INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 11, Number 3, Pages 567–586

ENERGY NORM ERROR ESTIMATES FOR AVERAGED DISCONTINUOUS GALERKIN METHODS IN 1 DIMENSION

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Abstract. Numerical solution of one-dimensional elliptic problems is investigated using an averaged discontinuous discretization. The corresponding numerical method can be performed using the favorable properties of the discontinuous Galerkin (dG) approach, while for the average an error estimation is obtained in the H^1-seminorm. We point out that this average can be regarded as a lower order modification of the average of a well-known overpenalized symmetric interior penalty (IP) method. This allows a natural derivation of the overpenalized IP methods.

Key words. discontinuous Galerkin method, smoothing technique, and error estimation.

1. Introduction

Discontinuous Galerkin (dG) methods have been intensively studied in the last decade. Due to the increasing need for highly accurate computation, these methods, allowing local refinement strategies, became very popular. Their unified mathematical analysis for elliptic boundary value problems was initiated in [2], and a number of articles have been published discussing its application to different problems. The theory was put later in a more general framework [10], [11], [12]. Regarding the practical computations, also some monographs have been appeared [15], [21]. The widespread results of the theoretical investigation for dG methods have been summarized recently in [8].

The error analysis for elliptic boundary value problems underwent a significant development. For the multidimensional case, extra smoothness of the analytic solution had been assumed in the original approach [2], which was alleviated in [14]. The a posteriori error analysis was initiated in [18] and [3] and was developed in [1] and [13] to obtain easily computable and guaranteed error bounds and an efficient a posteriori error estimator for a general 3-dimensional *hp*-adaptive algorithm has been derived in [22]. All of these results concern a so-called dG-norm which arises from the dG bilinear form. One can prove convergence also in a mesh-independent (BV) norm [4], [7], which can be used again to avoid the assumption on extra smoothness [8].

Several methods have been developed to obtain an error estimator in the L_2-norm and increase the accuracy of the dG approximation in negative Sobolev norms. The key idea is to apply a post-processing which is a smoothing technique using convolution with special kernels. This was first demonstrated in [6] for hyperbolic problems. These techniques have been developed in many aspects, for recent achievements see, *e.g.*, [17] and [19]. Similar results including superconvergence can be obtained for second-order elliptic problems in several space dimensions [5] using an element-by-element postprocessing in the L_2-norm.

Received by the editors September 3, 2013 and, in revised form, January 19 2014.
2000 *Mathematics Subject Classification.* 65N12, 65N15, 65N30 .
The authors thank the reviewers for their valuable comments. F. Izsák acknowledges the support of the Hungarian Research Fund OTKA (grants PD10441 and K104666).

The objective of this paper is to develop an error estimator *in the natural energy seminorm* between a postprocessed dG type approximation and the analytical solution in one space dimension for elliptic problems. In the main result (see Theorem 3), we provide an upper estimate for the error $\|\nabla(\eta_h * u_{\mathrm{IP}} - u)\|_{L_2}$, where the convolution gives the local average (a kind of postprocessing) and u_{IP} denotes an overpenalized version of the well-known symmetric interior penalty (IP) approximation.

We also throw new light upon a version of dG methods: we will point out that a postprocessed IP method can be regarded as a lower order modification of a continuous Galerkin method. In turn, this suggests a new derivation of a family of overpenalized IP methods, where instead of a heuristic choice the penalty term arises in a natural way.

These results are also confirmed in numerical experiments: the local average of the proposed method and that of the overpenalized IP method are really close to each other. Also, it will be verified that for the local averages the convergence in the H^1 seminorm is valid.

After the preliminaries, we introduce the finite element method which can be recognized both as a continuous and a discontinuous method. Then the corresponding bilinear form is analyzed first in a simple situation and then its relation with the interior penalty method is highlighted. Finally, we prove error estimation between the postprocessed solution and the analytic one. As a consequence, we obtain the above energy norm error estimation for a simple local average of the interior penalty approximation.

2. Mathematical preliminaries

We investigate the finite element solution of the one-dimensional elliptic boundary value problem

$$\begin{cases} -\Delta u = f \quad \text{in } \Omega = (a,b) \subset \mathbb{R} \\ u(a) = u(b) = 0, \end{cases} \tag{1}$$

where $f \in L_2(a,b)$ is given. For the numerical solution we consider a tessellation of the interval (a,b) into the disjoint subintervals $I_0, I_1, \dots, I_n$ such that

$$I_j = (\gamma_j, \gamma_{j+1}), \quad a = \gamma_0 < \gamma_1 < \dots \gamma_{n+1} = b.$$

The parameter h with

$$h = \min_{j \in \{1,2,\dots,n+1\}} (\gamma_j - \gamma_{j-1})$$

denotes the minimal length of the subintervals.

The vector space for the polynomials of maximal degree k on the interval I is denoted with $\mathcal{P}_k(I)$. For $\mathbf{k} = (k_0, k_1, \dots, k_n)$ we define

$$P_{h,\mathbf{k}}(a,b) = \mathcal{P}_{k_0}(I_0) \oplus \mathcal{P}_{k_1}(I_1) \oplus \dots \oplus \mathcal{P}_{k_n}(I_n),$$

the direct sum of the above polynomial spaces which corresponds to the piecewise polynomials with the given maximal degree $k_0, k_1, \dots, k_n$ on $I_0, I_1, \dots, I_n$.

The symbol $(\cdot,\cdot)_{I_*}$ refers to the $L_2(I_*)$ scalar product on $I_* \subset (a,b)$. If $I_* = (a,b)$ we omit the subscript. Accordingly, the generated $L_2(I_*)$-norm is denoted with $\|\cdot\|_{I_*}$, where $I_* = (a,b)$ will be omitted.

For the numerical solution we use the family of the average and jump operators $\{\!\{\cdot\}\!\}_j$ and $[\![\cdot]\!]_j$, which are defined by

$$\{\!\{u\}\!\}_j = \frac{1}{2} \cdot (\lim_{\gamma_j -} u + \lim_{\gamma_j +} u) \quad \text{and} \quad [\![u]\!]_j = \lim_{\gamma_j -} u - \lim_{\gamma_j +} u$$

and make sense for all piecewise polynomial functions u. We also use the piecewise gradient operator ∇_h.

For the analysis of the forthcoming bilinear form, we use that for each $u \in P_{h,\mathbf{k}}$

$$\nabla u = \nabla_h u - \sum_{j=1}^{n} [\![u]\!]_j \, \delta_{\gamma_j}, \tag{2}$$

where δ_{γ_j} denotes the Dirac (delta) distribution supported at γ_j, see, *e.g.*, [16], Theorem 2.10.

A well-known approach to the numerical solution of (1) is the IP method, which consists of finding $u_{\mathrm{IP}} \in P_{h,\mathbf{k}}(a,b)$ such that for all $v_h \in P_{h,\mathbf{k}}(a,b)$

$$a_{\mathrm{IP}}(u_{\mathrm{IP}}, v_h) = (f, v_h), \tag{3}$$

where the bilinear form $a_{\mathrm{IP}} : P_{h,\mathbf{k}}(a,b) \times P_{h,\mathbf{k}}(a,b) \to \mathbb{R}$ is given by

$$a_{\mathrm{IP}}(u_h, v_h) = (\nabla_h u_h, \nabla_h v_h) - \sum_{j=1}^{n} \{\!\{\nabla_h u_h\}\!\}_j \, [\![v_h]\!]_j + \{\!\{\nabla_h v_h\}\!\}_j \, [\![u_h]\!]_j + \sigma_h \sum_{j=1}^{n} [\![u_h]\!]_j \, [\![v_h]\!]_j \, ,$$

with the penalty parameter σ_h depending on h.

In the classical analysis $\sigma_h = \frac{\sigma}{h}$, where σ is some (sufficiently large) constant. This definition indeed, is justified in the multidimensional cases, where σ_h can be different on each interelement face F and is proportional to $h_F = \operatorname{diam} F$.

3. Results

As the main result, we point out a close relation between the average of an overpenalized IP method with $\sigma_h = \frac{1}{h^s}$ $(s > 1)$ and a suitable conforming numerical approximation.

3.1. Finite element discretization. To give the finite element space for the discretization of (1) we define $\eta_h = \frac{1}{2h^s}\chi_{[-h^s,h^s]}$, where χ_{I_*} denotes the characteristic function of the interval I_* and $s > 1$ is a given parameter. To streamline the notation we do not indicate the dependence on s. We define the convolution $\eta_h * w$ for a function $w \in L_2(a,b)$ with

$$\eta_h * w = \eta_h * w_0|_{(a-h^s, b+h^s)},$$

where w_0 denotes the zero extension of w to $(a - 2h^s, b + 2h^s)$. To compute convolutions, we use the notation $\tilde{I}_j = [\gamma_j + h^s, \gamma_{j+1} - h^s]$.

The finite element space $P_{h,\mathbf{k},s}$ is defined as

$$P_{h,\mathbf{k},s} = \{\eta_h * u_h : \ u_h \in P_{h,\mathbf{k}}\}$$

and the bilinear form $a : P_{h,\mathbf{k},s} \times P_{h,\mathbf{k},s}$ is defined with

$$a(\eta_h * u_h, \eta_h * v_h) = (\nabla(\eta_h * u_h), \nabla(\eta_h * v_h)),$$

which makes sense since $P_{h,\mathbf{k},s} \subset H_0^1(a - h^s, b + h^s)$. Accordingly, the finite element discretization of (1) is the following:
Find an element $\eta_h * u_h \in P_{h,\mathbf{k},s}$ such that for all $\eta_h * v_h \in P_{h,\mathbf{k},s}$ the following equality is valid:

$$a(\eta_h * u_h, \eta_h * v_h) = (f, \eta_h * v_h). \tag{4}$$

Since the above bilinear form a is bounded and coercive, (4) has a unique solution.

Note that a similar bilinear form $a_\eta : P_{h,\mathbf{k}} \times P_{h,\mathbf{k}} \to \mathbb{R}$ can be defined with

$$a_\eta(u_h, v_h) = (\nabla(\eta_h * u_h), \nabla(\eta_h * v_h))$$

and a corresponding variational problem can be constructed:
Find an element $u_h \in P_{h,\mathbf{k}}$ such that for all $v_h \in P_{h,\mathbf{k}}$ the following inequality is valid:

$$a_\eta(u_h, v_h) = (f, \eta_h * v_h). \tag{5}$$

Obviously, the solution u_h of (4) and (5) coincide. This makes possible the dual interpretation of the corresponding numerical method: if we compute u_h in (5), we can use all favorable properties of the dG methods. On the other hand, the local average $\eta_h * u_h$ in (4) can be regarded as a solution arising from a "continuous" Galerkin method and for this, a standard error analysis can be performed.

3.2. Analysis of the bilinear form. For the analysis, we first rewrite (4) as follows.

Proposition 1. *The bilinear form in* (4) *can be given as*

$$\begin{aligned}
&a(\eta_h * u, \eta_h * v) = (\nabla(\eta_h * u), \nabla(\eta_h * v)) \\
&= \sum_{j=0}^{n} \int_{\gamma_j + h^s}^{\gamma_{j+1} - h^s} \eta_h * \nabla_h u \cdot \eta_h * \nabla_h v + \sum_{j=0}^{n+1} \int_{\gamma_j - h^s}^{\gamma_j + h^s} \eta_h * \nabla_h u \cdot \eta_h * \nabla_h v \\
&- \sum_{j=1}^{n} [\![u]\!]_j \, \eta_h * (\eta_h * \nabla_h v)(\gamma_j) + [\![v]\!]_j \, \eta_h * (\eta_h * \nabla_h u)(\gamma_j) + \sum_{j=1}^{n} \frac{1}{2h^s} [\![u]\!]_j \, [\![v]\!]_j \, .
\end{aligned} \tag{6}$$

Proof. Using (2), the left hand side of (6) can be rewritten for $u, v \in P_{h,\mathbf{k}}$ as

$$\begin{aligned}
&a(\eta_h * u, \eta_h * v) = (\nabla(\eta_h * u), \nabla(\eta_h * v)) = (\eta_h * \nabla u, \eta_h * \nabla v) \\
&= (\eta_h * (\nabla_h u - \sum_{j=1}^{n} [\![u]\!]_j \, \delta_{\gamma_j}), \eta_h * (\nabla_h v - \sum_{j=1}^{n} [\![v]\!]_j \, \delta_{\gamma_j})) \\
&= (\eta_h * \nabla_h u, \eta_h * \nabla_h v) - (\eta_h * \sum_{j=1}^{n} [\![u]\!]_j \, \delta_{\gamma_j}, \eta_h * \nabla_h v) \\
&- (\eta_h * \sum_{j=1}^{n} [\![v]\!]_j \, \delta_{\gamma_j}, \eta_h * \nabla_h u) + (\eta_h * \sum_{j=1}^{n} [\![u]\!]_j \, \delta_{\gamma_j}, \eta_h * \sum_{j=1}^{n} [\![v]\!]_j \, \delta_{\gamma_j}).
\end{aligned} \tag{7}$$

To compute the second and third terms on the right hand side of (7), we note first that

$$\eta_h * \delta_{\gamma_j} = \eta_{h,\gamma_j} \quad \text{with} \quad \eta_{h,\gamma_j}(x) = \eta_h(x - \gamma_j),$$

which have disjoint support for different indices j. Hence we obtain

$$\begin{aligned}
&(\eta_h * \sum_{j=1}^{n} [\![u]\!]_j \, \delta_{\gamma_j}, \eta_h * \nabla_h v) = \sum_{j=1}^{n} [\![u]\!]_j \, (\eta_{h,\gamma_j}, \eta_h * \nabla_h v) \\
&= \sum_{j=1}^{n} [\![u]\!]_j \, \frac{1}{2h^s} \int_{\gamma_j - h^s}^{\gamma_j + h^s} \eta_h * \nabla_h v = \sum_{j=1}^{n} [\![u]\!]_j \, \eta_h * \eta_h * \nabla_h v(\gamma_j).
\end{aligned} \tag{8}$$

Using again the fact that the functions η_{h,γ_j}, $j = 1, 2, \ldots, n$ have disjoint support, we also have that

$$(\eta_h * \sum_{j=1}^{n} [\![u]\!]_j \, \delta_{\gamma_j}, \eta_h * \sum_{j=1}^{n} [\![v]\!]_j \, \delta_{\gamma_j}) = \sum_{j=1}^{n} [\![u]\!]_j \, [\![v]\!]_j \, (\eta_{h,\gamma_j}, \eta_{h,\gamma_j}) = \sum_{j=1}^{n} \frac{1}{2h^s} [\![u]\!]_j \, [\![v]\!]_j \, . \tag{9}$$

Using (8) and (9) in (7) we obtain the statement of the proposition. $\square$

Before we relate (10) to the IP bilinear form we give it more explicitly in a simple case.

Corollary 1. *If the tessellation $\mathcal{T}_h$ is uniform with the mesh size h and the local dG basis consists of locally first order polynomials then* (6) *can be given as*

$$\begin{aligned}(1-2h^{s-1})(\nabla_h u,\nabla_h v)&+\sum_{j=1}^{n}\frac{h^s}{6}\left[\!\left[\nabla_h u\right]\!\right]_j\left[\!\left[\nabla_h v\right]\!\right]_j+2h^s\left\{\!\!\left\{\nabla_h u\right\}\!\!\right\}_j\left\{\!\!\left\{\nabla_h v\right\}\!\!\right\}_j\\ -\sum_{j=1}^{n}\left[\!\left[u\right]\!\right]_j\left\{\!\!\left\{\nabla_h v\right\}\!\!\right\}_j&+\left[\!\left[v\right]\!\right]_j\left\{\!\!\left\{\nabla_h u\right\}\!\!\right\}_j+\sum_{j=1}^{n}\frac{1}{2h^s}\left[\!\left[u\right]\!\right]_j\left[\!\left[v\right]\!\right]_j.\end{aligned}\tag{10}$$

Proof. If u is piecewise linear then $\nabla_h u$ is piecewise constant and $\eta_h*\nabla_h u$ is piecewise linear with

$$\eta_h*\nabla_h u(x)=\begin{cases}\nabla_h u(x) & \text{if } |x-\gamma_j|\geq h^s\\ \left\{\!\!\left\{\nabla_h u\right\}\!\!\right\}_j+\frac{y}{2h^s}\left[\!\left[\nabla_h u\right]\!\right]_j & \text{if } y=\gamma_j-x \quad\text{with } |y|\leq h^s\end{cases}$$

for all $j\in\{0,1,\dots,n+1\}$ such that $\eta_h*\nabla_h u(x)$ is constant if x has a distance at least h^s from the interelement points. Therefore, the product $(\eta_h*\nabla_h u)\cdot(\eta_h*\nabla_h v)(x)$ is also constant if $|x-\gamma_j|\geq h^s$ for all j and is quadratic at the remaining regions with the values

$$(\eta_h*\nabla_h u)\cdot(\eta_h*\nabla_h v)(x)=\begin{cases}\nabla_h u(\gamma_j^-)\cdot\nabla_h v(\gamma_j^-) & \text{if } x=\gamma_j-h^s\\ \left\{\!\!\left\{\nabla_h u\right\}\!\!\right\}_j\cdot\left\{\!\!\left\{\nabla_h v\right\}\!\!\right\}_j & \text{if } x=\gamma_j\\ \nabla_h u(\gamma_j^+)\cdot\nabla_h v(\gamma_j^+) & \text{if } x=\gamma_j+h^s.\end{cases}$$

Using the identity

$$\begin{aligned}&\nabla_h u(\gamma_j^-)\nabla_h v(\gamma_j^-)+\nabla_h u(\gamma_j^+)\nabla_h v(\gamma_j^+)\\ &=\frac{1}{2}(\nabla_h u(\gamma_j^-)+\nabla_h u(\gamma_j^+))(\nabla_h v(\gamma_j^-)+\nabla_h v(\gamma_j^+))\\ &+\frac{1}{2}(\nabla_h u(\gamma_j^-)-\nabla_h u(\gamma_j^+))(\nabla_h v(\gamma_j^-)-\nabla_h v(\gamma_j^+))\\ &=2\left\{\!\!\left\{\nabla_h u\right\}\!\!\right\}_j\left\{\!\!\left\{\nabla_h v\right\}\!\!\right\}_j+\frac{1}{2}\left[\!\left[\nabla_h u\right]\!\right]_j\left[\!\left[\nabla_h v\right]\!\right]_j\end{aligned}$$

and the three point quadrature rule, we have that for all j and $u,v\in\mathcal{H}_h^{\mathbf{1}}$

$$\begin{aligned}&\int_{\gamma_j-h^s}^{\gamma_j+h^s}\eta_h*\nabla_h u\cdot\eta_h*\nabla_h v\\ &=\frac{2h^s}{6}(\nabla u(\gamma_j^-)\nabla_h v(\gamma_j^-)+4\left\{\!\!\left\{\nabla_h u\right\}\!\!\right\}_j\left\{\!\!\left\{\nabla_h v\right\}\!\!\right\}_j+\nabla_h u(\gamma_j^+)\nabla_h v(\gamma_j^+))\\ &=\frac{2h^2}{6}(\frac{1}{2}\left[\!\left[\nabla_h u\right]\!\right]_j\left[\!\left[\nabla_h v\right]\!\right]_j+4\left\{\!\!\left\{\nabla_h u\right\}\!\!\right\}_j\left\{\!\!\left\{\nabla_h v\right\}\!\!\right\}_j+2\left\{\!\!\left\{\nabla_h u\right\}\!\!\right\}_j\left\{\!\!\left\{\nabla_h v\right\}\!\!\right\}_j)\\ &=\frac{h^s}{6}\left[\!\left[\nabla_h u\right]\!\right]_j\left[\!\left[\nabla_h v\right]\!\right]_j+2h^s\left\{\!\!\left\{\nabla_h u\right\}\!\!\right\}_j\left\{\!\!\left\{\nabla_h v\right\}\!\!\right\}_j.\end{aligned}\tag{11}$$

Obviously, we also have the equalities

$$\int_{\gamma_{j-1}+h^s}^{\gamma_j-h^s}\nabla_h u\nabla_h v=(h-2h^s)\nabla_h u|_{K_j}\nabla_h v|_{K_j},\quad j=1,2,\dots,n,$$

which can be summed up to obtain

$$\sum_{j=1}^{n}\int_{\gamma_{j-1}+h^s}^{\gamma_j-h^s}\nabla_h u\nabla_h v=(1-2h^{s-1})(\nabla_h u,\nabla_h v).\tag{12}$$

Since $\eta_h * \nabla_h v$ is linear on $(\gamma_j - h^s, \gamma_j + h^s)$ with

$$\eta_h * \nabla_h v(\gamma_j - h^s) = \nabla_h v(\gamma_j -) \quad \text{and} \quad \eta_h * \nabla_h v(\gamma_j + h^s) = \nabla_h v(\gamma_j +),$$

its mean in γ_j becomes

$$\frac{1}{2h^s} \int_{\gamma_j - h^s}^{\gamma_j + h^s} \eta_h * \nabla_h v = \frac{\nabla_h v(\gamma_j -) + \nabla_h v(\gamma_j +)}{2} = \{\!\{\nabla_h v\}\!\}_j \,. \tag{13}$$

Substituting (11), (12) and (13) into (6), we obtain the bilinear form in (10). $\square$

3.3. Comparison with the IP bilinear form. The result in Corollary 1 motivates us to compare (6) with the IP bilinear form in a general situation.

Lemma 1. *Assume that $h^{s-1} \le \frac{1}{4}$. Then there is a constant c_0 depending on k but independent of h such that for all $p \in \mathcal{P}_k\left(-\frac{h}{2}, 0\right) \oplus \mathcal{P}_k\left(0, \frac{h}{2}\right)$ and $h \le \frac{1}{2}$ we have*

$$\max_{\left[-\frac{h}{2}, \frac{h}{2}\right]} p^2 \le \frac{c_0}{h} \int_{\left[-\frac{h}{2}, \frac{h}{2}\right] \setminus [-h^s, h^s]} p^2. \tag{14}$$

A similar statement holds for the estimation of the derivatives with the L_2-norm.

Lemma 2. *There is a constant c_1 depending on k but independent of $|I|$ such that for all $p \in \mathcal{P}_k(I)$ we have*

$$\|p'\|_{0,I} \le c_1 \frac{1}{|I|} \|p\|_{0,I}. \tag{15}$$

and

$$\max_I |p'| \le c_1 |I|^{-\frac{3}{2}} \|p\|_{0,I}. \tag{16}$$

The proof of the above statements is postponed to the appendix.

Using the above results we can estimate the second term on the right hand side of (6).

Proposition 2. *There is a constant c_0 depending on $\mathbf{k}$ but independent of h such that for all $u, v \in P_{h,\mathbf{k}}$ we have*

$$\sum_{j=0}^{n} \int_{\gamma_j - h^s}^{\gamma_j + h^s} \eta_h * \nabla_h u \cdot \eta_h * \nabla_h v \le 2c_0 h^{s-1} \sum_{j=0}^{n} \|\nabla_h u\|_{\tilde{I}_j} \|\nabla_h v\|_{\tilde{I}_j} \tag{17}$$

Proof. If $u \in C(-\frac{h}{2}, 0) \oplus C(0, \frac{h}{2})$ with a maximal polynomial degree k then according to Lemma 1 we obtain

$$\begin{aligned} \int_{-h^s}^{h^s} (\eta_h * \nabla_h u)^2 &\le 2h^s \max_{[-h^s, h^s]} |\eta_h * \nabla_h u|^2 \le 2h^s \max_{\left[-\frac{h}{2}, \frac{h}{2}\right]} |\nabla_h u|^2 \\ &\le c_0 \frac{2h^s}{h} \int_{\left[-\frac{h}{2}, \frac{h}{2}\right] \setminus [-h^s, h^s]} |\nabla_h u|^2. \end{aligned} \tag{18}$$

We note that (18) remains valid if the variable is transformed with $x \to x + \gamma_j$. Accordingly, using the notation $I_{j,s} = \left[\gamma_j - \frac{h_j}{2}, \gamma_j + \frac{h_{j+1}}{2}\right] \setminus [\gamma_j - h^s, \gamma_j + h^s]$ we have that for all $j = 1, 2, \dots, n$ the following inequality is valid:

(19)
$$\int_{\gamma_j-h^s}^{\gamma_j+h^s} \eta_h * \nabla_h u \cdot \eta_h * \nabla_h v \le \sqrt{\int_{\gamma_j-h^s}^{\gamma_j+h^s} |\eta_h * \nabla_h u|^2}\sqrt{\int_{\gamma_j-h^s}^{\gamma_j+h^s} |\eta_h * \nabla_h v|^2}$$
$$\le 2c_0 h^{s-1}\sqrt{\int_{\left[\gamma_j-\frac{h}{2},\gamma_j+\frac{h}{2}\right]\setminus[\gamma_j-h^s,\gamma_j+h^s]} |\nabla_h u|^2}\sqrt{\int_{\left[\gamma_j-\frac{h}{2},\gamma_j+\frac{h}{2}\right]\setminus[\gamma_j-h^s,\gamma_j+h^s]} |\nabla_h v|^2}$$
$$\le 2c_0 h^{s-1}\sqrt{\int_{I_{j,s}} |\nabla_h u|^2}\sqrt{\int_{I_{j,s}} |\nabla_h v|^2}.$$

Using the "discrete" Cauchy–Schwarz inequality

$$\sum_{j=0}^{n} a_j b_j \le \sqrt{\sum_{j=0}^{n} a_j^2 \sum_{j=0}^{n} b_j^2},$$

the summation of the terms in (19) gives that

$$\sum_{j=0}^{n}\int_{\gamma_j-h^s}^{\gamma_j+h^s} \eta_h * \nabla_h u \cdot \eta_h * \nabla_h v \le 2c_0 h^{s-1}\sum_{j=0}^{n}\sqrt{\int_{I_{j,s}} |\nabla_h u|^2}\sqrt{\int_{I_{j,s}} |\nabla_h v|^2}$$
$$\le 2c_0 h^{s-1}\sqrt{\sum_{j=0}^{n}\int_{I_{j,s}} |\nabla_h u|^2}\sqrt{\sum_{j=0}^{n}\int_{I_{j,s}} |\nabla_h v|^2} = 2c_0 h^{s-1}\sum_{j=0}^{n}\|\nabla_h u\|_{\tilde{I}_j}\|\nabla_h v\|_{\tilde{I}_j},$$

as we have stated. □

Remarks: The inequality in (18) can easily be rewritten into the following form: if $u \in C(-\frac{h}{2}, 0) \oplus C(0, \frac{h}{2})$ with a maximal polynomial degree k then for all $x \in \left(-\frac{h}{2} + h^s, \frac{h}{2} - h^s\right)$ we have

$$\left|\int_0^x |\eta_h * \nabla_h u|^2\right| \le c_0 \frac{x}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} |\nabla_h u|^2. \tag{20}$$

Similar derivations give that there is a constant $c_2 \in \mathbb{R}^+$ such that for all above functions u and parameters h, s with $\frac{h}{4} > h^s$ we have

$$\int_{-h^s}^{h^s} |\nabla_h u|^2 \le c_2 h^{s-1}\int_{\left(-\frac{h}{2},\frac{h}{2}\right)\setminus(-h^s,h^s)} |\nabla_h u|^2. \tag{21}$$

Lemma 3. *For arbitrary functions $u, v \in P_{h,\mathbf{k}}$ we have that*

$$\begin{aligned}&\left|(\nabla_h u, \nabla_h v) - \sum_{j=0}^{n}\int_{\tilde{I}_j} \eta_h * \nabla_h u \cdot \eta_h * \nabla_h v\right| \\ &\le \left(\frac{c_1^2}{4}h^{2s-2} + (c_1 + c_2)h^{s-1}\right)\sum_{j=0}^{n}\int_{\tilde{I}_j} \nabla_h u \cdot \nabla_h v.\end{aligned} \tag{22}$$

Remark: The inequality in (43) also implies that for all $u \in P_{k_j}(I_j)$ we have

$$\|\nabla_h u\|_{\tilde{I}_j}^2 \le \frac{1}{1 - \left(\frac{c_1^2}{4}h^{2s-2} + c_1 h^{s-1}\right)}\|\eta_h * \nabla_h u\|_{\tilde{I}_j}^2 \tag{23}$$

Lemma 4. *For all $v \in P_{h,\mathbf{k}}$ and $j = 1, 2, \ldots, n$ we have*

$$\left|\eta_h * (\eta_h * \nabla_h v)(\gamma_j) - \{\!\{\nabla_h v\}\!\}_j\right| \leq \frac{\sqrt{c_0}c_1}{2} h^{s-\frac{3}{2}} \sqrt{\int_{\gamma_j - \frac{h}{2}}^{\gamma_j + \frac{h}{2}} |\nabla_h v|^2}, \tag{24}$$

where c_0 is the same constant as in Lemma 1.

The proofs of Lemma 3 and Lemma 4 are postponed to the appendix.

Proposition 3. *For all $v \in P_{h,\mathbf{k}}$ we have*

(25)

$$\left|\sum_{j=1}^{n} [\![u]\!]_j \, \eta_h * (\eta_h * \nabla_h v)(\gamma_j) - \sum_{j=1}^{n} [\![u]\!]_j \, \{\!\{\nabla_h v\}\!\}_j\right| \leq C h^{s-1} \|\nabla(\eta_h * u)\| \|\nabla(\eta_h * v)\|,$$

where C is a mesh-independent constant.

Proof. We first introduce the functions $(\nabla_h v)_0 : \left(-\frac{h}{2}, \frac{h}{2}\right)$ with

$$(\nabla_h v)_0(x) = \begin{cases} \nabla_h v(x) & x \in \left(-\frac{h}{2}, \frac{h}{2}\right) \setminus (-h^s, h^s) \\ 0 & x \in (-h^s, h^s) \end{cases}$$

and $(\nabla_h v)_1 = \nabla_h v|_{\left(-\frac{h}{2}, \frac{h}{2}\right)} - (\nabla_h v)_0$. For these functions, $\|\cdot\|$ refers to the $L_2\left(-\frac{h}{2}, \frac{h}{2}\right)$ norm. Obviously, $(\nabla_h v)_1 \perp (\nabla_h v)_0$ and $[\![v]\!]_0 \, \eta_h \perp (\nabla_h v)_0$. The notations $(\eta_h * \nabla_h v)_1$ and $(\eta_h * \nabla_h v)_0$ will be used in a similar sense. We also note that a simple scaling argument implies the equivalence of the norms

$$c_3 \|(\eta_h * \nabla_h v)_0\| \geq \|(\nabla_h v)_0\| \geq \tilde{c}_4 \|(\eta_h * \nabla_h v)_0\| \tag{26}$$

with some mesh-independent constants c_3 and $\tilde{c}_4$. With the above notations, using also (21) and (26) we obtain that

$$\begin{aligned}
&\|\eta_h * \nabla_h v|_{\left(-\frac{h}{2}, \frac{h}{2}\right)} - [\![v]\!]_0 \, \eta_h\|^2 = \|(\eta_h * \nabla_h v)_1 - [\![v]\!]_0 \, \eta_h\|^2 + \|(\nabla_h v)_0\|^2 \\
&= \|(\eta_h * \nabla_h v)_1\|^2 + \| \, [\![v]\!]_0 \, \eta_h\|^2 - 2((\eta_h * \nabla_h v)_1, [\![v]\!]_0 \, \eta_h) + \|(\nabla_h v)_0\|^2 \\
&\geq \| \, [\![v]\!]_0 \, \eta_h\|^2 - 2((\eta_h * \nabla_h v)_1, [\![v]\!]_0 \, \eta_h) + \|(\nabla_h v)_0\|^2 \\
&\geq \| \, [\![v]\!]_0 \, \eta_h\|^2 + \|(\nabla_h v)_0\|^2 - 2\|(\eta_h * \nabla_h v)_1\| \| \, [\![v]\!]_0 \, \eta_h\| \\
&\geq \| \, [\![v]\!]_0 \, \eta_h\|^2 + \|(\nabla_h v)_0\|^2 - 2\sqrt{c_2 h^{s-1}} \|(\eta_h * \nabla_h v)_0\| \| \, [\![v]\!]_0 \, \eta_h\| \\
&\geq \| \, [\![v]\!]_0 \, \eta_h\|^2 + \|(\nabla_h v)_0\|^2 - \sqrt{c_2 h^{s-1}} (\|(\eta_h * \nabla_h v)_0\|^2 + \| \, [\![v]\!]_0 \, \eta_h\|^2) \\
&\geq (c_4 - \sqrt{c_2 h^{s-1}}) (\|(\eta_h * \nabla_h v)_0\|^2 + \| \, [\![v]\!]_0 \, \eta_h\|^2),
\end{aligned} \tag{27}$$

where $c_4 = \min\{1, \tilde{c}_4\}$. Note that using the above notations, (21) implies that

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} |\nabla_h v|^2 = \int_{-h^s}^{h^s} |\nabla_h v|^2 + \int_{\left(-\frac{h}{2}, \frac{h}{2}\right) \setminus (-h^s, h^s)} |\nabla_h v|^2 \leq (1 + c_2 h^{s-1}) \|(\nabla_h v)_0\|^2$$

and consequently, we also have that

$$\begin{aligned}
&| \, [\![u]\!]_0 \, | \sqrt{\int_{-\frac{h}{2}}^{\frac{h}{2}} |\nabla_h v|^2} \leq | \, [\![u]\!]_0 \, | \sqrt{1 + c_2 h^{s-1}} \sqrt{\|(\nabla_h v)_0\|^2} \\
&\leq \sqrt{1 + c_2 h^{s-1}} c_3 | \, [\![u]\!]_0 \, | \sqrt{\|(\eta_h * \nabla_h v)_0\|^2} \\
&= \sqrt{2h^s} c_3 \sqrt{1 + c_2 h^{s-1}} \| \, [\![u]\!]_0 \, \eta_h\| \|(\eta_h * \nabla_h v)_0\|.
\end{aligned} \tag{28}$$

Interchanging of u and v in (28), adding the result to (28), using the Cauchy–Schwarz inequality and (27) results in the following estimate

$$
\begin{aligned}
&|\,[\![u]\!]_0\,|\sqrt{\int_{-\frac{h}{2}}^{\frac{h}{2}} |\nabla_h v|^2} + |\,[\![v]\!]_0\,|\sqrt{\int_{-\frac{h}{2}}^{\frac{h}{2}} |\nabla_h u|^2} \\
&\le \sqrt{2h^s} c_3 \sqrt{1+c_2 h^{s-1}}\,(\|\,[\![u]\!]_0\,\eta_h\|\|(\eta_h * \nabla_h v)_0\| + \|\,[\![v]\!]_0\,\eta_h\|\|(\eta_h * \nabla_h u)_0\|) \\
&\le \sqrt{\frac{h^s}{2}} c_3 \sqrt{1+c_2 h^{s-1}} \sqrt{\|\,[\![u]\!]_0\,\eta_h\|^2 + \|(\eta_h * \nabla_h u)_0\|^2} \sqrt{\|\,[\![v]\!]_0\,\eta_h\|^2 + \|(\eta_h * \nabla_h v)_0\|^2} \\
&\le \sqrt{\frac{h^s}{2}} c_3 \frac{\sqrt{1+c_2 h^{s-1}}}{c_4 - c_2 h^{s-1}} \|\eta_h * \nabla_h u - [\![u]\!]_0\,\eta_h\| \cdot \|\eta_h * \nabla_h v - [\![v]\!]_0\,\eta_h\| \\
&= \sqrt{\frac{h^s}{2}} c_3 \frac{\sqrt{1+c_2 h^{s-1}}}{c_4 - c_2 h^{s-1}} \|\nabla(\eta_h * u)\| \cdot \|\nabla(\eta_h * v)\|.
\end{aligned}
$$

Transforming the variable we similarly get

$$
\begin{aligned}
&|\,[\![u]\!]_j\,|\sqrt{\int_{\gamma_j-\frac{h}{2}}^{\gamma_j+\frac{h}{2}} |\nabla_h v|^2} + |\,[\![v]\!]_j\,|\sqrt{\int_{\gamma_j-\frac{h}{2}}^{\gamma_j+\frac{h}{2}} |\nabla_h u|^2} \\
&\le \sqrt{\frac{h^s}{2}} \frac{\sqrt{1+c_2 h^{s-1}}}{c_4 - c_2 h^{s-1}} c_3 \sqrt{\int_{\gamma_j-h^s}^{\gamma_j+h^s} |\nabla(\eta_h * u)|^2} \sqrt{\int_{\gamma_j-h^s}^{\gamma_j+h^s} |\nabla(\eta_h * v)|^2}
\end{aligned}
$$

for $j = 1, 2, \ldots, n$ which, using the result of Lemma 4 and the discrete Cauchy–Schwarz inequality, gives the estimate

$$
\begin{aligned}
&\left|\sum_{j=1}^n [\![u]\!]_j\,\eta_h * (\eta_h * \nabla_h v)(\gamma_j) - \sum_{j=1}^n [\![u]\!]_j\,\{\!\{\nabla_h v\}\!\}_j\right| \\
&+ \left|\sum_{j=1}^n [\![v]\!]_j\,\eta_h * (\eta_h * \nabla_h u)(\gamma_j) - \sum_{j=1}^n [\![v]\!]_j\,\{\!\{\nabla_h u\}\!\}_j\right| \\
&\le \frac{\sqrt{c_0}}{2} h^{s-\frac{3}{2}} \sum_{j=1}^n |\,[\![u]\!]_j\,|\sqrt{\int_{\gamma_j-\frac{h}{2}}^{\gamma_j+\frac{h}{2}} |\nabla_h v|^2} + |\,[\![v]\!]_j\,|\sqrt{\int_{\gamma_j-\frac{h}{2}}^{\gamma_j+\frac{h}{2}} |\nabla_h u|^2} \\
&\le \frac{\sqrt{c_0}}{2} h^{s-\frac{3}{2}} \sqrt{\frac{h^s}{2}} c_3 \frac{\sqrt{1+c_2 h^{s-1}}}{c_4 - c_2 h^{s-1}} \sum_{j=1}^n \sqrt{\int_{\gamma_j-h^s}^{\gamma_j+h^s} |\nabla(\eta_h * u)|^2} \sqrt{\int_{\gamma_j-h^s}^{\gamma_j+h^s} |\nabla(\eta_h * v)|^2} \\
&\le \frac{\sqrt{c_0}}{2} h^{s-\frac{3}{2}} \sqrt{\frac{h^s}{2}} c_3 \frac{\sqrt{1+c_2 h^{s-1}}}{c_4 - c_2 h^{s-1}} \sqrt{\int_0^1 |\nabla(\eta_h * u)|^2} \sqrt{\int_0^1 |\nabla(\eta_h * v)|^2} \\
&\le \frac{\sqrt{c_0}}{2} h^{s-\frac{3}{2}} \sqrt{\frac{h^s}{2}} c_3 \frac{\sqrt{1+c_2 h^{s-1}}}{c_4 - c_2 h^{s-1}} \|\nabla(\eta_h * u)\|^2 \|\nabla(\eta_h * v)\|^2,
\end{aligned}
$$

which implies the statement in the proposition. $\square$

Using the above results, we can finally estimate the difference between the IP bilinear form and a_η. In the consecutive estimates, u_{IP} denotes the solution of (3) using the penalty coefficient $\sigma_h = \frac{1}{2h^s}$ and u_h denotes the solution of the problem in (4).

Theorem 1. *There is a constant C and a discretization parameter h_0 such that for all $h < h_0$ and $u, v \in \mathcal{P}_{h,\mathbf{k}}$ we have the following estimate:*

$$|a_{IP}(u,v) - a_\eta(u,v)| \le Ch^{s-1}\|\nabla(\eta_h * u)\|\|\nabla(\eta_h * v)\|. \tag{29}$$

Proof. Comparing the two bilinear forms we obtain that

$$\begin{aligned} a_{\mathrm{IP}}(u,v) - a_\eta(u,v) &= (\nabla_h u, \nabla_h v) - \sum_{j=0}^n \int_{\tilde{I}_j} \eta_h * \nabla_h u \cdot \eta_h * \nabla_h v \\ &- \sum_{j=0}^n \int_{\gamma_j - h^s}^{\gamma_j + h^s} \eta_h * \nabla_h u \cdot \eta_h * \nabla_h v + \sum_{j=0}^n [\![u]\!]_j \{\!\{\nabla_h v\}\!\}_j + [\![v]\!]_j \{\!\{\nabla_h u\}\!\}_j \\ &- \sum_{j=0}^n [\![u]\!]_j \, \eta_h * (\eta_h * \nabla_h v)(\gamma_j) + [\![v]\!]_j \, \eta_h * (\eta_h * \nabla_h u)(\gamma_j). \end{aligned} \tag{30}$$

Using the estimates in Lemma 3, in Proposition 2 in Proposition 3 and in (23) we obtain that

$$\begin{aligned} &|a_{\mathrm{IP}}(u,v) - a_\eta(u,v)| \\ &\le \left(\frac{c_1^2}{4} h^{2s-2} + (c_1 + c_2) h^{s-1}\right) \sum_{j=0}^n \int_{\tilde{I}_j} \nabla_h u \cdot \nabla_h v \\ &+ 2c_0 h^{s-1} \sum_{j=0}^n \|\nabla_h u\|_{\tilde{I}_j} \|\nabla_h v\|_{\tilde{I}_j} + 2c_2 h^{s-1} \|\nabla(\eta_h * u)\|\|\nabla(\eta_h * v)\| \\ &\le \left(\frac{c_1^2}{4} h^{2s-2} + (c_1 + c_2) h^{s-1} + 2c_0 h^{s-1}\right) \sum_{j=0}^n \|\nabla_h u\|_{\tilde{I}_j} \|\nabla_h v\|_{\tilde{I}_j} \\ &+ 2c_2 h^{s-1} \|\nabla(\eta_h * u)\|\|\nabla(\eta_h * v)\| \\ &\le \frac{\frac{c_1^2}{4} h^{2s-2} + (c_1 + c_2) h^{s-1} + 2c_0 h^{s-1}}{1 - \left(\frac{c_1^2}{4} h^{2s-2} + 2c_1 h^{s-1}\right)} \sum_{j=0}^n \|\eta_h * \nabla_h u\|_{\tilde{I}_j} \|\eta_h * \nabla_h v\|_{\tilde{I}_j} \\ &+ 2c_2 h^{s-1} \|\nabla(\eta_h * u)\|\|\nabla(\eta_h * v)\|. \end{aligned}$$

On the other hand, using again the discrete Cauchy–Schwarz inequality and the fact that $\nabla(\eta_h * u)|_{\tilde{I}_j} = (\eta_h * \nabla_h u)|_{\tilde{I}_j}$ we have

$$\begin{aligned} \sum_{j=0}^n \|\eta_h * \nabla_h u\|_{\tilde{I}_j} \|\eta_h * \nabla_h u\|_{\tilde{I}_j} &\le \sqrt{\sum_{j=0}^n |\eta_h * \nabla_h u|_{\tilde{I}_j}^2} \sqrt{\sum_{j=0}^n |\eta_h * \nabla_h v|_{\tilde{I}_j}^2} \\ &\le \|\nabla(\eta_h * u)\|\|\nabla(\eta_h * v)\| \end{aligned} \tag{31}$$

and therefore,

$$\begin{aligned} &|a_{\mathrm{IP}}(u,v) - a_\eta(u,v)| \\ &\le \left(\frac{\frac{c_1^2}{4} h^{2s-2} + (c_1 + c_2) h^{s-1} + 2c_0 h^{s-1}}{1 - \left(\frac{c_1^2}{4} h^{2s-2} + 2c_1 h^{s-1}\right)} + 2c_2 h^{s-1}\right) \|\nabla(\eta_h * u)\|\|\nabla(\eta_h * v)\|, \end{aligned} \tag{32}$$

which gives the statement in the theorem. $\square$

Remark: The result of this theorem shows that the present investigations do not cover the case of the classical IP method since for $s = 1$ the difference of the bilinear forms a_{IP} is not a lower order perturbation of a_η.

We can also estimate the local average of the solution arising from the IP bilinear form.

Theorem 2. *There is a constant C such that we have the following estimate:*

$$\|\nabla(\eta_h * u_{IP} - \eta_h * u_h)\| \le Ch^{s-1}\|\nabla(\eta_h * u_h)\| + c_3\|\eta_h * f - f\|.$$

Proof. Since u_h is the solution of (4), we have that for $v_h = u_h - u_{\mathrm{IP}}$ the following identity is valid:

$$(\nabla(\eta_h * u_h), \nabla(\eta_h * (u_h - u_{\mathrm{IP}}))) = (f, \eta_h * (u_h - u_{\mathrm{IP}})),$$

and therefore, a straightforward computation with the relation $\eta(-x) = \eta(x)$ gives that

$$\begin{aligned}
&(\nabla(\eta_h * (u_h - u_{\mathrm{IP}})), \nabla(\eta_h * (u_h - u_{\mathrm{IP}})))\\
&= (f, \eta_h * (u_h - u_{\mathrm{IP}})) - (\nabla(\eta_h * u_{\mathrm{IP}}), \nabla(\eta_h * (u_h - u_{\mathrm{IP}})))\\
&= (f, \eta_h * (u_h - u_{\mathrm{IP}})) - a_{\mathrm{IP}}(u_{\mathrm{IP}}, u_h - u_{\mathrm{IP}}) - (\nabla(\eta_h * u_{\mathrm{IP}}), \nabla(\eta_h * (u_h - u_{\mathrm{IP}})))\\
&+ a_{\mathrm{IP}}(u_{\mathrm{IP}}, u_h - u_{\mathrm{IP}})\\
&= (\eta_h * f, u_h - u_{\mathrm{IP}}) - (f, u_h - u_{\mathrm{IP}}) - (\nabla(\eta_h * (u_{\mathrm{IP}} - u_h)), \nabla(\eta_h * (u_h - u_{\mathrm{IP}})))\\
&+ a_{\mathrm{IP}}(u_{\mathrm{IP}} - u_h, u_h - u_{\mathrm{IP}}) - (\nabla(\eta_h * u_h), \nabla(\eta_h * (u_h - u_{\mathrm{IP}}))) + a_{\mathrm{IP}}(u_h, u_h - u_{\mathrm{IP}}).
\end{aligned}$$

Therefore, using the estimate in (29) for the last two pair of terms the equivalence in (26) and the Friedrichs inequality we obtain that

$$\begin{aligned}
&\|\nabla(\eta_h * (u_h - u_{\mathrm{IP}}))\|^2\\
&\le \|\eta_h * f - f\|\|u_h - u_{\mathrm{IP}}\| + Ch^{s-1}(\|\nabla(\eta_h * (u_{\mathrm{IP}} - u_h))\|^2\\
&+ \|\nabla(\eta_h * u_h)\|\|\nabla(\eta_h * (u_h - u_{\mathrm{IP}}))\|)\\
&\le c_3\|\eta_h * f - f\|\|\nabla(\eta_h * (u_h - u_{\mathrm{IP}}))\| + Ch^{s-1}\|\nabla(\eta_h * (u_{\mathrm{IP}} - u_h))\|^2\\
&+ Ch^{s-1}\|\nabla(\eta_h * u_h)\|\|\nabla(\eta_h * (u_h - u_{\mathrm{IP}}))\|
\end{aligned}$$

such that we finally get

$$(1 - Ch^{s-1})\|\nabla(\eta_h * (u_h - u_{\mathrm{IP}}))\| \le c_3\|\eta_h * f - f\| + Ch^{s-1}\|\nabla(\eta_h * u_h)\|,$$

which implies the estimate in the theorem. ☐

The final problem we face with is that the finite element space $P_{h,\mathbf{k},s}$ is not conforming, as possibly $\eta_h * u_h(0) \neq 0$ or $\eta_h * u_h(1) \neq 0$.

Lemma 5. *The approximation $\eta_h * u_h$ of u in (4) is quasi optimal in the sense that for some constant $C \in \mathbb{R}$ we have*

$$\|u - \eta_h * u_h\|_1 \le C \inf_{v_h \in P_{h,\mathbf{k}}} \|u - \eta_h * v_h\|_1 + \mathcal{O}(h^{s-1}).$$

Proof: We use the Strang lemma for the non-consistent finite element approximation, which states for our case that

$$\|u - \eta * u_h\|_{H^1} \le C_a \left(\inf_{v_h \in P_{h,\mathbf{k}}} \|u - \eta * v_h\|_{H^1} + \sup_{v_h \in P_{h,\mathbf{k}}} \frac{(f, \eta_h * v_h) - (\nabla u, \nabla(\eta_h * v_h))}{\|\eta * v_h\|_{H^1}} \right)$$

with a mesh-independent constant C_a. For the complete formula with a detailed explanation, we refer to [9].

We define a function $\tilde{v}_h : [0,1] \to \mathbb{R}$ with

$$\tilde{v}_h(x) = \begin{cases} \eta_h * v_h(x(1+2h^{s-1}) - h^s) & \text{for } x \in \left[0, \frac{h}{2}\right] \\ \eta_h * v_h(x) & \text{for } x \in \left(\frac{h}{2}, 1-\frac{h}{2}\right) \\ \eta_h * v_h(x(1+2h^{s-1}) + h^s - 2h^{s-1}) & \text{for } x \in \left[1-\frac{h}{2}, 1\right], \end{cases}$$

which is an x-dependent translation of $\eta_h * v_h$ and the differences of the variables satisfy

(33)
$$\max\left\{ \max_{x\in\left[0,\frac{h}{2}\right]} |x(1+2h^{s-1}) - h^s - x|, \max_{x\in\left[1-\frac{h}{2},1\right]} |x(1+2h^{s-1}) + h^s - 2h^{s-1} - x| \right\} \le h^s.$$

Note that $\operatorname{supp} \eta_h * v_h|_{[0,1]} - \tilde{v}_h = \left[0, \frac{h}{2}\right] \cup \left[1-\frac{h}{2}, 1\right]$ and $\tilde{v}_h \in H_0^1(0,1)$, therefore,

$$\begin{aligned} &\sup_{v_h\in P_{h,\mathbf{k}}} \frac{(f, \eta_h * v_h) - (\nabla u, \nabla(\eta_h * v_h))}{\|\eta * v_h\|_{H^1}} \\ &\sup_{v_h\in P_{h,\mathbf{k}}} \frac{(f, \eta_h * v_h|_{[0,1]}) - (\nabla u, \nabla(\eta_h * v_h))}{\|\eta * v_h\|_{H^1}} \\ &= \sup_{v_h\in P_{h,\mathbf{k}}} \frac{(f, \eta_h * v_h - \tilde{v}_h) - (\nabla u, \nabla(\eta_h * v_h - \tilde{v}_h))}{\|\eta * v_h\|_{H^1}}. \end{aligned} \tag{34}$$

We may assume that

$$\max_{x\in[0,h]\cup[1-h,1]} |v_h| = 1. \tag{35}$$

Then a continuity and a scaling argument gives the existence of $c_M \in \mathbb{R}^+$ (which may depend on $\mathbf{k}$ but not on the parameter h) such that $\eta_h * v_h(y) = c_M$ for some $y \in (0,h) \cup (1-h,1)$. Again, a scaling argument implies the inequality

$$\|\nabla(\eta_h * v_h)\| \ge \|\nabla(\eta_h * v_h)|_{(0,h)}\| + \|\nabla(\eta_h * v_h)|_{(h,1-h)}\| \ge \frac{c_M}{\sqrt{h}}. \tag{36}$$

On the other hand, the assumption in (35) gives that

$$\max_{x\in[0,h]\cup[1-h,1]} |v_h'| \le \frac{c_s}{h}$$

and similarly,

$$\max_{x\in[0,h]\cup[1-h,1]} |v_h''| \le \frac{c_s^2}{h^2}$$

for some constant c_s. Hence, with the aid of (33) we obtain that for all $x \in \operatorname{supp} \eta_h * v_h - \tilde{v}_h = \left[0, \frac{h}{2}\right] \cup \left[1-\frac{h}{2}, 1\right]$

$$|(\eta_h * v_h - \tilde{v}_h)(x)| \le c_s h^{s-1}$$

and similarly,

$$|\nabla(\eta_h * v_h - \tilde{v}_h)(x)| \le c_s h^{s-2},$$

which can be used together with (36) to complete the estimation in (34) to get

$$\begin{aligned}
&\sup_{v_h \in P_{h,\mathbf{k}}} \frac{(f, \eta_h * v_h) - (\nabla u, \nabla(\eta_h * v_h))}{\|\eta * v_h\|_{H^1}} \\
&\le \frac{\|f|_{[0,\frac{h}{2}]\cup[1-\frac{h}{2},1]}\|\|\eta_h * v_h - \tilde{v}_h\| + \|\nabla u|_{[0,\frac{h}{2}]\cup[1-\frac{h}{2},1]}\|\|\nabla(\eta_h * v_h - \tilde{v}_h)\|}{c_M \cdot h^{-\frac{1}{2}}} \\
&\le \frac{\|f|_{[0,\frac{h}{2}]\cup[1-\frac{h}{2},1]}\| c_s h^{\frac{1}{2}} h^{s-1} + \|\nabla u|_{[0,\frac{h}{2}]\cup[1-\frac{h}{2},1]}\| c_s h^{\frac{1}{2}} h^{s-2}}{c_M \cdot h^{-\frac{1}{2}}} \\
&\le c_s \left(\|f|_{[0,\frac{h}{2}]\cup[1-\frac{h}{2},1]}\| h^s + \|\nabla u|_{[0,\frac{h}{2}]\cup[1-\frac{h}{2},1]}\| h^{s-1} \right),
\end{aligned}$$

which proves the lemma. □

Remark: If the analytic solution is smooth in the sense that $\sup|\nabla u|$ is bounded then the error term in Lemma 5 becomes $\mathcal{O}(h^{s-\frac{1}{2}})$.

Theorem 3. *The averaged interior penalty approximation is quasi optimal in the sense that for some constant C we have the estimate*

$$\|\nabla(u - \eta_h * u_{IP})\| \le C \inf_{v_h \in P_{h,\mathbf{k}}} \|u - \eta_h * v_h\|_1 + \mathcal{O}(h^{s-1}) + C\|\eta_h * f - f\|.$$

Proof: A triangle inequality and the estimates in Theorem 2 and Lemma (5) imply that

$$\begin{aligned}
\|\nabla(u - \eta_h * u_{\mathrm{IP}})\|_1 &\le \|u - \eta_h * u_h\|_1 + \|\nabla(\eta_h * u_{\mathrm{IP}} - \eta_h * u_h)\| \\
&\le C \inf_{v_h \in P_{h,\mathbf{k}}} \|u - \eta_h * v_h\|_1 + \mathcal{O}(h^{s-1}) + C\|\eta_h * f - f\|,
\end{aligned}$$

as stated in the theorem. □

Remarks: The multidimensional generalization of the above theory would give the same benefits as presented here. First, the convergence of the local average of a possibly overpenalized IP method could be verified in the H^1-seminorm. The derivation would be free of the primal-dual formalism and introducing arbitrary numerical fluxes. Also the analysis in Section 3.1, which can be transferred unchanged, does not require any extra smoothness of the analytic solution as did in the original approach. We also guess that it is close link between the lifted dG methods, *e.g.*, the ones by Bassi and Rebay and the one by Brezzi et al., see Table 3.2 in [2] for their explicit form.
The main bottleneck is that the last term in the explicit form (6) can not be given in a simple form. In the one-dimensional case, the jump terms can be given with the linear combination of Dirac distributions (which is the distributional derivative of the jump function). These can be easily convolved with the averaging function η_h as δ is the identity element for the convolution. In the multi-dimensional case, the distributional derivative of the jump functions is not simply a Dirac distribution and even Proposition 1 is difficult to give in an explicit form.

4. Numerical experiments

In this section, we demonstrate the accuracy of the averaged numerical solution $\eta_h * u_h$ in (4) and relate it to other approximations. We also investigate experimentally the choice of the exponent s. According to (6), it can be recognized as a penalty parameter which has a crucial role in the computations. Too large penalty terms can result in ill-conditioned systems while small penalty terms can hinder the stability of the corresponding dG method, see [20].

TABLE 1. Computational error in different norms for the averaged approximations arising from the bilinear forms a_η and a_{IP}, respectively using locally first order elements and the parameter $s = 2$. In the last row, the estimated convergence order is given based on the finest grid.

Number of subintervals	max - norm		H^1 - seminorm	
	a_η	a_{IP}	a_η	a_{IP}
8	$1.3 \cdot 10^{-2}$	$1.8 \cdot 10^{-2}$	$2.5 \cdot 10^{-1}$	$2.5 \cdot 10^{-1}$
16	$3.7 \cdot 10^{-3}$	$4.6 \cdot 10^{-3}$	$1.4 \cdot 10^{-1}$	$1.4 \cdot 10^{-1}$
32	$1.0 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$	$7.2 \cdot 10^{-2}$	$7.2 \cdot 10^{-2}$
64	$2.8 \cdot 10^{-4}$	$2.9 \cdot 10^{-4}$	$3.6 \cdot 10^{-2}$	$3.6 \cdot 10^{-2}$
128	$7.1 \cdot 10^{-5}$	$7.3 \cdot 10^{-5}$	$1.9 \cdot 10^{-2}$	$1.9 \cdot 10^{-2}$
256	$1.8 \cdot 10^{-5}$	$1.8 \cdot 10^{-5}$	$9.3 \cdot 10^{-3}$	$9.3 \cdot 10^{-3}$
Convergence rate	2	2	1	1

The simple model problem we solve numerically is

$$\begin{cases} u''(x) = -\pi^2 \sin \pi x, & x \in (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

In the first series of experiments, we take a uniform tessellation of the interval $(0,1)$ and piecewise first order polynomials to give the finite element space $P_{h,\mathbf{1}}$. We compare the solution $\eta_h * u_h$ of (4) with the averaged IP solution $\eta_h * u_{\mathrm{IP}}$. The cornerstone of the implementation of the numerical method is to compute the local averages $\eta_h * u_h$ for the basis functions $u_h \in P_{h,\mathbf{k}}$.
If $\operatorname{supp} u_h = [x_j, x_{j+1}]$ then

$$\eta_h * \nabla_h u_h(x) = \begin{cases} \frac{1}{2h^s} \int_{x_j}^{x+h^s} \nabla_h u_h(y)\,\mathrm{d}y & \text{if } x \in [x_j - h^s, x_j + h^s] \\ \frac{1}{2h^s} \int_{x-h^s}^{x+h^s} \nabla_h u_h(y)\,\mathrm{d}y & \text{if } x \in [x_j + h^s, x_{j+1} - h^s] \\ \frac{1}{2h^s} \int_{x-h^s}^{x_{j+1}} \nabla_h u_h(y)\,\mathrm{d}y & \text{if } x \in [x_{j+1} - h^s, x_{j+1} + h^s], \end{cases} \tag{37}$$

which is a continuous, piecewise polynomial function. This can be given analytically. Similar statement holds for $\eta_h * u_h(x)$. Accordingly, the scalar product $\pi^2(\sin \pi x, \eta_h * u_h)$ on the right hand side of the variational problem (cf. with (5)) is computed piecewise for each basis function u_h performing a three point Gauss integration separately on the intervals $[x_i - h^s, x_i + h^s], [x_i + h^s, x_{i+1} - h^s]$ and $[x_{i+1} - h^s, x_{i+1} + h^s]$.

The computation of the stiffness matrix is based on Proposition 1. The entries containing the the term $\eta_h * \nabla_h u_h$ can be computed on the intervals $[x_j - h^s, x_j + h^s]$ and $[x_j + h^s, x_{j+1} - h^s]$ corresponding to the first two terms in (6). The computation of the third and fourth terms in (6) is based on the third expression of the equality in (8). Finally, the computation of the last term in (6) is straightforward. Even in case of second-order polynomials one can evaluate all the entries by hand.

As a result, the stiffness matrix becomes a band matrix with a larger band width compared to a classical IP stiffness matrix. In our case, if u_j and u_k are supported on neighboring intervals then in general the scalar product $(\eta_h * \nabla_h u_j, \eta_h * \nabla_h u_k)$ and the corresponding matrix entry $[j,k]$ is non-zero.

The result of the computations is presented in Table 1. The results confirm the convergence order stated in Theorem 3 and suggest superconvergence in the maximum norm.

In the second series of experiments, we use the finite element space $P_{h,\mathbf{2}}$ consisting of piecewise second order polynomials on the above tessellation. We compare the solution $\eta_h * u_h$ of (4) with $\eta_h * u_{\mathrm{IP}}$. In both cases, we obtained second order convergence in the H^1-seminorm as predicted in Lemma 5 and Theorem 3. The results are presented in Figure 1.

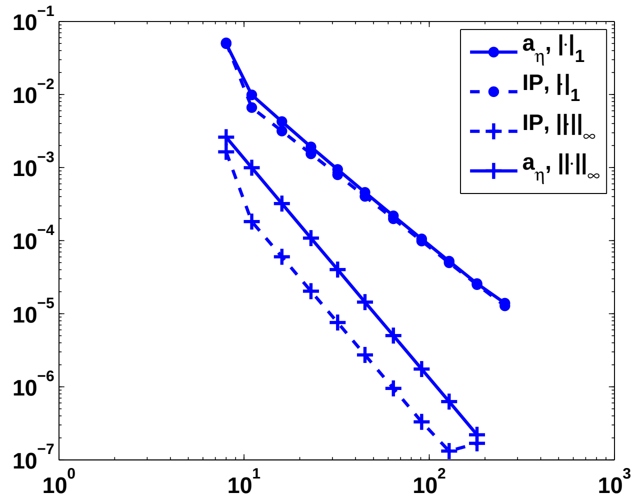

FIGURE 1. Computational error of the approximations $\eta_h * u_h$ (continuous line) and $\eta_h * u_{\mathrm{IP}}$ (dashed line) in the H^1-seminorm ($\bullet$) and in the maximum norm ($+$) vs. the number of subintervals using locally second order elements.

The results confirm again the predicted convergence rate: the IP method really seems a lower-order perturbation of the one given by a_η.

The only free parameter in the bilinear form a_η is the exponent s. We investigate its "optimal" choice leading to a minimal error in the H^1-seminorm.

In the above experiments, we applied the smallest exponent which is necessary to achieve the desired convergence rate. In concrete terms, $s = 2$ for $\mathbf{k} = \mathbf{1}$ and $s = 3$ for $\mathbf{k} = \mathbf{2}$. The experiments indicate that this is the optimal choice. Whenever larger values of s can result in smaller computational error for a coarse resolution, the condition number in the corresponding linear system grows rapidly, which deteriorates the optimal convergence rate. The results of the corresponding numerical experiments for $\mathbf{k} = \mathbf{2}$ are summarized in Table 2. At the maximal admissible resolution, the condition number is of order 10^{10}. One can observe this phenomenon in Fig. 1: the error in the $|\cdot|_\infty$-norm is increased at the finest resolution if the overpenalized IP method was applied.

Interestingly, we observed here that not only the coefficient of the penalty term but also the exponent of the discretization parameter should depend on the local polynomial degree.

Appendix

Proof of Lemma 1: The proof will be carried out using a standard scaling argument.

TABLE 2. H^1-seminorm of the computational error in case of coarse resolution for some parameters s and the maximal number of subintervals $N_{\max}$ until the optimal rate of convergence can be verified.

	k = 2				
	$s=3$	$s=3.5$	$s=4$	$s=4.5$	$s=5$
N=16	$4.26\cdot 10^{-3}$	$3.25\cdot 10^{-3}$	$3.19\cdot 10^{-3}$	$3.19\cdot 10^{-3}$	$3.19\cdot 10^{-3}$
N=32	$9.43\cdot 10^{-4}$	$8.02\cdot 10^{-4}$	$7.98\cdot 10^{-4}$	$7.98\cdot 10^{-4}$	$7.98\cdot 10^{-4}$
$N_{\max}$	256	144	128	72	64

If $h=1$ then $\mathcal{P}_k\left(-\frac{h}{2},0\right)\oplus\mathcal{P}_k\left(0,\frac{h}{2}\right)$ is a fixed finite dimensional subspace.Therefore, the norms given by

$$\|p_0\|_\infty = \max_{[-\frac12,\frac12]}|p_0| \quad\text{and}\quad \|p_0\|_{s,2}=\sqrt{\int_{[-\frac12,\frac12]\setminus[-\frac14,\frac14]}p_0^2} \tag{38}$$

on this space are equivalent. Hence we have a constant c_0 such that

$$\max_{[-\frac12,\frac12]}|p|\le\sqrt{c_0}\sqrt{\int_{[-\frac12,\frac12]\setminus[-\frac14,\frac14]}p^2}$$

which gives the statement of the lemma in case of $h=1$.

For an arbitrary parameter $h\le\frac12$ and $p\in\mathcal{P}_k\left(-\frac{h}{2},0\right)\oplus\mathcal{P}_k\left(0,\frac{h}{2}\right)$ we define $p_0\in C(-\frac12,0)\oplus C(0,\frac12)$ with $p_0(x)=p(hx)$. Therefore, using the statement for $h=1$ we obtain

$$\begin{aligned}\max_{[-\frac{h}{2},\frac{h}{2}]}p^2 &= \max_{[-\frac12,\frac12]}p_0^2\le c_0\int_{[-\frac12,\frac12]\setminus[-\frac14,\frac14]}p_0^2=\frac{c_0}{h}\int_{[-\frac{h}{2},\frac{h}{2}]\setminus[-\frac{h}{4},\frac{h}{4}]}p^2\\ &\le\frac{c_0}{h}\int_{[-\frac{h}{2},\frac{h}{2}]\setminus[-h^s,h^s]}p^2,\end{aligned}$$

as stated in the lemma. □

Proof of Lemma 2: A similar argument as in (38) implies that for the interval $[0,1]$ there is a constant c_1 such that for all polynomial p_0 of maximal degree k we have

$$\|p_0'\|^2\le c_1\|p_0\|^2 \tag{39}$$

and

$$\max_{x\in[0,1]}|p_0'|\le c_1\|p_0'\|. \tag{40}$$

Instead of taking a generic interval it is sufficient to prove the statements for intervals of type $I_{h_0}=[0,h_0]$. For an arbitrary $u\in\mathcal{P}_k$ we define p_0 with $p_0(x):=p(h_0x)$. Then using also (39) we have

$$\|p'\|^2_{I_{h_0}}=\int_0^{h_0}|p'|^2=h_0\cdot\frac{1}{h_0^2}\int_0^1|p_0'|^2\le c_1\frac{1}{h_0}\int_0^1|p_0|^2_{I_{h_0}}=c_1\frac{1}{h_0^2}\int_0^{h_0}|p|^2=c_1\frac{1}{h_0^2}\|p\|^2_{I_{h_0}}$$

such that (15) is proved.

Also, with the aid of (40) we obtain

$$\max_{x\in[0,h_0]}|p'|=\frac{1}{h_0}\max_{x\in[0,1]}|p_0'|\le c_1\frac{1}{h_0}\|p_0'\|=c_1\frac{1}{h_0}\|p'\|\frac{1}{\sqrt{h_0}}=c_1\frac{1}{\sqrt{h_0^3}}\|p'\|,$$

which proves the other statement in the lemma. $\square$

Proof of Lemma 3: We first use the estimate in Lemma 2 for the interval $\tilde{I}_j = [\gamma_j + h^s, \gamma_{j+1} - h^s]$ and a polynomial $u \in \mathcal{P}_{k_j}(I_j)$, which gives that for all $x \in I_j$ the following estimate is valid:

(41)
$$\begin{aligned}|\eta_h * u(x) - u(x)| &= \left| \frac{1}{2h^s} \int_{x-h^s}^{x+h^s} u(y)\,\mathrm{d}y \; - u(x) \right| \le \frac{1}{2h^s} \int_{-h^s}^{h^s} |u(x+y) - u(x)|\,\mathrm{d}y \\ &\le \frac{1}{2h^s} \max_{I_j} |u'| \int_{-h^s}^{h^s} |y|\,\mathrm{d}y \; \le \frac{h^s}{2} c_1 |I_j|^{-\frac{3}{2}} \|u\|_{I_j} \le h^{s-\frac{3}{2}} \frac{c_1}{2} \|u\|_{I_j}.\end{aligned}$$

It is also obvious that

$$\begin{aligned}&\left| (\eta_h * \nabla_h u, \eta_h * \nabla_h v)_{I_j} - (\nabla_h u, \nabla_h v)_{\tilde{I}_j} \right| \\ &\le \left| (\eta_h * \nabla_h u, \eta_h * \nabla_h v)_{\tilde{I}_j} - (\nabla_h u, \nabla_h v)_{\tilde{I}_j} \right| + |(\nabla_h u, \nabla_h v)_{I_j \setminus \tilde{I}_j}|.\end{aligned} \tag{42}$$

Using the result in (41), we have that for given $u, v \in P_{h,\mathbf{k}}$ the first term on the right hand side of (42) can be estimated for all $x \in I$ as

$$\eta_h * \nabla_h u(x) - \nabla_h u(x) \le \frac{c_1}{2} h^{s-\frac{3}{2}} \|\nabla_h u\|_{I_j}$$

and similar relation holds for v. Therefore, using the Cauchy–Schwarz inequality, the relation $h_0 < h$ and the short notation $\mathbb{1}$ for the constant one-function we obtain

(43)
$$\begin{aligned}&\left| (\eta_h * \nabla_h u, \eta_h * \nabla_h v)_{\tilde{I}_j} - (\nabla_h u, \nabla_h v)_{\tilde{I}_j} \right| \\ &\le \left| (\frac{c_1}{2} h^s \frac{1}{\sqrt{h_0^3}} \|\nabla_h u\|_{\tilde{I}_j} \cdot \mathbb{1} + \nabla_h u, \frac{c_1}{2} h^s \frac{1}{\sqrt{h_0^3}} \|\nabla_h v\|_{\tilde{I}_j} \cdot \mathbb{1} + \nabla_h v)_{\tilde{I}_j} - (\nabla_h u, \nabla_h v)_{\tilde{I}_j} \right| \\ &\le \frac{c_1^2}{4} h^{2s} \frac{1}{h_0^3} \|\nabla_h u\|_{\tilde{I}_j} \|\nabla_h v\|_{\tilde{I}_j} \|\mathbb{1}\|_{\tilde{I}_j} + \frac{c_1}{2} h^s \frac{1}{\sqrt{h_0^3}} \|\nabla_h u\|_{\tilde{I}_j} \left| (\nabla_h v, \mathbb{1})_{\tilde{I}_j} \right| \\ &+ \frac{c_1}{2} h^s \frac{1}{\sqrt{h_0^3}} \|\nabla_h v\|_{\tilde{I}_j} \left| (\nabla_h u, \mathbb{1})_{\tilde{I}_j} \right| \\ &\le \frac{c_1^2}{4} h^{2s} \frac{1}{h_0^2} \|\nabla_h u\|_{\tilde{I}_j} \|\nabla_h v\|_{\tilde{I}_j} + c_1 h^s \frac{1}{\sqrt{h_0^2}} \|\nabla_h u\|_{\tilde{I}_j} \|\nabla_h v\|_{\tilde{I}_j} \\ &\le \|\nabla_h u\|_{\tilde{I}_j} \|\nabla_h v\|_{\tilde{I}_j} \left(\frac{c_1^2}{4} h^{2s} \frac{1}{h^2} + c_1 h^s \frac{1}{h} \right).\end{aligned}$$

For the estimation of the second term on the right hand side of (42), we use (21) and obtain that

$$|(\nabla_h u, \nabla_h v)_{I_j \setminus \tilde{I}_j}| \le c_2 h^{s-1} |(\nabla_h u, \nabla_h v)_{\tilde{I}_j}| \le c_2 h^{s-1} \|\nabla_h u\|_{\tilde{I}_j} \|\nabla_h v\|_{\tilde{I}_j}. \tag{44}$$

The inequalities in (43) and (44) give that

$$\left| (\eta_h * \nabla_h u, \eta_h * \nabla_h v)_{\tilde{I}_j} - (\nabla_h u, \nabla_h v)_{I_j} \right| \le \left(\frac{c_1^2}{4} h^{2s-2} + (c_1 + c_2) h^{s-1} \right) (\nabla_h u, \nabla_h v)_{\tilde{I}_j}^2,$$

which can be summed up to arrive at the estimate (22). $\square$

Proof of Lemma 4: We first introduce the function $w : \left(-\frac{h}{2}, \frac{h}{2}\right)$ with

$$w = \nabla_h v - \{\!\!\{\nabla_h v\}\!\!\}_0 \nVdash + [\![\nabla_h v]\!]_0 \, (H - \frac{\nVdash}{2}),$$

where H denotes the Heaviside function. Note that

$$[\![w]\!]\,(0) = [\![\nabla_h v]\!]\,(0) + [\![\nabla_h v]\!]_0 \cdot (-1) = 0,$$

and

$$\lim w(0+) = \nabla_h v(0+) - \{\!\!\{\nabla_h v\}\!\!\}_0 \nVdash + \frac{1}{2}\,[\![\nabla_h v]\!]_0 = 0,$$

such that w is piecewise polynomial of maximal degree $\max \mathbf{k}$ and continuous on $\left[-\frac{h}{2}, \frac{h}{2}\right]$. We also have that

$$\nabla_h w = \nabla_h \nabla_h v, \tag{45}$$

where the piecewise derivative is taken in $\left(-\frac{h}{2}, 0\right) \cup \left(0, \frac{h}{2}\right)$. Then using the Newton–Leibniz formula, the Cauchy–Schwarz inequality, (45), (20) and the inverse inequality (15) we have the estimate

(46)

$$\begin{aligned}
&\left|\frac{1}{2h^s}\int_{-h^s}^{h^s} \eta_h * w\right| \\
&\le \frac{1}{2h^s}\left(\int_{-h^s}^{0}\left|\int_0^x \nabla[\eta_h * w](t)\,\mathrm{d}t\right|\,\mathrm{d}x + \int_0^{h^s}\left|\int_0^x \nabla[\eta_h * w](t)\,\mathrm{d}t\right|\,\mathrm{d}x\right) \\
&\le \frac{1}{2h^s}\left(\int_{-h^s}^{0}\sqrt{\int_x^0 |\nabla[\eta_h * w](t)|^2\,\mathrm{d}t}\sqrt{\int_x^0 1\,\mathrm{d}x} + \int_0^{h^s}\sqrt{\int_0^x |\nabla[\eta_h * w](t)|^2\,\mathrm{d}t}\sqrt{\int_0^x 1\,\mathrm{d}x}\right) \\
&\le \frac{1}{2h^s}\left(\int_{-h^s}^{0}\sqrt{\int_x^0 |\eta_h * \nabla_h w(t)|^2\,\mathrm{d}t}\sqrt{\int_x^0 1\,\mathrm{d}x} + \int_0^{h^s}\sqrt{\int_0^x |\eta_h * \nabla_h w(t)|^2\,\mathrm{d}t}\sqrt{\int_0^x 1\,\mathrm{d}x}\right) \\
&\le \frac{1}{2h^s}\left(\int_{-h^s}^{0}\sqrt{-x}\sqrt{c_0\frac{-x}{h}}\sqrt{\int_{-\frac{h}{2}}^{\frac{h}{2}} |\nabla_h\nabla_h v(t)|^2\,\mathrm{d}t}\,\mathrm{d}x\right. \\
&\left.+ \int_0^{h^s}\sqrt{x}\sqrt{c_0\frac{x}{h}}\sqrt{\int_{-\frac{h}{2}}^{\frac{h}{2}} |\nabla_h\nabla_h v(t)|^2\,\mathrm{d}t}\,\mathrm{d}x\right) \\
&= \frac{\sqrt{c_0}}{2h^{s+\frac{1}{2}}}\sqrt{\int_{-\frac{h}{2}}^{\frac{h}{2}} |\nabla_h\nabla_h v|^2}\left(\int_{-h^s}^0 -x\,\mathrm{d}x + \int_0^{h^s} x\,\mathrm{d}x\right) \\
&\le \frac{\sqrt{c_0}}{2h^{s+\frac{1}{2}}}\frac{c_1}{h}\sqrt{\int_{-\frac{h}{2}}^{\frac{h}{2}} |\nabla_h v|^2 h^{2s}} = \frac{\sqrt{c_0}c_1}{2}h^{s-\frac{3}{2}}\sqrt{\int_{-\frac{h}{2}}^{\frac{h}{2}} |\nabla_h v|^2}.
\end{aligned}$$

Since $H - \frac{\nVdash}{2}$ is an odd function, the same holds for $\eta_h * (H - \frac{\nVdash}{2})$ and therefore,

$$\int_{-h^s}^{h^s} \eta_h * (H - \frac{\nVdash}{2}) = 0,$$

which can be used to obtain

$$\begin{aligned}
\left|\frac{1}{2h^s}\int_{-h^s}^{h^s} \eta_h * w\right| &= \left|\frac{1}{2h^s}\int_{-h^s}^{h^s} \eta_h * \nabla_h v - \frac{1}{2h^s}\int_{-h^s}^{h^s} \{\!\!\{\nabla_h v\}\!\!\}_0\, \eta_h * \nVdash\right| \\
&= \left|\{\!\!\{\nabla_h v\}\!\!\}_0 - \frac{1}{2h^s}\int_{-h^s}^{h^s} \eta_h * \nabla_h v\right|.
\end{aligned}$$

Therefore, using (46) we get

$$\left| \{\!\{\nabla_h v\}\!\}_0 - \frac{1}{2h^s} \int_{-h^s}^{h^s} \eta_h * \nabla_h v \right| \leq \frac{\sqrt{c_0} c_1}{2} h^{s-\frac{3}{2}} \sqrt{\int_{-\frac{h}{2}}^{\frac{h}{2}} |\nabla_h v|^2}.$$

Similarly, for each grid point γ_j we have

$$\left| \{\!\{\nabla_h v\}\!\}_j - \frac{1}{2h^s} \int_{\gamma_j - h^s}^{\gamma_j + h^s} \eta_h * \nabla_h v \right| \leq \frac{\sqrt{c_0} c_1}{2} h^{s-\frac{3}{2}} \sqrt{\int_{\gamma_j - \frac{h}{2}}^{\gamma_j + \frac{h}{2}} \nabla_h v^2}, \tag{47}$$

which proves the lemma. □

References

[1] M. Ainsworth and R. Rankin, *Fully computable error bounds for discontinuous Galerkin finite element approximations on meshes with an arbitrary number of levels of hanging nodes*, SIAM J. Numer. Anal., 47 (2010), pp. 4112–4141.

[2] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal., 39 (2001/02), pp. 1749–1779 (electronic).

[3] R. Becker, P. Hansbo, and M. G. Larson, *Energy norm a posteriori error estimation for discontinuous Galerkin methods*, Comput. Methods Appl. Mech. Engrg., 192 (2003), pp. 723–733.

[4] A. Buffa and C. Ortner, *Compact embeddings of broken Sobolev spaces and applications*, IMA J. Numer. Anal., 29 (2009), pp. 827–855.

[5] B. Cockburn, J. Guzmán, and H. Wang, *Superconvergent discontinuous Galerkin methods for second-order elliptic problems*, Math. Comp., 78 (2009), pp. 1–24 (electronic).

[6] B. Cockburn, M. Luskin, C.-W. Shu, and E. Süli, *Enhanced accuracy by post-processing for finite element methods for hyperbolic equations*, Math. Comp., 72 (2003), pp. 577–606 (electronic).

[7] D. A. Di Pietro and A. Ern, *Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier-Stokes equations*, Math. Comp., 79 (2010), pp. 1303–1330.

[8] D. A. Di Pietro and A. Ern, *Mathematical Aspects of Discontinuous Galerkin Methods*, Springer-Verlag, Berlin, Heidelberg, 2012.

[9] A. Ern and J.-L. Guermond, *Theory and Practice of Finite Elements*, vol. 159 of Applied Mathematical Sciences, Springer-Verlag, New York, 2004.

[10] A. Ern and J.-L. Guermond, *Discontinuous Galerkin methods for Friedrichs' systems. I. General theory*, SIAM J. Numer. Anal., 44 (2006), pp. 753–778 (electronic).

[11] ———, *Discontinuous Galerkin methods for Friedrichs' systems. II. General theory*, SIAM J. Numer. Anal., 44 (2006), pp. 2363–2388 (electronic).

[12] A. Ern and J.-L. Guermond, *Discontinuous Galerkin methods for Friedrichs' systems. III. Multifield theories with partial coercivity*, SIAM J. Numer. Anal., 46 (2008), pp. 776–804.

[13] A. Ern, A. F. Stephansen, and M. Vohralík, *Guaranteed and robust discontinuous Galerkin a posteriori error estimates for convection-diffusion-reaction problems*, J. Comput. Appl. Math., 234 (2010), pp. 114–130.

[14] T. Gudi, *A new error analysis for discontinuous finite element methods for linear elliptic problems*, Math. Comp., 79 (2010), pp. 2169–2189.

[15] J. Hesthaven and T. Warburton, *Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications*, vol. 54 of Texts in Applied Mathematics, Springer, New York, first ed., 2007.

[16] F. Hirsch and G. Lacombe, *Elements of functional analysis*, vol. 192 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1999. Translated from the 1997 French original by Silvio Levy.

[17] L. Ji, Y. Xu, and J. K. Ryan, *Accuracy-enhancement of discontinuous Galerkin solutions for convection-diffusion equations in multiple-dimensions*, Math. Comp., 81 (2012), pp. 1929–1950 (electronic).

[18] O. A. Karakashian and F. Pascal, *A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems*, SIAM J. Numerical Analysis, 41 (2003), pp. 2374–2399.

[19] J. King, H. Mirzaee, J. K. Ryan, and R. M. Kirby, *Smoothness-Increasing Accuracy-Conserving (SIAC) Filtering for Discontinuous Galerkin Solutions: Improved Errors Versus Higher-Order Accuracy*, J. Sci. Comput., 53 (2012), pp. 129–149.
[20] D. Marazzina, *Stability properties of discontinuous Galerkin methods for 2D elliptic problems*, IMA J. Numer. Anal., 28 (2008), pp. 552–579.
[21] B. Rivière, *Discontinuous Galerkin methods for solving elliptic and parabolic equations*, vol. 35 of Frontiers in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008. Theory and implementation.
[22] L. Zhu, S. Giani, P. Houston, and D. Schötzau, *Energy norm a posteriori error estimation for hp-adaptive discontinuous Galerkin methods for elliptic problems in three dimensions*, Math. Models Methods Appl. Sci., 21 (2011), pp. 267–306.

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