimensions, with a uniform space mesh step size $\Delta x = \Delta y = h$, we have for all $\mathbf{F} \in \mathbb{V}_{E,0}$ the inequality ([3]),

$$||\mathrm{curl_h}\,\mathbf{F}||_H^2 \leq \frac{8}{h^2}\,||\mathbf{F}||_E^2\,, \tag{4.11}$$

from which we have $\forall\ \mathbf{F} \in \mathbb{V}_{E,0}$

$$\begin{aligned}(\mathcal{A}_h\mathbf{F},\mathbf{F})_E &= (\mathbf{F},\mathbf{F})_E - \frac{c_\infty^2\Delta t^2}{4}(\mathrm{curl_h}\,\mathbf{F},\mathrm{curl_h}\,\mathbf{F})_H\\ &= ||\mathbf{F}||_E^2 - \frac{c_\infty^2\Delta t^2}{4}||\,\mathrm{curl_h}\,\mathbf{F}||_H^2\\ &\geq ||\mathbf{F}||_E^2 - \frac{8c_\infty^2\Delta t^2}{4h^2}||\mathbf{F}||_E^2 = (1-2\nu^2)||\mathbf{F}||_E^2.\end{aligned} \tag{4.12}$$

□

Theorem 4.1 (Energy Decay for Maxwell-Debye)**.** *If the stability condition,*

$$1 - 2\nu^2 \geq \nu_0 > 0, \tag{4.13}$$

for the Courant number $\nu := \frac{c_\infty\Delta t}{h}$*, and constant* ν_0 *is satisfied, then the Yee scheme for the Maxwell-Debye System* (4.2) *satisfies the discrete identity*

$$\delta_t\mathcal{E}_{h,D}^{n+\frac{1}{2}} = \frac{-1}{\overline{\mathcal{E}}_{h,D}^{n+\frac{1}{2}}\tau\epsilon_0\epsilon_\infty(\epsilon_q-1)}||\epsilon_0\epsilon_\infty(\epsilon_q-1)\overline{\mathbf{E}}^{n+\frac{1}{2}} - \overline{\mathbf{P}}^{n+\frac{1}{2}}||_E^2, \tag{4.14}$$

for all $n \geq 0$ *where*

$$\mathcal{E}_{h,D}^n = \left\{\mu_0(H^{n+\frac{1}{2}},H^{n-\frac{1}{2}})_H + ||\sqrt{\epsilon_0\epsilon_\infty}\mathbf{E}^n||_E^2 + \left|\left|\frac{1}{\sqrt{\epsilon_0\epsilon_\infty(\epsilon_q-1)}}\mathbf{P}^n\right|\right|_E^2\right\}^{\frac{1}{2}} \tag{4.15}$$

defines a discrete energy.

Proof. We consider the average of (4.2a) at n and $n+1$, multiply with $\mu_0\Delta x\Delta y H^{n+\frac{1}{2}}$ and sum over all spatial nodes on τ_h^H to get

$$\mu_0(\overline{\delta_t H}^{n+\frac{1}{2}},H^{n+\frac{1}{2}})_H + (\overline{\mathrm{curl}_h\mathbf{E}}^{n+\frac{1}{2}},H^{n+\frac{1}{2}})_H = 0. \tag{4.16}$$

We can rewrite (4.16) as

$$\frac{\mu_0}{2\Delta t}\{(H^{n+\frac{3}{2}},H^{n+\frac{1}{2}})_H - (H^{n+\frac{1}{2}},H^{n-\frac{1}{2}})_H\} + (\overline{\mathrm{curl_h}\mathbf{E}}^{n+\frac{1}{2}},H^{n+\frac{1}{2}})_H = 0. \tag{4.17}$$

We multiply equation (4.2b) with $\epsilon_0\epsilon_\infty\Delta x\Delta y\overline{\mathbf{E}}^{n+\frac{1}{2}}$ and sum over all spatial nodes on $\tau_h^{E_x}\times\tau_h^{E_y}$ to get

$$\begin{aligned}\epsilon_0\epsilon_\infty(\delta_t\mathbf{E}^{n+\frac{1}{2}},\overline{\mathbf{E}}^{n+\frac{1}{2}})_E + \frac{\epsilon_0\epsilon_\infty(\epsilon_q-1)}{\tau}(\overline{\mathbf{E}}^{n+\frac{1}{2}},\overline{\mathbf{E}}^{n+\frac{1}{2}})_E - \frac{1}{\tau}(\overline{\mathbf{P}}^{n+\frac{1}{2}},\overline{\mathbf{E}}^{n+\frac{1}{2}})_E\\ = (\mathbf{curl}_h H^{n+\frac{1}{2}},\overline{\mathbf{E}}^{n+\frac{1}{2}})_E,\end{aligned} \tag{4.18}$$

which can be re-written as

$$\begin{aligned}\frac{\epsilon_0\epsilon_\infty}{2\Delta t}\left\{||\mathbf{E}^{n+1}||_E^2 - ||\mathbf{E}^n||_E^2\right\} + \frac{\epsilon_0\epsilon_\infty(\epsilon_q-1)}{\tau}||\overline{\mathbf{E}}^{n+\frac{1}{2}}||_E^2 - \frac{1}{\tau}(\overline{\mathbf{P}}^{n+\frac{1}{2}},\overline{\mathbf{E}}^{n+\frac{1}{2}})_E =\\ (\mathbf{curl}_h H^{n+\frac{1}{2}},\overline{\mathbf{E}}^{n+\frac{1}{2}})_E.\end{aligned} \tag{4.19}$$

Finally, we multiply equation (4.2c) by $\Delta x \Delta y \frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)}\overline{\mathbf{P}}^{n+\frac{1}{2}}$ and sum over all spatial nodes on $\tau_h^{E_x} \times \tau_h^{E_y}$ to get

$$\frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)}(\delta_t\mathbf{P}^{n+\frac{1}{2}},\overline{\mathbf{P}}^{n+\frac{1}{2}})_E = \frac{1}{\tau}(\overline{\mathbf{P}}^{n+\frac{1}{2}},\overline{\mathbf{E}}^{n+\frac{1}{2}})_E - \frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)\tau}||\overline{\mathbf{P}}^{n+\frac{1}{2}}||_E^2, \tag{4.20}$$

which can be re-written as

$$\frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)}\left\{||\mathbf{P}^{n+1}||_E^2 - ||\mathbf{P}^{n}||_E^2\right\} = \frac{1}{\tau}(\overline{\mathbf{P}}^{n+\frac{1}{2}},\overline{\mathbf{E}}^{n+\frac{1}{2}})_E - \frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)\tau}||\overline{\mathbf{P}}^{n+\frac{1}{2}}||_E^2. \tag{4.21}$$

Adding equations (4.17), (4.19), and (4.21), and using the definition (4.15) we have

$$\begin{aligned}\frac{1}{2\Delta t}\left\{(\mathcal{E}_{h,D}^{n+1})^2 - (\mathcal{E}_{h,D}^{n})^2\right\} = -\frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)\tau}\Big\{||\overline{\mathbf{P}}^{n+\frac{1}{2}}||_E^2 \\ -2\epsilon_0\epsilon_\infty(\epsilon_q-1)(\overline{\mathbf{P}}^{n+\frac{1}{2}},\overline{\mathbf{E}}^{n+\frac{1}{2}})_E + (\epsilon_0\epsilon_\infty(\epsilon_q-1))^2||\overline{\mathbf{E}}^{n+\frac{1}{2}}||_E^2\Big\}.\end{aligned} \tag{4.22}$$

We can rewrite this equation in the form

$$\frac{\mathcal{E}_{h,D}^{n+1} - \mathcal{E}_{h,D}^{n}}{\Delta t} = -\left(\frac{2}{\mathcal{E}_{h,D}^{n+1} + \mathcal{E}_{h,D}^{n}}\right)\frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)\tau}||\epsilon_0\epsilon_\infty(\epsilon_q-1)\overline{\mathbf{E}}^{n+\frac{1}{2}} - \overline{\mathbf{P}}^{n+\frac{1}{2}}||_E^2, \tag{4.23}$$

which on utilizing the definitions of the time differencing and averaging operators in (3.7), and (3.8), respectively, gives us the discrete identity (4.14) for Debye media. What is left to prove is that the quantity defined in (4.15) is a discrete energy, i.e., a positive function of the solution to the system (4.1).

Using the parallelogram law [3] we have

$$(H^{n+\frac{1}{2}}, H^{n-\frac{1}{2}})_H = \frac{1}{4}\left|\left|H^{n+\frac{1}{2}} + H^{n-\frac{1}{2}}\right|\right|_H^2 - \frac{1}{4}\left|\left|H^{n+\frac{1}{2}} - H^{n-\frac{1}{2}}\right|\right|_H^2. \tag{4.24}$$

Using (4.2a) and the definitions of the time differencing operator in (3.7) we can rewrite the second term in (4.24) as

$$\frac{1}{4}\left|\left|H^{n+\frac{1}{2}} - H^{n-\frac{1}{2}}\right|\right|_H^2 = \frac{\Delta t^2}{4}||\delta_t H^n||_H^2 = \frac{\Delta t^2}{4}||\text{curl}_h\mathbf{E}^n||_H^2. \tag{4.25}$$

Substituting equations (4.24) and (4.25) into the definition (4.15), and using the definition of the time averaging operator given in (3.8), we can re-write the discrete energy (4.15) as

$$\mathcal{E}_{h,D}^n = \left\{\mu_0||\overline{H}^n||_H^2 + \epsilon_0\epsilon_\infty(\mathbf{E}^n, \mathcal{A}_h\mathbf{E}^n)_E + \left|\left|\frac{1}{\sqrt{\epsilon_0\epsilon_\infty(\epsilon_q-1)}}\mathbf{P}^n\right|\right|_E^2\right\}^{\frac{1}{2}}, \tag{4.26}$$

with the operator $\mathcal{A}_h$ as defined in (4.9). If the stability condition (4.13) is satisfied, then $1-2\nu^2 \geq \nu_0 > 0$, the operator $\mathcal{A}_h$ is positive definite, i.e. $(\mathcal{A}_h\mathbf{F},\mathbf{F}) > 0, \forall\, \mathbf{F} \in \mathbb{V}_{E,0}$, and $\mathcal{E}_{h,D}^n$ defines a discrete energy. We note that this stability condition is the same for the Yee scheme applied to a non-dispersive dielectric with the same infinite frequency relative permittivity ϵ_∞ [31, 33]. □

Remark: For a nonuniform mesh the stability condition is again the same as for the non-dispersive case, i.e. $\nu < 1$ [33], with the Courant number

$$\nu = c_\infty \Delta t\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}}. \tag{4.27}$$

4.4. Convergence Analysis of the Yee scheme for the Maxwell-Debye Model. The technique to prove convergence of the Yee schemes is a classical one (see for e.g. [3, 17] and references therein) and employs the energy approach. To prove the convergence of the Yee scheme for the 2D Maxwell-Debye system for $0 \leq n \leq N$ we define the error quantities

$$\mathcal{H}^n = H^n - H(t^n), \tag{4.28a}$$

$$\mathcal{E}^n = \mathbf{E}^n - \mathbf{E}(t^n), \tag{4.28b}$$

$$\mathcal{P}^n = \mathbf{P}^n - \mathbf{P}(t^n). \tag{4.28c}$$

As was done for the discrete energy estimate in the proof of Theorem 4.1, we obtain the following identities for the Yee scheme for the Maxwell-Debye model in (4.2).

$$\mu_0\delta_t\mathcal{H}^n + \mathrm{curl_h}\mathcal{E}^n = \xi_H^n, \tag{4.29a}$$

$$\epsilon_0\epsilon_\infty\delta_t\mathcal{E}^{n+\frac{1}{2}} - \mathbf{curl}_h\mathcal{H}^{n+\frac{1}{2}} + \epsilon_0\epsilon_\infty\frac{(\epsilon_q-1)}{\tau}\overline{\mathcal{E}}^{n+\frac{1}{2}} - \frac{1}{\tau}\overline{\mathcal{P}}^{n+\frac{1}{2}} = \xi_E^{n+\frac{1}{2}}, \tag{4.29b}$$

$$\frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)}\delta_t\mathcal{P}^{n+\frac{1}{2}} + \frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)\tau}\overline{\mathcal{P}}^{n+\frac{1}{2}} - \frac{1}{\tau}\overline{\mathcal{E}}^{n+\frac{1}{2}} = \xi_P^{n+\frac{1}{2}}, \tag{4.29c}$$

where ξ_H^n, and $\xi_F^{n+\frac{1}{2}} = (\xi_{F_x}^{n+\frac{1}{2}}, \xi_{F_y}^{n+\frac{1}{2}})^T$, for $F = E, P$ are the local truncation errors for the Maxwell-Debye system as discussed in Lemma 4.1. We have the following result:

Theorem 4.2 (Convergence of Yee Scheme for 2D Maxwell-Debye)**.** *Suppose that the solutions to the two-dimensional Maxwell-Debye equations* (2.6) *or* (2.13) *satisfy the regularity conditions* $\mathbf{E} \in C^3([0,T];[C^3(\overline{\Omega})]^2)$, $\mathbf{P} \in C^3([0,T];[C(\overline{\Omega})]^2)$ *and* $H \in C^3\left([0,T];[C^3(\overline{\Omega})]\right)$*. For* $n \geq 0$*, let* $H^{n+\frac{1}{2}} \in \mathbb{V}_H$*,* $\mathbf{E}^n \in \mathbb{V}_{E,0}$ *and* $\mathbf{P}^n \in \mathbb{V}_E$ *be the solution to the Yee scheme for the Maxwell-Debye system* (4.2)*. Also, let* $\xi_H^n, \xi_{E_x}^{n+\frac{1}{2}}, \xi_{E_y}^{n+\frac{1}{2}}, \xi_{P_x}^{n+\frac{1}{2}}, \xi_{P_y}^{n+\frac{1}{2}}$ *be the truncation errors for the Yee scheme for Maxwell-Debye* (4.1) *or* (4.2) *satisfying the conditions of Lemma 4.1. Assume that the stability condition* (4.13) *is satisfied, then for any fixed* $T > 0, \exists$ *a positive constant* $C_D = C_D(\epsilon_0,\mu_0,\epsilon_\infty,\epsilon_q,\nu)$ *depending on the medium parameters and the Courant number* ν*, but independent of the mesh parameters* $\Delta t, \Delta x, \Delta y$*, such that*

$$\max_{0\leq n\leq N}\left\{\mathcal{ER}_{h,D}^n\right\} \leq \mathcal{ER}_{h,D}^0 + TC_D(\epsilon_0,\mu_0,\epsilon_\infty,\epsilon_q,\nu)\left(\Delta x^2+\Delta y^2+\Delta t^2\right), \tag{4.30}$$

where the energy of the error at time $t^n = n\Delta t$*,* $\mathcal{ER}_{h,D}^n$*, is defined as*

$$\mathcal{ER}_{h,D}^n = \left\{\mu_0(\mathcal{H}^{n+\frac{1}{2}},\mathcal{H}^{n-\frac{1}{2}})_H + ||\sqrt{\epsilon_0\epsilon_\infty}\mathcal{E}^n||_E^2 + \left|\left|\frac{1}{\sqrt{\epsilon_0\epsilon_\infty(\epsilon_q-1)}}\mathcal{P}^n\right|\right|_E^2\right\}^{\frac{1}{2}}. \tag{4.31}$$

Proof. We first note, based on the proof of Theorem 4.1, that the energy of the error (4.31) can be equivalently written in the form

$$\mathcal{ER}_{h,D}^n = \left\{||\sqrt{\mu_0}\overline{\mathcal{H}}^n||_H^2 + \epsilon_0\epsilon_\infty(\mathcal{E}^n,\mathcal{A}_h\mathcal{E}^n)_E + \left|\left|\frac{1}{\sqrt{\epsilon_0\epsilon_\infty(\epsilon_q-1)}}\mathcal{P}^n\right|\right|_E^2\right\}^{\frac{1}{2}}, \tag{4.32}$$

with the operator $\mathcal{A}_h$ as defined in equation (4.9).

Next, we follow a similar procedure to that done in the proof of Theorem 4.1. Multiplying the average of (4.29a) at n and $n+1$ by $\Delta x\Delta y\ \mathcal{H}^{n+\frac{1}{2}}$ and summing over all spatial nodes on τ_h^H, multiplying (4.29b) by $\Delta x\Delta y\ \overline{\mathcal{E}}^{n+\frac{1}{2}}$ and summing over all spatial nodes on $\tau_h^{E_x} \times \tau_h^{E_y}$, multiplying (4.29c) by $\Delta x \Delta y\ \overline{\mathcal{P}}^{n+\frac{1}{2}}$ on $\tau_h^{E_x} \times \tau_h^{E_y}$ and summing over all spatial nodes, and finally adding all the results we obtain

(4.33)
$$\delta_t \mathcal{ER}_{h,D}^{n+\frac{1}{2}} \overline{\mathcal{ER}_{h,D}}^{n+\frac{1}{2}} = \frac{-1}{\tau\epsilon_0\epsilon_\infty(\epsilon_q - 1)} ||\epsilon_0\epsilon_\infty(\epsilon_q-1)\overline{\mathcal{E}}^{n+\frac{1}{2}} - \overline{\mathcal{P}}^{n+\frac{1}{2}}||_E^2 + (\overline{\xi}_H^{n+\frac{1}{2}}, \mathcal{H}^{n+\frac{1}{2}})_H + (\xi_E^{n+\frac{1}{2}}, \overline{\mathcal{E}}^{n+\frac{1}{2}})_E + (\xi_P^{n+\frac{1}{2}}, \overline{\mathcal{P}}^{n+\frac{1}{2}})_E.$$

Dropping the first negative term we have

(4.34)
$$\delta_t \mathcal{ER}_{h,D}^{n+\frac{1}{2}} \overline{\mathcal{ER}_{h,D}}^{n+\frac{1}{2}} \le ||\overline{\xi}_H^{n+\frac{1}{2}}||_H ||\mathcal{H}^{n+\frac{1}{2}}||_H + \frac{1}{2}||\xi_E^{n+\frac{1}{2}}||_E \left(||\mathcal{E}^n||_E + ||\mathcal{E}^{n+1}||_E\right) + \frac{1}{2}||\xi_P^{n+\frac{1}{2}}||_E \left(||\mathcal{P}^n||_E + ||\mathcal{P}^{n+1}||_E\right).$$

From (4.32) we have the following bounds

$$||\overline{\mathcal{H}}^n||_H \le (\mu_0)^{-\frac{1}{2}} \mathcal{ER}_{h,D}^n, \tag{4.35a}$$

$$||\mathcal{E}^n||_E \le \left(\epsilon_0\epsilon_\infty(1-2\nu^2)\right)^{-\frac{1}{2}} \mathcal{ER}_{h,D}^n, \tag{4.35b}$$

$$||\mathcal{P}^n||_E \le \left(\epsilon_0\epsilon_\infty(\epsilon_q-1)\right)^{\frac{1}{2}} \mathcal{ER}_{h,D}^n. \tag{4.35c}$$

The inequalities in (4.35) give us bounds for the averaged electric and polarization terms as

$$\frac{1}{2}\left(||\mathcal{E}^n||_E + ||\mathcal{E}^{n+1}||_E\right) \le \left(\epsilon_0\epsilon_\infty(1-2\nu^2)\right)^{-\frac{1}{2}} \overline{\mathcal{ER}_{h,D}}^{n+\frac{1}{2}}, \tag{4.36a}$$

$$\frac{1}{2}\left(||\mathcal{P}^n||_E + ||\mathcal{P}^{n+1}||_E\right) \le \left(\epsilon_0\epsilon_\infty(\epsilon_q-1)\right)^{\frac{1}{2}} \overline{\mathcal{ER}_{h,D}}^{n+\frac{1}{2}}. \tag{4.36b}$$

For a bound on $||\mathcal{H}^{n+\frac{1}{2}}||_H$ we note that since $\mathcal{H}^{n+\frac{1}{2}} = \overline{\mathcal{H}}^n + \frac{\Delta t}{2}\delta_t\mathcal{H}^n$, we have

$$||\mathcal{H}^{n+\frac{1}{2}}||_H \le ||\overline{\mathcal{H}}^n||_H + \frac{\Delta t}{2}||\delta_t\mathcal{H}^n||_H. \tag{4.37}$$

Under the assumption of the stability condition (4.13) and the form of the (non-negative) energy of the error in (4.31) we have the inequality

$$-(\mathcal{H}^{n+\frac{1}{2}}, \mathcal{H}^{n-\frac{1}{2}})_H \le \frac{1}{\mu_0}\left(||\sqrt{\epsilon_0\epsilon_\infty}\mathcal{E}^n||_E^2 + \left|\left|\frac{1}{\sqrt{\epsilon_0\epsilon_\infty(\epsilon_q-1)}}\mathcal{P}^n\right|\right|_E^2\right). \tag{4.38}$$

Next, since $\frac{\Delta t^2}{4}||\delta_t\mathcal{H}^n||_H^2 = ||\overline{\mathcal{H}}^n||_H^2 - (\mathcal{H}^{n+\frac{1}{2}}, \mathcal{H}^{n-\frac{1}{2}})_H$ we have from (4.38)

(4.39)
$$\frac{\Delta t^2}{4}||\delta_t\mathcal{H}^n||_H^2 \le \frac{1}{\mu_0}\left(||\sqrt{\mu_0}\overline{\mathcal{H}}^n||_H^2 + ||\sqrt{\epsilon_0\epsilon_\infty}\mathcal{E}^n||_E^2 + \left|\left|\frac{1}{\sqrt{\epsilon_0\epsilon_\infty(\epsilon_q-1)}}\mathcal{P}^n\right|\right|_E^2\right).$$

Substituting inequalities (4.35) (4.36), (4.37) and (4.39) in (4.34) we have

(4.40)
$$\left(\delta_t \mathcal{ER}_{h,D}^{n+\frac{1}{2}}\right) \overline{\mathcal{ER}_{h,D}}^{n+\frac{1}{2}} \le C_D(\epsilon_0, \mu_0, \epsilon_\infty, \epsilon_q, \nu)\ \max\left\{||\overline{\xi}_H^{n+\frac{1}{2}}||_H, ||\xi_E^{n+\frac{1}{2}}||_E, ||\xi_P^{n+\frac{1}{2}}||_E\right\} \overline{\mathcal{ER}_{h,D}}^{n+\frac{1}{2}},$$

where $C_D = C_D(\epsilon_0, \mu_0, \epsilon_\infty, \epsilon_q, \nu)$ is a constant that depends on the medium parameters and the Courant number ν, but is independent of the mesh parameters $\Delta t, \Delta x, \Delta y$. Dividing by $\overline{\mathcal{ER}_{h,D}}^{n+\frac{1}{2}} \neq 0$, we get

(4.41)
$$\mathcal{ER}_{h,D}^{n+1} \leq \mathcal{ER}_{h,D}^{n} + \Delta t\ C_D(\epsilon_0, \mu_0, \epsilon_\infty, \epsilon_q, \nu) \max\left\{ ||\overline{\xi}_H^{n+\frac{1}{2}}||_H, ||\xi_E^{n+\frac{1}{2}}||_E, ||\xi_P^{n+\frac{1}{2}}||_E \right\}.$$

Recursively applying the inequality (4.41) from $n+1$ to 1 we have

$$\mathcal{ER}_{h,D}^{n+1} \leq \mathcal{ER}_{h,D}^{0} + \Delta t\ C_D(\epsilon_0, \mu_0, \epsilon_\infty, \epsilon_q, \nu) \left(\sum_{k=0}^{n} \max\left\{ ||\overline{\xi}_H^{k+\frac{1}{2}}||_H, ||\xi_E^{k+\frac{1}{2}}||_E, ||\xi_P^{k+\frac{1}{2}}||_E \right\} \right). \tag{4.42}$$

Using $\Delta t = T/N$ and $0 \leq n \leq N-1$, and noting that the sum in the second term on the right in (4.42) from Lemma 4.1 is $\mathcal{O}(\Delta t^2 + \Delta x^2 + \Delta y^2)$, we get

$$\mathcal{ER}_{h,D}^{n+1} \leq \mathcal{ER}_{h,D}^{0} + TC_D(\epsilon_0, \mu_0, \epsilon_\infty, \epsilon_q, \nu) \left(\Delta t^2 + \Delta y^2 + \Delta x^2 \right). \tag{4.43}$$

Note, we use C_D as a generic constant absorbing all constants that arise in the above inequalities. Finally taking the maximum for n from 0 to $N-1$ in (4.43) we obtain (4.30). □

Remark 4.1. *We note that the convergence result in Theorem 4.2 does not hold at the stability limit $\nu = \frac{1}{\sqrt{2}}$ (see also* [17]*). It is shown in* [27] *that the Yee scheme in a non-dispersive dielectric need not be stable at the stability limit. However,* (4.13) *provides a necessary and sufficient criteria for the stability of the Yee scheme. (Also see* [15]*).*

4.5. Discrete Divergence for the Maxwell-Debye Model. If the initial fields satisfy the Gauss divergence laws

$$\nabla \cdot \mathbf{D} = \rho, \tag{4.44a}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4.44b}$$

where ρ is the electric charge density, then one can show from the Maxwell curl equations that the Gauss divergence laws are satisfied for all time [29]. This is done by applying the divergence operator to the Maxwell curl equations, using the fact that the divergence of the curl is zero and utilizing the continuity equation (assume there are no sources, i.e. the source current density $\mathbf{J}_s = 0$), to get

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}_c = 0, \tag{4.45}$$

where $\mathbf{J}_c$ is the conduction current density. (Note: we have assumed in (2.1a) that the current density $\mathbf{J} = \mathbf{J}_c + \mathbf{J}_s = 0$). It is well known that the Yee scheme for the Maxwell equations in free space (linear media) preserves the divergence property of the solution at the discrete level [8]. We now show that this remains true for the Yee scheme applied to the Maxwell-Debye system (with $\rho = 0$).

4.5.1. Discrete Divergence Operator. To define the discrete divergence operator, we define the discrete mesh of interior vertices

$$\tau_h^0 := \{(x_\ell, y_j) \,|1 \leq \ell \leq L-1, 1 \leq j \leq J-1\}, \tag{4.46}$$

to be the set of spatial grid points on which the discrete divergence of an electric field, will be defined. On the mesh τ_h^0, we define the staggered l^2 normed space

$$\mathbb{V}_0 := \left\{ V = (V_{\ell,j}), (x_\ell, y_j) \in \tau_h^0,\ \|V\|_0^2 < \infty \right\}, \tag{4.47}$$

where the discrete grid norm $||\cdot||_0$ is defined as

$$\|V\|_0^2 = \Delta x\Delta y \sum_{\ell=1}^{L-1}\sum_{j=1}^{J-1}\Big(|V_{\ell,j}|^2\Big), \forall\, V\in\mathbb{V}_0, \tag{4.48}$$

with corresponding inner product

$$(V,W)_0 = \Delta x\Delta y \sum_{\ell=1}^{L-1}\sum_{j=1}^{J-1}\Big(V_{\ell,j}W_{\ell,j}\Big), \forall\, V,W\in\mathbb{V}_0, \tag{4.49}$$

We define the discrete divergence operator on $\mathbb{V}_{E,0}$ as

$$\begin{aligned}\operatorname{div_h}: \mathbb{V}_{E,0}\longrightarrow \mathbb{V}_0,\ (\operatorname{div_h}\mathbf{F})_{\ell,j} &:= (\delta_x F_x)_{\ell,j} + (\delta_y F_y)_{\ell,j},\\ &\forall\ 1\le \ell\le L-1,\ 1\le j\le J-1.\end{aligned} \tag{4.50}$$

Lemma 4.3 (Discrete Divergence for the Maxwell-Debye System)**.** *For the Yee scheme applied to the Maxwell-Debye system given in* (4.2) *the discrete divergence of the initial grid functions is preserved for all $n\ge 0$, i.e. we have the identity*

$$\operatorname{div_h}\mathbf{D}^n = \operatorname{div_h}\mathbf{D}^0,\ \textit{ on } \tau_h^0. \tag{4.51}$$

Proof. We note that the discrete derivative operators δ_t,δ_x and δ_y commute. From the Yee equations (4.2b) and (4.2c) we have on the mesh $\tau_h^{E_x}\times\tau_h^{E_y}$,

$$\delta_t\mathbf{D}^{n+\frac12} = \delta_t(\epsilon_0\epsilon_\infty\mathbf{E}^{n+\frac12} + \mathbf{P}^{n+\frac12}) = \mathbf{curl}_{\rm h}\, H. \tag{4.52}$$

Therefore $\operatorname{div_h}\mathbf{curl}_{\rm h}\,H = \delta_x\delta_y H - \delta_y\delta_x H = 0$. Thus, applying the discrete divergence operator $\operatorname{div_h}$ to (4.52) we obtain

$$\delta_t \operatorname{div_h}\mathbf{D}^{n+\frac12} = \delta_t(\epsilon_0\epsilon_\infty \operatorname{div_h}\mathbf{E}^{n+\frac12} + \operatorname{div_h}\mathbf{P}^{n+\frac12}) = \operatorname{div_h}\mathbf{curl}_{\rm h}\,H = 0,\ \text{ on } \tau_h^0. \tag{4.53}$$

We finally have

$$\operatorname{div_h}\mathbf{D}^{n+1} = \operatorname{div_h}\mathbf{D}^n,\ \text{ on } \tau_h^0. \tag{4.54}$$

Applying the identity (4.54) recursively in discrete time we obtain (4.51). □

Remark 4.2. *We note that for the 2D TE Maxwell model, the divergence of the magnetic flux density is not defined. Thus, the divergence free or solenoidal nature of the magnetic flux density is lost in the two dimensional model* [28].

5. Yee Scheme for the Maxwell-Lorentz System

5.1. Discretization. The discrete approximation of the 2D Maxwell-Lorentz system (2.16) by the Yee scheme is

$$\delta_t H^n_{\ell+\frac12,j+\frac12} = \frac{1}{\mu_0}\Big(\delta_y E^n_{x_{\ell+\frac12,j+\frac12}} - \delta_x E^n_{y_{\ell+\frac12,j+\frac12}}\Big), \tag{5.1a}$$

$$\delta_t E^{n+\frac12}_{x_{\ell+\frac12,j}} = \frac{1}{\epsilon_0\epsilon_\infty}\left(\delta_y H^{n+\frac12}_{\ell+\frac12,j} - \overline{J}^{n+\frac12}_{P_{x_{\ell+\frac12,j}}}\right), \tag{5.1b}$$

$$\delta_t E^{n+\frac12}_{y_{\ell,j+\frac12}} = -\frac{1}{\epsilon_0\epsilon_\infty}\left(\delta_x H^{n+\frac12}_{\ell,j+\frac12} + \overline{J}^{n+\frac12}_{P_{y_{\ell,j+\frac12}}}\right), \tag{5.1c}$$

$$\delta_t J^{n+\frac12}_{P_{x_{\ell+\frac12,j}}} = \epsilon_0\omega_p^2\overline{E}^{n+\frac12}_{x_{\ell+\frac12,j}} - \frac{1}{\tau}\overline{J}^{n+\frac12}_{P_{x_{\ell+\frac12,j}}} - \omega_0^2\overline{P}^{n+\frac12}_{x_{\ell+\frac12,j}}, \tag{5.1d}$$

$$\delta_t J^{n+\frac12}_{P_{y_{\ell,j+\frac12}}} = \epsilon_0\omega_p^2\overline{E}^{n+\frac12}_{y_{\ell,j+\frac12}} - \frac{1}{\tau}\overline{J}^{n+\frac12}_{P_{y_{\ell,j+\frac12}}} - \omega_0^2\overline{P}^{n+\frac12}_{y_{\ell,j+\frac12}}, \tag{5.1e}$$

$$\delta_t P^{n+\frac{1}{2}}_{x_{\ell+\frac{1}{2},j}} = \overline{J}^{n+\frac{1}{2}}_{P_{x_{\ell+\frac{1}{2},j}}}, \tag{5.1f}$$

$$\delta_t P^{n+\frac{1}{2}}_{y_{\ell,j+\frac{1}{2}}} = \overline{J}^{n+\frac{1}{2}}_{P_{y_{\ell,j+\frac{1}{2}}}}. \tag{5.1g}$$

Re-writing system (5.1) in vector form we consider the problem of solving the discrete Maxwell-Lorentz problem given by the Yee scheme as
Given $\mathbf{E}^0 \in \mathbb{V}_{E,0}$, $\mathbf{P}^0, \mathbf{J}^0_P \in \mathbb{V}_E$ and $H^{-\frac{1}{2}} \in \mathbb{V}_H$, find $\mathbf{E}^{n+1} \in \mathbb{V}_{E,0}$, $\mathbf{P}^{n+1}, \mathbf{J}^{n+1} \in \mathbb{V}_E$ and $H^{n+\frac{1}{2}} \in \mathbb{V}_H$ that satisfy

$$\delta_t H^n + \frac{1}{\mu_0}\left(\mathrm{curl}_h \mathbf{E}\right)^n = 0, \tag{5.2a}$$

$$\delta_t \mathbf{E}^{n+\frac{1}{2}} = \frac{1}{\epsilon_0\epsilon_\infty}\left(\mathbf{curl}_h H\right)^{n+\frac{1}{2}} - \overline{\mathbf{J}_P}^{n+\frac{1}{2}}, \tag{5.2b}$$

$$\delta_t \mathbf{J}^{n+\frac{1}{2}}_P = \epsilon_0\omega_p^2 \overline{\mathbf{E}}^{n+\frac{1}{2}} - \frac{1}{\tau}\overline{\mathbf{J}}^{n+\frac{1}{2}}_P - \omega_0^2 \overline{\mathbf{P}}^{n+\frac{1}{2}}, \tag{5.2c}$$

$$\delta_t \mathbf{P}^{n+\frac{1}{2}} = \overline{\mathbf{J}_P}^{n+\frac{1}{2}}. \tag{5.2d}$$

5.2. Accuracy: Truncation Error Analysis. The Yee scheme for the Maxwell-Lorentz system is also second-order accurate in both time and space.

Lemma 5.1 (Yee Scheme Truncation Errors for Maxwell-Lorentz)**.** *Suppose that the solutions to the two-dimensional Maxwell-Lorentz equations* (2.16) *or* (2.20) *satisfy the regularity conditions* $\mathbf{E} \in C^3([0,T];[C^3(\overline{\Omega})]^2)$, $\mathbf{P}, \mathbf{J}_P \in C^3([0,T];[C(\overline{\Omega})]^2)$ *and* $H \in C^3\left([0,T];[C^3(\overline{\Omega})]\right)$*. Let* ξ^n_H , $\xi^{n+\frac{1}{2}}_{E_x}, \xi^{n+\frac{1}{2}}_{E_y}, \xi^{n+\frac{1}{2}}_{J_{P_x}}, \xi^{n+\frac{1}{2}}_{J_{P_y}}, \xi^{n+\frac{1}{2}}_{P_x}, \xi^{n+\frac{1}{2}}_{P_y}$ *be the truncation errors of the Yee scheme for the Maxwell-Lorentz model* (5.1)*. Then*

$$\begin{aligned} max\left\{\left|\xi^n_H\right|, \left|\xi^{n+\frac{1}{2}}_{E_x}\right|, \left|\xi^{n+\frac{1}{2}}_{E_y}\right|, \left|\xi^{n+\frac{1}{2}}_{J_{P_x}}\right|, \left|\xi^{n+\frac{1}{2}}_{J_{P_y}}\right|, \left|\xi^{n+\frac{1}{2}}_{P_x}\right|, \left|\xi^{n+\frac{1}{2}}_{P_y}\right|\right\} \leq \\ C_L\left(\Delta x^2 + \Delta y^2 + \Delta t^2\right), \end{aligned} \tag{5.3}$$

where $C_L = C_L(\epsilon_0, \mu_0, \epsilon_\infty, \epsilon_q, \tau, \omega_0)$ *does not depend on the mesh sizes* Δx, Δy, *and* Δt*.*

Proof. We perform Taylor expansions and substitute the exact solution to obtain the truncation errors for the equations in system (5.1). We have

$$\begin{aligned} (\xi_H)^n_{i+\frac{1}{2},j+\frac{1}{2}} = &\frac{\Delta t^2}{24}\frac{\partial^3 H}{\partial t^3}(x_{\ell+\frac{1}{2}}, y_{j+\frac{1}{2}}, t_{11}) + \frac{\Delta y^2}{24\mu_0}\frac{\partial^3 E_x}{\partial y^3}(x_{\ell+\frac{1}{2}}, y_{11}, t^n) \\ &- \frac{\Delta x^2}{24\mu_0}\frac{\partial^3 E_y}{\partial x^3}(x_{11}, y_{j+\frac{1}{2}}, t^n), \end{aligned} \tag{5.4}$$

$$\begin{aligned} (\xi_{E_x})^{n+\frac{1}{2}}_{\ell+\frac{1}{2},j} = &\frac{\Delta t^2}{24}\frac{\partial^3 E_x}{\partial t^3}(x_{\ell+\frac{1}{2}}, y_j, t_{21}) + \frac{\Delta t^2}{24\epsilon_0\epsilon_\infty}\frac{\partial^3 P_x}{\partial t^3}(x_{\ell+\frac{1}{2}}, y_j, t_{22}) \\ &- \frac{\Delta y^2}{24\epsilon_0\epsilon_\infty}\frac{\partial^3 H}{\partial y^3}(x_{\ell+\frac{1}{2}}, y_{21}, t^{n+\frac{1}{2}}), \end{aligned} \tag{5.5}$$

$$\begin{aligned} (\xi_{E_y})^{n+\frac{1}{2}}_{\ell,j+\frac{1}{2}} = &\frac{\Delta t^2}{24}\frac{\partial^3 E_y}{\partial t^3}(x_\ell, y_{j+\frac{1}{2}}, t_{31}) + \frac{\Delta t^2}{24\epsilon_0\epsilon_\infty}\frac{\partial^3 P_y}{\partial t^3}(x_\ell, y_{j+\frac{1}{2}}, t_{32}) \\ &+ \frac{\Delta x^2}{24\epsilon_0\epsilon_\infty}\frac{\partial^3 H}{\partial x^3}(x_{31}, y_{j+\frac{1}{2}}, t^{n+\frac{1}{2}}), \end{aligned} \tag{5.6}$$

$$\begin{aligned}\left(\xi_{J_{P_x}}\right)^{n+\frac{1}{2}}_{\ell+\frac{1}{2},j} =& \frac{\Delta t^2}{24}\frac{\partial^3 J_{P_x}}{\partial t^3}(x_{\ell+\frac{1}{2}},y_j,t_{41}) - \epsilon_0\omega_p\frac{\Delta t^2}{8}\frac{\partial^2 E_x}{\partial t^2}(x_{\ell+\frac{1}{2}},y_j,t_{42})\\ &+\frac{\Delta t^2}{8\tau}\frac{\partial^2 J_{P_x}}{\partial t^2}(x_{\ell+\frac{1}{2}},y_j,t_{43}) + \frac{\omega_0^2\Delta t^2}{8}\frac{\partial^2 P_x}{\partial t^2}(x_{\ell+\frac{1}{2}},y_j,t_{44}),\end{aligned} \tag{5.7}$$

$$\begin{aligned}\left(\xi_{J_{P_y}}\right)^{n+\frac{1}{2}}_{\ell,j+\frac{1}{2}} =& \frac{\Delta t^2}{24}\frac{\partial^3 J_{P_y}}{\partial t^3}(x_\ell,y_{j+\frac{1}{2}},t_{51}) - \epsilon_0\omega_p^2\frac{\Delta t^2}{8}\frac{\partial^2 E_y}{\partial t^2}(x_\ell,y_{j+\frac{1}{2}},t_{52})\\ &+\frac{\Delta t^2}{8\tau}\frac{\partial^2 J_{P_y}}{\partial t^2}(x_{\ell+\frac{1}{2}},y_j,t_{53}) + \frac{\omega_0^2\Delta t^2}{8}\frac{\partial^2 P_y}{\partial t^2}(x_{\ell+\frac{1}{2}},y_j,t_{54}),\end{aligned} \tag{5.8}$$

$$\left(\xi_{P_x}\right)^{n+\frac{1}{2}}_{\ell+\frac{1}{2},j} = \frac{\Delta t^2}{24}\frac{\partial^3 P_x}{\partial t^3}(x_{\ell+\frac{1}{2}},y_j,t_{61}) - \frac{\Delta t^2}{8}\frac{\partial^2 J_{P_x}}{\partial t^2}(x_{\ell+\frac{1}{2}},y_j,t_{62}), \tag{5.9}$$

$$\left(\xi_{P_y}\right)^{n+\frac{1}{2}}_{\ell,j+\frac{1}{2}} = \frac{\Delta t^2}{24}\frac{\partial^3 P_y}{\partial t^3}(x_\ell,y_{j+\frac{1}{2}},t_{71}) - \frac{\Delta t^2}{8}\frac{\partial^2 J_{P_y}}{\partial t^2}(x_\ell,y_{j+\frac{1}{2}},t_{72}), \tag{5.10}$$

where $x_\ell \le x_{11} \le x_{\ell+1}$, $y_j \le y_{11} \le y_{j+1}$, and $t^{n-\frac{1}{2}} \le t_{11} \le t^{n+\frac{1}{2}}$. Next, $x_{\ell-\frac{1}{2}} \le x_{31} \le x_{\ell+\frac{1}{2}}$, $y_{j-\frac{1}{2}} \le y_{21} \le y_{j+\frac{1}{2}}$, $t^n \le t_{\ell i} \le t^{n+1}$ for $i \in \{1,2\}$, and $\ell = 2,3,\dots,7$, and $t^n \le t_{\ell i} \le t^{n+1}$ for $i \in \{3,4\}$, and $\ell = 4,5$ □

5.3. Discrete Energy Estimates for Lorentz media.

Theorem 5.1 (Energy Decay for Maxwell-Lorentz)**.** *If stability condition* (4.13) *is satisfied, then the Yee scheme for the Maxwell-Lorentz System* (5.2) *satisfies the discrete identity*

$$\delta_t\mathcal{E}^{n+\frac{1}{2}}_{h,L} = -\frac{1}{\overline{\mathcal{E}}^{n+\frac{1}{2}}_{h,L}\tau\epsilon_0\omega_0^2}||\overline{\mathbf{J}_P}^{n+\frac{1}{2}}||_E^2, \tag{5.11}$$

for all $n \ge 0$, where

$$\mathcal{E}^n_{h,L} = \left\{\mu_0(H^{n+\frac{1}{2}},H^{n-\frac{1}{2}})_H + ||\sqrt{\epsilon_0\epsilon_\infty}\mathbf{E}^n||_E^2 + \left|\left|\frac{1}{\sqrt{\epsilon_0\epsilon_\infty(\epsilon_q-1)}}\mathbf{P}^n\right|\right|_E^2 + \left|\left|\frac{1}{\sqrt{\epsilon_0}\omega_p}\mathbf{J}_P^n\right|\right|_E^2\right\}^{\frac{1}{2}}, \tag{5.12}$$

defines a discrete energy.

Proof. The proof is similar to the proof of Theorem 4.1 for the Yee scheme for the Maxwell-Debye system. We point out the differences here. We multiply equation (5.2b) with $\epsilon_0\epsilon_\infty\Delta x\Delta y\overline{\mathbf{E}}^{n+\frac{1}{2}}$ and sum over all spatial nodes on $\tau_h^{E_x}\times\tau_h^{E_y}$ to get

$$\epsilon_0\epsilon_\infty(\delta_t\mathbf{E}^{n+\frac{1}{2}},\overline{\mathbf{E}}^{n+\frac{1}{2}})_E + (\overline{\mathbf{J}}_P^{n+\frac{1}{2}},\overline{\mathbf{E}}^{n+\frac{1}{2}})_E = (\mathbf{curl}_h H^{n+\frac{1}{2}},\overline{\mathbf{E}}^{n+\frac{1}{2}})_E \tag{5.13}$$

which can be re-written as

$$\frac{\epsilon_0\epsilon_\infty}{2\Delta t}\left\{||\mathbf{E}^{n+1}||_E^2 - ||\mathbf{E}^n||_E^2\right\} + (\overline{\mathbf{J}}_P^{n+\frac{1}{2}},\overline{\mathbf{E}}^{n+\frac{1}{2}})_E = (\mathbf{curl}_h H^{n+\frac{1}{2}},\overline{\mathbf{E}}^{n+\frac{1}{2}})_E. \tag{5.14}$$

Next, we multiply equation (5.2c) by $\frac{\Delta x\Delta y}{\epsilon_0\omega_p^2}\overline{\mathbf{J}_P}^{n+\frac{1}{2}}$ and sum over all spatial nodes on $\tau_h^{E_x}\times\tau_h^{E_y}$ to get

$$\begin{aligned}&\frac{1}{\epsilon_0\omega_p^2}(\delta_t\mathbf{J}_P^{n+\frac{1}{2}},\overline{\mathbf{J}_P}^{n+\frac{1}{2}})_E =\\ &(\overline{\mathbf{E}}^{n+\frac{1}{2}},\overline{\mathbf{J}_P}^{n+\frac{1}{2}})_E - \frac{1}{\tau\epsilon_0\omega_p^2}(\overline{\mathbf{J}}_P^{n+\frac{1}{2}},\overline{\mathbf{J}}_P^{n+\frac{1}{2}})_E - \frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)}(\overline{\mathbf{J}_P}^{n+\frac{1}{2}},\overline{\mathbf{P}}^{n+\frac{1}{2}})_E,\end{aligned} \tag{5.15}$$

which we re-write as

$$\frac{1}{\epsilon_0\omega_p^2}\left\{||\mathbf{J}_P^{n+1}||_E^2 - ||\mathbf{J}_P^n||_E^2\right\} - (\overline{\mathbf{E}}^{n+\frac{1}{2}}, \overline{\mathbf{J}_P}^{n+\frac{1}{2}})_E + \frac{1}{\tau\epsilon_0\omega_p^2}||\overline{\mathbf{J}}_P^{n+\frac{1}{2}}||_E^2 + \frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)}(\overline{\mathbf{J}_P}^{n+\frac{1}{2}}, \overline{\mathbf{P}}^{n+\frac{1}{2}})_E = 0. \tag{5.16}$$

Finally, we multiply equation (5.2d) by $\frac{\Delta x \Delta y}{\epsilon_0\epsilon_\infty(\epsilon_q-1)}\overline{\mathbf{P}}^{n+\frac{1}{2}}$ and sum over all spatial nodes on $\tau_h^{E_x} \times \tau_h^{E_y}$ to get

$$\frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)}(\delta_t\mathbf{P}^{n+\frac{1}{2}}, \overline{\mathbf{P}}^{n+\frac{1}{2}})_E - \frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)}(\overline{\mathbf{J}_P}^{n+\frac{1}{2}}, \overline{\mathbf{P}}^{n+\frac{1}{2}})_E = 0, \tag{5.17}$$

which can be re-written as

$$\frac{1}{2\Delta t\epsilon_0\epsilon_\infty(\epsilon_q-1)}\left\{||\mathbf{P}^{n+1}||_E^2 - ||\mathbf{P}^n||_E^2\right\} - \frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)}(\overline{\mathbf{J}_P}^{n+\frac{1}{2}}, \overline{\mathbf{P}}^{n+\frac{1}{2}})_E = 0. \tag{5.18}$$

Adding equations (4.17), (5.14), (5.16), and (5.18), and using the definition (5.12) we have

$$\frac{1}{2\Delta t}\left\{(\mathcal{E}_{h,L}^{n+1})^2 - (\mathcal{E}_{h,L}^n)^2\right\} = -\frac{1}{\epsilon_0\tau\omega_p^2}\left\{||\overline{\mathbf{J}_P}^{n+\frac{1}{2}}||_E^2\right\}. \tag{5.19}$$

We can rewrite this equation in the form

$$\frac{\mathcal{E}_{h,L}^{n+1} - \mathcal{E}_{h,L}^n}{\Delta t} = -\left(\frac{2}{\mathcal{E}_{h,D}^{n+1} + \mathcal{E}_{h,D}^n}\right)\frac{1}{\epsilon_0\tau\omega_p^2}||\overline{\mathbf{J}_P}^{n+\frac{1}{2}}||_E^2, \tag{5.20}$$

which on utilizing the definitions of the time differencing and averaging operators in (3.7), and (3.8), respectively, gives us the discrete identity (5.11) for Lorentz media. As for the case of Debye media, if the stability condition (4.13) is satisfied, the quantity defined in (5.12) is a discrete energy, i.e. a nonnegative function of the solution to the system (5.2). The rest of the proof is similar to the proof of Theorem 4.1 for the case of Debye media. □

5.4. Convergence Analysis of the Yee scheme for the Maxwell-Lorentz Model. For $0 \le n \le N$, define the error quantities

$$\mathcal{H}^n = H^n - H(t^n), \tag{5.21a}$$

$$\mathcal{E}^n = \mathbf{E}^n - \mathbf{E}(t^n), \tag{5.21b}$$

$$\mathcal{J}_P^n = \mathbf{J}_P^n - \mathbf{J}_P(t^n), \tag{5.21c}$$

$$\mathcal{P}^n = \mathbf{P}^n - \mathbf{P}(t^n). \tag{5.21d}$$

From the Yee scheme for the Maxwell-Lorentz model (5.2), we obtain the identities

$$\mu_0\delta_t\mathcal{H}^n + (\mathrm{curl}_h\mathcal{E})^n = \xi_H^n, \tag{5.22a}$$

$$\epsilon_0\epsilon_\infty\delta_t\mathcal{E}^{n+\frac{1}{2}} - (\mathbf{curl}_h\mathcal{H})^{n+\frac{1}{2}} + \overline{\mathcal{J}_P}^{n+\frac{1}{2}} = \xi_E^{n+\frac{1}{2}}, \tag{5.22b}$$

$$\frac{1}{\epsilon_0\omega_p^2}\delta_t\mathcal{J}_P^{n+\frac{1}{2}} - \overline{\mathcal{E}}^{n+\frac{1}{2}} + \frac{1}{\epsilon_0\omega_p^2\tau}\overline{\mathcal{J}}_P^{n+\frac{1}{2}} + \frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)}\overline{\mathcal{P}}^{n+\frac{1}{2}} = \xi_J^{n+\frac{1}{2}}, \tag{5.22c}$$

$$\frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)}\delta_t\mathcal{P}^{n+\frac{1}{2}} - \frac{1}{\epsilon_0\epsilon_\infty(\epsilon_q-1)}\overline{\mathcal{J}}_P^{n+\frac{1}{2}} = \xi_P^{n+\frac{1}{2}}. \tag{5.22d}$$

For the Maxwell-Lorentz system we have the following result:

Theorem 5.2 (Convergence of Yee Scheme for 2D Maxwell-Lorentz)**.** *Suppose that the solutions to the two-dimensional Maxwell-Lorentz equations* (2.16) *satisfy the regularity conditions* $\mathbf{E} \in C^3([0,T];[C^3(\overline{\Omega})]^2)$, $\mathbf{P}, \mathbf{J}_P \in C^3([0,T];[C(\overline{\Omega})]^2)$ *and* $H \in C^3\left([0,T];[C^3(\overline{\Omega})]\right)$. *For* $n \geq 0$, *let* $H^{n+\frac{1}{2}} \in \mathbb{V}_H$, $\mathbf{E}^n \in \mathbb{V}_{E,0}$, $\mathbf{J}_P^n \in \mathbb{V}_E$ *and* $\mathbf{P}^n \in \mathbb{V}_E$ *be the solution to the Yee scheme for the Maxwell-Lorentz system* (5.2), *and let* $\xi_H^n, \xi_E^{n+\frac{1}{2}}, \xi_J^{n+\frac{1}{2}}, \xi_P^{n+\frac{1}{2}}$ *be the truncation errors with* $\xi_F^k = (\xi_{F_x}^k, \xi_{F_y}^k)^T, F = E, J, P,$ *satisfying the conditions of Lemma 5.1. If the stability condition* (4.13) *is satisfied, then for any fixed* $T > 0, \exists$ *a positive constant* $C_L = C_L(\epsilon_0, \mu_0, \epsilon_\infty, \epsilon_q, \tau, \omega_0, \nu)$ *depending on the medium parameters and the Courant number* ν, *but independent of the mesh parameters* $\Delta t, \Delta x, \Delta y$, *such that*

$$\max_{0 \leq n \leq N} \left\{\mathcal{ER}_{h,L}^n\right\} \leq \mathcal{ER}_{h,L}^0 + TC_L(\epsilon_0, \mu_0, \epsilon_\infty, \epsilon_q, \omega_0, \nu)\left(\Delta t^2 + \Delta y^2 + \Delta x^2\right), \tag{5.23}$$

where the energy of the error at time $t^n = n\Delta t$, $\mathcal{ER}_{h,L}^n$, *is defined as*

$$\begin{aligned} \mathcal{ER}_{h,L}^n = &\Bigg\{ \mu_0(\mathcal{H}^{n+\frac{1}{2}}, \mathcal{H}^{n-\frac{1}{2}})_H + ||\sqrt{\epsilon_0\epsilon_\infty}\mathcal{E}^n||_E^2 + \left|\left|\frac{1}{\sqrt{\epsilon_0\epsilon_\infty(\epsilon_q - 1)}}\mathcal{P}^n\right|\right|_E^2 \\ &+ \left|\left|\frac{1}{\sqrt{\epsilon_0\omega_p}}\mathcal{J}_P^n\right|\right|_E^2 \Bigg\}^{\frac{1}{2}}. \end{aligned} \tag{5.24}$$

Proof. We follow a similar procedure to the convergence analysis for the Yee scheme for the Maxwell- Debye model in Theorem 4.2. We multiply the average of (5.22a) at n and $n+1$ by $\Delta x\Delta y\mathcal{H}^{n+\frac{1}{2}}$ and sum over all spatial nodes on τ_h^H, multiply (5.22b) by $\Delta x\Delta y\overline{\mathcal{E}}^{n+\frac{1}{2}}$ and sum over all spatial nodes on $\tau_h^{E_x} \times \tau_h^{E_y}$, multiply (5.22c) by $\Delta x\Delta y\overline{\mathcal{J}}_P^{n+\frac{1}{2}}$ and sum over all spatial nodes on $\tau_h^{E_x} \times \tau_h^{E_y}$, multiply (5.22d) by $\Delta x\Delta y\overline{\mathcal{P}}^{n+\frac{1}{2}}$ and sum over all spatial nodes on $\tau_h^{E_x} \times \tau_h^{E_y}$ and add the results to obtain

$$\begin{aligned} \delta_t\mathcal{ER}_{h,L}^{n+\frac{1}{2}}\overline{\mathcal{ER}_{h,L}}^{n+\frac{1}{2}} = &\frac{-1}{\tau\epsilon_0\omega_p^2}||\overline{\mathcal{J}}_P^{n+\frac{1}{2}}||_E^2 + (\overline{\xi}_H^{n+\frac{1}{2}}, \mathcal{H}^{n+\frac{1}{2}})_H \\ &+ (\xi_E^{n+\frac{1}{2}}, \overline{\mathcal{E}}^{n+\frac{1}{2}})_E + (\xi_J^{n+\frac{1}{2}}, \overline{\mathcal{J}}_P^{n+\frac{1}{2}})_E, +(\xi_P^{n+\frac{1}{2}}, \overline{\mathcal{P}}^{n+\frac{1}{2}})_E, \end{aligned} \tag{5.25}$$

Expanding (5.25) we have

$$\begin{aligned} \delta_t\mathcal{ER}_{h,L}^{n+\frac{1}{2}}\overline{\mathcal{ER}_{h,L}}^{n+\frac{1}{2}} \leq &||\overline{\xi}_H^{n+\frac{1}{2}}||_H||\mathcal{H}^{n+\frac{1}{2}}||_H + \frac{1}{2}||\xi_E^{n+\frac{1}{2}}||_E\left(||\mathcal{E}^n||_E + ||\mathcal{E}^{n+1}||_E\right) \\ &+ \frac{1}{2}||\xi_J^{n+\frac{1}{2}}||_E\left(||\mathcal{J}_P^n||_E + ||\mathcal{J}_P^{n+1}||_E\right) + \frac{1}{2}||\xi_P^{n+\frac{1}{2}}||_E\left(||\mathcal{P}^n||_E + ||\mathcal{P}^{n+1}||_E\right). \end{aligned} \tag{5.26}$$

Substituting inequalities (4.35), (4.36), (4.37) and (4.39) in (5.26) and using the bound

$$\frac{1}{2}\left(||\mathcal{J}_P^n||_E + ||\mathcal{J}_P^{n+1}||_E\right) \leq \epsilon_0\omega_p^2\overline{\mathcal{ER}_{h,L}}^{n+\frac{1}{2}}, \tag{5.27}$$

we have

$$\left(\delta_t\mathcal{ER}_{h,L}^{n+\frac{1}{2}}\right)\overline{\mathcal{ER}_{h,L}}^{n+\frac{1}{2}} \leq C_L\max\left\{||\overline{\xi}_H^{n+\frac{1}{2}}||, ||\xi_E^{n+\frac{1}{2}}||, ||\xi_J^{n+\frac{1}{2}}||, ||\xi_P^{n+\frac{1}{2}}||\right\}\overline{\mathcal{ER}_{h,L}}^{n+\frac{1}{2}}, \tag{5.28}$$

where $C_L = C_L(\epsilon_0, \mu_0, \epsilon_\infty, \epsilon_q, \tau, \omega_0, \nu)$ is a constant that depends on the medium parameters and the Courant number ν, but is independent of the mesh parameters

$\Delta t, \Delta x, \Delta y$. Dividing by $\overline{\mathcal{ER}_{h,L}}^{n+\frac{1}{2}} \neq 0$, using recursion in time from $n+1$ to 1, $\Delta t = T/N$, the results of Lemma 5.1, and taking the maximum for n from 0 to $N-1$ we finally obtain

$$\mathcal{ER}_{h,L}^{n+1} \leq \mathcal{ER}_{h,L}^{0} + TC_L(\epsilon_0, \mu_0, \epsilon_\infty, \epsilon_q, \omega_0, \tau, \nu)\left(\Delta t^2 + \Delta y^2 + \Delta x^2\right). \tag{5.29}$$

□

Remark 5.1. *As noted in Remark 4.1 for the convergence result in Theorem 4.2 for the 2D Yee Maxwell-Debye scheme, the convergence result in Theorem 5.2 for the 2D Yee Maxwell-Lorentz scheme does not hold at the stability limit* $\nu = \frac{1}{\sqrt{2}}$.

5.5. Discrete Divergence for the Maxwell-Lorentz Model.

Lemma 5.2 (Discrete Divergence for the Maxwell-Lorentz System)**.** *For the Yee scheme applied to the Maxwell-Lorentz system given in* (5.2) *the discrete divergence of the initial grid functions is preserved for all* $n \geq 0$, *i.e. we have the identity*

$$\operatorname{div_h} \mathbf{D}^n = \operatorname{div_h} \mathbf{D}^0, \ \textit{on } \tau_h^0. \tag{5.30}$$

Proof. The proof is the same as the proof of Lemma 4.3 for Debye media. □

6. Numerical Simulations of the Yee Scheme for the Maxwell-Debye Model

We perform numerical simulations of system (4.2) on the domain $\Omega = [0,1] \times [0,1]$. For our simulation we assume a uniform mesh with $\Delta x = \Delta y = h$. We use $T = 1$, parameter values $\mu_0 = 1$, $\epsilon_0 = 1$ (i.e. $c_0 = 1$), $\epsilon_\infty = 1, \epsilon_q = 2$ ($\epsilon_s = 2$), and $\tau = 1$.

6.1. An Exact Solution for the Maxwell-Debye Model. We use an exact solution, introduced in [7], to the Maxwell-Debye system (2.13) along with PEC boundary conditions (2.1d), which we use to initialize our simulations. We define the wave vector as $\mathbf{k} = (k_x, k_y)^T$, where $k_x = \pi \tilde{k}_x$, $k_y = \pi \tilde{k}_y$, and the corresponding wave number is $|\mathbf{k}| = \sqrt{k_x^2 + k_y^2}$. We also define the function $\alpha_D(\theta, |\mathbf{k}|) := \theta^2 - \theta + |\mathbf{k}|^2$. The exact solution to the Maxwell-Debye system (2.13) with (2.1d) is

$$H = \frac{|\mathbf{k}|^2}{\pi} e^{-\theta t} \cos(k_x x) \cos(k_y y), \tag{6.1a}$$

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} -\frac{\theta}{\pi} k_y e^{-\theta t} \cos(k_x x) \sin(k_y y) \\ \frac{\theta}{\pi} k_x e^{-\theta t} \sin(k_x x) \cos(k_y y) \end{pmatrix}, \tag{6.1b}$$

$$\mathbf{P} = \begin{pmatrix} P_x \\ P_y \end{pmatrix} = \begin{pmatrix} \frac{k_y}{\pi} \alpha_D(\theta, |\mathbf{k}|) e^{-\theta t} \cos(k_x x) \sin(k_y y) \\ -\frac{k_x}{\pi} \alpha_D(\theta, |\mathbf{k}|) e^{-\theta t} \sin(k_x x) \cos(k_y y) \end{pmatrix}, \tag{6.1c}$$

where the parameter θ is a real number. We note that for $\tilde{k}_x$ and $\tilde{k}_y$ integers, the exact solution (6.1) satisfies the perfect conductor conditions (2.1d) on the boundary of the domain Ω, and the electric and polarization fields are divergence free on Ω.

The wave number $|\mathbf{k}|$ and parameter θ are related by the equation

$$\theta^3 - 2\theta^2 + |\mathbf{k}|^2 \theta - |\mathbf{k}|^2 = 0. \tag{6.2}$$

The energy defined in (2.12) for the exact solution (6.1) can be computed to be

$$\mathcal{E}_D(t) = \frac{|\mathbf{k}|\mathrm{e}^{-\theta t}}{2\pi}\sqrt{(|\mathbf{k}|^2+\theta^2+\alpha(\theta,|\mathbf{k}|)^2)}. \tag{6.3}$$

In our simulations we use various values of the Courant number ν, defined in (4.13), and various values of $k_x = k_y = k$. The real root of equation (6.2) depends on the value of the wave number $|\mathbf{k}| = \sqrt{2}k$. In particular, for $\tilde{k} = \tilde{k_x} = \tilde{k_y} = 1$, $\theta \approx 1.0532$.

The exact dispersion relation for Debye media [5, 31] relating the wave number $|\mathbf{k}|$ to the angular frequency ω is

$$|\mathbf{k}| = \frac{\omega}{c_0}\sqrt{\frac{\epsilon_s - \mathrm{i}\omega\tau\epsilon_\infty}{1-\mathrm{i}\omega\tau}}. \tag{6.4}$$

Thus, using the chosen values for the parameters $c_0 = \tau = \epsilon_\infty = 1$, and $\epsilon_s = \epsilon_q = 2$ in (6.4) and squaring both sides, the dispersion relation can be written as

$$(\mathrm{i}\omega)^3 - 2(\mathrm{i}\omega)^2 + |\mathbf{k}|^2\,\mathrm{i}\omega - |\mathbf{k}|^2 = 0. \tag{6.5}$$

As noted in [7], comparing the dispersion relation (6.5) to the relation (6.2) for real θ, we note that the exact solution (6.1) corresponds to a solution for the Maxwell-Debye system (2.13) for a purely imaginary angular frequency $\omega = -\mathrm{i}\theta$.

6.2. Relative and Energy Errors. For the discrete solution produced we compute relative errors defined as

$$\mathcal{E}_{R,D}(t^n) = \left(\|\mathbf{E}(t^n)-\mathbf{E}^n\|_E^2 + \|H(t^n)-\overline{H}^n\|_H^2 + \|\mathbf{P}(t^n)-\mathbf{P}^n\|_E^2\right)^{\frac{1}{2}}, \tag{6.6}$$

$$\text{relative error} = \max_{0\le n\le N-1}\left(\frac{\mathcal{E}_{R,D}(t^n)}{\mathcal{E}_D(t^n)}\right), \tag{6.7}$$

where the grid norms $\|\cdot\|_E$, and $\|\cdot\|_H$ are defined in (3.14) and (3.15), respectively. We also define the energy error for the discrete solutions as

$$\text{energy error} = \max_{0\le n\le N-1}\left(\left|\frac{\frac{d\mathcal{E}_D}{dt}(t^{n+\frac{1}{2}}) - \delta_t\mathcal{E}_{h,D}^{n+\frac{1}{2}}}{\frac{d\mathcal{E}_D}{dt}(t^{n+\frac{1}{2}})}\right|\right), \tag{6.8}$$

where the discrete energy $\mathcal{E}_{h,D}^n$ is defined in (4.15) and $\frac{d\mathcal{E}_D}{dt}\left(t^{n+\frac{1}{2}}\right)$ is the time derivative of the exact energy (6.3) computed at the time point $t^{n+\frac{1}{2}}$.

Table 1 presents the relative errors (6.7) and confirms the second order accuracy of the Yee scheme for various values of Δt, h, k and ν. The variable $N \in \mathbb{N}$ is the number of time steps performed, thus $N\Delta t = T$. We note that the largest value of Δt chosen (0.02) is such that $\Delta t/\tau$ ($\tau = 1$ in this example) is $\mathcal{O}(10^{-2})$ or lower. This is in agreement with results obtained in [31] which indicate that to resolve all time scales in the problem we must choose $\Delta t = \mathcal{O}(10^{-2}\tau)$ for Debye media. Table 2 presents the energy errors (6.8). The results in this table indicate that the energy error decreases in a second order accurate manner, and provides another confirmation of the second order accuracy of the Yee scheme.

In Figures 1 and 2 we plot the relative errors (6.6) and the energy errors (6.8), respectively, for the various values of N, ν and k as presented in the corresponding tables. An $\mathcal{O}(h^2)$ reference is provided to visually confirm the second order accuracy of the Yee scheme.

TABLE 1. Relative Errors for the 2D Yee Maxwell-Debye scheme.

$k = 1\pi$						
N	$\nu = 0.3$		$\nu = 0.5$		$\nu = 0.7$	
	Error	Rate	Error	Rate	Error	Rate
50	1.20×10^{-3}		4.57×10^{-4}		2.53×10^{-4}	
100	2.99×10^{-4}	2.01	1.14×10^{-4}	2.01	6.30×10^{-5}	2.00
200	7.46×10^{-5}	2.00	2.84×10^{-5}	2.00	1.57×10^{-5}	2.00
400	1.86×10^{-5}	2.00	7.10×10^{-6}	2.00	3.93×10^{-6}	2.00
800	4.65×10^{-6}	2.00	1.77×10^{-6}	2.00	9.83×10^{-7}	2.00
$k = 5\pi$						
N	$\nu = 0.3$		$\nu = 0.5$		$\nu = 0.7$	
	Error	Rate	Error	Rate	Error	Rate
50	5.39×10^{-3}		2.01×10^{-3}		1.02×10^{-3}	
100	1.39×10^{-4}	1.96	4.97×10^{-4}	2.02	2.54×10^{-4}	2.01
200	3.44×10^{-4}	2.01	1.24×10^{-4}	2.01	6.34×10^{-5}	2.00
400	8.57×10^{-5}	2.00	3.09×10^{-5}	2.00	1.58×10^{-5}	2.00
800	2.14×10^{-5}	2.00	7.72×10^{-6}	2.00	3.95×10^{-6}	2.00
$k = 10\pi$						
N	$\nu = 0.3$		$\nu = 0.5$		$\nu = 0.7$	
	Error	Rate	Error	Rate	Error	Rate
50	1.23×10^{-2}		4.08×10^{-3}		2.02×10^{-3}	
100	2.79×10^{-3}	2.14	9.75×10^{-4}	2.06	4.94×10^{-4}	2.03
200	6.74×10^{-4}	2.05	2.41×10^{-4}	2.02	1.23×10^{-4}	2.02
400	1.67×10^{-4}	2.01	6.00×10^{-5}	2.00	3.06×10^{-5}	2.00
800	4.16×10^{-5}	2.00	1.50×10^{-5}	2.00	7.66×10^{-6}	2.00

TABLE 2. Energy Errors for the 2D Yee Maxwell-Debye scheme.

$k = 1\pi$						
N	$\nu = 0.3$		$\nu = 0.5$		$\nu = 0.7$	
	Error	Rate	Error	Rate	Error	Rate
50	1.67×10^{-3}		6.44×10^{-4}		3.60×10^{-4}	
100	4.16×10^{-4}	2.01	1.60×10^{-4}	2.01	8.97×10^{-5}	2.01
200	1.04×10^{-4}	2.00	4.00×10^{-5}	2.00	2.24×10^{-5}	2.00
400	2.59×10^{-5}	2.00	9.99×10^{-6}	2.00	5.59×10^{-6}	2.00
800	6.48×10^{-6}	2.00	2.50×10^{-6}	2.00	1.40×10^{-6}	2.00
$k = 5\pi$						
N	$\nu = 0.3$		$\nu = 0.5$		$\nu = 0.7$	
	Error	Rate	Error	Rate	Error	Rate
50	6.68×10^{-3}		2.32×10^{-3}		1.20×10^{-3}	
100	1.61×10^{-3}	2.06	5.79×10^{-4}	2.01	2.99×10^{-4}	2.00
200	3.98×10^{-4}	2.02	1.44×10^{-4}	2.01	7.44×10^{-5}	2.01
400	9.91×10^{-5}	2.00	3.60×10^{-5}	2.00	1.86×10^{-5}	2.00
800	2.47×10^{-5}	2.00	9.89×10^{-6}	2.00	4.64×10^{-6}	2.00
$k = 10\pi$						
N	$\nu = 0.3$		$\nu = 0.5$		$\nu = 0.7$	
	Error	Rate	Error	Rate	Error	Rate
50	1.46×10^{-2}		5.24×10^{-3}		2.52×10^{-3}	
100	3.53×10^{-3}	2.05	1.23×10^{-3}	2.09	6.30×10^{-4}	2.00
200	8.55×10^{-4}	2.04	3.06×10^{-4}	2.01	1.56×10^{-4}	2.01
400	2.12×10^{-4}	2.01	7.62×10^{-5}	2.00	3.90×10^{-5}	2.00
800	5.27×10^{-5}	2.00	1.90×10^{-5}	2.00	9.75×10^{-6}	2.00

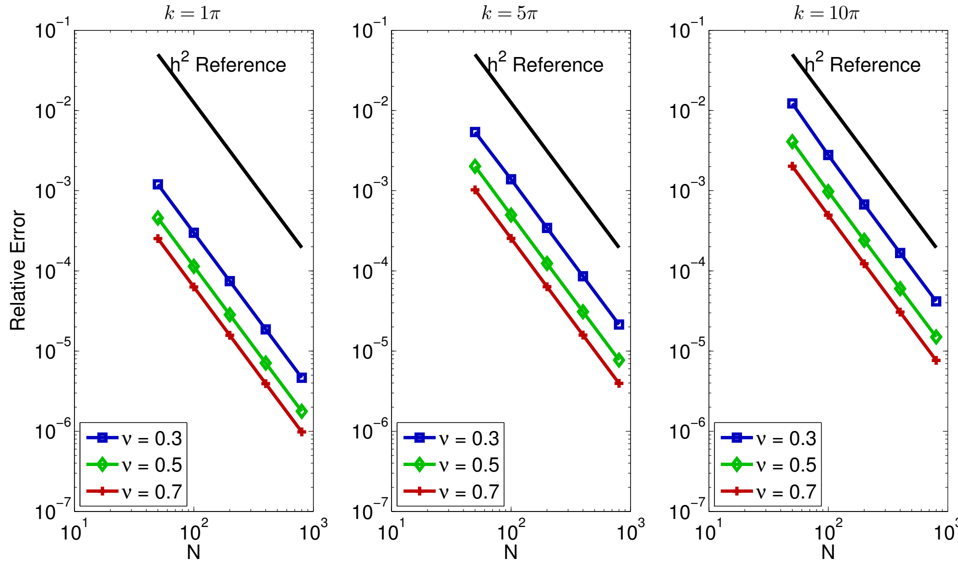

FIGURE 1. Relative errors for different wave numbers ($k_x = k_y = k$) in the 2D Yee Maxwell-Debye scheme for Courant numbers $\nu = 0.3$, 0.5 and 0.7.

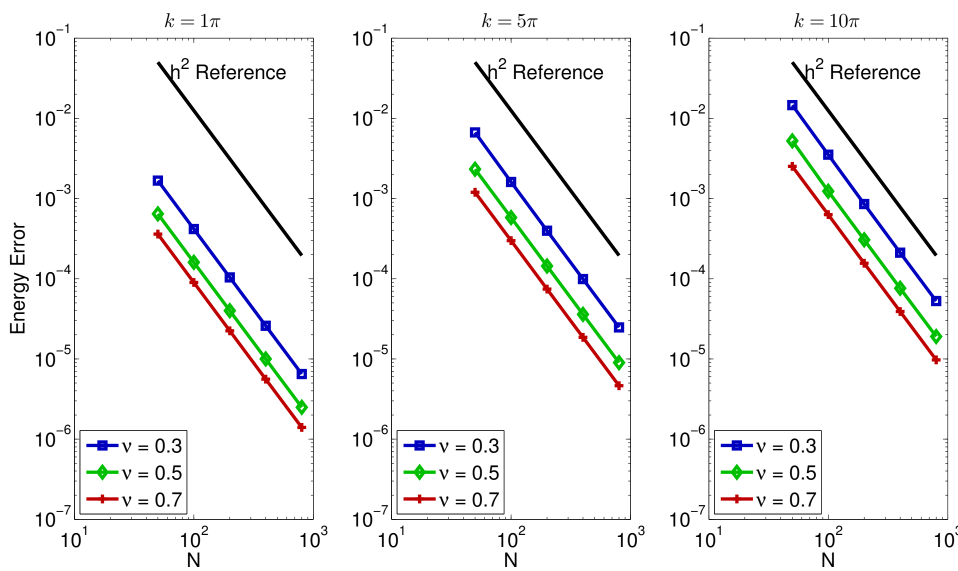

FIGURE 2. Energy errors for different wave numbers ($k_x = k_y = k$) in the 2D Yee Maxwell-Debye scheme for Courant numbers $\nu = 0.3$, 0.5 and 0.7.

6.3. Convergence Analysis of the Discrete Divergence. We verify the identity in (4.51) by computing the maximum absolute grid error in the discrete divergence as follows

$$\max_{0\leq n\leq N} \| \operatorname{div}_{\mathrm{h}} \mathbf{D}^n - \operatorname{div}_{\mathrm{h}} \mathbf{D}^0 \|_0, \tag{6.9}$$

where the grid norm $\|\cdot\|_0$ is defined in (4.48). Table 3 presents the absolute errors (6.9) of the 2D Yee Maxwell-Debye scheme for various values of Δt, h, k and ν. Again, $N \in \mathbb{N}$, with $N\Delta t = T$, refers to the number of time steps performed. All errors are sufficiently small to suggest that they are due to roundoff.

TABLE 3. Discrete Divergence Errors for the 2D Yee Maxwell-Debye scheme.

$k = 1\pi$			
N	$\nu = 0.3$	$\nu = 0.5$	$\nu = 0.7$
50	9.47×10^{-14}	1.26×10^{-13}	2.25×10^{-13}
100	2.43×10^{-13}	4.05×10^{-13}	5.82×10^{-13}
200	7.12×10^{-13}	1.15×10^{-12}	1.61×10^{-12}
400	2.04×10^{-12}	3.24×10^{-12}	4.70×10^{-12}
800	5.76×10^{-12}	9.66×10^{-12}	1.34×10^{-11}
$k = 5\pi$			
N	$\nu = 0.3$	$\nu = 0.5$	$\nu = 0.7$
50	4.66×10^{-12}	1.68×10^{-11}	1.47×10^{-11}
100	2.99×10^{-11}	5.48×10^{-11}	7.16×10^{-11}
200	7.29×10^{-11}	1.30×10^{-10}	1.98×10^{-10}
400	2.31×10^{-10}	3.83×10^{-10}	6.19×10^{-10}
800	6.86×10^{-10}	1.17×10^{-9}	1.71×10^{-9}
$k = 10\pi$			
N	$\nu = 0.3$	$\nu = 0.5$	$\nu = 0.7$
50	6.99×10^{-11}	1.14×10^{-10}	1.89×10^{-10}
100	1.15×10^{-10}	2.48×10^{-10}	5.40×10^{-10}
200	3.91×10^{-10}	1.09×10^{-9}	1.31×10^{-9}
400	1.71×10^{-9}	3.35×10^{-9}	4.91×10^{-9}
800	6.44×10^{-9}	8.73×10^{-9}	1.25×10^{-8}

7. Numerical Simulations of the Yee Scheme for the Maxwell-Lorentz Model

We perform numerical simulations of system (5.2) on the domain $\Omega = [0,1]\times[0,1]$ using exact solutions for which $\epsilon_0 = \mu_0 = \epsilon_\infty = \omega_0 = 1, \tau = 0.4$, and $\epsilon_s = \epsilon_q = 2$.

7.1. An Exact Solution for the Maxwell-Lorentz Model. We define the functions $\alpha_L(\theta, |\mathbf{k}|) := \theta^2 + 2\theta + |\mathbf{k}|^2 - 1$, and $\beta_L(\theta, |\mathbf{k}|) := \theta^2 + |\mathbf{k}|^2$. We consider the following exact solution to the Maxwell-Lorentz system (2.16) along with PEC boundary conditions (2.1d)

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} -\dfrac{\theta}{\pi} k_y e^{-\theta t} \cos(k_x x)\sin(k_y y) \\ \dfrac{\theta}{\pi} k_x e^{-\theta t} \sin(k_x x)\cos(k_y y) \end{pmatrix}, \tag{7.1a}$$

$$H = \frac{|\mathbf{k}|^2}{\pi} e^{-\theta t} \cos(k_x x)\cos(k_y y), \tag{7.1b}$$

$$\mathbf{P} = \begin{pmatrix} P_x \\ P_y \end{pmatrix} = \begin{pmatrix} \dfrac{-k_y}{\pi}\alpha_L(\theta,|\mathbf{k}|)e^{-\theta t}\cos(k_x x)\sin(k_y y) \\ \dfrac{k_x}{\pi}\alpha_L(\theta,|\mathbf{k}|)e^{-\theta t}\sin(k_x x)\cos(k_y y) \end{pmatrix}, \tag{7.1c}$$

$$\mathbf{J}_P = \begin{pmatrix} J_{P,x} \\ J_{P,y} \end{pmatrix} = \begin{pmatrix} \dfrac{-k_y}{\pi}\beta_L(\theta,|\mathbf{k}|)e^{-\theta t}\cos(k_x x)\sin(k_y y) \\ \dfrac{k_x}{\pi}\beta_L(\theta,|\mathbf{k}|)e^{-\theta t}\sin(k_x x)\cos(k_y y) \end{pmatrix}. \tag{7.1d}$$

In the above, the wave number and parameter θ are related by the equation

$$\theta^4 - \frac{1}{\tau}\theta^3 + (2 + |\mathbf{k}|^2)\theta^2 - \frac{|\mathbf{k}|^2}{\tau}\theta + |\mathbf{k}|^2 = 0. \tag{7.2}$$

As in the Debye model, for $\tilde{k}_x$ and $\tilde{k}_y$ integers, the exact solution (7.1) satisfies the perfect conductor conditions on the boundary of the domain Ω, and the electric and polarization fields are divergence free on Ω. The energy defined in (2.19) for the exact solution (7.1) can be computed to be

$$\mathcal{E}(t) = \frac{|\mathbf{k}|\mathrm{e}^{-\theta t}}{2\pi}\sqrt{\beta_L\,(1+\beta_L) + \alpha_L(\theta,|\mathbf{k}|)^2}. \tag{7.3}$$

The real root of equation (7.2) depends on the value of the wave number $|\mathbf{k}|$. In particular, for $\tilde{k} = 1$, $\theta \approx 0.5087$.

The exact dispersion relation for Lorentz media [5, 31] relating the wave number $|\mathbf{k}|$ to the angular frequency ω for the chosen values of parameters is

$$|\mathbf{k}| = \frac{\omega}{c_0}\sqrt{\frac{(\omega^2-2)\tau + \mathrm{i}\omega}{(\omega^2-1)\tau + \mathrm{i}\omega}}. \tag{7.4}$$

Squaring both sides, the dispersion relation can be written as

$$(\mathrm{i}\omega)^4 - \frac{1}{\tau}(\mathrm{i}\omega)^3 + (2+|\mathbf{k}|^2)(\mathrm{i}\omega)^2 - \frac{|\mathbf{k}|^2}{\tau}(\mathrm{i}\omega) + |\mathbf{k}|^2 = 0. \tag{7.5}$$

Comparing the disersion relation (7.5) to the relation (7.2) for real θ, we note that the exact solution (7.1) corresponds to a solution for the Maxwell-Lorentz system (2.16) for a purely imaginary angular frequency $\omega = -\mathrm{i}\theta$.

7.2. Relative and Energy Errors. For the discrete solution produced we compute relative errors defined as

$$\mathcal{E}_{R,L}(t^n) = \Big(\|\mathbf{E}(t^n) - \mathbf{E}^n\|_E^2 + \|H(t^n) - \overline{H}^n\|_H^2 + \|\mathbf{P}(t^n) - \mathbf{P}^n\|_E^2 + \|\mathbf{J}_P(t^n) - \mathbf{J}_P^n\|_E^2\Big)^{\frac{1}{2}}, \tag{7.6}$$

$$\text{relative error} = \max_{0\le n\le N}\left(\frac{\mathcal{E}_{R,L}(t^n)}{\mathcal{E}_L(t^n)}\right), \tag{7.7}$$

where the grid norms $\|\cdot\|_E$, and $\|\cdot\|_H$ are defined in (3.14) and (3.15), respectively. We also define the energy error for the discrete solutions as

$$\text{energy error} = \max_{0\le n\le N}\left(\left|\frac{\dfrac{d\mathcal{E}_L}{dt}(t^{n+\frac{1}{2}}) - \delta_t\mathcal{E}_{h,L}^{n+\frac{1}{2}}}{\dfrac{d\mathcal{E}_L}{dt}(t^{n+\frac{1}{2}})}\right|\right), \tag{7.8}$$

where the discrete energy $\mathcal{E}_{h,L}^n$ is defined in (5.12) and $\frac{d\mathcal{E}_L}{dt}\left(t^{n+\frac{1}{2}}\right)$ is the time derivative of the exact energy (7.3) computed at the time point $t^{n+\frac{1}{2}}$.

Table 4 presents the relative errors (7.7) and confirms the second order accuracy of the Yee scheme for various values of Δt, h, k and ν. The number of time steps performed is $N \in \mathbb{N}$ with $N\Delta t = T$. Table 5 presents the energy errors (7.8).

TABLE 4. Relative Errors for the 2D Yee Maxwell-Lorentz scheme.

$k = 1\pi$						
N	$\nu = 0.3$		$\nu = 0.5$		$\nu = 0.7$	
	Error	Rate	Error	Rate	Error	Rate
50	4.43×10^{-4}		1.62×10^{-4}		8.45×10^{-5}	
100	1.11×10^{-4}	2.00	4.04×10^{-5}	2.00	2.11×10^{-5}	2.00
200	2.76×10^{-5}	2.00	1.01×10^{-5}	2.00	5.27×10^{-6}	2.00
400	6.90×10^{-5}	2.00	2.52×10^{-6}	2.00	1.32×10^{-6}	2.00
800	1.72×10^{-6}	2.00	6.30×10^{-6}	2.00	3.29×10^{-7}	2.00
$k = 5\pi$						
N	$\nu = 0.3$		$\nu = 0.5$		$\nu = 0.7$	
	Error	Rate	Error	Rate	Error	Rate
50	2.42×10^{-3}		8.89×10^{-4}		4.49×10^{-4}	
100	6.14×10^{-4}	1.98	2.19×10^{-4}	2.02	1.12×10^{-4}	2.00
200	1.52×10^{-4}	2.01	5.47×10^{-5}	2.00	2.79×10^{-5}	2.00
400	3.79×10^{-5}	2.00	1.37×10^{-5}	2.00	6.97×10^{-6}	2.00
800	9.47×10^{-6}	2.00	3.41×10^{-6}	2.00	1.74×10^{-6}	2.00
$k = 10\pi$						
N	$\nu = 0.3$		$\nu = 0.5$		$\nu = 0.7$	
	Error	Rate	Error	Rate	Error	Rate
50	5.45×10^{-3}		1.81×10^{-3}		8.97×10^{-4}	
100	1.24×10^{-3}	2.14	4.34×10^{-4}	2.06	2.20×10^{-4}	2.03
200	3.00×10^{-4}	2.04	1.07×10^{-4}	2.01	5.47×10^{-5}	2.01
400	7.45×10^{-5}	2.01	2.68×10^{-5}	2.00	1.37×10^{-5}	2.00
800	1.86×10^{-5}	2.00	6.68×10^{-6}	2.00	3.41×10^{-6}	2.00

The results in this table indicate that the energy error decreases in a second order accurate manner, and provides another confirmation of the second order accuracy of the Yee scheme.

In Figures 3 and 4 we plot the relative errors (7.6) and the energy errors (7.8), respectively, for the various values of N, ν and k as presented in the corresponding tables. An $\mathcal{O}(h^2)$ reference is provided to visually confirm the second order accuracy of the Yee scheme.

7.3. Convergence Analysis of Discrete Divergence. Finally, we verify the identity in (5.30) by computing the maximum absolute grid error in the discrete divergence as defined in (6.9). Table 6 presents the absolute errors in the discrete divergence of solutions to the 2D Yee Maxwell-Lorentz scheme for various values of Δt, h, k and ν. Again, $N \in \mathbb{N}$, with $N\Delta t = T$, refers to the number of time steps performed. All errors are sufficiently small to suggest that they are due to roundoff.

8. Conclusions

In this paper, we have presented an accuracy, stability and convergence analysis of the Yee scheme for Maxwell's equations in Debye and Lorentz dispersive media using energy techniques. This research fills an important gap in the literature on Yee methods for dispersive media models by explicitly computing energy decay

TABLE 5. Energy Errors for the 2D Yee Maxwell-Lorentz scheme.

$k = 1\pi$						
N	$\nu = 0.3$		$\nu = 0.5$		$\nu = 0.7$	
	Error	Rate	Error	Rate	Error	Rate
50	2.00×10^{-4}		6.59×10^{-5}		3.55×10^{-5}	
100	4.97×10^{-5}	2.01	1.64×10^{-5}	2.01	9.14×10^{-6}	1.96
200	1.24×10^{-5}	2.00	4.10×10^{-6}	2.00	2.32×10^{-6}	1.98
400	3.10×10^{-6}	2.00	1.02×10^{-6}	2.00	5.84×10^{-7}	1.99
800	7.74×10^{-7}	2.00	2.56×10^{-7}	2.00	1.47×10^{-7}	2.00
$k = 5\pi$						
N	$\nu = 0.3$		$\nu = 0.5$		$\nu = 0.7$	
	Error	Rate	Error	Rate	Error	Rate
50	3.37×10^{-4}		1.20×10^{-4}		6.43×10^{-5}	
100	7.99×10^{-5}	2.08	2.99×10^{-5}	2.01	1.63×10^{-5}	1.98
200	1.97×10^{-5}	2.02	7.44×10^{-6}	2.00	4.07×10^{-6}	2.00
400	4.92×10^{-6}	2.00	1.86×10^{-6}	2.00	1.02×10^{-6}	2.00
800	1.23×10^{-6}	2.00	4.65×10^{-7}	2.00	2.55×10^{-7}	2.00
$k = 10\pi$						
N	$\nu = 0.3$		$\nu = 0.5$		$\nu = 0.7$	
	Error	Rate	Error	Rate	Error	Rate
50	4.29×10^{-4}		1.21×10^{-4}		6.91×10^{-5}	
100	8.86×10^{-5}	2.28	3.31×10^{-5}	1.87	1.75×10^{-5}	1.98
200	2.23×10^{-5}	1.99	8.22×10^{-6}	2.01	4.44×10^{-6}	1.98
400	5.50×10^{-6}	2.02	2.06×10^{-6}	2.00	1.11×10^{-6}	2.00
800	1.37×10^{-6}	2.01	5.13×10^{-7}	2.00	2.78×10^{-7}	2.00

TABLE 6. Discrete Divergence Errors for the 2D Yee Maxwell-Lorentz scheme.

$k = 1\pi$			
N	$\nu = 0.3$	$\nu = 0.5$	$\nu = 0.7$
50	1.71×10^{-13}	2.96×10^{-13}	3.59×10^{-13}
100	4.10×10^{-13}	6.91×10^{-13}	1.02×10^{-12}
200	1.22×10^{-12}	2.05×10^{-12}	2.83×10^{-12}
400	3.34×10^{-12}	5.78×10^{-12}	8.06×10^{-12}
800	9.82×10^{-12}	1.62×10^{-11}	2.30×10^{-11}
$k = 5\pi$			
N	$\nu = 0.3$	$\nu = 0.5$	$\nu = 0.7$
50	9.59×10^{-12}	2.35×10^{-11}	4.37×10^{-11}
100	3.39×10^{-11}	7.80×10^{-11}	1.11×10^{-10}
200	1.61×10^{-10}	2.32×10^{-10}	3.61×10^{-10}
400	5.19×10^{-10}	8.36×10^{-10}	9.91×10^{-10}
800	1.18×10^{-9}	1.96×10^{-9}	2.77×10^{-9}
$k = 10\pi$			
N	$\nu = 0.3$	$\nu = 0.5$	$\nu = 0.7$
50	1.18×10^{-10}	2.25×10^{-10}	2.66×10^{-10}
100	3.62×10^{-10}	3.83×10^{-10}	8.60×10^{-10}
200	1.18×10^{-9}	1.64×10^{-9}	2.63×10^{-9}
400	2.85×10^{-9}	5.09×10^{-9}	8.09×10^{-9}
800	9.96×10^{-9}	1.55×10^{-8}	2.29×10^{-8}

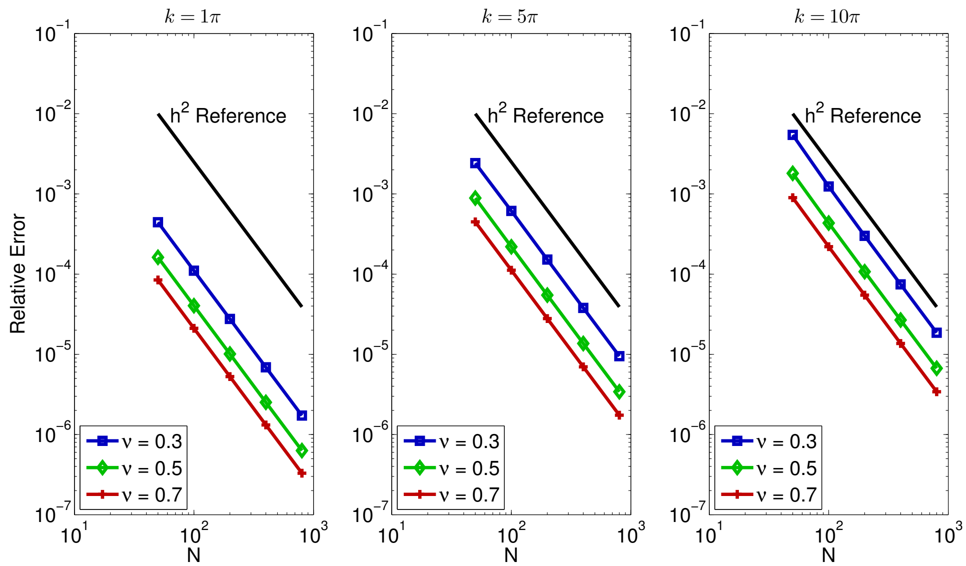

FIGURE 3. Relative errors for different wave numbers ($k_x = k_y = k$) in the 2D Yee Maxwell-Lorentz scheme for Courant numbers $\nu = 0.3$, 0.5 and 0.7.

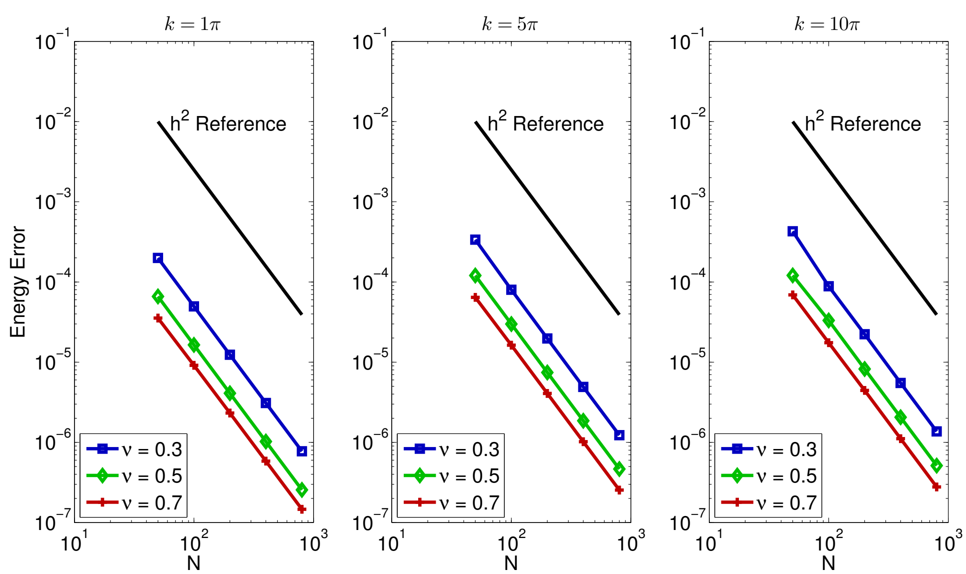

FIGURE 4. Energy errors for different wave numbers ($k_x = k_y = k$) in the 2D Yee Maxwell-Lorentz scheme for Courant numbers $\nu = 0.3$, 0.5 and 0.7.

inequalities for these methods which aid in a convergence analysis of the numerical schemes. Our analysis assumes dispersive media parameters that are constant. However, the generality of the energy analysis will allow an extension of our results to the case of parameters that are functions of space and/or time.

We have constructed novel exact solutions for the Maxwell-Debye and Maxwell-Lorentz models that illustrate our analytical results. These exact solutions will also be helpful to illustrate analyses of other numerical techniques.

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