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TWO-LEVEL PENALTY FINITE ELEMENT METHODS FOR NAVIER-STOKES EQUATIONS WITH NONLINEAR SLIP BOUNDARY CONDITIONS

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Abstract. The two-level penalty finite element methods for Navier-Stokes equations with nonlinear slip boundary conditions are investigated in this paper, whose variational formulation is the Navier-Stokes type variational inequality problem of the second kind. The basic idea is to solve the Navier-Stokes type variational inequality problem on a coarse mesh with mesh size H in combining with solving a Stokes type variational inequality problem for simple iteration or solving a Oseen type variational inequality problem for Oseen iteration on a fine mesh with mesh size h. The error estimate obtained in this paper shows that if $H = O(h^{5/9})$, then the two-level penalty methods have the same convergence orders as the usual one-level penalty finite element method, which is only solving a large Navier-Stokes type variational inequality problem on the fine mesh. Hence, our methods can save a amount of computational work.

Key words. Navier-Stokes Equations; Nonlinear Slip Boundary Conditions; Variational Inequality Problem; Penalty Finite Element Method; Two-Level Methods.

1. Introduction

Constructing efficient algorithms for solving Navier-Stokes equations is a fundamental and important problem. A difficulty lies in that the velocity and the pressure are coupled by the solenoidal condition. The popular technique to overcome this difficulty is relaxing the solenoidal condition in an appropriate method and resulting in a pesudo-compressible system, such as the penalty method introduced by Temam in [1,2], the locally stabilized methods introduced by Kechkar in [3], the pressure projection stabilized methods introduced by Bochev in [4] and Li in [5] and the references cited therein.

The other difficulty is that the Navier-Stokes equations are nonlinear. The two-level method is a very popular technique for solving the numerical solutions of the nonlinear equations. Its main idea is to solve a nonlinear problem on a coarse mesh and solving a linear problem on a fine mesh, which saves computational work for solving a nonlinear problem. There are a large amount of papers about the two-level method, such as for nonlinear partial differential equations [6-11] and especially for Navier-Stokes equations with homogeneous Dirichlet boundary conditions [12-21].

In this paper, we will consider the two-level penalty finite element methods for Navier-Stokes equations with nonlinear slip boundary conditions. Since the nonlinear boundary conditions are from the subdifferential property on the part boundary, the weak variational formulation is the variational inequality problem of the second kind with Navier-Stokes operator which is called the Navier-Stokes type variational inequality problem. This nonlinear slip boundary conditions are firstly

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introduced by Fujita in [22] and appear in the modeling of blood flow in a vein of an arterial sclerosis patient. The approach consists solving the Navier-Stokes type variational inequality problem on a coarse mesh with mesh size H in combining with solving a Stokes type variational inequality problem for simple iteration or solving a Oseen type variational inequality problem for Oseen iteration on a fine mesh with mesh size h. Denote $u^{\varepsilon h}$ and $p^{\varepsilon h}$ the penalty approximation solutions on the fine mesh. The error estimate derived in this paper is

$$||u-u^{\varepsilon h}||_V + ||p-p^{\varepsilon h}|| \leq c(\varepsilon + h^{5/4} + H^{9/4}),$$

where $c > 0$ is independent of h and H. This error estimate shows that if $H = O(h^{5/9})$ and ε is sufficiently small, the two-level penalty finite element methods have the same convergence orders as the usual one-level penalty finite element methods studied in [23]. Hence, our methods can save the CPU time and improve the computational efficiency.

2. Navier-Stokes Equations with Nonlinear Slip Boundary Conditions

Let $\psi : \mathbb{R}^2 \to \overline{\mathbb{R}} = (-\infty, +\infty]$ be a given function possessing the properties of convexity and weak semi-continuity from below (ψ is not identical with $+\infty$). The subdifferential set $\partial\psi(a)$ denotes a subdifferential of the function ψ at the point a:

$$\partial\psi(a) = \{b \in \mathbb{R}^2 : \psi(t) - \psi(a) \geq b \cdot (t-a), \quad \forall\, t \in \mathbb{R}^2\}.$$

Consider the steady Navier-Stokes equations

$$\begin{cases} -\mu\Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega \end{cases} \tag{1}$$

with the following nonlinear slip boundary conditions [22]:

$$\begin{cases} u = 0, & \text{on } \Gamma, \\ u_n = 0, \quad -\sigma_\tau(u) \in g\partial|u_\tau| & \text{on } S, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^2$, is a bounded convex domain. $\Gamma \cap S = \emptyset, \overline{\Gamma \cup S} = \partial\Omega$. The viscous coefficient $\mu > 0$ is a positive constant. g is the scalar functions; $u_n = u \cdot n$ and $u_\tau = u - u_n n$ are the normal and tangential components of the velocity, where n stands for the unit vector of the external normal to S; $\sigma_\tau(u) = \sigma - \sigma_n n$, independent of p, is the tangential component of the stress vector σ which is defined by $\sigma_i = \sigma_i(u,p) = (\mu e_{ij}(u) - p\delta_{ij})n_j$, where $e_{ij}(u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, i,j = 1,2$. From the definition of the subdifferential property, we note that the variational formulation of (1)-(2) is the variational inequality problem of the second kind with Navier-Stokes operator.

To give the variational formulation, we introduce some spaces which we will need later in this paper. Denote

$$V = \{u \in H^1(\Omega)^2,\ u|_\Gamma = 0,\ u\cdot n|_S = 0\}, \quad V_0 = H_0^1(\Omega)^2,$$

$$V_\sigma = \{u \in V, \quad \operatorname{div} u = 0\}, \qquad M = L_0^2(\Omega) = \{q \in L^2(\Omega), \int_\Omega q \mathrm{d}x = 0\}.$$

Let $||\cdot||_k$ be the norm in Hilbert space $H^k(\Omega)^2$. Let $(\cdot,\cdot)$ and $||\cdot||$ be the inner product and the norm in $L^2(\Omega)^2$. Then we can define the inner product and the norm in V by $(\nabla\cdot,\nabla\cdot)$ and $||\cdot||_V = ||\nabla\cdot||$, respectively, because $||\nabla\cdot||$ is equivalent to $||\cdot||_1$. Let $\mathbb{X}$ be a Banach space. Denote $\mathbb{X}'$ the dual space of $\mathbb{X}$ and $<\cdot,\cdot>$ be the dual pairing in $\mathbb{X}\times\mathbb{X}'$.

Introduce the following bilinear forms and trilinear form:

$$\begin{cases} a(u,v) = \mu(\nabla u, \nabla v) & \forall\ u,v \in V, \\ d(v,p) = (p, \mathrm{div} v) & \forall\ v \in V, p \in M, \\ b(u,v,w) = \displaystyle\int_\Omega (u\cdot\nabla)v\cdot w \mathrm{d}x & \forall\ u,v,w \in V. \end{cases}$$

Moreover, if $\mathrm{div} u = 0$, it is easy to verify the trilinear form $b(\cdot,\cdot,\cdot)$ satisfies

$$\begin{aligned} b(u,v,w) &= ((u\cdot\nabla)v, w) + \frac{1}{2}((\mathrm{div} u)v, w) \\ &= \frac{1}{2}((u\cdot\nabla)v, w) - \frac{1}{2}((u\cdot\nabla)w, v) \qquad \forall\ u,v,w \in V. \end{aligned}$$

Thus we have the following antisymmetric property:

$$b(u,v,w) = -b(u,w,v) \qquad \forall\ u,v,w \in V.$$

Especially, if $v = w$, there holds

$$b(u,v,v) \equiv 0 \qquad \forall\ u,v \in V.$$

Denote

$$N = \sup_{u,v,w\in V} \frac{b(u,v,w)}{\|u\|_V\|v\|_V\|w\|_V},$$

then we have

$$b(u,v,w) \le N||u||_V||v||_V||w||_V \qquad \forall\ u,v,w \in V.$$

The weak formulation associated with the problem (1)-(2) is the following variational inequality problem of the second kind with Navier-Stokes operator:

$$(3) \qquad \begin{cases} \text{find } (u,p) \in V\times M \text{ such that} \\ a(u,v-u) + b(u,u,v-u) + j(v_\tau) - j(u_\tau) \\ \qquad\qquad -d(v-u,p) \ge (f, v-u) \quad \forall\ v \in V, \\ d(u,q) = 0 \qquad \forall\ q \in M, \end{cases}$$

where $j(\eta) = \displaystyle\int_S g|\eta|\mathrm{d}s$. We call (3) the Navier-Stokes type variational inequality problem. In [24], Saito shows that there exists some positive $\beta > 0$ such that

$$\beta||q|| \le \sup_{v\in V} \frac{d(v,q)}{||v||_V},$$

then the variational inequality (3) is equivalent to

$$(4) \qquad \begin{cases} \text{find } u \in V_\sigma \text{ such that} \\ a(u,v-u) + b(u,u,v-u) + j(v_\tau) - j(u_\tau) \ge (f, v-u) \qquad \forall\ v \in V_\sigma. \end{cases}$$

We recall the following existence and uniqueness theorem of the solution of the variational inequality problem (4) established in [23]. Moreover, the condition (5) is called the uniqueness condition.

Theorem 2.1 *Given $f \in L^2(\Omega)^2$ and $g \in L^2(S)$. If*

$$(5) \qquad \frac{4\kappa_1 N(\|f\| + \|g\|_{L^2(S)})}{\mu^2} < 1,$$

then the variational inequality problem (4) has a unique solution $u \in K_\sigma = \{v \in V_\sigma :\ \|v\|_V \le \frac{2\kappa_1}{\mu}(\|f\| + \|g\|_{L^2(S)})\}$, where $\kappa_1 > 0$ satisfies

$$|(f,v) - j(v_\tau)| \le \kappa_1(||f|| + ||g||_{L^2(S)})||v||_V \qquad \forall\ v \in V_\sigma.$$

In the problem (4), the space V_σ is the kernel space of the divergence operator, which brings the difficulty to the numerical computations. In the problem (3), the second equation doesn't include the pressure p. Thus, the total stiff matrix from the finite element approximation is a non positive definite matrix, which also is difficult in the numerical computations. If we add a positive definite term in the second equation of (3) which is associated with the pressure p, then the difficulty can be overcome. Now, we give the penalty problem of the problem (1)-(2), which is to approximate the solution (u, p) by $(u^\varepsilon, p^\varepsilon)$ satisfying the following penalty Navier-Stokes equations with nonlinear slip boundary conditions

$$\left\{\begin{array}{ll} -\mu\Delta u^\varepsilon + (u^\varepsilon\cdot\nabla)u^\varepsilon + \nabla p^\varepsilon = f & \text{in } \Omega, \\ \varepsilon p^\varepsilon + \mathrm{div} u^\varepsilon = 0 & \text{in } \Omega \\ u^\varepsilon = 0, & \text{on } \Gamma, \\ u^\varepsilon_n = 0, \quad -\sigma_\tau(u^\varepsilon) \in g\partial|u^\varepsilon_\tau| & \text{on } S, \end{array}\right. \tag{6}$$

where $0 < \varepsilon < 1$ is the penalty parameter. Then the weak variational formulation associated with the penalty problem (6) is

$$\left\{\begin{array}{l} \text{find } (u^\varepsilon, p^\varepsilon) \in V\times M \text{ such that} \\ a(u^\varepsilon, v-u^\varepsilon) + b(u^\varepsilon, u^\varepsilon, v-u^\varepsilon) - d(v-u^\varepsilon, p^\varepsilon) \\ \qquad\qquad + j(v_\tau) - j(u^\varepsilon_\tau) \ge (f, v-u^\varepsilon) \quad \forall\, v\in V, \\ \varepsilon c(p^\varepsilon, q) + d(u^\varepsilon, q) = 0 \qquad \forall\, q\in M, \end{array}\right. \tag{7}$$

where

$$c(p^\varepsilon, q) = \int_\Omega p^\varepsilon q \mathrm{d}x \qquad \forall\, q\in M.$$

The following error between the solutions (u, p) and the penalty approximation solution $(u^\varepsilon, p^\varepsilon)$ has been showed with respect to the penalty parameter ε in [23]:

$$||u-u^\varepsilon||_V + ||p-p^\varepsilon|| \le c\varepsilon, \tag{8}$$

where $c > 0$ is independent of ε.

3. Penalty Finite Element Approximation

In this section, we will give the conforming finite element approximation for the variational inequality problem (7). Let τ_h be a family of regular partitions of Ω into triangles of diameter not greater than $0 < h < 1$.

Let $P_r(K)$ be the space of the polynomials on K of degree at most r. The finite element subspaces of V and M are defined by

$$V_h = V\cap W_h \quad \text{with} \quad W_h = \{v_h \in C(\overline{\Omega})^2 \ : \ v_h|_K \in [P_2(K)]^2 \quad \forall\, K\in\tau_h\}$$

and

$$M_h = \{q_h \in C(\overline{\Omega}) \ : \ q_h|_K \in P_1(K) \quad \forall\, K\in\tau_h, \ \int_\Omega q_h dx = 0\}.$$

These finite element spaces of the velocity and the pressure are composed of the Taylor-Hood element. It is well known that the bilinear form $d(\cdot,\cdot) : V_h\times M_h \longrightarrow \mathbb{R}$ satisfies the discrete inf-sup condition. The finite element approximation formulation of the variational inequality problem (7) is

$$\left\{\begin{array}{l} \text{find } (u^\varepsilon_h, p^\varepsilon_h) \in V_h\times M_h \text{ such that} \\ a(u^\varepsilon_h, v_h-u^\varepsilon_h) + b(u^\varepsilon_h, u^\varepsilon_h, v_h-u^\varepsilon_h) + j(v_{h\tau}) - j(u^\varepsilon_{h\tau}) \\ \qquad\qquad -d(v_h-u^\varepsilon_h, p^\varepsilon_h) \ge (f, v_h-u^\varepsilon_h) \qquad \forall\, v_h\in V_h, \\ d(u^\varepsilon_h, q_h) + \varepsilon c(p^\varepsilon_h, q_h) = 0 \quad \forall\, q_h\in M_h, \end{array}\right. \tag{9}$$

Let

$$\mathcal{B}_h(u^\varepsilon_h, p^\varepsilon_h; v_h, q_h) = a(u^\varepsilon_h, v_h) - d(v_h, p^\varepsilon_h) + d(u^\varepsilon_h, q_h) + \varepsilon c(p^\varepsilon_h, q_h),$$

then the discrete problem (9) can be rewritten as

$$\mathcal{B}_h(u_h^\varepsilon, p_h^\varepsilon; v_h - u_h^\varepsilon, q_h - p_h^\varepsilon) + b(u_h^\varepsilon, u_h^\varepsilon, v_h - u_h^\varepsilon) + j(v_{h\tau}) - j(u_{h\tau}^\varepsilon) \geq (f, v_h - u_h^\varepsilon). \tag{10}$$

The existence and uniqueness of the solution of the discrete problem (10) and the error estimate between $(u^\varepsilon, p^\varepsilon)$ and $(u_h^\varepsilon, p_h^\varepsilon)$ have been showed in [23].

Theorem 3.1 *Suppose that the uniqueness condition (5) holds, then the discrete problem (10) has a unique solution* $(u_h^\varepsilon, p_h^\varepsilon) \in K_h$, *where*

$$K_h = \Big\{(v_h, q_h) \in (V_h, M_h), \quad ||v_h||_V \leq \frac{2\kappa_1}{\mu}(||f|| + ||g||_{L^2(S)}),$$

$$\varepsilon^{\frac{1}{2}}||q_h|| \leq \frac{2\kappa_1(||f|| + ||g||_{L^2(S)})}{\mu^{\frac{1}{2}}}\Big\}.$$

Theorem 3.2 *Let* $(u^\varepsilon, p^\varepsilon) \in V \times M$ *and* $(u_h^\varepsilon, p_h^\varepsilon) \in V_h \times M_h$ *be the weak solution of (7) and (10), respectively. Suppose that the solution* $(u^\varepsilon, p^\varepsilon)$ *satisfies* $(u^\varepsilon, p^\varepsilon) \in H^3(\Omega)^2 \cap V \times H^2(\Omega) \cap M$, *then there holds*

$$||u^\varepsilon - u_h^\varepsilon|| + h\|u^\varepsilon - u_h^\varepsilon\|_V + h\|p^\varepsilon - p_h^\varepsilon\| \leq ch^{9/4}, \tag{11}$$

where $c > 0$ *is independent of* h.

4. Two-Level Penalty Finite Element Methods

In this section, based on the simple iteration method and the Oseen iteration method on the fine mesh, the two-level penalty finite element methods are constructed. Let τ_H and τ_h be the family of regular partitions of Ω into triangles of diameter not great than H and h, where $0 < h < H < 1$. The finite element approximation spaces (V_H, M_H) and (V_h, M_h) with respect to the partition τ_H and τ_h, respectively, are constructed as in Section 3. Firstly, we consider the following simple two-level penalty finite element method.

Algorithm 4.1 Simple Two-Level Penalty Finite Element Method

The simple two-level penalty finite element method is constructed in terms of the simple iteration for solving Navier-Stokes problem.

Step I: Solve the Navier-Stokes type variational inequality problem on the coarse mesh, i.e., find $(u_H^\varepsilon, p_H^\varepsilon) \in (V_H, M_H)$ such that for all $(v_H, q_H) \in (V_H, M_H)$, there holds

$$\begin{aligned}\mathcal{B}_H(u_H^\varepsilon, p_H^\varepsilon; v_H - u_H^\varepsilon, q_H - p_H^\varepsilon) &+ b(u_H^\varepsilon, u_H^\varepsilon, v_H - u_H^\varepsilon) \\ &+ j(v_{H\tau}) - j(u_{H\tau}^\varepsilon) \geq (f, v_H - u_H^\varepsilon).\end{aligned} \tag{12}$$

Step II: Solve the Stokes type variational inequality problem on the fine mesh, i.e., find $(u^{\varepsilon h}, p^{\varepsilon h}) \in (V_h, M_h)$ such that for all $(v_h, q_h) \in (V_h, M_h)$, there holds

$$\begin{aligned}\mathcal{B}_h(u^{\varepsilon h}, p^{\varepsilon h}; v_h - u^{\varepsilon h}, q_h - p^{\varepsilon h}) &+ b(u_H^\varepsilon, u_H^\varepsilon, v_h - u^{\varepsilon h}) \\ &+ j(v_{h\tau}) - j(u_\tau^{\varepsilon h}) \geq (f, v_h - u^{\varepsilon h}).\end{aligned} \tag{13}$$

According to Theorem 3.2, for sufficiently smooth solution u^ε and p^ε, the following error estimate with respect to the coarse mesh size H holds:

$$||u^\varepsilon - u_H^\varepsilon|| + H||u^\varepsilon - u_H^\varepsilon||_V + H||p^\varepsilon - p_H^\varepsilon|| \leq cH^{9/4}, \tag{14}$$

where $c > 0$ is independent of H. The main objective of two-level methods is approximating u^ε and p^ε by the approximation solution $u^{\varepsilon h}$ and $p^{\varepsilon h}$ on the fine mesh. Thus, we will study the convergence order of $(u^{\varepsilon h}, p^{\varepsilon h})$ to $(u^\varepsilon, p^\varepsilon)$.

Theorem 4.1 *Under the conditions in Theorem 2.1 and Theorem 3.1-3.2 for H and h, if the solution $(u^\varepsilon, p^\varepsilon)$ of the problem (7) satisfies $(u^\varepsilon, p^\varepsilon) \in H^3(\Omega)^2 \cap V \times H^2(\Omega) \cap M$, then the following error estimate holds:*

$$||u^\varepsilon - u^{\varepsilon h}||_V + ||p^\varepsilon - p^{\varepsilon h}|| \le c(h^{5/4} + H^{9/4}), \tag{15}$$

where $u^{\varepsilon h}$ and $p^{\varepsilon h}$ are the solution of (13), $c > 0$ is independent of h and H.

Proof For all $v_h \in V_h, q_h \in M_h$, by the definition of the bilinear form $\mathcal{B}_h$, we have

$$\begin{aligned}
&\mu||u^{\varepsilon h} - v_h||_V^2 + \varepsilon||p^{\varepsilon h} - q_h||^2 \\
&= \mathcal{B}_h(u^{\varepsilon h} - v_h, p^{\varepsilon h} - q_h; u^{\varepsilon h} - v_h, p^{\varepsilon h} - q_h) \\
&= \mathcal{B}_h(u^{\varepsilon h}, p^{\varepsilon h}; u^{\varepsilon h} - v_h, p^{\varepsilon h} - q_h) - \mathcal{B}_h(v_h, q_h; u^{\varepsilon h} - v_h, p^{\varepsilon h} - q_h) \\
&= a(u^{\varepsilon h}, u^{\varepsilon h} - v_h) - d(u^{\varepsilon h} - v_h, p^{\varepsilon h}) + d(u^{\varepsilon h}, p^{\varepsilon h} - q_h) \\
&\quad + \varepsilon c(p^{\varepsilon h}, p^{\varepsilon h} - q_h) - \mathcal{B}_h(v_h, q_h; u^{\varepsilon h} - v_h, p^{\varepsilon h} - q_h) \\
&\le (f, u^{\varepsilon h} - v_h) + b(u_H^\varepsilon, u_H^\varepsilon, v_h - u^{\varepsilon h}) + j(v_{h\tau}) - j(u_\tau^{\varepsilon h}) \\
&\quad - \mathcal{B}_h(v_h, q_h; u^{\varepsilon h} - v_h, p^{\varepsilon h} - q_h).
\end{aligned} \tag{16}$$

Setting $v = u^{\varepsilon h}$ and $v = 2u^\varepsilon - v_h$ in (7), one has

$$a(u^\varepsilon, u^{\varepsilon h} - u^\varepsilon) + b(u^\varepsilon, u^\varepsilon, u^{\varepsilon h} - u^\varepsilon) + j(u_\tau^{\varepsilon h}) - j(u_\tau^\varepsilon) - d(u^{\varepsilon h} - u^\varepsilon, p^\varepsilon) \ge (f, u^{\varepsilon h} - u^\varepsilon)$$

and

$$a(u^\varepsilon, u^\varepsilon - v_h) + b(u^\varepsilon, u^\varepsilon, u^\varepsilon - v_h) - d(u^\varepsilon - v_h, p^\varepsilon) + j(2u_\tau^\varepsilon - v_{h\tau}) - j(u_\tau^\varepsilon) \ge (f, u^\varepsilon - v_h),$$

which gives

$$\begin{aligned}
a(u^\varepsilon, u^{\varepsilon h} - v_h) + b(u^\varepsilon, u^\varepsilon, u^{\varepsilon h} - v_h) + j(2u_\tau^\varepsilon - v_{h\tau}) - 2j(u_\tau^\varepsilon) + j(u_\tau^{\varepsilon h}) \\
- d(u^{\varepsilon h} - v_h, p^\varepsilon) \ge (f, u^{\varepsilon h} - v_h).
\end{aligned}$$

Substituting the above inequality into (16) yields

$$\begin{aligned}
&\mu||u^{\varepsilon h} - v_h||_V^2 + \varepsilon||p^{\varepsilon h} - q_h||^2 \\
&\le a(u^\varepsilon, u^{\varepsilon h} - v_h) + b(u^\varepsilon, u^\varepsilon, u^{\varepsilon h} - v_h) - d(u^{\varepsilon h} - v_h, p^\varepsilon) \\
&\quad + j(v_{h\tau}) - 2j(u_\tau^\varepsilon) + j(2u_\tau^\varepsilon - v_{h\tau}) \\
&\quad + b(u_H^\varepsilon, u_H^\varepsilon, v_h - u^{\varepsilon h}) - a(v_h, u^{\varepsilon h} - v_h) \\
&\quad + d(u^{\varepsilon h} - v_h, q_h) - d(v_h, p^{\varepsilon h} - q_h) - \varepsilon c(q_h, p^{\varepsilon h} - q_h) \\
&= a(u^\varepsilon - v_h, u^{\varepsilon h} - v_h) - d(u^{\varepsilon h} - v_h, p^\varepsilon - q_h) + d(u^\varepsilon - v_h, p^{\varepsilon h} - q_h) \\
&\quad + \varepsilon c(p^\varepsilon - q_h, p^{\varepsilon h} - q_h) + b(u^\varepsilon, u^\varepsilon, u^{\varepsilon h} - v_h) - b(u_H^\varepsilon, u_H^\varepsilon, u^{\varepsilon h} - v_h) \\
&\quad + j(v_{h\tau}) - 2j(u_\tau^\varepsilon) + j(2u_\tau^\varepsilon - v_{h\tau}) \\
&\le |a(u^\varepsilon - v_h, u^{\varepsilon h} - v_h)| + |d(u^\varepsilon - v_h, p^{\varepsilon h} - q_h) - d(u^{\varepsilon h} - v_h, p^\varepsilon - q_h)| \\
&\quad + \varepsilon c(p^\varepsilon - q_h, p^{\varepsilon h} - q_h) \\
&\quad + |b(u^\varepsilon, u^\varepsilon, u^{\varepsilon h} - v_h) - b(u_H^\varepsilon, u_H^\varepsilon, u^{\varepsilon h} - v_h)| \\
&\quad + |j(v_{h\tau}) - 2j(u_\tau^\varepsilon) + j(2u_\tau^\varepsilon - v_{h\tau})| \\
&= I_1 + \cdots + I_5,
\end{aligned} \tag{17}$$

where we use $d(u^\varepsilon, q_h) + \varepsilon c(p^\varepsilon, q_h) = 0$ for all $q_h \in M_h$. Next, we estimate the five terms of the right side of (17). According to Young inequality, we estimate I_1 as follows:

$$I_1 \le \mu||u^\varepsilon - v_h||_V||u^{\varepsilon h} - v_h||_V \le \frac{\mu}{4}||u^{\varepsilon h} - v_h||_V^2 + \mu||u^\varepsilon - v_h||_V^2, \tag{18}$$

Similarly, we estimate I_2 as follows:

$$\begin{aligned} I_2 &\le ||u^\varepsilon - v_h||_V||p^{\varepsilon h} - q_h|| + ||u^{\varepsilon h} - v_h||_V||p^\varepsilon - q_h|| \\ &\le \alpha||p^{\varepsilon h} - q_h||^2 + \frac{1}{4\alpha}||u^\varepsilon - v_h||_V^2 + \frac{\mu}{4}||u^{\varepsilon h} - v_h||_V^2 + \frac{1}{\mu}||p^\varepsilon - q_h||^2. \end{aligned} \tag{19}$$

where $\alpha > 0$ is a sufficiently small constant. Next we estimate I_3 by

$$\begin{aligned} I_3 &= \varepsilon c(p^\varepsilon - q_h, p^{\varepsilon h} - q_h) \\ &\le \frac{3\varepsilon}{4}||p^{\varepsilon h} - q_h||^2 + \frac{\varepsilon}{3}||p^\varepsilon - q_h||^2. \end{aligned} \tag{20}$$

We estimate I_4 as follows:

$$\begin{aligned} I_4 &= |b(u^\varepsilon, u^\varepsilon, u^{\varepsilon h} - v_h) - b(u_H^\varepsilon, u_H^\varepsilon, u^{\varepsilon h} - v_h)| \\ &= |b(u^\varepsilon - u_H^\varepsilon, u^\varepsilon, u^{\varepsilon h} - v_h) - b(u^\varepsilon, u^\varepsilon - u_H^\varepsilon, u^{\varepsilon h} - v_h) \\ &\quad - b(u^\varepsilon - u_H^\varepsilon, u^\varepsilon - u_H^\varepsilon, u^{\varepsilon h} - v_h)| \\ &= |b(u^\varepsilon - u_H^\varepsilon, u^\varepsilon, u^{\varepsilon h} - v_h) + b(u^\varepsilon, u^{\varepsilon h} - v_h, u^\varepsilon - u_H^\varepsilon) \\ &\quad - b(u^\varepsilon - u_H^\varepsilon, u^\varepsilon - u_H^\varepsilon, u^{\varepsilon h} - v_h)| \\ &\le (||\nabla u^\varepsilon||_{L^\infty(\Omega)} + ||u^\varepsilon||_{L^\infty(\Omega)})||u^\varepsilon - u_H^\varepsilon|| \cdot ||u^{\varepsilon h} - v_h||_V \\ &\quad + N||u^\varepsilon - u_H^\varepsilon||_V^2||u^{\varepsilon h} - v_h||_V \\ &\le c(||u^\varepsilon - u_H^\varepsilon|| + ||u^\varepsilon - u_H^\varepsilon||_V^2)||u^{\varepsilon h} - v_h||_V \\ &\le \frac{\mu}{4}||u^{\varepsilon h} - v_h||_V^2 + c(||u^\varepsilon - u_H^\varepsilon||^2 + ||u^\varepsilon - u_H^\varepsilon||_V^4), \end{aligned} \tag{21}$$

where $c > 0$ is independent of h and H. For I_5, we can easily obtain

$$I_5 \le c||u^\varepsilon - v_h||_{L^2(S)}. \tag{22}$$

Substituting (18)-(22) into (17), we can obtain

$$\begin{aligned} &\frac{\mu}{4}||u^{\varepsilon h} - v_h||_V^2 + \frac{\varepsilon}{4}||p^{\varepsilon h} - q_h||^2 \\ &\le (\mu + \frac{1}{4\alpha})||u^\varepsilon - v_h||_V^2 + (\frac{1}{\mu} + \frac{\varepsilon}{3})||p^\varepsilon - q_h||^2 + \alpha||p^{\varepsilon h} - q_h||^2 \\ &\quad + c(||u^\varepsilon - u_H^\varepsilon||^2 + ||u^\varepsilon - u_H^\varepsilon||_V^4 + ||u^\varepsilon - v_h||_{L^2(S)}), \end{aligned}$$

By triangle inequality ,we have

$$\begin{aligned} &\mu||u^\varepsilon - u^{\varepsilon h}||_V^2 + \varepsilon||p^\varepsilon - p^{\varepsilon h}||^2 \\ &\le 2\mu||u^\varepsilon - v_h||_V^2 + 2\mu||u^{\varepsilon h} - v_h||_V^2 + 2\varepsilon||p^\varepsilon - q_h||^2 + 2\varepsilon||p^{\varepsilon h} - q_h||^2 \\ &\le (10\mu + \frac{2}{\alpha})||u^\varepsilon - v_h||_V^2 + (\frac{8}{\mu} + \frac{14\varepsilon}{3})||p^\varepsilon - q_h||^2 + 8\alpha||p^{\varepsilon h} - q_h||^2 \\ &\quad + c(||u^\varepsilon - u_H^\varepsilon||^2 + ||u^\varepsilon - u_H^\varepsilon||_V^4 + ||u^\varepsilon - v_h||_{L^2(S)}) \\ &\le c(||u^\varepsilon - v_h||_V^2 + ||p^\varepsilon - q_h||^2 + ||u^\varepsilon - u_H^\varepsilon||^2 \\ &\quad + ||u^\varepsilon - u_H^\varepsilon||_V^4 + ||u^\varepsilon - v_h||_{L^2(S)}) + 8\alpha||p^{\varepsilon h} - q_h||^2, \end{aligned} \tag{23}$$

where $c > 0$ is a constant which depends on μ. Now, we estimate $||p^{\varepsilon h} - q_h||$. Let $w_h \in \widetilde{V}_h = W_h \cap V_0$. Setting $v = u^\varepsilon \pm w_h$ in (7) yields

$$a(u^\varepsilon, w_h) + b(u^\varepsilon, u^\varepsilon, w_h) - d(w_h, p^\varepsilon) \ge (f, w_h) \qquad \forall\ w_h \in \widetilde{V}_h.$$

and

$$a(u^\varepsilon, -w_h) + b(u^\varepsilon, u^\varepsilon, -w_h) - d(-w_h, p^\varepsilon) \geq (f, -w_h) \qquad \forall\ w_h \in \widetilde{V}_h.$$

Thus

$$a(u^\varepsilon, w_h) + b(u^\varepsilon, u^\varepsilon, w_h) - d(w_h, p^\varepsilon) = (f, w_h) \qquad \forall\ w_h \in \widetilde{V}_h.$$

Similarly, from (13) one has

$$a(u^{\varepsilon h}, w_h) + b(u_H^\varepsilon, u_H^\varepsilon, w_h) - d(w_h, p^{\varepsilon h}) = (f, w_h) \qquad \forall\ w_h \in \widetilde{V}_h.$$

Hence

$$a(u^\varepsilon - u^{\varepsilon h}, w_h) + b(u^\varepsilon, u^\varepsilon, w_h) - b(u_H^\varepsilon, u_H^\varepsilon, w_h) = d(w_h, p^\varepsilon - p^{\varepsilon h}) \qquad \forall\ w_h \in \widetilde{V}_h.$$

So, for all $w_h \in \widetilde{V}_h$,

$$\begin{aligned}
&d(w_h, p^{\varepsilon h} - q_h) \\
&= d(w_h, p^{\varepsilon h} - p^\varepsilon) + d(w_h, p^\varepsilon - q_h) \\
&= a(u^{\varepsilon h} - u^\varepsilon, w_h) + b(u_H^\varepsilon, u_H^\varepsilon, w_h) - b(u^\varepsilon, u^\varepsilon, w_h) + d(w_h, p^\varepsilon - q_h) \\
&= a(u^{\varepsilon h} - u^\varepsilon, w_h) + b(u^\varepsilon - u_H^\varepsilon, u^\varepsilon - u_H^\varepsilon, w_h) - b(u^\varepsilon - u_H^\varepsilon, u^\varepsilon, w_h) \\
&\quad - b(u^\varepsilon, w_h, u^\varepsilon - u_H^\varepsilon) + d(w_h, p^\varepsilon - q_h) \\
&\leq \mu||u^{\varepsilon h} - u^\varepsilon||_V||w_h||_V + c(||u^\varepsilon - u_H^\varepsilon|| + ||u^\varepsilon - u_H^\varepsilon||_V^2)||w_h||_V + ||p^\varepsilon - q_h||||w_h||_V.
\end{aligned}$$

In addition, for all $q_h \in M_h$, in terms of the discrete inf-sup condition, we have

$$\begin{aligned}
\beta||p^{\varepsilon h} - q_h|| &\leq \sup_{w_h \in \widetilde{V}_h} \frac{d(w_h, p^{\varepsilon h} - q_h)}{||w_h||_V} \\
&\leq \mu||u^{\varepsilon h} - u^\varepsilon||_V + c(||u^\varepsilon - u_H^\varepsilon|| + ||u^\varepsilon - u_H^\varepsilon||_V^2) + ||p^\varepsilon - q_h||.
\end{aligned} \tag{24}$$

Substituting (24) into (23), it yields

$$\begin{aligned}
||u^\varepsilon - u^{\varepsilon h}||_V &\leq c(||u^\varepsilon - v_h||_V + ||p^\varepsilon - q_h|| + ||u^\varepsilon - u_H^\varepsilon|| \\
&\quad + ||u^\varepsilon - u_H^\varepsilon||_V^2 + ||u^\varepsilon - v_h||_{L^2(S)}^{\frac{1}{2}}) + c\alpha^{\frac{1}{2}}||u^{\varepsilon h} - u^\varepsilon||_V.
\end{aligned} \tag{25}$$

Then for sufficiently small $\alpha > 0$, one has

$$\begin{aligned}
||u^\varepsilon - u^{\varepsilon h}||_V &\leq c(||u^\varepsilon - v_h||_V + ||p^\varepsilon - q_h|| + ||u^\varepsilon - u_H^\varepsilon|| \\
&\quad + ||u^\varepsilon - u_H^\varepsilon||_V^2 + ||u^\varepsilon - v_h||_{L^2(S)}^{\frac{1}{2}}) \\
&\leq c(h^{5/4} + H^{9/4}),
\end{aligned} \tag{26}$$

where we use

$$||u^\varepsilon - v_h||_{L^2(S)} \leq c||u^\varepsilon - v_h||^{1/2}||u^\varepsilon - v_h||_V^{1/2}.$$

From (24) and triangle inequality, we have

$$\begin{aligned}
||p^\varepsilon - p^{\varepsilon h}|| &\leq ||p^\varepsilon - q_h|| + ||p^{\varepsilon h} - q_h|| \\
&\leq c(||u^\varepsilon - v_h||_V + ||p^\varepsilon - q_h|| + ||u^\varepsilon - u_H^\varepsilon|| \\
&\quad + ||u^\varepsilon - u_H^\varepsilon||_V^2 + ||u^\varepsilon - v_h||_{L^2(S)}^{\frac{1}{2}}) \\
&\leq c(h^{5/4} + H^{9/4}). \qquad \square
\end{aligned}$$

Algorithm 4.2 Oseen Two-Level Penalty Finite Element Method

The second two-level penalty finite element method is constructed in terms of the Oseen iteration for solving Navier-Stokes problem.

Step I: Solve the Navier-Stokes type variational inequality problem on the coarse mesh, i.e., find $(u^\varepsilon_H, p^\varepsilon_H) \in (V_H, M_H)$ such that for all $(v_H, q_H) \in (V_H, M_H)$ there holds

$$\mathcal{B}_H(u^\varepsilon_H, p^\varepsilon_H; v_H - u^\varepsilon_H, q_H - p^\varepsilon_H) + b(u^\varepsilon_H, u^\varepsilon_H, v_H - u^\varepsilon_H) \\ + j(v_{H\tau}) - j(u^\varepsilon_{H\tau}) \geq (f, v_H - u^\varepsilon_H). \tag{27}$$

Step II: Solve the Oseen type variational inequality problem on the fine mesh, i.e., find $(u^{\varepsilon h}, p^{\varepsilon h}) \in (V_h, M_h)$ such that for all $(v_h, q_h) \in (V_h, M_h)$ there holds

$$\mathcal{B}_h(u^{\varepsilon h}, p^{\varepsilon h}; v_h - u^{\varepsilon h}, q_h - p^{\varepsilon h}) + b(u^\varepsilon_H, u^{\varepsilon h}, v_h - u^{\varepsilon h}) \\ + j(v_{h\tau}) - j(u^{\varepsilon h}_\tau) \geq (f, v_h - u^{\varepsilon h}). \tag{28}$$

For the Oseen two-level penalty finite element method, we obtain the following convergence order of $(u^{\varepsilon h}, p^{\varepsilon h})$ to $(u^\varepsilon, p^\varepsilon)$.

Theorem 4.2 *Under the conditions in Theorem 2.1 and Theorem 3.1-3.2 for H and h, if the solution $(u^\varepsilon, p^\varepsilon)$ of the problem (7) satisfies $(u^\varepsilon, p^\varepsilon) \in H^3(\Omega)^2 \cap V \times H^2(\Omega) \cap M$, then the following error estimate holds:*

$$||u^\varepsilon - u^{\varepsilon h}||_V + ||p^\varepsilon - p^{\varepsilon h}|| \leq c(h^{5/4} + H^{9/4}), \tag{29}$$

where $u^{\varepsilon h}$ and $p^{\varepsilon h}$ are the solution of (28), $c > 0$ is independent of h and H.

Proof Similar to the proof of Theorem 4.1, we have

$$\begin{aligned} &\mu||u^{\varepsilon h} - v_h||_V^2 + \varepsilon||p^{\varepsilon h} - q_h||^2 \\ &\leq |a(u^\varepsilon - v_h, u^{\varepsilon h} - v_h)| + |d(u^\varepsilon - v_h, p^{\varepsilon h} - q_h) - d(u^{\varepsilon h} - v_h, p^\varepsilon - q_h)| \\ &\quad + \varepsilon c(p^\varepsilon - q_h, p^{\varepsilon h} - q_h) + |b(u^\varepsilon, u^\varepsilon, u^{\varepsilon h} - v_h) - b(u^\varepsilon_H, u^{\varepsilon h}, u^{\varepsilon h} - v_h)| \\ &\quad + |j(v_{h\tau}) - 2j(u^\varepsilon_\tau) + j(2u^\varepsilon_\tau - v_{h\tau})| \\ &= I_6 + \cdots + I_{10}, \end{aligned} \tag{30}$$

As before, we estimate the five terms on the right hand side of (30) separately.

$$I_6 \leq \mu||u^\varepsilon - v_h||_V||u^{\varepsilon h} - v_h||_V \leq \frac{\mu}{12}||u^{\varepsilon h} - v_h||_V^2 + 3\mu||u^\varepsilon - v_h||_V^2, \tag{31}$$

Similarly, we estimate I_7 as follows:

$$\begin{aligned} I_7 &\leq ||u^\varepsilon - v_h||_V||p^{\varepsilon h} - q_h|| + ||u^{\varepsilon h} - v_h||_V||p^\varepsilon - q_h|| \\ &\leq \alpha||p^{\varepsilon h} - q_h||^2 + \frac{1}{4\alpha}||u^\varepsilon - v_h||_V^2 + \frac{\mu}{12}||u^{\varepsilon h} - v_h||_V^2 + \frac{3}{\mu}||p^\varepsilon - q_h||^2. \end{aligned} \tag{32}$$

where $\alpha > 0$ is a sufficiently small constant. For I_8 we have the estimate:

$$\begin{aligned} I_8 &= \varepsilon c(p^\varepsilon - q_h, p^{\varepsilon h} - q_h) \\ &\leq \frac{3\varepsilon}{4}||p^{\varepsilon h} - q_h||^2 + \frac{\varepsilon}{3}||p^\varepsilon - q_h||^2. \end{aligned} \tag{33}$$

We estimate I_9 as follows:

$$\begin{aligned} I_9 &= |b(u^\varepsilon, u^\varepsilon, u^{\varepsilon h} - v_h) - b(u^\varepsilon_H, u^{\varepsilon h}, u^{\varepsilon h} - v_h)| \\ &= |b(u^\varepsilon - u^\varepsilon_H, u^\varepsilon, u^{\varepsilon h} - v_h) - b(u^\varepsilon_H, u^\varepsilon - v_h, u^{\varepsilon h} - v_h) \\ &\quad - b(u^\varepsilon_H, v_h - u^{\varepsilon h}, u^{\varepsilon h} - v_h)| \\ &\leq c||\nabla u^\varepsilon||_{L^\infty(\Omega)}||u^\varepsilon - u^\varepsilon_H|| \cdot ||u^{\varepsilon h} - v_h||_V + N||u^\varepsilon_H||_V||u^{\varepsilon h} - v_h||_V^2 \\ &\quad + N||u^\varepsilon_H||_V||u^\varepsilon - v_h||_V||u^{\varepsilon h} - v_h||_V \\ &\leq \frac{\mu}{12}||u^{\varepsilon h} - v_h||_V^2 + c||u^\varepsilon - u^\varepsilon_H||^2 + \frac{\mu}{2}||u^{\varepsilon h} - v_h||_V^2 + c||u^\varepsilon - v_h||_V^2, \end{aligned} \tag{34}$$

where $c > 0$ is independent of h and H. For I_{10}, we can easily obtain

$$I_{10} \leq c||u^\varepsilon - v_h||_{L^2(S)}. \tag{35}$$

Substituting (31)-(35) into (30), we can obtain

$$\begin{aligned}
&\frac{\mu}{4}||u^{\varepsilon h} - v_h||_V^2 + \frac{\varepsilon}{4}||p^{\varepsilon h} - q_h||^2 \\
&\leq (3\mu + \frac{1}{4\alpha})||u^\varepsilon - v_h||_V^2 + (\frac{3}{\mu} + \frac{\varepsilon}{3})||p^\varepsilon - q_h||^2 + \alpha||p^{\varepsilon h} - q_h||^2 \\
&\quad + c(||u^\varepsilon - u_H^\varepsilon||^2 + ||u^\varepsilon - v_h||_V^2 + ||u^\varepsilon - v_h||_{L^2(S)}),
\end{aligned}$$

Using triangle inequality, we get

$$\begin{aligned}
&\mu||u^\varepsilon - u^{\varepsilon h}||_V^2 + \varepsilon||p^\varepsilon - p^{\varepsilon h}||^2 \\
&\leq 2\mu||u^\varepsilon - v_h||_V^2 + 2\mu||u^{\varepsilon h} - v_h||_V^2 + 2\varepsilon||p^\varepsilon - q_h||^2 + 2\varepsilon||p^{\varepsilon h} - q_h||^2 \\
&\leq (26\mu + \frac{2}{\alpha})||u^\varepsilon - v_h||_V^2 + (\frac{24}{\mu} + \frac{14\varepsilon}{3})||p^\varepsilon - q_h||^2 + 8\alpha||p^{\varepsilon h} - q_h||^2 \\
&\quad + c(||u^\varepsilon - u_H^\varepsilon||^2 + ||u^\varepsilon - v_h||_V^2 + ||u^\varepsilon - v_h||_{L^2(S)}) \\
&\leq c(||u^\varepsilon - v_h||_V^2 + ||p^\varepsilon - q_h||^2 + ||u^\varepsilon - u_H^\varepsilon||^2 \\
&\quad + ||u^\varepsilon - v_h||_{L^2(S)}) + 8\alpha||p^{\varepsilon h} - q_h||^2,
\end{aligned} \tag{36}$$

where $c > 0$ is a constant which is dependent of μ. Now, we estimate $||p^{\varepsilon h} - q_h||$. Using similar arguments as in the proof of Theorem 4.1, it follows that

$$a(u^\varepsilon - u^{\varepsilon h}, w_h) + b(u^\varepsilon, u^\varepsilon, w_h) - b(u_H^\varepsilon, u^{\varepsilon h}, w_h) = d(w_h, p^\varepsilon - p^{\varepsilon h}) \qquad \forall\ w_h \in \widetilde{V}_h.$$

Thus, for all $w_h \in \widetilde{V}_h$,

$$\begin{aligned}
&d(w_h, p^{\varepsilon h} - q_h) \\
&= d(w_h, p^{\varepsilon h} - p^\varepsilon) + d(w_h, p^\varepsilon - q_h) \\
&= a(u^{\varepsilon h} - u^\varepsilon, w_h) + b(u_H^\varepsilon, u^{\varepsilon h}, w_h) - b(u^\varepsilon, u^\varepsilon, w_h) + d(w_h, p^\varepsilon - q_h) \\
&= a(u^{\varepsilon h} - u^\varepsilon, w_h) + b(u_H^\varepsilon - u^\varepsilon, u^\varepsilon, w_h) - b(u_H^\varepsilon, u^\varepsilon - v_h, w_h) \\
&\quad - b(u_H^\varepsilon, v_h - u_H^\varepsilon, w_h) + d(w_h, p^\varepsilon - q_h) \\
&\leq \mu||u^{\varepsilon h} - u^\varepsilon||_V||w_h||_V + c||\nabla u^\varepsilon||_{L^\infty(\Omega)}||u^\varepsilon - u_H^\varepsilon|| \cdot ||w_h||_V \\
&\quad + N||u_H^\varepsilon||_V(||u^\varepsilon - v_h||_V + ||u^{\varepsilon h} - v_h||_V)||w_h||_V + ||p^\varepsilon - q_h||||w_h||_V.
\end{aligned}$$

In addition, for all $q_h \in M_h$, in terms of the discrete inf-sup condition, we have

$$\begin{aligned}
\beta||p^{\varepsilon h} - q_h|| &\leq \sup_{w_h \in \widetilde{V}_h} \frac{d(w_h, p^{\varepsilon h} - q_h)}{||w_h||_V} \\
&\leq c(||u^\varepsilon - u_H^\varepsilon|| + ||u^\varepsilon - v_h||_V + ||u^\varepsilon - u^{\varepsilon h}||_V) + ||p^\varepsilon - q_h||.
\end{aligned} \tag{37}$$

Substituting (37) into (36),

$$\begin{aligned}
||u^\varepsilon - u^{\varepsilon h}||_V &\leq c(||u^\varepsilon - v_h||_V + ||p^\varepsilon - q_h|| + ||u^\varepsilon - u_H^\varepsilon|| \\
&\quad + ||u^\varepsilon - v_h||_{L^2(S)}^{\frac{1}{2}}) + c\alpha^{\frac{1}{2}}||u^{\varepsilon h} - u^\varepsilon||_V.
\end{aligned} \tag{38}$$

Then for sufficiently small $\alpha > 0$, one has

$$\begin{aligned} &||u^\varepsilon - u^{\varepsilon h}||_V \\ &\le c(||u^\varepsilon - v_h||_V + ||p^\varepsilon - q_h|| + ||u^\varepsilon - u_H^\varepsilon|| + ||u^\varepsilon - v_h||_{L^2(S)}^{\frac{1}{2}}) \\ &\le c(h^{5/4} + H^{9/4}). \end{aligned} \tag{39}$$

Finally, from (37) by using triangle inequality, we get

$$\begin{aligned} ||p^\varepsilon - p^{\varepsilon h}|| &\le ||p^\varepsilon - q_h|| + ||p^{\varepsilon h} - q_h|| \\ &\le c(||u^\varepsilon - v_h||_V + ||p^\varepsilon - q_h|| + ||u^\varepsilon - u_H^\varepsilon|| + ||u^\varepsilon - v_h||_{L^2(S)}^{\frac{1}{2}}) \\ &\le c(h^{5/4} + H^{9/4}). \qquad \square \end{aligned}$$

5. Numerical Results

In this section, we will give numerical results to confirm the error analysis obtained in Section 4 and to show the advantage of the two-level methods. Since the two-level methods (12)-(13) and (27)-(28) are given in the form of the variational inequality problems which are not directly solved, the appropriate iteration algorithm must be constructed. In [25], we give the Uzawa iteration algorithm to solve the Stokes type variational inequality problem. Moreover, the numerical results show that this Uzawa iteration algorithm is stable and convergent.

For simplicity, we only give the Uzawa iteration method for solving the variational inequality problem (3). Similar schemes can be used to solve the two-level methods (12)-(13) and (27)-(28). First, there exists a multiplier $\lambda \in \Lambda$ such that the variational inequality problem (3) is equivalent to the following variational identity problem:

$$\begin{cases} a(u,v) + b(u,u,v) - d(v,p) + \int_S \lambda g v_\tau \mathrm{d}s = (f,v), & \forall\, v \in V, \\ d(u,q) = 0, & \forall\, q \in M, \\ \lambda u_\tau = |u_\tau|, & \text{a.e. on } S, \end{cases} \tag{40}$$

where $\lambda \in \Lambda = \{\gamma \in L^2(S) : \ |\gamma(x)| \le 1 \text{ a.e. on } S\}$. In this case, we can solve the problem (40) by the following Uzawa iteration scheme:

$$\lambda^0 \in \Lambda \quad \text{is given }, \tag{41}$$

then λ^n is known, we compute (u^n, p^n) and λ^{n+1} by

$$\begin{cases} a(u^n,v) + b(u^n,u^n,v) - d(v,p^n) = (f,v) - \int_S \lambda^n g v_\tau \mathrm{d}s, & \forall\, v \in V, \\ d(u^n,q) = 0, \qquad \forall\, q \in M, \end{cases} \tag{42}$$

and

$$\lambda^{n+1} = P_\Lambda(\lambda^n + \rho g u_\tau^n), \qquad \rho > 0, \tag{43}$$

where

$$P_\Lambda(\gamma) = \sup(-1, \inf(1,\gamma)), \qquad \forall\, \gamma \in L^2(S).$$

Consider the problem (1)-(2) in the fixed square domain $(0,1)\times(0,1)$ (see Figure 1). Let $\mu = 0.1$. The external force f is chosen such that the exact solution (u,p) is

$$u(x,y) = (u_1(x,y), u_2(x,y)), \quad p(x,y) = (2x-1)(2y-1),$$
$$u_1(x,y) = -x^2y(x-1)(3y-2), \quad u_2(x,y) = xy^2(y-1)(3x-2).$$

It is easy to verify the exact solution u satisfies $u = 0$ on Γ, $u \cdot \vec{n} = u_1 = 0, u_2 \neq 0$ on S_1 and $u_1 \neq 0, u \cdot \vec{n} = u_2 = 0$ on S_2. Moreover, the tangential vector τ on S_1 and S_2 are $(0, 1)$ and $(-1, 0)$. Thus, we have

$$\begin{cases} \sigma_\tau = 4\mu y^2(y-1) & \text{on } S_1, \\ \sigma_\tau = 4\mu x^2(x-1) & \text{on } S_2. \end{cases}$$

On the other hand, from the nonlinear slip boundary conditions (2), there holds

$$|\sigma_\tau| \leq g,$$

then the function g can be chosen as $g = -\sigma_\tau \geq 0$ on S_1 and S_2.

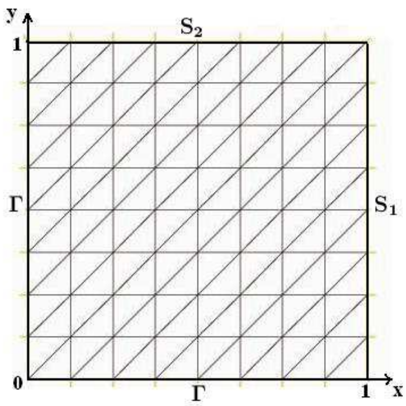

Figure 1 the domain Ω

In order to show the advantage of the two-level methods, we compare the numerical accuracy and the computational efficiency of the one-level penalty method with the two-level penalty methods. For one-level penalty method, we will solve a large Navier-Stokes type variational inequality problem on the fine mesh h. Let the iteration initial value $\lambda^0 = 1$ and the parameter $\varepsilon = 10^{-7}$, $\mu = 0.1$ and $\rho = \frac{\mu}{2}$. We pick nine coarse mesh size values, i.e.. $H = \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots, \frac{1}{20}$. In terms of the estimate (8), Theorem 4.1 and 4.2, the error estimate between the solution (u, p) and the penalty approximation solution $(u^{\varepsilon h}, p^{\varepsilon h})$ on the fine mesh is

$$||u - u^{\varepsilon h}||_V + ||p - p^{\varepsilon h}|| \leq c(\varepsilon + h^{5/4} + H^{9/4}). \tag{44}$$

In Table 1, the scaling between $1/H$ and $1/h = (1/H)^{9/5}$ is compared.

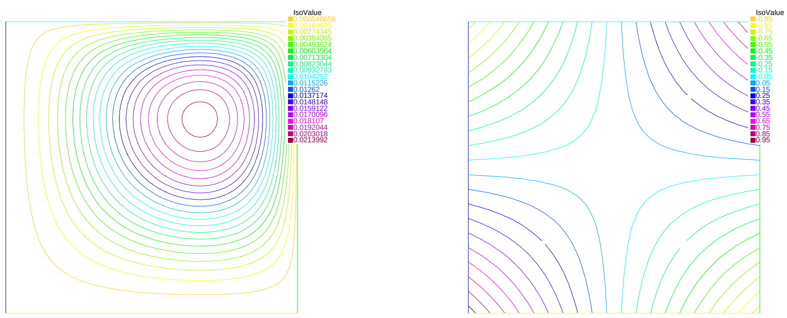

Figure 2 Exact solution

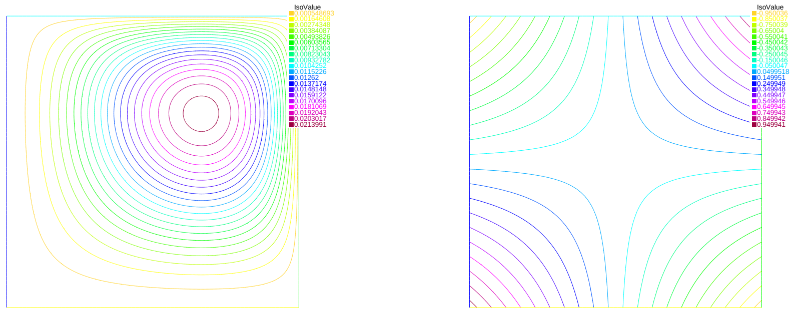

Figure 3 Simple two-level method

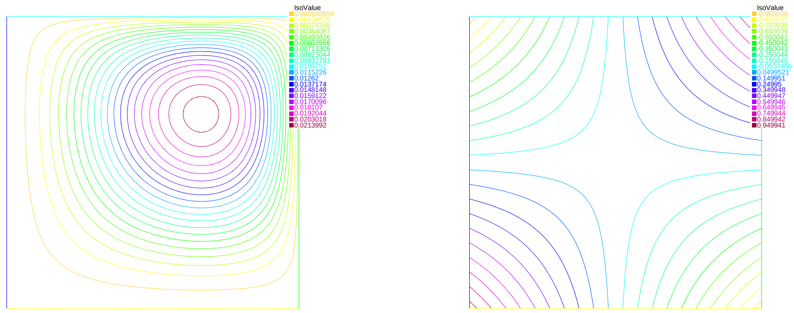

Figure 4 Oseen two-level method

Table 1. Comparison of the scaling between $1/H$ and $1/h = (1/H)^{9/5}$

$1/H$	4	6	8	10	12	14	16	18	20
$1/h$	12.125	25.157	42.224	63.095	87.604	115.619	147.033	181.756	219.712

Thus, we choose the fine mesh size $H \approx h^{5/9}$ in the numerical experiments. Then, the error estimate (44) becomes the following error estimate:

$$||u - u^{\varepsilon h}||_V + ||p - p^{\varepsilon h}|| \le ch^{5/4}.$$

Table 2. Simple two-level penalty method

$1/H$	$1/h$	$\dfrac{\lVert u-u^{\varepsilon h}\rVert_V}{\lVert u\rVert_V}$	Order	$\dfrac{\lVert p-p^{\varepsilon h}\rVert}{\lVert p\rVert}$	Order	CPU
4	12	1.51475×10^{-2}	/	5.41186×10^{-3}	/	0.234
6	25	3.29947×10^{-3}	2.076	1.25008×10^{-3}	1.996	0.807
8	42	1.14444×10^{-3}	2.041	4.44010×10^{-4}	1.995	2.177
10	63	5.05641×10^{-4}	2.015	1.97792×10^{-4}	1.994	4.801
12	87	2.65317×10^{-4}	1.998	1.03934×10^{-4}	1.994	9.219
14	115	1.52629×10^{-4}	1.981	5.96418×10^{-5}	1.991	17.481
16	147	9.42968×10^{-5}	1.962	3.66411×10^{-5}	1.984	35.569
18	181	6.27777×10^{-5}	1.955	2.42769×10^{-5}	1.978	52.625
20	219	4.34967×10^{-5}	1.925	1.67076×10^{-5}	1.961	71.955

Table 3. Oseen two-level penalty method

$1/H$	$1/h$	$\dfrac{\|\|u-u^{\varepsilon h}\|\|_V}{\|\|u\|\|_V}$	Order	$\dfrac{\|\|p-p^{\varepsilon h}\|\|}{\|\|p\|\|}$	Order	CPU
4	12	1.49636×10^{-2}	/	5.38462×10^{-3}	/	0.271
6	25	3.22149×10^{-3}	2.092	1.24190×10^{-3}	1.999	0.907
8	42	1.11106×10^{-3}	2.052	4.40321×10^{-4}	1.999	2.503
10	63	4.87693×10^{-4}	2.031	1.95827×10^{-4}	1.998	5.523
12	87	2.54326×10^{-4}	2.017	1.02739×10^{-4}	1.998	10.572
14	115	1.45272×10^{-4}	2.007	5.88402×10^{-5}	1.998	19.421
16	147	8.89883×10^{-5}	1.996	3.60465×10^{-5}	1.996	39.396
18	181	5.87691×10^{-5}	1.994	2.38038×10^{-5}	1.994	58.010
20	219	4.03241×10^{-5}	1.976	1.62917×10^{-5}	1.990	86.192

Tables 2-4 display the relative H^1 errors of the velocity and the relative L^2 errors of the pressure and their convergence orders and CPU time when we use the simple two-level penalty method, the Oseen two-level penalty method and the one-level penalty method, respectively. For one-level penalty method, when $h = \dfrac{1}{181}$, this method doesn't work and the computer display "out of memory". However, two-level penalty methods can obtain the desired numerical results. From these tables, we observe the predicted optimal convergence orders. Moreover, the two-level methods are over two times faster than the one-level method when we compare the CPU time. In addition, as we predicted, the simple two-level method is faster than the Oseen two-level method. The reason is that the term $u_H^\varepsilon \cdot \nabla u^{\varepsilon h}$ in Oseen method causes more consumed work.

Figure 2-4 show the streamline of flow and the pressure contour of the numerical solution by the simple and Oseen two-level methods and the exact solution, respectively. Figure 5 show the H^1 and L^2 convergence rates of the velocity and the pressure for two-level methods.

In conclusion, the two-level penalty finite element methods for the Navier-Stokes type variational inequality problem are the high-performance algorithm and improve the computational efficiency.

Table 4. One-level penalty method

$1/h$	$\dfrac{\|\|u-u^{\varepsilon h}\|\|_V}{\|\|u\|\|_V}$	Order	$\dfrac{\|\|p-p^{\varepsilon h}\|\|}{\|\|p\|\|}$	Order	CPU
12	1.48679×10^{-2}	/	5.37527×10^{-3}	/	0.486
25	3.19099×10^{-3}	2.097	1.23915×10^{-3}	1.999	2.053
42	1.09618×10^{-3}	2.060	4.39093×10^{-4}	1.999	6.070
63	4.79382×10^{-4}	2.040	1.95176×10^{-4}	1.999	13.918
87	2.49113×10^{-4}	2.028	1.02363×10^{-4}	1.999	27.489
115	1.41805×10^{-4}	2.019	5.86162×10^{-5}	1.998	52.079
147	8.65696×10^{-5}	2.010	3.59207×10^{-5}	1.995	108.293
181	out of memory	/	out of memory	/	

6. Conclusion

In this paper, we study the simple and Oseen two-level penalty finite element methods for the variational inequality problem of the second kind with Navier-Stokes operator. The error estimates obtained in Theorem 4.1 and 4.2 show that if $H = O(h^{5/9})$, then these two-level penalty finite element methods have the same convergence orders as the usual one-level penalty finite element method. In particular, there holds

$$||u - u^{\varepsilon h}||_V + ||p - p^{\varepsilon h}|| \leq c(\varepsilon + h^{5/4}).$$

Thus, our two-level methods can save a amount of computational work. Although we deal with the two dimensional Navier-Stokes equations with nonlinear slip boundary conditions, however, from the proof of Theorem 4.1 and Theorem 4.2, we conclude that these theoretical results can be extended to the three dimensional problem.

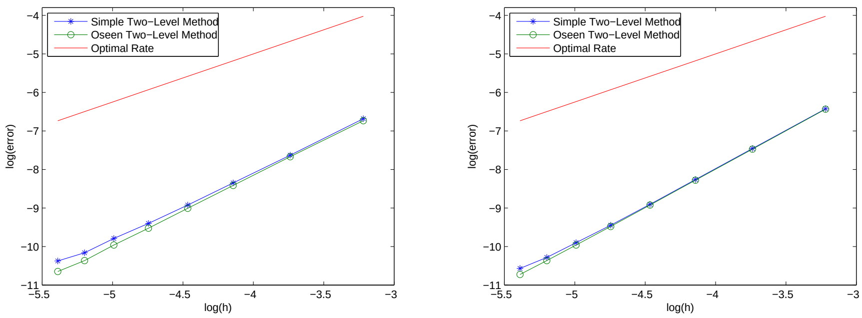

Figure 5 H^1 and L^2 convergence rates of velocity and pressure

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