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# GLOBAL $H^2$-REGULARITY RESULTS OF THE 3D PRIMITIVE EQUATIONS OF THE OCEAN

YINNIAN HE AND JIANHUA WU

**Abstract.** In this article, we consider the 3D viscous primitive equations (PEs for brevity) of the ocean under two physically relevant boundary conditions for the $H^1$ and $H^2$ smooth initial data, respectively. The $H^2$ regularity result of the solution for the viscous PEs of the ocean has been unknown since the work by Cao and Titi [3], and Kobelkov [26]. In this article we provide the global $H^2$-regularity results of the solution and its time derivatives for the 3D viscous primitive equations of the ocean by using the $L^6$ estimates developed in [3] and some new energy estimate techniques.

**Key words.** Primitive equations, ocean, regularity.

## 1. Introduction

Given a smooth bounded domain $\omega \subset \mathbb{R}^2$ and the cylindrical domain $\Omega = \omega \times (-d, 0) \subset \mathbb{R}^3$, we consider in $\Omega$ the following 3D viscous PEs of the ocean with rigid lid approximation and in the presence of one stratification:

$$u_t + L_1 u + (u \cdot \nabla)u + w\partial_z u + \nabla P + f\vec{k} \times u = F_1, \tag{1.1}$$

$$\theta_t + L_2\theta + (u \cdot \nabla)\theta + w\partial_z\theta - \sigma w = F_2 \tag{1.2}$$

$$\nabla \cdot u + \partial_z w = 0, \tag{1.3}$$

$$\partial_z P + \gamma\theta = 0. \tag{1.4}$$

The unknowns for the 3D viscous PEs are the fluid velocity field $(u, w) = (u_1, u_2, w) \in \mathbb{R}^3$ with $u = (u_1, u_2)$ being the horizontal velocity, the density $\theta$ and the pressure $P$. Here $f = f_0(\beta + y)$ is the given Coriolis rotation frequency with $\beta$-plane approximation, $F_1$ and $F_2$ are two given functions and $\vec{k}$ is the vertical unit vector, $\sigma > 0$ is the stratification constant of the ocean and $\gamma > 0$ is the gravitational constant. The elliptic operators $L_1$ and $L_2$ are given respectively as the following:

$$L_i = -\nu_i\Delta - \mu_i\partial_z^2, \ i = 1, \ 2.$$

Here the positive constants $\nu_1, \ \mu_1$ are the horizontal and vertical viscosity coefficients; while the positive constants $\nu_2, \ \mu_2$ are the horizontal and vertical thermal diffusivity coefficients and

$$\nabla = (\partial_x, \partial_y), \ \Delta = \partial_{xx} + \partial_{yy}, \ \partial_{x_i} = \frac{\partial}{\partial x_i}, \ \partial_{x_i x_i} = \partial^2_{x_i},$$

with $i = 1, 2, 3$ and $(x_1, x_2, x_3) = (x, y, z)$.

For more details on the PEs of the ocean, the reader is referred to [7, 28, 29, 34] for the physical aspect, and to [3, 20, 25, 22, 23, 24, 31, 32, 16, 17, 18, 19, 25] for the mathematical aspect.

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Here and after, we use the following notations:

$$\eta_t = \frac{\partial \eta}{\partial t},\ \phi_{x_i} = \partial_{x_i}\phi,\ \psi_{x_i x_i} = \partial_{x_i x_i}\psi,$$

with $i = 1, 2, 3$ for any $\eta(t) \in H^1(0, \infty)$, $\phi(x, y, z) \in H^1(\Omega)$ and $\psi(x, y, z) \in H^2(\Omega)$.

We partition the boundary of $\Omega$ into the following three parts:

$$\Gamma_u = \{(x, y, z) \in \bar{\Omega}; z = 0\},$$

$$\Gamma_b = \{(x, y, z) \in \bar{\Omega}; z = -d\},$$

$$\Gamma_s = \{(x, y, z) \in \bar{\Omega}; (x, y) \in \partial\omega,\ -d \le z \le 0\}.$$

Next, we provide the system (1.1)-(1.4) with the following boundary conditions-with the wind-driven on the top surface and non-slip and non-heat flux on the side walls and the bottom (see, e.g., page 246 in [3], page 160 in [20] and page 1037 in [25]):

$$\begin{aligned}
&\text{on } \Gamma_u,\ \partial_z u = d\,\tau^*,\ w = 0,\ \partial_z\theta = -\alpha(\theta - \theta^*);\\
&\text{on}\Gamma_b,\ \partial_z u = 0,\ w = 0, \partial_z\theta = 0;\\
&\text{on } \Gamma_s,\ u\cdot\mathbf{n} = 0,\ \frac{\partial u}{\partial \mathbf{n}} \times \mathbf{n} = \mathbf{0},\ \frac{\partial \theta}{\partial \mathbf{n}} = \mathbf{0},\\
&\text{or on } \Gamma_s,\ u = 0,\ \frac{\partial \theta}{\partial \mathbf{n}} = 0,
\end{aligned}$$

where $\tau^* = \tau^*(x, y)$ is the wind stress on the ocean surface, $\alpha$ is a positive constant, $\mathbf{n}$ is the normal vector of $\Gamma_s$ and $\theta^* = \theta^*(x, y)$ is the typical density distribution of the top surface of the ocean. Based on the above condition, it is natural to assume that $\tau^*(x, y)$ and $\theta^*(x, y)$ satisfy

$$\tau^* \cdot \mathbf{n} = \mathbf{0},\ \frac{\partial \tau^*}{\partial \mathbf{n}} \times \mathbf{n} = \mathbf{0}, \quad \text{or } \tau^* = \mathbf{0},\ \text{and } \frac{\partial \theta^*}{\partial \mathbf{n}} = \mathbf{0} \text{ on } \partial\omega.$$

Due to this condition, we can convert the previous boundary condition into the homogeneous by replacing $(u, \theta)$ by $(u + \frac{1}{2}([(z+d)^2 - \frac{1}{3}d^3]\tau^*, \theta + \theta^*)$ (refer to page 248 in [3]).

Hence, we consider the following boundary conditions for the 3D viscous PEs:

$$w|_{\Gamma_u \cup \Gamma_b} = 0. \tag{1.5}$$

$$\partial_z u|_{\Gamma_u \cup \Gamma_b} = 0,\ u\cdot\mathbf{n}|_{\Gamma_s} = 0,\ \frac{\partial u}{\partial \mathbf{n}} \times \mathbf{n}|_{\mathbf{\Gamma_s}} = \mathbf{0}; \tag{1.6-1}$$

$$\text{or } \partial_z u|_{\Gamma_u \cup \Gamma_b} = 0,\ u|_{\Gamma_s} = 0; \tag{1.6-2}$$

$$\partial_z\theta|_{\Gamma_b} = (\partial_z\theta + \alpha\theta)|_{\Gamma_u} = 0,\ \frac{\partial \theta}{\partial \mathbf{n}}|_{\Gamma_s} = 0, \tag{1.7}$$

refer to (28)-(29) of page 248 in [3] and (1.3)-(1.4) of page 160 in [20] for the boundary condition (1.5), (1.6-1) and (1.7), and Remark 2.1 of page 1038 in [25] for the boundary condition (1.5), (1.6-2) and (1.7).

Also, the initial conditions of $u(x, y, z, t)$ and $\theta(x, y, z, t)$ should be given by

$$u(x, y, z, 0) = u_0(x, y, z),\ \theta(x, y, z, 0) = \theta_0(x, y, z). \tag{1.8}$$

Using the Dirichlet boundary condition (1.5) of $w$ on $\Gamma_u \cap \Gamma_b$ and (1.3)-(1.4), we have

$$w(x, y, z, t) = -\int_{-d}^{z} \nabla \cdot u(x, y, \xi, t)d\xi,\ \int_{-d}^{0} \nabla \cdot u(x, y, \xi, t)d\xi = 0,$$

$$P(x, y, z, t) = p(x, y, t) - \gamma\int_{-d}^{z} \theta(x, y, \xi, t)d\xi.$$

With the above statements, one obtains the initial boundary value problem of the 3D viscous PEs:

$$u_t + L_1 u + \nabla p(x,y,t) - \gamma \int_{-d}^{z} \nabla\theta(x,y,\xi,t)d\xi + f\vec{k}\times u$$

$$+ (u\cdot\nabla)u - (\int_{-d}^{z} \nabla\cdot u(x,y,\xi,t)d\xi)\partial_z u = F_1, \tag{1.9}$$

$$\theta_t + L_2\theta + \sigma\int_{-d}^{z} \nabla\cdot u(x,y,\xi,t)d\xi + (u\cdot\nabla)\theta - (\int_{-d}^{z}\nabla\cdot u(x,y,\xi,t)d\xi)\partial_z\theta$$

$$= F_2, \tag{1.10}$$

$$\nabla\cdot\bar{u} = 0, \tag{1.11}$$

together with the boundary condition (1.6)-(1.7) and the initial condition (1.8), where

$$\bar{\phi}(x,y) = \frac{1}{d}\int_{-d}^{0}\phi(x,y,z)dz,\ \tilde{\phi} = \phi - \bar{\phi},$$

for any function $\phi(x,y,z)$ in $\Omega$.

**Remark 1.1.** Recall [3, 20], $F_1 = 0$ and $\gamma = 1$ in (1.1)and (1.4) and $\sigma = 0$ in (1.2) and (1.10).

The 3D viscous PEs are very important research subjects in the field of geophysical fluid dynamics, at both the theoretical and numerical levels. There are some well-known difficulties associated with this fundamental equation for 3D oceanic model since their strong nonlinearity.

The mathematical study of the PEs originates in a series of articles, by Lions, Temam and Wang in the early 1990s: see, for instance, [22, 23, 24], where the mathematical formulation of the PEs, which resembles that of the Navier-Stokes equations, was established. They defined the notions of weak and strong solutions. They also proved the existence of the weak solutions, but the uniqueness of weak solutions remains unresolved, and it is still an open problem for now. The existence of the strong solutions, locally in time, and their uniqueness were obtained in [6] and [32]. The existence and uniqueness of the strong solutions, globally in time, to the 3D viscous PEs in thin domains for a large set of initial data whose sizes depend inversely on the thickness of the domain were established in [17]. Also, the asymptotic analysis and the finite dimensional behavior of the 3D viscous PEs in thin domain as the depth of the domain goes to zero were studied in [18, 19]. For a more extensive discussion and review on this subject, the reader is referred to the recent articles [31] and [32].

It has appeared that the problem of the global existence and uniqueness of the strong solutions for the 3D viscous PEs might be harder than the 3D Navier-Stokes equations since the PEs have more complicated nonlinear terms. However, Cao and Titi resolved this problem positively in their recent article [3]; and another different proof of this result was given by Kobelkov [26]. They also proved that the strong solutions depend continuously on the initial data in the $L^2$-norms. One natural question arising from this global existence result is the dynamical behavior of the strong solutions of the 3D viscous PEs [33]. Recently, Ju in his work [20] first obtained the uniform $H^1$- bounds with respect to time $t$ for the strong solutions under the assumption that $F_2 \in L^2(\Omega)$. These uniform $H^1$-bounds allowed the author to obtain the absorbing balls for the solutions in the $H^1$-space. The radii of these absorbing balls are independent of the initial data. Finally, the author has proved the existence of the global attractor for the strong solutions of the 3D viscous PEs in $H^1$-space and that the attractor is both compact and connected in $H^1$-space.

Another natural question arising from the global existence of the strong solutions and the global attractor is the $H^2$-regularity of the strong solutions of the 3D viscous PEs [20], which is the focus of this article. The $H^k(k \geq 2)$ as well as Gevery class regularity of the solution for the 3D PEs with the periodic boundary condition case is proved by Petcu by use of the odd-symmetry (due to the Neumann conditions) in [30]. Inspired by the methods developed in [3, 20] for the existence of the strong solutions and the global attractor of the 3D viscous PEs, we extend these studies to the $H^2$-regularity of the solution and some of its time derivatives for the 3D viscous PEs under two different boundary conditions (1.6-1)-(1.7) and (1.6-2)-(1.7). Due to the stronger nonlinear terms in the 3D viscous PEs, we can not obtain the $H^2$-regularity of the solution from its $H^1$-regularity even when its initial data is $H^2$-regular. Then, for this purpose, we first obtain the $L^6$-regularity of the derivative of the solution with respect to $z$-variable. From this regularity and the $H^1$-regularity of the solution, we deduce the $H^2$-regularity of the solution when its initial data is $H^2$-regular. If the initial data is $H^1_+$-regularity, that $(u_0, \theta_0)$ is $H^1$-regular and $(u_{0z}, \theta_{0z})$ is $L^6$-regular, we can not prove that the second-order spatial derivatives of the solution are bounded near $t = 0$. Similar difficulty also appears in the analysis of the regularity in the case of the Navier-Stokes equations. Using a special technique used in the regularity estimates of the Navier-Stokes equations by Heywood and Rannarcher [14], we obtain the $H^2$-regularity of the product of the solution and the smooth factor $\tau^{\frac{1}{2}}(t)$, where $\tau(t) = \min\{1, t\}$. Similar techniques have been used to obtain the error analysis of the numerical solutions for the Navier-Stokes equations, see, for example, [14, 15, 8, 9, 10, 11, 12]. Similarly, we deduce the $H^2$-regularity of the product of the time first-order derivative of solution and the smooth factor $\tau^{\beta}(t)$, where $\beta = 1$ when the initial data is $H^2$-regular and $\beta = \frac{3}{2}$ when the initial data has $H^1_+$-regularity. Finally, we deduce the $H^2$-regularity of the product of the second-order time derivative of the solution and the smooth factor $\tau^{1+\beta}(t)$.

Our main results in this paper are included in the following theorem.

**Theorem 1.1.** Assume that the initial data $(u_0, \theta_0)$ is $H^k_*$-regular with $k = 1,\ 2$, and

$$(\partial_t^i F_1, \partial_t^i F_2) \in L^\infty(\mathbb{R}^+; L^2(\Omega)^2) \times L^\infty(\mathbb{R}^+; L^2(\Omega)),\ i = 0, 1, \cdots, m,$$

where $H^1_* = H^1_+$ (which will be defined in Section 2) and $H^2_* = H^2$. Then, the solution $(u, p, \theta)$ of the 3D viscous PEs satisfies the following bounds:

$$\tau^{2m-k}(t)[\|\partial_t^m u(t)\|^2_{L^2} + \|\partial_t^m \theta(t)\|^2_{L^2} + \tau(t)(\|\partial_t^m u(t)\|^2_{H^1} + \|\partial_t^m \theta(t)\|^2_{H^1} + \|\partial_t^{m-1} u(t)\|^2_{H^2} + \|\partial_t^{m-1} \theta(t)\|^2_{H^2} + \|\partial_t^{m-1} p(t)\|^2_{H^1})] \leq \kappa, \tag{1.12}$$

$$\int_0^t e^{\alpha_2(s-t)} \tau^{2m-k}(s)[\|\partial_t^m u\|^2_{H^1} + \|\partial_t^m \theta\|^2_{H^1} + \tau(s)(\|\partial_t^m u\|^2_{H^2} + \|\partial_t^m \theta\|^2_{H^2} + \|\partial_t^m p\|^2_{H^1})] ds \leq \kappa, \tag{1.13}$$

$$\int_0^t e^{\alpha_2(s-t)} \tau^{2m+1-k}(s)(\|\partial_t^{m+1} u\|^2_{L^2} + \|\partial_t^{m+1} \theta\|^2_{L^2}) ds \leq \kappa, \tag{1.14}$$

for all $t \geq 0$ and $m = 1, 2, 3$, where $\tau(t) = \min\{1, t\}$ and $\alpha_2 > 0$ is a fixed constant and $\kappa$ is a general positive constant depending on the data $(\Omega, \alpha, \sigma, \gamma, d, \nu_1, \mu_1, \nu_2, \mu_2, f, F_1, F_2, u_0, \theta_0)$, which can take different values at its different occurrences.

**Remark 1.2.** The regularity of the time derivatives of the solution with $m = 1, 2, 3$ is useful to the error estimate of the $m$-th order time discrete scheme of the PEs of the ocean, see [13].

This paper is organized as follows. In §2, some basic mathematical setting and important inequalities are recalled and some basic lemmas and some estimates of the nonlinear terms are provided. In §3, the $H^1$-regularity of the solution is concluded and the $L^6$-regularity of the $z$-derivative of the solution is proved. In §4, the $H^2$-regularity of the solution $(u, p, \theta)$ and the $H^1$-regularity of the first-order time derivative $(u_t, p_t, \theta_t)$ are obtained. In §5, the $H^2$-regularity of the first-order time derivative $(u_t, p_t, \theta_t)$ and the $H^1$-regularity of the second-order time derivative $(u_{tt}, p_{tt}, \theta_{tt})$ are provided. In §6, the $H^2$-regularity of the second-order time derivative $(u_{tt}, p_{tt}, \theta_{tt})$ and the $H^1$-regularity of the three-order time derivative $(u_{ttt}, p_{ttt}, \theta_{ttt})$ are proved.

## 2. Preliminaries

For the 3D domain $\Omega$ and 2D domain $\omega$ and $m \geq 0,\ p \geq 1$, we introduce the standard Sobolev spaces $H^m(\Omega)$ and $H^m(\omega)$ or $H^m(\Omega)^2$ and $H^m(\omega)^2$ with the norms $\|\cdot\|_{m,\Omega}$ and $\|\cdot\|_{m,\omega}$ and semi-norms $|\cdot|_{m,\Omega}$ and $|\cdot|_{m,\omega}$, respectively. For some details of the Sobolev spaces, the reader can refer to Adams [1].

Set

$$H_2 = L^2(\Omega),\ V_2 = H^1(\Omega),$$
$$H_1 = \{v \in L^2(\Omega)^2; \text{div } \bar{v} = 0,\ v \cdot \mathbf{n}|_{\Gamma_s} = 0\},$$
$$V_1 = \{v \in H^1(\Omega)^2; \text{div } \bar{v} = 0,\ v \cdot \mathbf{n}|_{\Gamma_s} = 0\},$$

for the boundary condition (1.6-1) and

$$V_1 = \{v \in H^1(\Omega)^2; \text{div } \bar{v} = 0,\ v|_{\Gamma_s} = 0\},$$

for the boundary condition (1.6-2).

For the domain $\omega$, we also introduce the following Sobolev spaces

$$H_0 = \{v \in L^2(\omega)^2; \text{div } v = 0,\ v \cdot \mathbf{n}|_{\Gamma_s} = 0\},\ L_0^2(\omega) = \{q \in L^2(\omega); \int_\omega q dx dy = 0\},$$

$$V_0 = \{v \in H^1(\omega)^2; \text{div } v = 0,\ v \cdot \mathbf{n}|_{\partial\omega} = 0\},$$

for the boundary condition $v \cdot \mathbf{n}|_{\partial\omega} = 0,\ \frac{\partial v}{\partial \mathbf{n}} \times \mathbf{n}|_{\partial\omega} = 0$ and

$$V_0 = \{v \in H^1(\omega)^2; \text{div } v = 0,\ v|_{\partial\omega} = 0\},$$

for the boundary condition $v|_{\partial\omega} = 0$.

We also use $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\omega$ to denote the inner product in $L^2(\Omega)$ and $L^2(\omega)$ or $L^2(\Omega)^2$ and $L^2(\omega)^2$ or $L^2(\Omega)^4$ and $L^2(\omega)^4$, respectively.

We denote by $A_1$ the Stokes-type operator associated with the primitive equations (see [3, 20, 27]), that is $A_1 = PL_1$, where $P$ is the $L^2$-orthogonal projection from $L^2(\Omega)^2$ to $H_1$. Also, we write $A_2 = L_2$. Therefore, we define the bilinear forms $a_i : V_i \times V_i \to R,\ i = 1, 2$ as follows:

$$a_1(u, v) = \nu_1(\nabla u, \nabla v)_\Omega + \mu_1(u_z, v_z)_\Omega = (A_1^{\frac{1}{2}} u, A_1^{\frac{1}{2}} v)_\Omega,$$

$$a_2(\theta, \eta) = \nu_2(\nabla\theta, \nabla\eta)_\Omega + \mu_2(\theta_z, \eta_z)_\Omega + \mu_2\alpha(\theta(z = 0), \eta(z = 0))_\omega = (A_2^{\frac{1}{2}}\theta, A_2^{\frac{1}{2}}\eta)_\Omega.$$

Also, define

$$D(A_i) = \{\phi \in H_i; A_i\phi \in H_i\},\ i = 1, 2,$$

with the norm $\|A_i \cdot\|_{0,\Omega}$.

We have the following Poincaré inequalities

$$2\alpha_1\|u\|_{0,\Omega}^2 \le \|A_1^{\frac{1}{2}}u\|_{0,\Omega}^2, \tag{2.1}$$

for any $u \in H^1(\Omega)^2$ satisfying (1.6-1) or (1.6-2), and

$$2\frac{\mu_2}{4d(d+\alpha^{-1})}\|\theta\|_{0,\Omega}^2 \le \|A_2^{\frac{1}{2}}\theta\|_{0,\Omega}^2, \tag{2.2}$$

for any $\theta \in H^1(\Omega)$ satisfying (1.7). For (2.1), the reader is referred to [20, 3, 5] with the boundary condition (1.6-1) and [1] with the boundary condition (1.6-2); and for (2.2), the reader is referred to [3]. Therefore, there exist two positive constants $c_0$ and $c_1$ such that

(2.3)
$$c_0\|A_1^{\frac{1}{2}}u\|_{0,\Omega}^2 \le \|u\|_{1,\Omega}^2 \le c_1\|A_1^{\frac{1}{2}}u\|_{0,\Omega}^2, \qquad c_0\|A_2^{\frac{1}{2}}\theta\|_{0,\Omega}^2 \le \|\theta\|_{1,\Omega}^2 \le c_1\|A_2^{\frac{1}{2}}\theta\|_{0,\Omega}^2,$$

for any $u \in H^1(\Omega)^2$ satisfying (1.6-1) or (1.6-2) and for any $\theta \in H^1(\Omega)$ satisfying (1.7). Here and after, we shall use the letters $C$ or $c$ (with or without subscripts) to denote a general positive constant depending on the data $(\Omega, \alpha, \sigma, \gamma, d, \nu_1, \mu_1, \nu_2, \mu_2)$, which can take different values at its different occurrences.

Also, we need a further assumption on the regularity results of the solution of the Stokes-type system associated with the primitive equations of ocean and the modified Poisson equation when the domain $\omega$ is sufficient smooth.

(**A1**) For a given $g_1 \in H^k(\Omega)^2$, the steady modified Stokes-type system

$$-\nu_1\Delta v - \mu_1\partial_{zz}v + \nabla q(x,y) = g_1 \text{ in } \Omega, \ \operatorname{div} \bar{v}(x,y) = 0 \text{ in } \omega, \tag{2.4}$$

admits a unique solution $(v,q) \in (H^2(\Omega)^2 \cap V_1) \times (L_0^2(\omega) \cap H^1(\omega))$ for the boundary conditions $\partial_z v|_{\Gamma_u \cup \Gamma_b} = 0$ and $v \cdot \mathbf{n}|_{\Gamma_s} = 0$, $\frac{\partial v}{\partial \mathbf{n}} \times \mathbf{n}|_{\mathbf{\Gamma_s}} = \mathbf{0}$ or $v|_{\Gamma_s} = 0$ such that

$$\|v\|_{2,\Omega}^2 + \|q\|_{1,\omega}^2 \le c\|g_1\|_{0,\Omega}^2; \tag{2.5}$$

and for a given $g_2 \in L^2(\Omega)$, the elliptic equation

$$-\nu_2\Delta\phi - \mu_2\partial_{zz}\phi = g_2 \text{ in } \Omega, \tag{2.6}$$

admits a unique solution $\phi \in H^2(\Omega)$ for the boundary condition

$$\partial_z\phi|_{\Gamma_b} = (\partial_z\phi + \alpha\phi)|_{\Gamma_u} = 0, \ \left.\frac{\partial\phi}{\partial\mathbf{n}}\right|_{\Gamma_s} = 0, \tag{2.7}$$

such that

$$\|\phi\|_{2,\Omega} \le c\|g_2\|_{0,\Omega}^2. \tag{2.8}$$

The second part in the assumption (**A1**) is the classical results. Some details of the first part in the assumption (**A1**) can be found in pages 2740-2741 in [27], pages 56-57 in [35] and pages 308 and 311 in [36] in the case of the boundary conditions (1.5), (1.6-2) and (1.7). In the case of the boundary conditions (1.5), (1.6-1) and (1.7), some results in the assumption (**A1**) can be proved by the similar manner as in [27, 35, 36].

Note that the assumption (**A1**) implies

$$c_0\|A_1 v\|_{0,\Omega}^2 \le \|v\|_{2,\Omega}^2 \le c_1\|A_1 v\|_{0,\Omega}^2, \tag{2.9}$$

$$c_0\|A_2\psi\|_{0,\Omega}^2 \le \|\psi\|_{2,\Omega}^2 \le c_1\|A_2\psi\|_{0,\Omega}^2, \tag{2.10}$$

for $v \in D(A_1)$, $\psi \in D(A_2)$.

Moreover, we define the bilinear operator:

$$B(v,\phi)=(v\cdot\nabla)\phi-(\int_{-d}^{z}\nabla\cdot v(x,y,\xi)d\xi)\partial_z\phi.$$

It is easy to see that

$$(B(v,\phi),\phi)_\Omega=0\ \forall v\in V_1,\ \phi\in H^1(\Omega)\text{ or }H^1(\Omega)^2,$$

$$(f\vec{k}\times v,v)_\Omega=0\ \forall v\in L^2(\Omega)^2, \tag{2.11}$$

$$(\int_{-d}^{z}\nabla\cdot v(x,y,\xi)d\xi,\phi)_\Omega-(\int_{-d}^{z}\nabla\phi(x,y,\xi)d\xi,v)_\Omega=0\ \forall v\in V_1,\ \phi\in V_2. \tag{2.12}$$

We usually make the following assumption about the prescribed data for problem (1.6)-(1.11):

(**A2**) The initial data $(u_0,\theta_0)\in D_*(A_1^{\frac{k}{2}})\times D_*(A_2^{\frac{k}{2}})$ and $(F_1,F_2),\ (F_{1z},F_{2z})\in L^\infty(\mathbb{R}^+;L^2(\Omega)^2)\times L^\infty(\mathbb{R}^+;L^2(\Omega))$ are such that for some positive constant $C_0$,

$$\begin{aligned}&\|A_1^{\frac{k}{2}}u_0\|_{0,\Omega}^2+\|A_2^{\frac{k}{2}}\theta_0\|_{0,\Omega}^2\\&\quad+\sup_{t\ge 0}\{\|F_1(t)\|_{0,\Omega}^2+\|F_2(t)\|_{0,\Omega}^2+\|F_{1z}(t)\|_{0,\Omega}^2+\|F_{2z}(t)\|_{0,\Omega}^2\}\le C_0,\end{aligned} \tag{2.13}$$

with $k=0,1,2$, where $D_*(A_1^{\frac{k}{2}})\times D_*(A_2^{\frac{k}{2}})=D(A_1^{\frac{k}{2}})\times D(A_2^{\frac{k}{2}})$ for $k=0,\ 2$ and $D_*(A_1^{\frac{1}{2}})\times D_*(A_2^{\frac{1}{2}})=H_+^1$ and

$$H_+^1=\{(u,\theta)\in V_1\times V_2;(u_z,\theta_z)\in L^6(\Omega)^2\times L^6(\Omega)\}.$$

Also, we recall the following important inequality(see [2, 3]):

$$\begin{aligned}&\int_\omega\int_{-d}^{0}|\nabla u(x,y,\xi)|d\xi\int_{-d}^{0}|\phi_z||w|dzdxdy\\&\le c\|\nabla u\|_{0,\Omega}^{\frac{1}{2}}\|\Delta u\|_{0,\Omega}^{\frac{1}{2}}\|\phi_z\|_{0,\Omega}^{\frac{1}{2}}\|\nabla\phi_z\|_{0,\Omega}^{\frac{1}{2}}\|w\|_{0,\Omega},\end{aligned} \tag{2.14}$$

for $u\in D(A_1)$, $(\phi,w)\in D(A_1)\times L^2(\Omega)^2$ or $(\phi,w)\in D(A_2)\times L^2(\Omega)$ and the following Sobolev and Ladyzhenskaya inequalities [1, 3, 4, 5, 21]:

$$\|\phi\|_{L^3}\le c\|\phi\|_{0,\Omega}^{\frac{1}{2}}\|\phi\|_{1,\Omega}^{\frac{1}{2}},\ \|\phi\|_{L^6}\le c\|\phi\|_{1,\Omega}, \tag{2.15}$$

for all $\phi\in H^1(\Omega)$, where the norm $\|\cdot\|_{L^q}$ denotes $\|\cdot\|_{L^q(\Omega)^2}$ or $\|\cdot\|_{L^q(\Omega)}$.

In order to obtain the regularity results of $(u,p,\theta)$ of the PEs, we consider the following abstract PEs:

$$v_t+L_1v+\nabla q(x,y,t)=G=f_1+\gamma\int_{-d}^{z}\nabla\eta(x,y,\xi,t)d\xi-f\vec{k}\times v-G_1(v,v), \tag{2.16}$$

$$\eta_t+L_2\eta=f_2-\sigma\int_{-d}^{z}\nabla\cdot v(x,y,\xi,t)d\xi-G_2(v,\eta), \tag{2.17}$$

$$\nabla\cdot\bar{v}=0, \tag{2.18}$$

together with the boundary conditions

$$v_z|_{\Gamma_u\cup\Gamma_b}=0,\ v\cdot\mathbf{n}|_{\Gamma_s}=0,\ \frac{\partial v}{\partial\mathbf{n}}\times\mathbf{n}|_{\Gamma_s}=0; \tag{2.19-1}$$

$$\text{or } v_z|_{\Gamma_u\cup\Gamma_b}=0,\ v|_{\Gamma_s}=0; \tag{2.19-2}$$

$$\eta_z|_{\Gamma_b}=(\eta_z+\alpha\eta)|_{\Gamma_u}=0,\ \frac{\partial\eta}{\partial\mathbf{n}}|_{\Gamma_s}=0, \tag{2.20}$$

where $G_1(v,\eta)$ and $G_2(v,\eta)$ are functions of $(v,\eta)$ and of their derivatives. Also, the initial conditions of $v(x,y,z,t)$ and $\eta(x,y,z,t)$ should be given by

$$v(x,y,z,0)=v_0(x,y,z),\ \ \eta(x,y,z,0)=\eta_0(x,y,z). \tag{2.21}$$

**Lemma 2.1.** Under Assumptions (**A1**) and (**A2**), the solution $(v,q,\eta)$ of (2.16)-(2.21) satisfies the following bounds:

$$\begin{aligned}
&\frac{d}{dt}(\gamma^{-1}\|v\|_{0,\Omega}^2+\sigma^{-1}\|\eta\|_{0,\Omega}^2)+\alpha_2(\gamma^{-1}\|v\|_{0,\Omega}^2+\sigma^{-1}\|\eta\|_{0,\Omega}^2)\\
&\qquad+\frac{5}{4}\gamma^{-1}\|A_1^{\frac12}v\|_{0,\Omega}^2+\frac{5}{4}\sigma^{-1}\|A_2^{\frac12}\eta\|_{0,\Omega}^2\\
&\qquad\le\frac{4}{\alpha_2}(\gamma^{-1}\|f_1\|_{0,\Omega}^2+\sigma^{-1}\|f_2\|_{0,\Omega}^2)\\
&\qquad-2\gamma^{-1}(G_1(v,v),v)_\Omega-2\sigma^{-1}(G_2(v,\eta),\eta)_\Omega,
\end{aligned}\tag{2.22}$$

$$\begin{aligned}
&\|A_1v\|_{0,\Omega}^2+\|A_2\eta\|_{0,\Omega}^2+\|\nabla q\|_{0,\Omega}^2\le c(\|v_t\|_{0,\Omega}^2+\|\eta_t\|_{0,\Omega}^2)+c(\|f_1\|_{0,\Omega}^2+\|f_2\|_{0,\Omega}^2)\\
&\qquad+c(\|A_1^{\frac12}v\|_{0,\Omega}^2+c\|A_2^{\frac12}\eta\|_{0,\Omega}^2)+c(\|G_1(v,v)\|_{0,\Omega}^2+\|G_2(v,\eta)\|_{0,\Omega}^2),
\end{aligned}\tag{2.23}$$

$$\begin{aligned}
&\frac{d}{dt}(\|A_1^{\frac12}v\|_{0,\Omega}^2+\|A_2^{\frac12}\eta\|_{0,\Omega}^2)+\alpha_2(\|A_1^{\frac12}v\|_{0,\Omega}^2+\|A_2^{\frac12}\eta\|_{0,\Omega}^2)+\frac54\|A_1v\|_{0,\Omega}^2+\frac54\|A_2\eta\|_{0,\Omega}^2\\
&\qquad\le c(\|f_1\|_{0,\Omega}^2+\|f_2\|_{0,\Omega}^2)+c(\|A_1^{\frac12}v\|_{0,\Omega}^2+\|A_2^{\frac12}\eta\|_{0,\Omega}^2)\\
&\qquad+c(\|G_1(v,v)\|_{0,\Omega}^2+\|G_2(v,\eta)\|_{0,\Omega}^2),
\end{aligned}\tag{2.24}$$

$$\begin{aligned}
&\|v_t\|_{0,\Omega}^2+\|\eta_t\|_{0,\Omega}^2+\|\nabla q\|_{0,\Omega}^2\le c(\|A_1v\|_{0,\Omega}^2+\|A_2\eta\|_{0,\Omega}^2)\\
&\qquad+c(\|f_1\|_{0,\Omega}^2+\|f_2\|_{0,\Omega}^2)+c(\|A_1^{\frac12}v\|_{0,\Omega}^2+c\|A_2^{\frac12}\eta\|_{0,\Omega}^2)\\
&\qquad+c(\|G_1(v,v)\|_{0,\Omega}^2+\|G_2(v,\eta)\|_{0,\Omega}^2),
\end{aligned}\tag{2.25}$$

where $\alpha_2=\min\{\alpha_1,\frac{\mu_2}{4d(d+\alpha^{-1})}\}$.

**Proof.** Taking the inner products of (2.16) with $\gamma^{-1}v$ in $L^2(\Omega)^2$ and (2.17) with $\sigma^{-1}\eta$ in $L^2(\Omega)$, adding these two relations and using (2.11)-(2.12) and (2.1)-(2.2), we obtain

$$\begin{aligned}
&\frac12\frac{d}{dt}(\gamma^{-1}\|v\|_{0,\Omega}^2+\sigma^{-1}\|\eta\|_{0,\Omega}^2)+\frac12\alpha_2(\gamma^{-1}\|v\|_{0,\Omega}^2+\sigma^{-1}\|\eta\|_{0,\Omega}^2)\\
&+\frac34\gamma^{-1}\|A_1^{\frac12}v\|_{0,\Omega}^2+\frac34\sigma^{-1}\|A_2^{\frac12}\eta\|_{0,\Omega}^2\\
&=\gamma^{-1}(f_1,v)_\Omega+\sigma^{-1}(f_2,\eta)-(G_1(v,v),v)_\Omega-\sigma^{-1}(G_2(v,\eta),\eta)_\Omega\\
&\le\frac18(\gamma^{-1}\|A_1^{\frac12}v\|_{0,\Omega}^2+\sigma^{-1}\|A_2^{\frac12}\eta\|_{0,\Omega}^2)+\frac{2}{\alpha_2}(\gamma^{-1}\|f_1\|_{0,\Omega}^2+\sigma^{-1}\|f_2\|_{0,\Omega}^2)\\
&-\gamma^{-1}(G_1(v,v),v)_\Omega-\sigma^{-1}(G_2(v,\eta),\eta)_\Omega,
\end{aligned}$$

which yields (2.22).

Next, applying the assumption (**A1**) to (2.16)-(2.20), we obtain

$$\begin{aligned}
&\|A_1v\|_{0,\Omega}^2+\|\nabla q\|_{0,\Omega}^2\le c\|G-v_t\|_{0,\Omega}^2\\
&\qquad\le c\|v_t\|_{0,\Omega}^2+c\|f_1\|_{0,\Omega}^2+c(\|A_1^{\frac12}v\|_{0,\Omega}^2+c\|A_2^{\frac12}\eta\|_{0,\Omega}^2)+c\|G_1(v,v)\|_{0,\Omega}^2,\\
&\|A_2\eta\|_{0,\Omega}^2\le\|f_2-\sigma\int_{-d}^z\nabla\cdot v(x,y,\xi,t)d\xi-G_2(v,\eta)-\eta_t\|_{0,\Omega}^2\\
&\qquad\le c\|\eta_t\|_{0,\Omega}^2+c\|f_2\|_{0,\Omega}^2+c\|A_1^{\frac12}v\|_{0,\Omega}^2+c\|G_2(v,\eta)\|_{0,\Omega}^2,
\end{aligned}$$

which yield (2.23).

Then, by taking the inner products of (2.16) with $A_1 v \in D(A_1)$ in $L^2(\Omega)^2$, (2.17) with $A_2\eta$ in $L^2(\Omega)$, noting that $\nabla\cdot A_1\bar{v}=0$ and $A_1\bar{v}\cdot n|_{\partial\omega}=0$, and using (2.3), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A_1^{\frac{1}{2}}v\|_{0,\Omega}^2+\|A_1v\|_{0,\Omega}^2&=(G,A_1v)_\Omega\\&\leq\frac{1}{8}\|A_1v\|_{0,\Omega}^2+c\|f_1\|_{0,\Omega}^2+c\|A_1^{\frac{1}{2}}v\|_{0,\Omega}^2+c\|A_2^{\frac{1}{2}}\eta\|_{0,\Omega}^2+c\|G_1(v,v)\|_{0,\Omega}^2,\\ \frac{1}{2}\frac{d}{dt}\|A_2^{\frac{1}{2}}\eta\|_{0,\Omega}^2+\|A_2\eta\|_{0,\Omega}^2&=(f_2-\sigma\int_{-d}^{z}\nabla\cdot v(x,y,\xi,t)d\xi-G_2(v,\eta),A_2\eta)_\Omega\\&\leq\frac{1}{8}\|A_2\eta\|_{0,\Omega}^2++c\|f_2\|_{0,\Omega}^2+c\|A_1^{\frac{1}{2}}v\|_{0,\Omega}^2+c\|G_2(v,\eta)\|_{0,\Omega}^2.\end{aligned}$$

Adding these two inequalities and using (2.1)-(2.2), we obtain (2.24).

Finally, taking the inner products of (2.16) with $v_t+\nabla q(x,y,t)$ in $L^2(\Omega)^2$ and (2.17) with $\eta_t$, using (2.3) and (2.18), we obtain

$$\begin{aligned}\|v_t\|_{0,\Omega}^2+\|\nabla q\|_{0,\Omega}^2&\leq\|f_1+\gamma\int_{-d}^{z}\nabla\eta(x,y,\xi,t)d\xi-f\vec{k}\times v-G_1(v,v)-L_1v\|_{0,\Omega}^2\\&\leq c\|A_1v\|_{0,\Omega}^2+c\|f_1\|_{0,\Omega}^2+c\|A_1^{\frac{1}{2}}v\|_{0,\Omega}^2+c\|A_2^{\frac{1}{2}}\eta\|_{0,\Omega}^2+c\|G_1(v,v)\|_{0,\Omega}^2,\\ \|\eta_t\|_{0,\Omega}^2&\leq\|f_2-\sigma\int_{-d}^{z}\nabla\cdot v(x,y,\xi,t)d\xi)-G_2(v,\eta)-L_2\eta\|_{0,\Omega}^2\\&\leq c\|A_2\eta\|_{0,\Omega}^2+c\|f_2\|_{0,\Omega}^2+c\|A_1^{\frac{1}{2}}v\|_{0,\Omega}^2+c\|G_2(v,\eta)\|_{0,\Omega}^2.\end{aligned}$$

Adding these two inequalities yields (2.25).

Also, we often use the following Gronwall lemma in this paper.

**Lemma 2.2.** Let $g$, $h$ and $y$ be three non-negative local integrable functions on $[t_0,\infty)$ and $\beta>0$ is a constant such that

$$\frac{dy}{dt}+\beta y+g\leq h\ \forall t\geq t_0.$$

Then,

$$y(t)+\int_{t_0}^{t}e^{\beta(s-t)}g(s)ds\leq e^{-\beta(t-t_0)}y(t_0)+\int_{t_0}^{t}e^{\beta(s-t)}h(s)ds.$$

This proof is straightforward and shall be omitted.

**Lemma 2.3.** Under the assumptions (**A1**) and (**A2**) with $k=0$, the solution $(u,p,\theta)$ of (1.6)-(1.11) satisfies the following bound:

$$\gamma^{-1}\|u(t)\|_{0,\Omega}^2+\sigma^{-1}\|\theta(t)\|_{0,\Omega}^2+\int_0^t e^{\alpha_2(s-t)}[\gamma^{-1}\|A_1^{\frac{1}{2}}u\|_{0,\Omega}^2+\sigma^{-1}\|A_2^{\frac{1}{2}}\theta\|_{0,\Omega}^2]ds\leq\kappa.$$

**Proof.** Setting

$$(u,p,\theta)=(v,q,\eta),\ (F_1,F_2)=(f_1,f_2),\ B(u,u)=G_1(v,v),\ B(u,\theta)=G_2(v,\eta),$$

in (1.9)-(1.11) and using (2.11)-(2.12) and (2.22) in Lemma 2.1 , we obtain

$$\begin{aligned}\frac{d}{dt}(\gamma^{-1}\|u\|_{0,\Omega}^2+\sigma^{-1}\|\theta\|_{0,\Omega}^2)&+\alpha_2(\gamma^{-1}\|u\|_{0,\Omega}^2+\sigma^{-1}\|\theta\|_{0,\Omega}^2)\\&+\gamma^{-1}\|A_1^{\frac{1}{2}}u\|_{0,\Omega}^2+\sigma^{-1}\|A_2^{\frac{1}{2}}\theta\|_{0,\Omega}^2\\&\leq\frac{4}{\alpha_2}(\gamma^{-1}\|F_1\|_{0,\Omega}^2+\sigma^{-1}\|F_2\|_{0,\Omega}^2).\end{aligned}$$

Applying Lemma 2.2 to the above inequality has completed the proof of Lemma 2.3.

Finally, from (2.3), (2.9)-(2.10) and (2.14)-(2.15), we deduce some inequalities about $B(\cdot,\cdot)$ which will often be used.

It follows from (2.3), (2.9)-(2.10) and (2.14)-(2.15) that for $i = 1,\ 2,\ 3,$

$$\begin{aligned}
&\|B(\partial_t^i u, u)\|_{0,\Omega}^2 + \|B(u, \partial_t^i u)\|_{0,\Omega}^2 \\
&\quad \le c[\|\nabla u\|_{L^3}\|\partial_t^i u\|_{L^6} + \|\nabla\partial_t^i u\|_{0,\Omega}^{\frac12}\|A_1\partial_t^i u\|_{0,\Omega}^{\frac12}\|u_z\|_{0,\Omega}^{\frac12}\|\nabla u_z\|_{0,\Omega}^{\frac12}]^2 \\
&\quad + c[\|u\|_{L^6}\|\nabla\partial_t^i u\|_{L^3} + \|\nabla u\|_{0,\Omega}^{\frac12}\|A_1 u\|_{0,\Omega}^{\frac12}\|\partial_t^i u_z\|_{0,\Omega}^{\frac12}\|\nabla\partial_t^i u_z\|_{0,\Omega}^{\frac12}]^2 \\
&\quad \le c\|A_1 u\|_{0,\Omega}\|A_1^{\frac12} u\|_{0,\Omega}\|A_1^{\frac12}\partial_t^i u\|_{0,\Omega}\|A_1\partial_t^i u\|_{0,\Omega} \\
&\quad \le \frac{1}{16}\|A_1\partial_t^i u\|_{0,\Omega}^2 + c\|A_1 u\|_{0,\Omega}^2\|A_1^{\frac12} u\|_{0,\Omega}^2\|A_1^{\frac12}\partial_t^i u\|_{0,\Omega}^2, \qquad (2.26)
\end{aligned}$$

$$\begin{aligned}
\|B(\partial_t^i u, \theta)\|_{0,\Omega}^2 &\le c[\|\nabla\theta\|_{L^3}\|\partial_t^i u\|_{L^6} + \|\nabla\partial_t^i u\|_{0,\Omega}^{\frac12}\|A_1\partial_t^i u\|_{0,\Omega}^{\frac12}\|\theta_z\|_{0,\Omega}^{\frac12}\|\nabla\theta_z\|_{0,\Omega}^{\frac12}]^2 \\
&\le c\|A_2\theta\|_{0,\Omega}\|A_2^{\frac12}\theta\|_{0,\Omega}\|A_1^{\frac12}\partial_t^i u\|_{0,\Omega}\|A_1\partial_t^i u\|_{0,\Omega} \\
&\le \frac{1}{16}\|A_1\partial_t^i u\|_{0,\Omega}^2 + c\|A_2\theta\|_{0,\Omega}^2\|A_2^{\frac12}\theta\|_{0,\Omega}^2\|A_1^{\frac12}\partial_t^i u\|_{0,\Omega}^2, \qquad (2.27)
\end{aligned}$$

$$\begin{aligned}
\|B(u, \partial_t^i\theta)\|_{0,\Omega}^2 &\le c[\|u\|_{L^6}\|\nabla\partial_t^i\theta\|_{L^3} + \|\nabla u\|_{0,\Omega}^{\frac12}\|A_1 u\|_{0,\Omega}^{\frac12}\|\partial_t^i\theta_z\|_{0,\Omega}^{\frac12}\|\nabla\partial_t^i\theta_z\|_{0,\Omega}^{\frac12}]^2 \\
&\le c\|A_1 u\|_{0,\Omega}\|A_1^{\frac12} u\|_{0,\Omega}\|A_2^{\frac12}\partial_t^i\theta\|_{0,\Omega}\|A_2\partial_t^i\theta\|_{0,\Omega} \\
&\le \frac{1}{16}\|A_2\partial_t^i\theta\|_{0,\Omega}^2 + c\|A_1 u\|_{0,\Omega}^2\|A_1^{\frac12} u\|_{0,\Omega}^2\|A_2^{\frac12}\partial_t^i\theta\|_{0,\Omega}^2. \qquad (2.28)
\end{aligned}$$

$$\begin{aligned}
\|B(u_t, u_t)\|_{0,\Omega} &\le c\|u_t\|_{L^6}\|\nabla u_t\|_{L^3} + c\|\nabla u_t\|_{0,\Omega}^{\frac12}\|A_1 u_t\|_{0,\Omega}^{\frac12}\|u_{zt}\|_{0,\Omega}^{\frac12}\|\nabla u_{zt}\|_{0,\Omega}^{\frac12} \\
&\le c\|A_1 u_t\|_{0,\Omega}\|A_1^{\frac12} u_t\|_{0,\Omega}, \qquad (2.29)
\end{aligned}$$

$$\begin{aligned}
\|B(u_t, \theta_t)\|_{0,\Omega} &\le c\|u_t\|_{L^6}\|\nabla\theta_t\|_{L^3} + c\|\nabla u_t\|_{0,\Omega}^{\frac12}\|A_1 u_t\|_{0,\Omega}^{\frac12}\|\theta_{zt}\|_{0,\Omega}^{\frac12}\|\nabla\theta_{zt}\|_{0,\Omega}^{\frac12} \\
&\le c[\|A_1 u_t\|_{0,\Omega}\|A_1^{\frac12} u_t\|_{0,\Omega}\|A_2\theta_t\|_{0,\Omega}\|A_2^{\frac12}\theta_t\|_{0,\Omega}]^{\frac12} \\
&\le c\|A_1 u_t\|_{0,\Omega}\|A_1^{\frac12} u_t\|_{0,\Omega} + c\|A_2\theta_t\|_{0,\Omega}\|A_2^{\frac12}\theta_t\|_{0,\Omega}, \qquad (2.30)
\end{aligned}$$

$$\begin{aligned}
\|B(u_{tt}, u_t)\|_{0,\Omega} &\le c\|u_{tt}\|_{L^6}\|\nabla u_t\|_{L^3} + c\|\nabla u_{tt}\|_{0,\Omega}^{\frac12}\|A_1 u_{tt}\|_{0,\Omega}^{\frac12}\|u_{zt}\|_{0,\Omega}^{\frac12}\|\nabla u_{zt}\|_{0,\Omega}^{\frac12} \\
&\le c[\|A_1 u_{tt}\|_{0,\Omega}\|A_1^{\frac12} u_{tt}\|_{0,\Omega}\|A_1 u_t\|_{0,\Omega}\|A_1^{\frac12} u_t\|_{0,\Omega}]^{\frac12}, \qquad (2.31)
\end{aligned}$$

$$\begin{aligned}
\|B(u_t, u_{tt})\|_{0,\Omega} &\le c\|u_t\|_{L^6}\|\nabla u_{tt}\|_{L^3} + c\|\nabla u_t\|_{0,\Omega}^{\frac12}\|A_1 u_t\|_{0,\Omega}^{\frac12}\|u_{ztt}\|_{0,\Omega}^{\frac12}\|\nabla u_{ztt}\|_{0,\Omega}^{\frac12} \\
&\le c[\|A_1 u_{tt}\|_{0,\Omega}\|A_1^{\frac12} u_{tt}\|_{0,\Omega}\|A_1 u_t\|_{0,\Omega}\|A_1^{\frac12} u_t\|_{0,\Omega}]^{\frac12}, \qquad (2.32)
\end{aligned}$$

$$\begin{aligned}
\|B(u_{tt}, \theta_t)\|_{0,\Omega} &\le c\|u_{tt}\|_{L^6}\|\nabla\theta_t\|_{L^3} + c\|\nabla u_{tt}\|_{0,\Omega}^{\frac12}\|A_1 u_{tt}\|_{0,\Omega}^{\frac12}\|\theta_{zt}\|_{0,\Omega}^{\frac12}\|\nabla\theta_{zt}\|_{0,\Omega}^{\frac12} \\
&\le c[\|A_1 u_{tt}\|_{0,\Omega}\|A_1^{\frac12} u_{tt}\|_{0,\Omega}\|A_2\theta_t\|_{0,\Omega}\|A_2^{\frac12}\theta_t\|_{0,\Omega}]^{\frac12}, \qquad (2.33)
\end{aligned}$$

$$\begin{aligned}
\|B(u_t, \theta_{tt})\|_{0,\Omega} &\le c\|u_t\|_{L^6}\|\nabla\theta_{tt}\|_{L^3} + c\|\nabla u_t\|_{0,\Omega}^{\frac12}\|A_1 u_t\|_{0,\Omega}^{\frac12}\|\theta_{ztt}\|_{0,\Omega}^{\frac12}\|\nabla\theta_{ztt}\|_{0,\Omega}^{\frac12} \\
&\le c[\|A_2\theta_{tt}\|_{0,\Omega}\|A_2^{\frac12}\theta_{tt}\|_{0,\Omega}\|A_1 u_t\|_{0,\Omega}\|A_1^{\frac12} u_t\|_{0,\Omega}]^{\frac12}, \qquad (2.34)
\end{aligned}$$

$$\|B(\partial_t^i u, u)\|_{L^{\frac{3}{2}}} \le c\|\nabla u\|_{0,\Omega}\|\partial_t^i u\|_{L^6} + c\|\nabla \partial_t^i u\|_{0,\Omega}\|u_z\|_{L^6}$$
$$\le c(\|\nabla u\|_{0,\Omega} + \|u_z\|_{L^6})\|A_1^{\frac{1}{2}}\partial_t^i u\|_{0,\Omega}, \tag{2.35}$$

$$\|B(\partial_t^i u, \theta)\|_{L^{\frac{3}{2}}} \le c\|\nabla \theta\|_{0,\Omega}\|\partial_t^i u\|_{L^6} + c\|\nabla \partial_t^i u\|_{0,\Omega}\|\theta_z\|_{L^6}$$
$$\le c(\|\nabla \theta\|_{0,\Omega} + \|\theta_z\|_{L^6})\|A_1^{\frac{1}{2}}\partial_t^i u\|_{0,\Omega}. \tag{2.36}$$

## 3. $H^1$-estimates

It is well known that the $H^1$ -estimates of the solution $(u, p, \theta)$ of PEs with $\sigma = 0$ and $F_1 = 0$ have been given in [3, 20]. Under Assumptions (**A1**) and (**A2**) with $k = 1$, by changing slightly the arguments used in [3, 20], we can obtain:

$$\|A_1^{\frac{1}{2}} u(t)\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\theta(t)\|_{0,\Omega}^2 \le \kappa, \tag{3.1}$$

for $t \ge 0$ in the case of $\sigma$ and $F_1$ being non zero.

**Lemma 3.1.** Under the assumptions (**A1**) and (**A2**) with $k = 1$, the solution $(u, p, \theta)$ satisfies the following bounds:

$$\|A_1^{\frac{1}{2}} u(t)\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\theta(t)\|_{0,\Omega}^2 + \int_0^t e^{\alpha_2(s-t)}[\|A_1 u\|_{0,\Omega}^2 + \|A_2\theta\|_{0,\Omega}^2]ds \le \kappa, \tag{3.2}$$

$$\int_0^t e^{\alpha_2(s-t)}[\|u_t\|_{0,\Omega}^2 + \|\theta_t\|_{0,\Omega}^2 + \|\nabla p\|_{0,\Omega}^2]ds \le \kappa. \tag{3.3}$$

**Proof.** First, using some very similar arguments given in [3, 20] and (2.1)-(2.2), we have

$$\frac{d}{dt}\|A_1^{\frac{1}{2}} u(t)\|_{0,\Omega}^2 + \alpha_2\|A_1^{\frac{1}{2}} u\|_{0,\Omega}^2 + \|A_1 u\|_{0,\Omega}^2$$
$$\le c\|u\|_{L^6}^4\|A_1^{\frac{1}{2}} u\|_{0,\Omega}^2 + c\|A_1^{\frac{1}{2}} u\|_{0,\Omega}^6 + c\|A_2^{\frac{1}{2}}\theta\|_{0,\Omega}^2 + c\|F_1\|_{0,\Omega}^2, \tag{3.4}$$

$$\frac{d}{dt}\|A_2^{\frac{1}{2}}\theta(t)\|_{0,\Omega}^2 + \alpha_2\|A_2^{\frac{1}{2}}\theta\|_{0,\Omega}^2 + \|A_2\theta\|_{0,\Omega}^2$$
$$\le c(\|u\|_{L^6}^4 + \|A_1^{\frac{1}{2}} u\|_{0,\Omega}^2\|A_1 u\|_{0,\Omega}^2)\|A_2^{\frac{1}{2}}\theta\|_{0,\Omega}^2 + c\|A_1^{\frac{1}{2}} u\|_{0,\Omega}^2 + c\|F_2\|_{0,\Omega}^2. \tag{3.5}$$

Applying Lemma 2.2 to (3.4) and using (2.3), (2.15) and (3.1), we obtain

$$\|A_1^{\frac{1}{2}} u(t)\|_{0,\Omega}^2 + \int_0^t e^{\alpha_2(s-t)}\|A_1 u\|_{0,\Omega}^2 ds \le \kappa. \tag{3.6}$$

Again, applying Lemma 2.2 to (3.5) and using (2.3), (2.15), (3.1) and (3.6), we obtain

$$\|A_2^{\frac{1}{2}}\theta(t)\|_{0,\Omega}^2 + \int_0^t e^{\alpha_2(s-t)}\|A_2\theta\|_{0,\Omega}^2 ds \le \kappa,$$

which, together with (3.6), gives (3.2).

Setting

$$(u, p, \theta) = (v, q, \eta),\ (F_1, F_2) = (f_1, f_2),\ B(u, u) = G_1(v, v),\ B(u, \theta) = G_2(v, \eta),$$

in (1.9)-(1.11) and using (2.25) in Lemma 2.1 , we obtain

$$\|u_t\|_{0,\Omega}^2 + \|\theta_t\|_{0,\Omega}^2 + \|\nabla p\|_{0,\Omega}^2 \le c(\|A_1 u\|_{0,\Omega}^2 + \|A_2\theta\|_{0,\Omega}^2) + c(\|F_1\|_{0,\Omega}^2 + \|F_2\|_{0,\Omega}^2)$$
$$+ c\|A_1^{\frac{1}{2}} v\|_{0,\Omega}^2 + c\|A_2^{\frac{1}{2}}\theta\|_{0,\Omega}^2 + c\|B(u, u)\|_{0,\Omega}^2 + c\|B(u, \theta)\|_{0,\Omega}^2. \tag{3.7}$$

It follows from (2.14)-(2.15), (2.3) and (2.10) that

$$\begin{aligned} c\|B(u,u)\|_{0,\Omega}^2 &\le c[\|u\|_{L^6}\|\nabla u\|_{0,\Omega} + c\|\nabla u\|_{0,\Omega}^{\frac12}\|A_1u\|_{0,\Omega}^{\frac12}\|u_z\|_{0,\Omega}^{\frac12}\|\nabla u_z\|_{0,\Omega}^{\frac12}]^2 \\ &\le c\|A_1^{\frac12}u\|_{0,\Omega}^2\|A_1u\|_{0,\Omega}^2, \\ c\|B(u,\theta)\|_{0,\Omega}^2 &\le c[\|u\|_{L^6}\|\nabla\theta\|_{0,\Omega} + c\|\nabla u\|_{0,\Omega}^{\frac12}\|A_1u\|_{0,\Omega}^{\frac12}\|\theta_z\|_{0,\Omega}^{\frac12}\|\nabla\theta_z\|_{0,\Omega}^{\frac12}]^2 \\ &\le c\|A_1^{\frac12}u\|_{0,\Omega}\|A_1u\|_{0,\Omega}\|A_2^{\frac12}\theta\|_{0,\Omega}\|A_2\theta\|_{0,\Omega} \\ &\le c\|A_1^{\frac12}u\|_{0,\Omega}^2\|A_1u\|_{0,\Omega}^2 + c\|A_2^{\frac12}\theta\|_{0,\Omega}^2\|A_2\theta\|_{0,\Omega}^2. \end{aligned}$$

Combining these inequalities with (3.7), we get

$$\begin{aligned} &\|u_t\|_{0,\Omega}^2 + \|\theta_t\|_{0,\Omega}^2 + \|\nabla p\|_{0,\Omega}^2 \\ &\qquad \le c(1 + \|A_1^{\frac12}u\|_{0,\Omega}^2 + \|A_2^{\frac12}\theta\|_{0,\Omega}^2)(1 + \|A_1u\|_{0,\Omega}^2 + \|A_2\theta\|_{0,\Omega}^2) \\ &\qquad + c(\|F_1\|_{0,\Omega}^2 + \|F_2\|_{0,\Omega}^2). \end{aligned}$$

Combining the above inequality with (3.2) yields (3.3).

In order to gain the $L^6$-regularity results of $(u_z, \theta_z)$, we differentiate (1.9)-(1.10) with respect to $z$, we obtain

$$\begin{aligned} &u_{zt} + L_1u_z + (u\cdot\nabla)u_z - (\int_{-d}^z \nabla\cdot u(x,y,\xi,t)d\xi)\partial_z u_z \\ &\quad + (u_z\cdot\nabla)u - (\nabla\cdot u)u_z - \gamma\nabla\theta + f\vec{k}\times u_z = F_{1z}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\theta_{zt} + L_2\theta_z + (u\cdot\nabla)\theta_z - (\int_{-d}^z \nabla\cdot u(x,y,\xi,t)d\xi)\partial_z\theta_z \\ &\quad + (u_z\cdot\nabla)\theta - (\nabla\cdot u)\theta_z + \sigma\nabla\cdot u = F_{2z}, \end{aligned} \tag{3.9}$$

Moreover, by the boundary conditions (1.6)-(1.7) and the fact that $\partial_z\mathbf{n}|_{\Gamma_s} = 0$, we obtain the boundary conditions of $u_z$ and $\theta_z$:

$$u_z|_{\Gamma_u\cup\Gamma_b} = 0,\ u_z\cdot\mathbf{n}|_{\Gamma_s} = 0,\ \frac{\partial u_z}{\partial\mathbf{n}}\times\mathbf{n}|_{\Gamma_s} = 0, \tag{3.10-1}$$

or

$$u_z|_{\Gamma_u\cup\Gamma_b} = 0,\ u_z|_{\Gamma_s} = 0, \tag{3.10-2}$$

$$(\theta_z + \alpha\theta)|_{\Gamma_u} = \theta_z|_{\Gamma_b} = 0,\ \frac{\partial\theta_z}{\partial\mathbf{n}}|_{\Gamma_s} = 0, \tag{3.11}$$

and the initial conditions:

$$u_z(x,y,z,0) = u_{0z}(x,y,z),\ \theta_z(x,y,z.0) = \theta_{0z}(x,y,z). \tag{3.12}$$

**Lemma 3.2.** Under the assumptions (**A1**) and (**A2**) with $k = 1,\ 2$, the solution $(u,p,\theta)$ of the 3D viscous PEs of the ocean satisfies the following bounds:

$$\|u_z(t)\|_{L^6}^2 + \int_0^t e^{\alpha_2(s-t)}\|A_1^{\frac12}(u_z|u_z|^2)\|_{0,\Omega}^2 ds \le \kappa, \tag{3.13}$$

$$\|\theta_z(t)\|_{L^6}^2 + \int_0^t e^{\alpha_2(s-t)}\|A_2^{\frac12}\theta_z^3\|_{0,\Omega}^2 ds \le \kappa. \tag{3.14}$$

**Proof.** Taking the inner products of (3.8) with $u_z|u_z|^4$ in $L^2(\Omega)^2$ and (3.9) with $\theta_z|\theta_z|^4$ in $L^2(\Omega)$ and using (2.11)-(2.12) and (3.10)-(3.11), we obtain

$$\frac{1}{6}\frac{d}{dt}\|u_z\|_{L^6}^6+\frac{5}{6}\|A_1^{\frac{1}{2}}(u_z|u_z|^2)\|_{0,\Omega}^2+((u_z\cdot\nabla)u,u_z|u_z|^4)_\Omega-((\nabla\cdot u)u_z,u_z|u_z|^4)_\Omega$$
$$=(F_{1z}+\gamma\nabla\theta,u_z|u_z|^4)_\Omega, \tag{3.15}$$
$$\frac{1}{6}\frac{d}{dt}\|\theta_z\|_{L^6}^6+\frac{5}{6}\|A_2^{\frac{1}{2}}\theta_z^3\|_{0,\Omega}^2+((u_z\cdot\nabla)\theta,\theta_z^5)_\Omega-(\nabla\cdot u,\theta_z^6)_\Omega+\sigma(\nabla\cdot u,\theta_z^5)_\Omega$$
$$\le(F_{2z},\theta_z^5)_\Omega. \tag{3.16}$$

It follows from (2.3), (2.9)-(2.10), (2.15) and (3.10)-(3.11) that

$$\begin{aligned}
|((u_z\cdot\nabla)u,u_z|u_z|^4)_\Omega-(\nabla\cdot u,|u_z|^6)_\Omega| &\le c\int_\Omega|\nabla u||u_z|^{\frac{3}{2}}|u_z|^{\frac{9}{2}}dxdydz\\
&\le c\|\nabla u\|_{0,\Omega}\|u_z\|_{L^6}^{\frac{3}{2}}\|u_z|u_z|^2\|_{L^6}^{\frac{3}{2}}\\
&\le\frac{1}{16}\|A_1^{\frac{1}{2}}(u_z|u_z|^2)\|_{0,\Omega}^2+c\|\nabla u\|_{0,\Omega}^4\|u_z\|_{L^6}^6,\\
\gamma|(\nabla\theta,u_z|u_z|^4)_\Omega| &\le\gamma\|\nabla\theta\|_{0,\Omega}\|u_z\|_{L^6}^2\|u_z|u_z|^2\|_{L^6}\\
&\le\frac{1}{16}\|A_1^{\frac{1}{2}}(u_z|u_z|^2)\|_{0,\Omega}^2+c\|\nabla\theta\|_{0,\Omega}^2\|u_z\|_{L^6}^4,\\
(F_{1z},u_z|u_z|^4)_\Omega &\le\|F_{1z}\|_{0,\Omega}\|u_z\|_{L^6}^2\|u_z|u_z|^2\|_{L^6}\\
&\le\frac{1}{16}\|A_1^{\frac{1}{2}}(u_z|u_z|^2)\|_{0,\Omega}^2+c\|F_{1z}\|_{0,\Omega}^2\|u_z\|_{L^6}^4,\\
|((\nabla\cdot u)\theta_z,\theta_z^5)_\Omega| &\le\sqrt{2}\int_\Omega|\nabla u||\theta_z|^{\frac{3}{2}}|\theta_z|^{\frac{9}{2}}dxdydz\\
&\le\sqrt{2}\|\nabla u\|_{0,\Omega}\|\theta_z\|_{L^6}^{\frac{3}{2}}\|\theta_z^3\|_{L^6}^{\frac{3}{2}}\\
&\le\frac{1}{16}\|A_2^{\frac{1}{2}}\theta_z^3\|_{0,\Omega}^2+c\|\nabla u\|_{0,\Omega}^4\|\theta_z\|_{L^6}^6,\\
|((u_z\cdot\nabla)\theta,\theta_z^5)_\Omega| &\le\|u_z\|_{L^6}\|\nabla\theta\|_{L^3}\|\theta_z\|_{L^6}^2\|\theta_z^3\|_{L^6}\\
&\le\frac{1}{16}\|A_2^{\frac{1}{2}}\theta_z^3\|_{0,\Omega}^2+c\|u_z\|_{L^6}^2\|\nabla\theta\|_{L^3}^2\|\theta_z\|_{L^6}^4,\\
\sigma|(\nabla\cdot u,\theta_z^5)_\Omega| &\le c\|\nabla u\|_{0,\Omega}\|\theta_z\|_{L^6}^2\|\theta_z^3\|_{L^6}\\
&\le\frac{1}{16}\|A_1^{\frac{1}{2}}\theta_z^3\|_{0,\Omega}^2+c\|\nabla u\|_{0,\Omega}^2\|\theta_z\|_{L^6}^4,\\
(F_{2z},\theta_z^5)_\Omega &\le\|F_{2z}\|_{0,\Omega}\|\theta_z\|_{L^6}^2\|\theta_z^3\|_{L^6}\\
&\le\frac{1}{16}\|A_2^{\frac{1}{2}}\theta_z^3\|_{0,\Omega}^2+c\|F_{2z}\|_{0,\Omega}^2\|\theta_z\|_{L^6}^4.
\end{aligned}$$

Combining (3.15) and (3.16) with the above inequalities and using (2.3), (2.9)-(2.10) and (2.15), we obtain

$$\frac{1}{6}\frac{d}{dt}\|u_z\|_{L^6}^6+\frac{1}{2}\alpha_2\|u_z\|_{L^6}^6+\|A_1^{\frac{1}{2}}(u_z|u_z|^2)\|_{0,\Omega}^2$$
$$\le c(\|\nabla u\|_{0,\Omega}^4\|A_1u\|_{0,\Omega}^2+\|\nabla\theta\|_{0,\Omega}^2+\|F_{1z}\|_{0,\Omega}^2)\|u_z\|_{L^6}^4, \tag{3.17}$$
$$\frac{1}{6}\frac{d}{dt}\|\theta_z\|_{L^6}^6+\frac{1}{2}\alpha_2\|\theta_z\|_{L^6}^6+\|A_2^{\frac{1}{2}}\theta_z^3\|_{0,\Omega}^2$$
$$\le c(\|\nabla u\|_{0,\Omega}^4\|A_2\theta\|_{0,\Omega}^2+\|u_z\|_{L^6}^2\|A_2\theta\|_{0,\Omega}^2+\|\nabla u\|_{0,\Omega}^2)\|\theta_z\|_{L^6}^4$$
$$+c\|F_{2z}\|_{0,\Omega}^2\|\theta_z\|_{L^6}^4. \tag{3.18}$$

Therefore,

$$\frac{d}{dt}\|u_z\|_{L^6}^2 + \alpha_2\|u_z\|_{L^6}^2 \le c\|\nabla u\|_{0,\Omega}^4\|A_1 u\|_{0,\Omega}^2 + c\|\nabla\theta\|_{0,\Omega}^2 + c\|F_{1z}\|_{0,\Omega}^2, \tag{3.19}$$

$$\begin{aligned}\frac{d}{dt}\|\theta_z\|_{L^6}^2 + \alpha_2\|\theta_z\|_{L^6}^2 &\le c\|\nabla u\|_{0,\Omega}^4\|A_2\theta\|_{0,\Omega}^2 + c\|u_z\|_{L^6}^2\|A_2\theta\|_{0,\Omega}^2 \\ &\quad + c\|\nabla u\|_{0,\Omega}^2 + c\|F_{2z}\|_{0,\Omega}^2.\end{aligned} \tag{3.20}$$

Thus, applying Lemma 2.2 to (3.19) and using (3.2), we get

$$\|u_z(t)\|_{L^6}^2 \le e^{-\alpha_2 t}\|u_{0z}\|_{L^6}^2 + \kappa \le \kappa. \tag{3.21}$$

Then, applying Lemma 2.2 to (3.20) and using Lemma 3.1 and (3.21), we obtain

$$\|\theta_z(t)\|_{L^6}^2 \le e^{-\alpha_2 t}\|\theta_{0z}\|_{L^6}^2 + \kappa \le \kappa. \tag{3.22}$$

Finally, applying Lemma 2.2 to (3.17) and (3.18) and using (3.21)-(3.22) and Lemma 3.1, we obtain (3.13)-(3.14).

## 4. $H^2$-estimates of $(u,\theta)$

Differentiating (1.9)-(1.10) with respect to $t$, we obtain

$$\begin{aligned}&u_{tt} + L_1 u_t + \nabla p_t - \gamma\nabla\int_{-d}^{z}\theta_t(x,y,\xi,t)d\xi + f\vec{k}\times u_t + B(u_t,u) + B(u,u_t)\\ &\quad = F_{1t},\end{aligned} \tag{4.1}$$

$$\theta_{tt} + L_2\theta_t + \sigma\int_{-d}^{z}\nabla\cdot u_t(x,y,\xi,t)d\xi + B(u_t,\theta) + B(u,\theta_t) = F_{2t}. \tag{4.2}$$

**Lemma 4.1.** Assume that the assumptions (**A1**) and (**A2**) with $k=1,\ 2$ hold and $(F_{1t},F_{2t}) \in L^\infty(\mathbb{R}^+;L^2(\Omega)^2)\times L^\infty(\mathbb{R}^+;L^2(\Omega))$. The solution $(u,p,\theta)$ of the 3D viscous PEs of the ocean satisfies the following bounds:

$$\begin{aligned}&\gamma^{-1}\tau^{2-k}(t)\|u_t(t)\|_{0,\Omega}^2 + \sigma^{-1}\tau^{2-k}(t)\|\theta_t(t)\|_{0,\Omega}^2\\ &+\int_0^t e^{\alpha_2(s-t)}\tau^{2-k}(s)[\gamma^{-1}\|A_1^{\frac12}u_t\|_{0,\Omega}^2 + \sigma^{-1}\|A_2^{\frac12}\theta_t\|_{0,\Omega}^2]ds \le \kappa,\end{aligned} \tag{4.3}$$

$$\tau^{2-k}(t)\|A_1u(t)\|_{0,\Omega}^2 + \tau^{2-k}(t)\|A_2\theta(t)\|_{0,\Omega}^2 + \tau^{2-k}(t)\|\nabla p(t)\|_{0,\Omega}^2 \le \kappa, \tag{4.4}$$

where $\tau(t) = \min\{1,t\}$.

**Proof.** Setting

$$(u_t,p_t,\theta_t) = (v,q,\eta),\ (F_{1t},F_{2t}) = (f_1,f_2),$$

$$B(u_t,u)+B(u,u_t) = G_1(v,v),\ B(u_t,\theta)+B(u,\theta_t) = G_2(v,\eta),$$

in (4.1)-(4.2), we deduce from (2.22) in Lemma 2.1 that

$$\begin{aligned}&\frac{d}{dt}(\gamma^{-1}\|u_t\|_{0,\Omega}^2 + \sigma^{-1}\|\theta_t\|_{0,\Omega}^2) + \alpha_2(\gamma^{-1}\|u_t\|_{0,\Omega}^2 + \sigma^{-1}\|\theta_t\|_{0,\Omega}^2)\\ &\qquad + \frac54\gamma^{-1}\|A_1^{\frac12}u_t\|_{0,\Omega}^2 + \frac54\sigma^{-1}\|A_2^{\frac12}\theta_t\|_{0,\Omega}^2\\ &\qquad \le \frac{4}{\alpha_2}(\gamma^{-1}\|F_{1t}\|_{0,\Omega}^2 + \sigma^{-1}\|F_{2t}\|_{0,\Omega}^2) - 2\gamma^{-1}(B(u_t,u),u_t)_\Omega\\ &\qquad - 2\sigma^{-1}(B(u_t,\theta),\theta_t)_\Omega.\end{aligned} \tag{4.5}$$

It follows from (2.3), (2.15) and (2.35)-(2.36) that

$$\begin{aligned}
c|(B(u_t,u),u_t)_\Omega| &\le c\|B(u_t,u)\|_{L^{\frac{3}{2}}}\|u_t\|_{L^3}\\
&\le c(\|\nabla u\|_{0,\Omega}+\|u_z\|_{L^6})\|u_t\|_{0,\Omega}^{\frac{1}{2}}\|A_1^{\frac{1}{2}}u_t\|_{0,\Omega}^{\frac{3}{2}}\\
&\le \frac{1}{16}\|A_1^{\frac{1}{2}}u_t\|_{0,\Omega}^2+c(\|\nabla u\|_{0,\Omega}^4+\|u_z\|_{L^6}^4)\|u_t\|_{0,\Omega}^2,\\
c|(B(u_t,\theta),\theta_t)_\Omega| &\le c\|B(u_t,\theta)\|_{L^{\frac{3}{2}}}\|\theta_t\|_{L^3}\\
&\le c(\|\nabla \theta\|_{0,\Omega}+\|\theta_z\|_{L^6})\|A_1^{\frac{1}{2}}u_t\|_{0,\Omega}\|\theta_t\|_{0,\Omega}^{\frac{1}{2}}\|A_2^{\frac{1}{2}}\theta_t\|_{0,\Omega}^{\frac{1}{2}}\\
&\le \frac{1}{16}(\|A_1^{\frac{1}{2}}u_t\|_{0,\Omega}^2+\|A_2^{\frac{1}{2}}\theta_t\|_{0,\Omega}^2)+c(\|\nabla\theta\|_{0,\Omega}^4+\|\theta_z\|_{L^6}^4)\|u_t\|_{0,\Omega}^2.
\end{aligned}$$

Combining these inequalities with (4.5), we obtain

$$\begin{aligned}
&\frac{d}{dt}(\gamma^{-1}\|u_t\|_{0,\Omega}^2+\sigma^{-1}\|\theta_t\|_{0,\Omega}^2)+\alpha_2(\gamma^{-1}\|u_t\|_{0,\Omega}^2+\sigma^{-1}\|\theta_t\|_{0,\Omega}^2)\\
&\qquad+(\gamma^{-1}\|A_1^{\frac{1}{2}}u_t\|_{0,\Omega}^2+\sigma^{-1}\|A_2^{\frac{1}{2}}\theta_t\|_{0,\Omega}^2)\\
&\qquad\le c(\|F_{1t}\|_{0,\Omega}^2+\|F_{2t}\|_{0,\Omega}^2)+c(\|\nabla u\|_{0,\Omega}^4+\|\theta_z\|_{L^6}^4)(\|u_t\|_{0,\Omega}^2+\|\theta_t\|_{0,\Omega}^2).
\end{aligned}\tag{4.6}$$

Applying Lemma 2.2 to (4.6) and using Lemma 3.1 and Lemma 3.2, we get (4.3) for $k=2$.

For $k=1$, from (3.3) in Lemma 3.1, we have

$$\int_0^1 e^{\alpha_2 s}[\gamma^{-1}\|u_t\|_{0,\Omega}^2+\sigma^{-1}\|\theta_t\|_{0,\Omega}^2]ds\le\kappa.$$

Then, there exists a sequence $\varepsilon_n\to 0$, such that

$$\tau(\varepsilon_n)(\gamma^{-1}\|u_t(\varepsilon_n)\|_{0,\Omega}^2+\sigma^{-1}\|\theta_t(\varepsilon_n)\|_{0,\Omega}^2)\to 0.$$

Multiplying (4.6) by $\tau(t)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}[\tau(t)(\gamma^{-1}\|u_t\|_{0,\Omega}^2+\sigma^{-1}\|\theta_t\|_{0,\Omega}^2)]+\alpha_2\tau(t)(\gamma^{-1}\|u_t\|_{0,\Omega}^2+\sigma^{-1}\|\theta_t\|_{0,\Omega}^2)\\
&+\tau(t)(\gamma^{-1}\|A_1^{\frac{1}{2}}u_t\|_{0,\Omega}^2+\sigma^{-1}\|A_2^{\frac{1}{2}}\theta_t\|_{0,\Omega}^2)\\
&\le c(\|F_{1t}\|_{0,\Omega}^2+\|F_{2t}\|_{0,\Omega}^2)+c(1+\|\nabla u\|_{0,\Omega}^4+\|\theta_z\|_{L^6}^4)(\|u_t\|_{0,\Omega}^2+\|\theta_t\|_{0,\Omega}^2).
\end{aligned}$$

Therefore, applying Lemma 2.2 with $t_0=\varepsilon_n$ to the above inequality and letting $\varepsilon_n\to 0$ and using Lemma 3.1 and Lemma 3.2, we arrive at (4.3) for $k=1$.

Moreover, by setting

$$(u,p,\theta)=(v,q,\eta),\ (F_1,F_2)=(f_1,f_2),\ B(u,u)=G_1(v,v),\ B(u,\theta)=G_2(v,\eta),$$

in (1.9)-(1.11), we deduce from (2.23) in Lemma 2.1 that

$$\begin{aligned}
&\|A_1u\|_{0,\Omega}^2+\|A_2\theta\|_{0,\Omega}^2+\|\nabla p\|_{0,\Omega}^2\le c(\|u_t\|_{0,\Omega}^2+\|\theta_t\|_{0,\Omega}^2)+c(\|F_1\|_{0,\Omega}^2+\|F_2\|_{0,\Omega}^2)\\
&\qquad+c(\|A_1^{\frac{1}{2}}u\|_{0,\Omega}^2+c\|A_2^{\frac{1}{2}}\theta\|_{0,\Omega}^2)+c\|B(u,u)\|_{0,\Omega}^2+c\|B(u,\theta)\|_{0,\Omega}^2.
\end{aligned}\tag{4.7}$$

It follows from (2.3), (2.9)-(2.10) and (2.15) that

$$\begin{aligned}
c\|(B(u,u)\|_{0,\Omega}^2 &\le c[\|\nabla u\|_{L^3}\|u\|_{L^6} + \|\nabla u\|_{L^3}\|u_z\|_{L^6}]^2\\
&\le c(\|u\|_{L^6}^2 + \|u_z\|_{L^6}^2)\|\nabla u\|_{0,\Omega}\|A_1 u\|_{0,\Omega}\\
&\le \frac{1}{16}\|A_1 u\|_{0,\Omega}^2 + c(\|A_1^{\frac12}u\|_{0,\Omega}^4 + \|u_z\|_{L^6}^4)\|\nabla u\|_{0,\Omega}^2,\\
c\|(B(u,\theta)\|_{0,\Omega}^2 &\le c[\|\nabla\theta\|_{L^3}\|u\|_{L^6} + \|\nabla u\|_{L^3}\|\theta_z\|_{L^6}]^2\\
&\le c(\|u\|_{L^6}^2 + \|\theta_z\|_{L^6}^2)(\|\nabla u\|_{0,\Omega}\|A_1 u\|_{0,\Omega} + \|\nabla\theta\|_{0,\Omega}\|A_2\theta\|_{0,\Omega})\\
&\le \frac{1}{16}(\|A_1 u\|_{0,\Omega}^2 + \|A_2\theta\|_{0,\Omega}^2)\\
&\quad + c(\|A_1^{\frac12}u\|_{0,\Omega}^4 + \|\theta_z\|_{L^6}^4)(\|\nabla u\|_{0,\Omega}^2 + \|\nabla\theta\|_{0,\Omega}^2).
\end{aligned}$$

Combining these inequalities with (4.7) yields

$$\begin{aligned}
&\|A_1 u(t)\|_{0,\Omega}^2 + \|A_2\theta(t)\|^2 + \|\nabla p(t)\|_{0,\Omega}^2\\
&\quad\le c(\|u_t\|_{0,\Omega}^2 + \|\theta_t\|_{0,\Omega}^2) + c(\|F_1\|_{0,\Omega}^2 + \|F_2\|_{0,\Omega}^2)\\
&\quad + c(1 + \|A_1^{\frac12}u\|_{0,\Omega}^4 + \|\theta_z\|_{L^6}^4 + \|u_z\|_{L^4}^4)(\|A_1^{\frac12}u\|_{0,\Omega}^2 + c\|A_2^{\frac12}\theta\|_{0,\Omega}^2).
\end{aligned}\tag{4.8}$$

Combining (4.8) with (4.3) and using (3.2) and Lemma 3.2 yields (4.4).

**Lemma 4.2.** Under Assumptions of Lemma 4.1, the solution $(u,p,\theta)$ of the 3D viscous PEs of the ocean satisfies the following bounds:

$$\begin{aligned}
&\tau^{3-k}(t)\|A_1^{\frac12}u_t(t)\|_{0,\Omega}^2 + \tau^{3-k}(t)\|A_2^{\frac12}\theta_t(t)\|_{0,\Omega}^2\\
&+\int_0^t e^{\alpha_2(s-t)}\tau^{3-k}(s)[\|A_1 u_t\|_{0,\Omega}^2 + \|A_2\theta_t\|_{0,\Omega}^2]^2 ds \le \kappa,
\end{aligned}\tag{4.9}$$

$$\int_0^t e^{\alpha_2(s-t)}\tau^{3-k}(s)[\|u_{tt}\|_{0,\Omega}^2 + \|\theta_{tt}\|_{0,\Omega}^2 + \|\nabla p_t\|_{0,\Omega}^2]ds \le \kappa.\tag{4.10}$$

**Proof.** Setting

$$(u_t, p_t, \theta_t) = (v, q, \eta),\ \ (F_{1t}, F_{2t}) = (f_1, f_2),$$

$$B(u_t,u) + B(u,u_t) = G_1(v,v),\ \ B(u_t,\theta) + B(u,\theta_t) = G_2(v,\eta),$$

in (4.1)-(4.2), we deduce from (2.24) in Lemma 2.1 that

$$\begin{aligned}
&\frac{d}{dt}(\|A_1^{\frac12}u_t\|_{0,\Omega}^2 + \|A_2^{\frac12}\theta_t\|_{0,\Omega}^2) + \alpha_2(\|A_1^{\frac12}u_t\|_{0,\Omega}^2 + \|A_2^{\frac12}\theta_t\|_{0,\Omega}^2)\\
&\quad + \frac54\|A_1 u_t\|_{0,\Omega}^2 + \frac54\|A_2\theta_t\|_{0,\Omega}^2\\
&\quad \le c(\|F_{1t}\|_{0,\Omega}^2 + \|F_{2t}\|_{0,\Omega}^2) + c(\|A_1^{\frac12}u_t\|_{0,\Omega}^2 + \|A_2^{\frac12}\theta_t\|_{0,\Omega}^2)\\
&\quad + c(\|B(u_t,u) + B(u,u_t)\|_{0,\Omega}^2 + \|B(u_t,\theta) + B(u,\theta_t)\|_{0,\Omega}^2),
\end{aligned}\tag{4.11}$$

$$\begin{aligned}
&\|u_{tt}\|_{0,\Omega}^2 + \|\theta_{tt}\|_{0,\Omega}^2 + \|\nabla p_t\|_{0,\Omega}^2 \le c(\|A_1 u_t\|_{0,\Omega}^2 + \|A_2\theta_t\|_{0,\Omega}^2)\\
&\quad + c(\|F_{1t}\|_{0,\Omega}^2 + \|F_{2t}\|_{0,\Omega}^2) + c(\|A_1^{\frac12}u_t\|_{0,\Omega}^2 + \|A_2^{\frac12}\theta_t\|_{0,\Omega}^2)\\
&\quad + c(\|B(u_t,u) + B(u,u_t)\|_{0,\Omega}^2 + \|B(u_t,\theta) + B(u,\theta_t)\|_{0,\Omega}^2).
\end{aligned}\tag{4.12}$$

Using (2.26)-(2.28) in (4.11)-(4.12) yields

$$\begin{aligned}\frac{d}{dt}(\|A_1^{\frac12}u_t\|_{0,\Omega}^2+\|A_2^{\frac12}\theta_t\|_{0,\Omega}^2)&+\alpha_2(\|A_1^{\frac12}u_t\|_{0,\Omega}^2+\|A_2^{\frac12}\theta_t\|_{0,\Omega}^2)\\&+(\|A_1u_t\|_{0,\Omega}^2+\|A_2\theta_t\|_{0,\Omega}^2)\\&\le c(1+\|A_1^{\frac12}u\|_{0,\Omega}^2\|A_1u\|_{0,\Omega}^2+\|A_2\theta\|_{0,\Omega}^2\|A_2^{\frac12}\theta\|_{0,\Omega}^2)\\&\times(\|A_1^{\frac12}u_t\|_{0,\Omega}^2+\|A_2^{\frac12}\theta_t\|_{0,\Omega}^2)+c(\|F_{1t}\|_{0,\Omega}^2+\|F_{2t}\|_{0,\Omega}^2),\end{aligned}\tag{4.13}$$

and

$$\begin{aligned}\|u_{tt}\|_{0,\Omega}^2&+\|\theta_{tt}\|_{0,\Omega}^2+\|\nabla p_t\|_{0,\Omega}^2\\&\le c(\|F_{1t}\|_{0,\Omega}^2+\|F_{2t}\|_{0,\Omega}^2)+c(\|A_1u_t\|_{0,\Omega}^2+\|A_2\theta_t\|_{0,\Omega}^2)\\&+c(1+\|A_1^{\frac12}u\|_{0,\Omega}^2\|A_1u\|_{0,\Omega}^2+\|A_2\theta\|_{0,\Omega}^2\|A_2^{\frac12}\theta\|_{0,\Omega}^2)\\&\times(\|A_1^{\frac12}u_t\|_{0,\Omega}^2+\|A_2^{\frac12}\theta_t\|_{0,\Omega}^2).\end{aligned}\tag{4.14}$$

From (4.3) in Lemma 4.1, we have

$$\int_0^1 e^{\alpha_2 s}\tau^{2-k}(s)[\|A_1^{\frac12}u_t\|_{0,\Omega}^2+\|A_2^{\frac12}\theta_t\|_{0,\Omega}^2]ds\le\kappa.$$

Then, there exists a sequence $\varepsilon_n\to0$, such that

$$\tau^{3-k}(\varepsilon_n)(\|A_1^{\frac12}u_t(\varepsilon_n)\|_{0,\Omega}^2+\|A_2^{\frac12}\theta_t(\varepsilon_n)\|_{0,\Omega}^2)\to0.$$

Therefore, multiplying (4.13) by $\tau^{3-k}(t)$, applying Lemma 2.2 with $t_0=\varepsilon_n$, letting $\varepsilon_n\to0$ and using Lemma 3.1 and Lemma 4.1, we arrive (4.9).

Moreover, Combining (4.14) with (4.9) and using Lemma 3.1 and Lemma 4.1, we get (4.10) and complete the proof of Lemma 4.2.

Combining Lemma 4.1 with Lemma 4.2 has completed the proof of Theorem 1.1 for $m=1$.

## 5. $H^2$-estimates of $(u_t,\theta_t)$

Differentiating (4.1)-(4.2) with respect to $t$, we obtain

$$u_{ttt}+L_1u_{tt}-\gamma\nabla\int_{-d}^z\theta_{tt}(x,y,\xi,t)d\xi+f\vec{k}\times u_{tt}+\partial_t^2B(u,u)=F_{1tt},\tag{5.1}$$

$$\theta_{ttt}+L_2\theta_{tt}+\sigma\int_{-d}^z\nabla\cdot u_{tt}(x,y,\xi,t)d\xi+\partial_t^2B(u,\theta)=F_{2tt},\tag{5.2}$$

where

$$\partial_t^2B(u,u)=B(u_{tt},u)+B(u,u_{tt})+2B(u_t,u_t),$$
$$\partial_t^2B(u,\theta)=B(u_{tt},\theta)+B(u,\theta_{tt})+2B(u_t,\theta_t).$$

**Lemma 5.1.** Assume that the assumptions (**A1**) and (**A2**) with $k=1,\ 2$ hold and

$$(F_{1t},F_{2t}),\ (F_{1tt},F_{2tt})\in L^\infty(\mathbb{R}^+;L^2(\Omega)^2)\times L^\infty(\mathbb{R}^+;L^2(\Omega)).$$

Then, the solution $(u,p,\theta)$ of the 3D PEs of the ocean satisfies the following bounds:

$$\begin{aligned}\gamma^{-1}\tau^{4-k}(t)\|u_{tt}(t)\|_{0,\Omega}^2&+\sigma^{-1}\tau^{4-k}(t)\|\theta_{tt}(t)\|_{0,\Omega}^2\\&+\int_0^t e^{\alpha_2(s-t)}\tau^{4-k}(s)[\gamma^{-1}\|A_1^{\frac12}u_{tt}\|_{0,\Omega}^2+\sigma^{-1}\|A_2^{\frac12}\theta_{tt}\|_{0,\Omega}^2]ds\le\kappa,\end{aligned}\tag{5.3}$$

$$\tau^{4-k}(t)[\|A_1u_t(t)\|_{0,\Omega}^2+\|A_2\theta_t(t)\|_{0,\Omega}^2+\|\nabla p_t(t)\|_{0,\Omega}^2]\le\kappa.\tag{5.4}$$

**Proof.** Setting

$$(u_{tt}, p_{tt}, \theta_{tt}) = (v, q, \eta),\ (F_{1tt}, F_{2tt}) = (f_1, f_2),$$

$$\partial_t^2 B(u,u) = G_1(v,v),\ \partial_t^2 B(u,\theta) = G_2(v,\eta),$$

in (5.1)-(5.2), we deduce from (2.11) and (2.22) in Lemma 2.1 that

$$\begin{aligned}
&\frac{d}{dt}(\gamma^{-1}\|u_{tt}\|_{0,\Omega}^2 + \sigma^{-1}\|\theta_{tt}\|_{0,\Omega}^2) + \alpha_2(\gamma^{-1}\|u_{tt}\|_{0,\Omega}^2 + \sigma^{-1}\|\theta_{tt}\|_{0,\Omega}^2) \\
&\qquad + \frac{5}{4}(\gamma^{-1}\|A_1^{\frac12}u_{tt}\|_{0,\Omega}^2 + \sigma^{-1}\|A_2^{\frac12}\theta_{tt}\|_{0,\Omega}^2) \\
&\qquad \le \frac{4}{\alpha_2}(\gamma^{-1}\|F_{1tt}\|_{0,\Omega}^2 + \sigma^{-1}\|F_{2tt}\|_{0,\Omega}^2) \\
&\qquad - 2\gamma^{-1}(B(u_{tt},u) + 2B(u_t,u_t), u_{tt})_\Omega \\
&\qquad - 2\sigma^{-1}(B(u_{tt},\theta) + 2B(u_t,\theta_t), \theta_{tt})_\Omega. \qquad (5.5)
\end{aligned}$$

It follows from (2.3), (2.15), (2.29)-(2.30) and (2.35)-(2.36) that

$$\begin{aligned}
&2\gamma^{-1}|(B(u_{tt},u), u_{tt})_\Omega| \le c\|B(u_{tt},u)\|_{L^{\frac32}}\|u_{tt}\|_{L^3} \\
&\le c(\|\nabla u\|_{0,\Omega} + \|u_z\|_{L^6})\|u_{tt}\|_{0,\Omega}^{\frac12}\|A_1^{\frac12}u_{tt}\|_{0,\Omega}^{\frac32} \\
&\le \frac{1}{16}\|A_1^{\frac12}u_{tt}\|_{0,\Omega}^2 + c(\|\nabla u\|_{0,\Omega}^4 + \|u_z\|_{L^6}^4)\|u_{tt}\|_{0,\Omega}^2, \\
&2\sigma^{-1}|(B(u_{tt},\theta), \theta_{tt})_\Omega| \le c\|B(u_{tt},\theta)\|_{L^{\frac32}}\|\theta_{tt}\|_{L^3} \\
&\le c(\|\nabla\theta\|_{0,\Omega} + \|\theta_z\|_{L^6})\|A_1^{\frac12}u_{tt}\|_{0,\Omega}\|\theta_{tt}\|_{0,\Omega}^{\frac12}\|A_2^{\frac12}\theta_{tt}\|_{0,\Omega}^{\frac12} \\
&\le \frac{1}{16}(\|A_1^{\frac12}u_{tt}\|_{0,\Omega}^2 + \|A_2^{\frac12}\theta_{tt}\|_{0,\Omega}^2) + c(\|\nabla\theta\|_{0,\Omega}^4 + \|\theta_z\|_{L^6}^4)\|\theta_{tt}\|_{0,\Omega}^2, \\
&4\gamma^{-1}|(B(u_t,u_t), u_{tt})_\Omega| \le \|B(u_t,u_t)\|_{0,\Omega}\|u_{tt}\|_{0,\Omega} \\
&\le c\|A_1u_t\|_{0,\Omega}\|A_1^{\frac12}u_t\|_{0,\Omega}\|u_{tt}\|_{0,\Omega} \\
&\le c\|A_1^{\frac12}u_t\|_{0,\Omega}(\|A_1u_t\|_{0,\Omega}^2 + \|u_{tt}\|_{0,\Omega}^2), \\
&4\sigma^{-1}|(B(u_t,\theta_t), \theta_{tt})_\Omega| \le c\|B(u_t,\theta)\|_{0,\Omega}\|\theta_{tt}\|_{0,\Omega} \\
&\le c\|A_1u_t\|_{0,\Omega}^{\frac12}\|A_1^{\frac12}u_t\|_{0,\Omega}^{\frac12}\|A_2\theta_t\|_{0,\Omega}^{\frac12}\|A_2^{\frac12}\theta_t\|_{0,\Omega}^{\frac12}\|\theta_{tt}\|_{0,\Omega} \\
&\le c(\|A_1^{\frac12}u_t\|_{0,\Omega} + \|A_2^{\frac12}\theta_t\|_{0,\Omega})(\|A_1u_t\|_{0,\Omega}^2 + \|A_2\theta_t\|_{0,\Omega}^2 + \|\theta_{tt}\|_{0,\Omega}^2).
\end{aligned}$$

Combining these inequalities with (4.5), we obtain

$$\begin{aligned}
&\frac{d}{dt}(\gamma^{-1}\|u_{tt}\|_{0,\Omega}^2 + \sigma^{-1}\|\theta_{tt}\|_{0,\Omega}^2) + \alpha_2(\gamma^{-1}\|u_{tt}\|_{0,\Omega}^2 + \sigma^{-1}\|\theta_{tt}\|_{0,\Omega}^2) \\
&+ (\gamma^{-1}\|A_1^{\frac12}u_{tt}\|_{0,\Omega}^2 + \sigma^{-1}\|A_2^{\frac12}\theta_{tt}\|_{0,\Omega}^2) \\
&\le c(\|F_{1tt}\|_{0,\Omega}^2 + \|F_{2tt}\|_{0,\Omega}^2) \\
&+ c(\|A_1u_t\|_{0,\Omega}^2 + \|A_2\theta_t\|_{0,\Omega}^2 + \|u_{tt}\|_{0,\Omega}^2 + \|\theta_{tt}\|_{0,\Omega}^2) \\
&\times (\|A_1^{\frac12}u_t\|_{0,\Omega} + \|A_2^{\frac12}\theta_t\|_{0,\Omega} + \|\nabla u\|_{0,\Omega}^4 + \|\nabla\theta\|_{0,\Omega}^4 + \|u_z\|_{L^6}^4 + \|\theta_z\|_{L^6}^4). \qquad (5.6)
\end{aligned}$$

From (4.10) in Lemma 4.2, we have

$$\int_0^1 e^{\alpha_2 s}\tau^{3-k}(s)[\gamma^{-1}\|u_{tt}\|_{0,\Omega}^2 + \sigma^{-1}\|\theta_{tt}\|_{0,\Omega}^2]ds \le \kappa.$$

Then, there exists a sequence $\varepsilon_n \to 0$, such that

$$\tau^{4-k}(\varepsilon_n)(\gamma^{-1}\|u_{tt}(\varepsilon_n)\|_{0,\Omega}^2 + \sigma^{-1}\|\theta_{tt}(\varepsilon_n)\|_{0,\Omega}^2) \to 0.$$

Therefore, multiplying (5.6) by $\tau^{4-k}(t)$, applying Lemma 2.2 with $t_0 = \varepsilon_n$, letting $\varepsilon_n \to 0$ and using Lemmas 3.1, 3.2 and 4.2, we get (5.3).

Moreover, by setting

$$(u_t, p_t, \theta_t) = (v, q, \eta),\ (F_{1t}, F_{2t}) = (f_1, f_2),$$
$$B(u_t, u) + B(u, u_t) = G_1(v, v),\ B(u_t, \theta) + B(u, \theta_t) = G_2(v, \eta),$$

in (4.1)-(4.2), we deduce from (2.23) in Lemma 2.1 that

$$\begin{aligned}\|A_1 u_t\|_{0,\Omega}^2 &+ \|A_2\theta_t\|_{0,\Omega}^2 + \|\nabla p_t\|_{0,\Omega}^2 \\ &\le c(\|u_{tt}\|_{0,\Omega}^2 + \|\theta_{tt}\|_{0,\Omega}^2) + c(\|F_{1t}\|_{0,\Omega}^2 + \|F_{2t}\|_{0,\Omega}^2) \\ &+ c(\|A_1^{\frac12}u_t\|_{0,\Omega}^2 + c\|A_2^{\frac12}\theta_t\|_{0,\Omega}^2) + c\|B(u_t,u) + B(u,u_t)\|_{0,\Omega}^2 \\ &+ c\|B(u_t,\theta) + B(u,\theta_t)\|_{0,\Omega}^2. \end{aligned} \tag{5.7}$$

Using (2.26)-(2.28) in (5.7) yields

$$\begin{aligned}&\|A_1 u_t(t)\|_{0,\Omega}^2 + \|A_2\theta_t(t)\|^2 + \|\nabla p_t(t)\|_{0,\Omega}^2 \\ &\le c(\|u_{tt}\|_{0,\Omega}^2 + \|\theta_{tt}\|_{0,\Omega}^2) + c(\|F_{1t}\|_{0,\Omega}^2 + \|F_{2t}\|_{0,\Omega}^2) \\ &+ c(1 + \|A_1^{\frac12}u\|_{0,\Omega}^2\|A_1u\|_{0,\Omega}^2 + \|A_2\theta\|_{0,\Omega}^2\|A_2^{\frac12}\theta\|_{0,\Omega}^2) \\ &\times (\|A_1^{\frac12}u_t\|_{0,\Omega}^2 + \|A_2^{\frac12}\theta_t\|_{0,\Omega}^2). \end{aligned} \tag{5.8}$$

Multiplying (5.8) by $\tau^{4-k}(t)$ and using (5.3), Lemmas 3.1, 4.1 and 4.2 yields (5.4).

**Lemma 5.2.** Under the same assumptions of Lemma 5.1, the solution $(u, p, \theta)$ of the 3D viscous PEs of the ocean satisfies the following bounds:

$$\begin{aligned}&\tau^{5-k}(t)\|A_1^{\frac12}u_{tt}(t)\|_{0,\Omega}^2 + \tau^{5-k}(t)\|A_2^{\frac12}\theta_{tt}(t)\|_{0,\Omega}^2 \\ &+ \int_0^t e^{\alpha_2(s-t)}\tau^{5-k}(s)[\|A_1u_{tt}\|_{0,\Omega}^2 + \|A_2\theta_{tt}\|_{0,\Omega}^2]^2 ds \le \kappa, \end{aligned} \tag{5.9}$$

$$\int_0^t e^{\alpha_2(s-t)}\tau^{5-k}(s)[\|u_{ttt}\|_{0,\Omega}^2 + \|\theta_{ttt}\|_{0,\Omega}^2 + \|\nabla p_{tt}\|_{0,\Omega}^2] ds \le \kappa. \tag{5.10}$$

**Proof.** Setting

$$(u_{tt}, p_{tt}, \theta_{tt}) = (v, q, \eta),\ (F_{1tt}, F_{2tt}) = (f_1, f_2),$$
$$B(u_{tt}, u) + B(u, u_{tt}) + 2B(u_t, u_t) = G_1(v, v),$$
$$B(u_{tt}, \theta) + B(u, \theta_{tt}) + 2B(u_t, \theta_t) = G_2(v, \eta),$$

in (5.1)-(5.2), we deduce from (2.24) in Lemma 2.1 that

$$\begin{aligned}&\frac{d}{dt}(\|A_1^{\frac12}u_{tt}\|_{0,\Omega}^2 + \|A_2^{\frac12}\theta_{tt}\|_{0,\Omega}^2) + \alpha_2(\|A_1^{\frac12}u_{tt}\|_{0,\Omega}^2 + \|A_2^{\frac12}\theta_{tt}\|_{0,\Omega}^2) \\ &\quad + \frac54(\|A_1u_{tt}\|_{0,\Omega}^2 + \|A_2\theta_{tt}\|_{0,\Omega}^2) \\ &\quad \le c(\|F_{1tt}\|_{0,\Omega}^2 + \|F_{2tt}\|_{0,\Omega}^2) + c(\|A_1^{\frac12}u_{tt}\|_{0,\Omega}^2 + \|A_2^{\frac12}\theta_{tt}\|_{0,\Omega}^2) \\ &\quad + c\|\partial_t^2 B(u,u)\|_{0,\Omega}^2 + c\|\partial_t^2 B(u,\theta)\|_{0,\Omega}^2, \end{aligned} \tag{5.11}$$

$$\begin{aligned}\|u_{ttt}\|_{0,\Omega}^2 &+ \|\theta_{ttt}\|_{0,\Omega}^2 + \|\nabla p_{tt}\|_{0,\Omega}^2 \\ &\le c(\|A_1u_{tt}\|_{0,\Omega}^2 + \|A_2\theta_{tt}\|_{0,\Omega}^2) + (\|F_{1tt}\|_{0,\Omega}^2 + \|F_{2tt}\|_{0,\Omega}^2) \\ &+ c(\|A_1^{\frac12}u_{tt}\|_{0,\Omega}^2 + \|A_2^{\frac12}\theta_{tt}\|_{0,\Omega}^2) + c(\|\partial_t^2 B(u,u)\|_{0,\Omega}^2 + \|\partial_t^2 B(u,\theta)\|_{0,\Omega}^2). \end{aligned} \tag{5.12}$$

Using (2.26)-(2.30) in (5.11)-(5.12) yields

$$\begin{aligned}
&\frac{d}{dt}(\|A_1^{\frac{1}{2}}u_{tt}\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\theta_{tt}\|_{0,\Omega}^2) + \alpha_2(\|A_1^{\frac{1}{2}}u_{tt}\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\theta_{tt}\|_{0,\Omega}^2)\\
&+ (\|A_1u_{tt}\|_{0,\Omega}^2 + \|A_2\theta_{tt}\|_{0,\Omega}^2) \le c(\|F_{1tt}\|_{0,\Omega}^2 + \|F_{2tt}\|_{0,\Omega}^2)\\
&+ c\|A_1u_t\|_{0,\Omega}^2\|A_1^{\frac{1}{2}}u_t\|_{0,\Omega}^2 + c\|A_2\theta_t\|_{0,\Omega}^2\|A_2^{\frac{1}{2}}\theta_t\|_{0,\Omega}^2\\
&+ c(1 + \|A_1^{\frac{1}{2}}u\|_{0,\Omega}^2\|A_1u\|_{0,\Omega}^2 + \|A_2\theta\|_{0,\Omega}^2\|A_2^{\frac{1}{2}}\theta\|_{0,\Omega}^2)\\
&\times (\|A_1^{\frac{1}{2}}u_{tt}\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\theta_{tt}\|_{0,\Omega}^2), \qquad (5.13)\\
&\|u_{tt}\|_{0,\Omega}^2 + \|\theta_{tt}\|_{0,\Omega}^2 + \|\nabla p_t\|_{0,\Omega}^2\\
&\le c(\|F_{1tt}\|_{0,\Omega}^2 + \|F_{2tt}\|_{0,\Omega}^2) + c(\|A_1u_{tt}\|_{0,\Omega}^2 + \|A_2\theta_{tt}\|_{0,\Omega}^2)\\
&+ c(1 + \|A_1^{\frac{1}{2}}u\|_{0,\Omega}^2\|A_1u\|_{0,\Omega}^2 + \|A_2\theta\|_{0,\Omega}^2\|A_2^{\frac{1}{2}}\theta\|_{0,\Omega}^2)\\
&\times (\|A_1^{\frac{1}{2}}u_{tt}\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\theta_{tt}\|_{0,\Omega}^2)\\
&+ c\|A_1u_t\|_{0,\Omega}^2\|A_1^{\frac{1}{2}}u_t\|_{0,\Omega}^2 + c\|A_2\theta_t\|_{0,\Omega}^2\|A_2^{\frac{1}{2}}\theta_t\|_{0,\Omega}^2. \qquad (5.14)
\end{aligned}$$

From (5.3) in Lemma 5.1, we have

$$\int_0^1 e^{\alpha_2 s}\tau^{4-k}(s)[\|A_1^{\frac{1}{2}}u_{tt}\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\theta_{tt}\|_{0,\Omega}^2]ds \le \kappa.$$

Then, there exists a sequence $\varepsilon_n \to 0$, such that

$$\tau^{5-k}(\varepsilon_n)(\|A_1^{\frac{1}{2}}u_{tt}(\varepsilon_n)\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\theta_{tt}(\varepsilon_n)\|_{0,\Omega}^2) \to 0.$$

Therefore, multiplying (5.13) by $\tau^{5-k}(t)$, applying Lemma 2.2 with $t_0 = \varepsilon_n$, letting $\varepsilon_n \to 0$ and using Lemmas 3.1, 4.1, 4.2 and 5.1 and noting

$$\begin{aligned}
&c\int_0^t e^{\alpha_2(s-t)}\tau^{5-k}(s)[\|A_1u_t\|_{0,\Omega}^2\|A_1^{\frac{1}{2}}u_t\|_{0,\Omega}^2 + \|A_2\theta_t\|_{0,\Omega}^2\|A_2^{\frac{1}{2}}\theta_t\|_{0,\Omega}^2]ds\\
&+ c\int_0^t e^{\alpha_2(s-t)}\tau^{k-1}(s)[\tau^{6-2k}(s)\|A_1u_t\|_{0,\Omega}^2\|A_1^{\frac{1}{2}}u_t\|_{0,\Omega}^2\\
&+ \tau^{6-2k}(s)\|A_2\theta_t\|_{0,\Omega}^2\|A_2^{\frac{1}{2}}\theta_t\|_{0,\Omega}^2]ds \le \kappa, \qquad (5.15)
\end{aligned}$$

we get (5.9). Moreover, combining (5.14) with (5.9) and using (5.15) and Lemmas 3.1, 4.1, 4.2 and 5.1, we get (5.10). The proof of Lemma 5.2 is complete. Also combining Lemma 5.1 with Lemma 5.2 finishes the proof of Theorem 1.1 for $m = 2$.

## 6. $H^2$-estimates of $(u_{tt}, \theta_{tt})$

Differentiating (5.1)-(5.2) with respect to $t$, we obtain

$$\begin{aligned}
&u_{tttt} + L_1u_{ttt} + \nabla p_{ttt} - \gamma\nabla\int_{-d}^z \nabla\theta_{ttt}(x,y,\xi,t)d\xi + f\vec{k}\times u_{ttt}\\
&+ \partial_t^3B(u,u) = F_{1ttt}, \qquad (6.1)
\end{aligned}$$

$$\theta_{tttt} + L_2\theta_{ttt} + \sigma\int_{-d}^z \nabla\cdot u_{ttt}(x,y,\xi,t)d\xi + \partial_t^3B(u,\theta) = F_{2ttt}, \qquad (6.2)$$

where

$$\partial_t^3B(u,u) = B(u_{ttt},u) + B(u,u_{ttt}) + 3B(u_{tt},u_t) + 3B(u_t,u_{tt}),$$

$$\partial_t^3B(u,\theta) = B(u_{ttt},\theta) + B(u,\theta_{ttt}) + 3B(u_{tt},\theta_t) + B(u_t,\theta_{tt}).$$

**Lemma 6.1.** Assume that the assumptions (**A1**) and (**A2**) with $k = 1,\ 2$ hold and

$$(F_{1t}, F_{2t}),\ (F_{1tt}, F_{2tt}), (F_{1ttt}, F_{2ttt}) \in L^\infty(\mathbb{R}^+; L^2(\Omega)^2) \times L^\infty(\mathbb{R}^+; L^2(\Omega)).$$

Then, the solution $(u, p, \theta)$ of the 3D PEs of the ocean satisfies the following bounds:

$$\gamma^{-1}\tau^{6-k}(t)\|u_{ttt}(t)\|_{0,\Omega}^2 + \sigma^{-1}\tau^{6-k}(t)\|\theta_{ttt}(t)\|_{0,\Omega}^2 + \int_0^t e^{\alpha_2(s-t)}\tau^{6-k}(s)[\gamma^{-1}\|A_1^{\frac12}u_{ttt}\|_{0,\Omega}^2 + \sigma^{-1}\|A_2^{\frac12}\theta_{ttt}\|_{0,\Omega}^2]ds \le \kappa, \tag{6.3}$$

$$\tau^{6-k}(t)[\|A_1u_{tt}(t)\|_{0,\Omega}^2 + \|A_2\theta_{tt}(t)\|_{0,\Omega}^2 + \|\nabla p_{tt}(t)\|_{0,\Omega}^2] \le \kappa. \tag{6.4}$$

**Proof.** Setting

$$(u_{ttt}, p_{ttt}, \theta_{ttt}) = (v, q, \eta),\ (F_{1ttt}, F_{2ttt}) = (f_1, f_2),$$

$$\partial_t^3 B(u, u) = G_1(v, v),\ \partial_t^3 B(u, \theta) = G_2(v, \eta),$$

in (6.1)-(6.2), we deduce from (2.11) and (2.22) in Lemma 2.1 that

$$\begin{aligned}
&\frac{d}{dt}(\gamma^{-1}\|u_{ttt}\|_{0,\Omega}^2 + \sigma^{-1}\|\theta_{ttt}\|_{0,\Omega}^2) + \alpha_2(\gamma^{-1}\|u_{ttt}\|_{0,\Omega}^2 + \sigma^{-1}\|\theta_{ttt}\|_{0,\Omega}^2)\\
&\quad + \frac54(\gamma^{-1}\|A_1^{\frac12}u_{ttt}\|_{0,\Omega}^2 + \sigma^{-1}\|A_2^{\frac12}\theta_{ttt}\|_{0,\Omega}^2)\\
&\quad \le \frac{4}{\alpha_2}(\gamma^{-1}\|F_{1ttt}\|_{0,\Omega}^2 + \sigma^{-1}\|F_{2ttt}\|_{0,\Omega}^2)\\
&\quad - 2\gamma^{-1}(B(u_{ttt}, u) + 3B(u_{tt}, u_t) + 3B(u_t, u_{tt}), u_{ttt})_\Omega\\
&\quad - 2\sigma^{-1}(B(u_{ttt}, \theta) + 3B(u_{tt}, \theta_t) + 3B(u_t, \theta_{tt}), \theta_{ttt})_\Omega.
\end{aligned} \tag{6.5}$$

It follows from (2.3), (2.15) and (2.31)-(2.36) that

$$\begin{aligned}
2\gamma^{-1}|(B(u_{ttt}, u), u_{ttt})_\Omega| &\le c\|B(u_{ttt}, u)\|_{L^{\frac32}}\|u_{ttt}\|_{L^3}\\
&\le c(\|\nabla u\|_{0,\Omega} + \|u_z\|_{L^6})\|u_{ttt}\|_{0,\Omega}^{\frac12}\|A_1^{\frac12}u_{ttt}\|_{0,\Omega}^{\frac32}\\
&\le \frac{1}{16}\|A_1^{\frac12}u_{ttt}\|_{0,\Omega}^2 + c(\|\nabla u\|_{0,\Omega}^4 + \|u_z\|_{L^6}^4)\|u_{ttt}\|_{0,\Omega}^2,\\
2\sigma^{-1}|(B(u_{ttt}, \theta), \theta_{ttt})_\Omega| &\le c\|B(u_{ttt}, \theta)\|_{L^{\frac32}}\|\theta_{ttt}\|_{L^3}\\
&\le c(\|\nabla \theta\|_{0,\Omega} + \|\theta_z\|_{L^6})\|\theta_{ttt}\|_{0,\Omega}^{\frac12}\|A_2^{\frac12}\theta_{ttt}\|_{0,\Omega}^{\frac12}\|A_1u_{ttt}\|_{0,\Omega}\\
&\le \frac{1}{16}(\|A_1^{\frac12}u_{ttt}\|_{0,\Omega}^2 + \|A_2^{\frac12}\theta_{ttt}\|_{0,\Omega}^2)\\
&\quad + c(\|\nabla \theta\|_{0,\Omega}^4 + \|\theta_z\|_{L^6}^4)\|\theta_{ttt}\|_{0,\Omega}^2,\\
6\gamma^{-1}|(B(u_{tt}, u_t), u_{ttt})_\Omega| &\le c\|A_1u_{tt}\|_{0,\Omega}^{\frac12}\|A_1^{\frac12}u_{tt}\|_{0,\Omega}^{\frac12}\|A_1u_t\|_{0,\Omega}^{\frac12}\|A_1^{\frac12}u_t\|_{0,\Omega}^{\frac12}\|u_{ttt}\|_{0,\Omega},\\
6\gamma^{-1}|(B(u_t, u_{tt}), u_{ttt})_\Omega| &\le c\|A_1u_{tt}\|_{0,\Omega}^{\frac12}\|A_1^{\frac12}u_{tt}\|_{0,\Omega}^{\frac12}\|A_1u_t\|_{0,\Omega}^{\frac12}\|A_1^{\frac12}u_t\|_{0,\Omega}^{\frac12}\|u_{ttt}\|_{0,\Omega},\\
6\sigma^{-1}|(B(u_{tt}, \theta_t), \theta_{ttt})_\Omega| &\le c\|A_1u_{tt}\|_{0,\Omega}^{\frac12}\|A_1^{\frac12}u_{tt}\|_{0,\Omega}^{\frac12}\|A_2\theta_t\|_{0,\Omega}^{\frac12}\|A_2^{\frac12}\theta_t\|_{0,\Omega}^{\frac12}\|\theta_{ttt}\|_{0,\Omega},\\
6\sigma^{-1}|(B(u_t, \theta_{tt}), \theta_{ttt})_\Omega| &\le c\|A_2\theta_{tt}\|_{0,\Omega}^{\frac12}\|A_2^{\frac12}\theta_{tt}\|_{0,\Omega}^{\frac12}\|A_1u_t\|_{0,\Omega}^{\frac12}\|A_1^{\frac12}u_t\|_{0,\Omega}^{\frac12}\|\theta_{ttt}\|_{0,\Omega}.
\end{aligned}$$

Combining these inequalities with (6.5) and using the Young inequality, we obtain

$$\begin{aligned}\frac{d}{dt}(\gamma^{-1}\|u_{ttt}\|^2_{0,\Omega}+\sigma^{-1}\|\theta_{ttt}\|^2_{0,\Omega})+\alpha_2(\gamma^{-1}\|u_{ttt}\|^2_{0,\Omega}+\sigma^{-1}\|\theta_{ttt}\|^2_{0,\Omega})\\ +\gamma^{-1}\|A_1^{\frac12}u_{ttt}\|^2_{0,\Omega}+\sigma^{-1}\|A_2^{\frac12}\theta_{ttt}\|^2_{0,\Omega}\\ \le c(\|F_{1ttt}\|^2_{0,\Omega}+\|F_{2ttt}\|^2_{0,\Omega})+c(\|\nabla u\|^4_{0,\Omega}+\|u_z\|^4_{L^6})\|u_{ttt}\|^2_{0,\Omega}\\ +c(\|\nabla\theta\|^4_{0,\Omega}+\|\theta_z\|^4_{L^6})\|\theta_{ttt}\|^2_{0,\Omega}+I(t),\end{aligned}\tag{6.6}$$

where

$$\begin{aligned}I(t)=c(\|u_{ttt}\|_{0,\Omega}+\|\theta_{ttt}\|_{0,\Omega})\times(\|A_1u_{tt}\|_{0,\Omega}\|A_1^{\frac12}u_{tt}\|_{0,\Omega}+\|A_2\theta_{tt}\|_{0,\Omega}\|A_2^{\frac12}\theta_{tt}\|_{0,\Omega})^{\frac12}\\ \times(\|A_1u_t\|_{0,\Omega}\|A_1^{\frac12}u_t\|_{0,\Omega}+\|A_2\theta_t\|_{0,\Omega}\|A_2^{\frac12}\theta_t\|_{0,\Omega})^{\frac12}.\end{aligned}$$

From (5.10) in Lemma 5.2, we have

$$\int_0^1 e^{\alpha_2 s}\tau^{5-k}(s)[\|u_{ttt}\|^2_{0,\Omega}+\|\theta_{ttt}\|^2_{0,\Omega}]ds\le\kappa.$$

Then, there exists a sequence $\varepsilon_n\to 0$, such that

$$\tau^{6-k}(\varepsilon_n)(\|u_{ttt}(\varepsilon_n)\|^2_{0,\Omega}+\|\theta_{ttt}(\varepsilon_n)\|^2_{0,\Omega})\to 0.$$

Therefore, multiplying (6.6) by $\tau^{6-k}(t)$, applying Lemma 2.2 with $t_0=\varepsilon_n$, letting $\varepsilon_n\to 0$, and using Lemmas 3.1, 3.2 and 4.1, 4.2, 5.1 and 5.2 and noting

$$\begin{aligned}&\int_0^t e^{\alpha_2(s-t)}\tau^{6-k}(s)I(s)ds\le c[\int_0^t e^{\alpha_2(s-t)}\tau^{5-k}(s)(\|u_{ttt}\|^2_{0,\Omega}+\|\theta_{ttt}\|^2_{0,\Omega})ds]^{\frac12}\\ &\times[\int_0^t e^{\alpha_2(s-t)}\tau^{10-2k}(s)(\|A_1u_{tt}\|^2_{0,\Omega}\|A_1^{\frac12}u_{tt}\|^2_{0,\Omega}+\|A_2\theta_{tt}\|^2_{0,\Omega}\|A_2^{\frac12}\theta_{tt}\|^2_{0,\Omega})ds]^{\frac14}\\ &\times[\int_0^t e^{\alpha_2(s-t)}\tau^{6-k+2(k-1)}(s)(\|A_1u_t\|^2_{0,\Omega}\|A_1^{\frac12}u_t\|^2_{0,\Omega}+\|A_2\theta_t\|^2_{0,\Omega}\|A_2^{\frac12}\theta_t\|^2_{0,\Omega})ds]^{\frac14}\\ &\le\kappa,\end{aligned}$$

we get (6.3).

Moreover, by setting

$$(u_{tt},p_{tt},\theta_{tt})=(v,q,\eta),\ (F_{1tt},F_{2tt})=(f_1,f_2),$$

$$B_{tt}(u,u)=G_1(v,v),\ B_{tt}(u,\theta)=G_2(v,\eta),$$

in (5.1)-(5.2), we deduce from (2.23) in Lemma 2.1 that

$$\begin{aligned}\|A_1u_{tt}\|^2_{0,\Omega}+\|A_2\theta_{tt}\|^2_{0,\Omega}+\|\nabla p_{tt}\|^2_{0,\Omega}\le c(\|u_{ttt}\|^2_{0,\Omega}+\|\theta_{ttt}\|^2_{0,\Omega})\\ +c(\|F_{1tt}\|^2_{0,\Omega}+\|F_{2tt}\|^2_{0,\Omega})\\ +c\|A_1^{\frac12}u_{tt}\|^2_{0,\Omega}+c\|A_2^{\frac12}\theta_{tt}\|^2_{0,\Omega}+c\|B(u_{tt},u)+2B(u_t,u_t)+B(u,u_{tt})\|^2_{0,\Omega}\\ +c\|B(u_{tt},\theta)+2B(u_t,\theta_t)+B(u,\theta_{tt})\|^2_{0,\Omega}.\end{aligned}\tag{6.7}$$

Using (2.26)-(2.30) in (6.7) yields

$$\begin{aligned}\|A_1u_{tt}(t)\|_{0,\Omega}^2+\|A_2\theta_{tt}(t)\|^2+\|\nabla p_{tt}(t)\|_{0,\Omega}^2 &\le c(\|u_{ttt}\|_{0,\Omega}^2+\|\theta_{ttt}\|_{0,\Omega}^2)\\ &+c(\|F_{1tt}\|_{0,\Omega}^2+\|F_{2tt}\|_{0,\Omega}^2)\\ &+c(1+\|A_1^{\frac12}u\|_{0,\Omega}^2\|A_1u\|_{0,\Omega}^2+\|A_2\theta\|_{0,\Omega}^2\|A_2^{\frac12}\theta\|_{0,\Omega}^2)\\ &\times(\|A_1^{\frac12}u_{tt}\|_{0,\Omega}^2+\|A_2^{\frac12}\theta_{tt}\|_{0,\Omega}^2)\\ &+c\|A_1u_t\|_{0,\Omega}^2\|A_1^{\frac12}u_t\|_{0,\Omega}^2+c\|A_2\theta_t\|_{0,\Omega}^2\|A_2^{\frac12}\theta_t\|_{0,\Omega}^2. \end{aligned}\tag{6.8}$$

Multiplying (6.8) by $\tau^{6-k}(t)$ and using (6.3), Lemmas 3.1, 4.1 and 4.2, 5.1 and 5.2 and noting

$$\begin{aligned}&c\tau^{6-k}(t)[\|A_1u_t(t)\|_{0,\Omega}^2\|A_1^{\frac12}u_t(t)\|_{0,\Omega}^2+\|A_2\theta_t(t)\|_{0,\Omega}^2\|A_2^{\frac12}\theta_t(t)\|_{0,\Omega}^2]\\ &=c\tau^{k-1}(t)\tau^{4-k}(t)\|A_1u_t(t)\|_{0,\Omega}^2\tau^{3-k}(t)\|A_1^{\frac12}u_t(t)\|_{0,\Omega}^2\\ &+c\tau^{k-1}(t)\tau^{4-k}(t)\|A_2\theta_t\|_{0,\Omega}^2\tau^{3-k}(t)\|A_2^{\frac12}\theta_t\|_{0,\Omega}^2\le\kappa,\end{aligned}$$

we obtain (6.4).

**Lemma 6.2.** Under the assumptions of Lemma 6.1, the solution $(u,p,\theta)$ of the 3D viscous PEs of the ocean satisfies the following bounds:

$$\begin{aligned}&\tau^{7-k}(t)\|A_1^{\frac12}u_{ttt}(t)\|_{0,\Omega}^2+\tau^{7-k}(t)\|A_2^{\frac12}\theta_{ttt}(t)\|_{0,\Omega}^2\\ &+\int_0^t e^{\alpha_2(s-t)}\tau^{7-k}(s)[\|A_1u_{ttt}\|_{0,\Omega}^2+\|A_2\theta_{ttt}\|_{0,\Omega}^2]^2ds\le\kappa,\end{aligned}\tag{6.9}$$

$$\int_0^t e^{\alpha_2(s-t)}\tau^{7-k}(s)[\|u_{tttt}\|_{0,\Omega}^2+\|\theta_{tttt}\|_{0,\Omega}^2+\|\nabla p_{ttt}\|_{0,\Omega}^2]ds\le\kappa.\tag{6.10}$$

**Proof.** Setting

$$(u_{ttt},p_{ttt},\theta_{ttt})=(v,q,\eta),\ (F_{1ttt},F_{2ttt})=(f_1,f_2),$$

$$\partial_t^3B(u,u)=G_1(v,v),\ \partial_t^3B(u,,\theta)=G_2(v,\eta),$$

in (6.1)-(6.2), we deduce from (2.24) in Lemma 2.1 that

$$\begin{aligned}\frac{d}{dt}(\|A_1^{\frac12}u_{ttt}\|_{0,\Omega}^2+\|A_2^{\frac12}\theta_{ttt}\|_{0,\Omega}^2)&+\alpha_2(\|A_1^{\frac12}u_{ttt}\|_{0,\Omega}^2+\|A_2^{\frac12}\theta_{ttt}\|_{0,\Omega}^2)\\ &+\frac54\|A_1u_{ttt}\|_{0,\Omega}^2+\frac54\|A_2\theta_{ttt}\|_{0,\Omega}^2\\ &\le c(\|F_{1ttt}\|_{0,\Omega}^2+\|F_{2ttt}\|_{0,\Omega}^2)+c(\|A_1^{\frac12}u_{ttt}\|_{0,\Omega}^2+\|A_2^{\frac12}\theta_{ttt}\|_{0,\Omega}^2)\\ &+c(\|\partial_t^3B(u,u)\|_{0,\Omega}^2+\|\partial_t^3B(u,\theta)\|_{0,\Omega}^2),\end{aligned}\tag{6.11}$$

$$\begin{aligned}\|u_{tttt}\|_{0,\Omega}^2+\|\theta_{tttt}\|_{0,\Omega}^2+\|\nabla p_{ttt}\|_{0,\Omega}^2 &\le c(\|A_1u_{ttt}\|_{0,\Omega}^2+\|A_2\theta_{ttt}\|_{0,\Omega}^2)+c(\|F_{1ttt}\|_{0,\Omega}^2+\|F_{2ttt}\|_{0,\Omega}^2)\\ &+c(\|A_1^{\frac12}u_{ttt}\|_{0,\Omega}^2+\|A_2^{\frac12}\theta_{ttt}\|_{0,\Omega}^2)\\ &+c(\|\partial_t^3B(u,u)\|_{0,\Omega}^2+\|\partial_t^3B(u,\theta)\|_{0,\Omega}^2).\end{aligned}\tag{6.12}$$

Using (2.26)-(2.28) and (2.31)-(2.34) in (6.11)-(6.12) yields

$$\begin{aligned}
&\frac{d}{dt}(\|A_1^{\frac{1}{2}}u_{ttt}\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\theta_{ttt}\|_{0,\Omega}^2) + \alpha_2(\|A_1^{\frac{1}{2}}u_{ttt}\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\theta_{ttt}\|_{0,\Omega}^2)\\
&\quad + (\|A_1u_{ttt}\|_{0,\Omega}^2 + \|A_2\theta_{ttt}\|_{0,\Omega}^2) \le c(\|F_{1ttt}\|_{0,\Omega}^2 + \|F_{2ttt}\|_{0,\Omega}^2)\\
&\quad + c(1 + \|A_1u\|_{0,\Omega}^2\|A_1^{\frac{1}{2}}u\|_{0,\Omega}^2 + \|A_2\theta\|_{0,\Omega}^2\|A_2^{\frac{1}{2}}\theta\|_{0,\Omega}^2)\\
&\quad \times (\|A_1^{\frac{1}{2}}u_{ttt}\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\theta_{ttt}\|_{0,\Omega}^2)\\
&\quad + c(\|A_1u_{tt}\|_{0,\Omega}\|A_1^{\frac{1}{2}}u_{tt}\|_{0,\Omega} + \|A_2\theta_{tt}\|_{0,\Omega}\|A_2^{\frac{1}{2}}\theta_{tt}\|_{0,\Omega})\\
&\quad \times (\|A_1u_t\|_{0,\Omega}\|A_1^{\frac{1}{2}}u_t\|_{0,\Omega} + \|A_2\theta_t\|_{0,\Omega}\|A_2^{\frac{1}{2}}\theta_t\|_{0,\Omega}),
\end{aligned}\tag{6.13}$$

$$\begin{aligned}
&\|u_{tttt}\|_{0,\Omega}^2 + \|\theta_{tttt}\|_{0,\Omega}^2 + \|\nabla p_{ttt}\|_{0,\Omega}^2\\
&\quad \le c(\|F_{1ttt}\|_{0,\Omega}^2 + \|F_{2ttt}\|_{0,\Omega}^2) + c(\|A_1u_{ttt}\|_{0,\Omega}^2 + \|A_2\theta_{ttt}\|_{0,\Omega}^2)\\
&\quad + c(1 + \|A_1u\|_{0,\Omega}^2\|A_1^{\frac{1}{2}}u\|_{0,\Omega}^2 + \|A_2\theta\|_{0,\Omega}^2\|A_2^{\frac{1}{2}}\theta\|_{0,\Omega}^2)\\
&\quad \times (\|A_1^{\frac{1}{2}}u_{ttt}\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\theta_{ttt}\|_{0,\Omega}^2)\\
&\quad + c(\|A_1u_{tt}\|_{0,\Omega}\|A_1^{\frac{1}{2}}u_{tt}\|_{0,\Omega} + \|A_2\theta_{tt}\|_{0,\Omega}\|A_2^{\frac{1}{2}}\theta_{tt}\|_{0,\Omega})\\
&\quad \times (\|A_1u_t\|_{0,\Omega}\|A_1^{\frac{1}{2}}u_t\|_{0,\Omega} + \|A_2\theta_t\|_{0,\Omega}\|A_2^{\frac{1}{2}}\theta_t\|_{0,\Omega}).
\end{aligned}\tag{6.14}$$

From (6.3) in Lemma 6.1, we have

$$\int_0^1 e^{\alpha_2 s}\tau^{6-k}(s)[\|A_1^{\frac{1}{2}}u_{ttt}\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\theta_{ttt}\|_{0,\Omega}^2]ds \le \kappa.$$

Then, there exists a sequence $\varepsilon_n \to 0$, such that

$$\tau^{7-k}(\varepsilon_n)(\|A_1^{\frac{1}{2}}u_{ttt}(\varepsilon_n)\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\theta_{ttt}(\varepsilon_n)\|_{0,\Omega}^2) \to 0.$$

Therefore, multiplying (6.13) by $\tau^{7-k}(t)$, applying Lemma 2.2 with $t_0 = \varepsilon_n$, letting $\varepsilon_n \to 0$, using Lemmas 3.1, 4.1, 4.2, 5.1, 5.2 and 6.1 and noting

$$\begin{aligned}
&\int_0^t e^{\alpha_2(s-t)}\tau^{7-k}(s)[(\|A_1u_{tt}\|_{0,\Omega}\|A_1^{\frac{1}{2}}u_{tt}\|_{0,\Omega} + \|A_2\theta_{tt}\|_{0,\Omega}\|A_2^{\frac{1}{2}}\theta_{tt}\|_{0,\Omega})\\
&\times (\|A_1u_t\|_{0,\Omega}\|A_1^{\frac{1}{2}}u_t\|_{0,\Omega} + \|A_2\theta_t\|_{0,\Omega}\|A_2^{\frac{1}{2}}\theta_t\|_{0,\Omega})]ds\\
&\le [\int_0^t e^{\alpha_2(s-t)}\tau^{10-2k}(s)(\|A_1u_{tt}\|_{0,\Omega}^2\|A_1^{\frac{1}{2}}u_{tt}\|_{0,\Omega}^2 + \|A_2\theta_{tt}\|_{0,\Omega}^2\|A_2^{\frac{1}{2}}\theta_{tt}\|_{0,\Omega}^2)ds]^{\frac{1}{2}}\\
&\times [\int_0^t e^{\alpha_2(s-t)}\tau^{6-2k+2(k-1)}(s)(\|A_1u_t\|_{0,\Omega}^2\|A_1^{\frac{1}{2}}u_t\|_{0,\Omega}^2 + \|A_2\theta_t\|_{0,\Omega}^2\|A_2^{\frac{1}{2}}\theta_t\|_{0,\Omega}^2)ds]^{\frac{1}{2}}\\
&\le \kappa,
\end{aligned}$$

we arrive at (6.9). Moreover, combining the above inequality with (6.14) and (6.9), using Lemmas 3.1, 4.1, 4.2, 5.1, 5.2 and 6.1, we obtain (6.10). The proof of Lemma 6.2 is complete. Combining Lemma 6.1 with Lemma 6.2 will close up the proof of Theorem 1.1 for $m = 3$.

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Center for Computational Geosciences, School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, P.R. China.
*E-mail*: heyn@mail.xjtu.edu.cn

College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, P. R. China.
*E-mail*: jianhuaw@snnu.edu.cn