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ON A NUMERICAL TECHNIQUE TO STUDY DIFFERENCE SCHEMES FOR SINGULARLY PERTURBED PARABOLIC REACTION-DIFFUSION EQUATIONS

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This paper is dedicated to the 65th birthday of Professor Francisco J. Lisbona

Abstract. A new technique to study special difference schemes numerically for a Dirichlet problem on a rectangular domain (in x, t) is considered for a singularly perturbed parabolic reaction-diffusion equation with a perturbation parameter ε; $\varepsilon \in (0, 1]$. A well known difference scheme on a piecewise-uniform grid is used to solve the problem. Such a scheme converges ε-uniformly in the maximum norm at the rate $\mathcal{O}(N^{-2} \ln^2 N + N_0^{-1})$ as N, $N_0 \to \infty$, where $N + 1$ and $N_0 + 1$ are the numbers of nodes in the spatial and time meshes, respectively; for $\varepsilon \geq m \ln^{-1} N$ the scheme converges at the rate $\mathcal{O}(N^{-2} + N_0^{-1})$. In this paper we elaborate a new approach based on the consideration of **regularized errors** in discrete solutions, i.e., **total errors** (with respect to both variables x and t), and also **fractional errors** (in x and in t) generated in the approximation of differential derivatives by grid derivatives. The regularized total errors agree well with known theoretical estimates for actual errors and their convergence rate orders. It is also shown that a "standard" approach based on the "fine grid technique" turns out inefficient for numerical study of difference schemes because this technique brings to large errors already when estimating the total actual error.

Key words. parabolic reaction-diffusion equation, perturbation parameter, boundary layer, difference scheme, piecewise-uniform grids, ε-uniform convergence, numerical experiments, total error, fractional errors, regularized errors.

1. Introduction

At present, a series of theoretically justified numerical methods convergent ε-uniformly has been elaborated for representative classes of singularly perturbed problems (see, e.g., [3, 5] and the bibliography therein). We also know some "heuristic approaches" (they are widely used when solving applied problems) whose justification is rather problematic (see, e.g., [1, 7]). At the same time, nobody knows good experimental methods to study the efficiency of available special grid methods (both theoretical and "heuristic" ones).

Thus, the development of experimental methods for numerical study that allow us to reveal the quality of special schemes is an actual problem in the construction of reliable ε-uniformly convergent grid methods for wide classes of singularly perturbed problems.

Here we could mention only some interesting numerical researches in [1, 7]. In [1], the two-mesh difference technique has been considered for numerical study of difference schemes on piecewise-uniform grids. This technique has been applied, in particular, to solve a two-dimensional elliptic equation on a rectangle in the case when the convective term includes the derivative along the horizontal axis. The schemes have been considered on meshes with the same number of nodes in both

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variables, and this did not allow to reveal the distinct character in the behavior of the errors in different directions. Schemes for singularly perturbed parabolic equations were not considered in [1]. In [2], a high-order scheme in time based on the defect correction technique has been constructed for a parabolic convection-diffusion equation. In the numerical study of this scheme, in order to compute the "exact solution", the main term of the singular solution component (two first terms) written in explicit form was used, and the remainder (smooth part) of the solution was approximated by the corresponding scheme. This approach allowed us to find the errors in x and t (fractional errors). However, even in the case of the *reaction*-diffusion equation such an approach to investigate the errors turns out inapplicable because it is very difficult to write out the main term of the singular component in explicit form.

In the present paper, a parabolic reaction-diffusion equation is considered for which *the convergence rates* of a scheme on a piecewise-uniform grid *are essentially different* for each of variables x and t. The difference scheme converges with the second order up to a logarithmic factor in the spatial variable and with the first order in the time variable that significantly complicates the numerical analysis of the scheme. To study solutions of such discrete problems, a new technique is needed which could allow us to distinguish the behaviour of the errors in each of the variables x and t, i.e., to reveal the fractional errors.

In the present paper, we propose a new approach based on the consideration of *regularized errors* for the discrete solutions, i.e., *total errors* (with respect to both variables x and t), and also *fractional errors* (in x and in t) generated in the approximation of differential derivatives by difference derivatives. The new approach allows us to study effectively special difference schemes on piecewise-uniform grids.

Contents of the paper. The formulation of the initial-boundary value problem for a singularly perturbed parabolic reaction-diffusion equation and the aim of the research are presented in Section 2. Standard difference schemes on uniform and piecewise-uniform grids and estimates of the convergence rates for these schemes are given in Section 3. Some definitions and notations for standard errors (total and fractional errors in space and in time) are introduced in Section 4; the corresponding numerical experiments are discussed in Section 5. Regularized fractional and total errors are introduced in Section 6. Technics for computing the regularized errors and their convergence orders and the corresponding numerical experiments are also considered in Section 6. Conclusions are given in Section 7.

2. Problem formulation and aim of research

On the set $\overline{G}$ with the boundary S

$$\overline{G} = G \cup S, \quad G = D \times (0, T], \quad D = (0, d), \tag{2.1}$$

we consider the initial-boundary value problem for a singularly perturbed parabolic reaction-diffusion equation

$$L\, u(x,t) = f(x,t), \quad (x,t) \in G, \qquad u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{2.2}$$

Here* $L_{(2.2)} \equiv \varepsilon^2\, a(x,t)\, \dfrac{\partial^2}{\partial x^2} - c(x,t) - p(x,t)\, \dfrac{\partial}{\partial t}$, $(x,t) \in G$, the functions $a(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$ and $\varphi(x,t)$ are assumed to be sufficiently smooth on $\overline{G}$ and

* Notation $L_{(j.k)}$ ($\overline{G}_{(j.k)}, M_{(j.k)}$) means that this operator (domain, constant) is introduced in the formula $(j.k)$.

S, respectively, moreover, †

(2.3) $a_0 \le a(x,t) \le a^0, \quad c_0 \le c(x,t) \le c^0, \quad p_0 \le p(x,t) \le p^0, \quad (x,t) \in \overline{G};$

$$|f(x,t)| \le M, \quad (x,t) \in \overline{G}; \quad |\varphi(x,t)| \le M, \quad (x,t) \in S; \quad a_0,\, c_0,\, p_0 > 0;$$

the parameter ε takes arbitrary values in the open-closed interval $(0,1]$. For small values of ε, a parabolic boundary layer appears in a neighbourhood of the lateral part of the boundary S^L, $S^L = \{(x,t) :\ x = 0,\ d,\ 0 < t \le T\}$. We denote by S_0 the lower part of the boundary, $S = S_0 \cup S^L$.

Our aim is to elaborate a new approach based on numerical experiments to study convergence of grid solutions for initial-boundary value problem (2.2), (2.1).

3. Difference schemes on uniform and piecewise-uniform grids for problem (2.2), (2.1)

We consider difference schemes constructed on the basis of classical approximations of problem (2.2), (2.1) on uniform and piecewise-uniform grids.

On the set $\overline{G}$ we introduce the rectangular grid

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{3.1}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are arbitrary, in general, nonuniform meshes on the intervals $[0,d]$ and $[0,T]$, respectively. Let $h^i = x^{i+1} - x^i$, x^i, $x^{i+1} \in \overline{\omega}$, $h = \max_i h^i$, and $h_t^k = t^{k+1} - t^k$, t^k, $t^{k+1} \in \overline{\omega}_0$, $h_t = \max_k h_t^k$. We assume that the following conditions hold: $h \le M\,N^{-1}$, $h_t \le M\,N_0^{-1}$, where $N+1$ and N_0+1 are the numbers of the nodes in the meshes $\overline{\omega}$ and $\overline{\omega}_0$, respectively.

We approximate problem (2.2), (2.1) by the monotone difference scheme [4]

$$\Lambda z(x,t) = f(x,t), \quad (x,t) \in G_h, \qquad z(x,t) = \varphi(x,t), \quad (x,t) \in S_h. \tag{3.2}$$

Here $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$, $\Lambda \equiv \varepsilon^2\, a(x,t)\, \delta_{\overline{x}\widehat{x}} - c(x,t) - p(x,t)\, \delta_{\overline{t}}$, $\delta_{\overline{x}\widehat{x}}\, z(x,t)$ is the central second order difference derivative on a nonuniform grid, $\delta_{\overline{x}\widehat{x}}\, z(x,t) = 2(h^i + h^{i-1})^{-1}[\delta_x\, z(x,t) - \delta_{\overline{x}}\, z(x,t)]$, $(x,t) = (x^i,t) \in G_h$; $\delta_x\, z(x,t)$ and $\delta_{\overline{x}}\, z(x,t)$ are the first-order (forward and backward) difference derivatives.

In the case when the grid $\overline{G}_h$ is uniform in both variables:

$$\overline{G}_h = \overline{G}_h^u \equiv \overline{\omega} \times \overline{\omega}_0, \tag{3.3}$$

using the maximum principle, we get the estimate

$$|u(x,t) - z(x,t)| \le M\, \left[\left(\varepsilon + N^{-1}\right)^{-2} N^{-2} + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h. \tag{3.4}$$

The scheme (3.2), (3.3) converges under the condition

$$N^{-1} = o(\varepsilon), \quad N_0^{-1} = o(1). \tag{3.5}$$

Now on the set $\overline{G}$, we introduce the piecewise-uniform grid (see, e.g., [3, 5]).

$$\overline{G}_h = \overline{G}_h^s \equiv \overline{\omega}^{\,s} \times \overline{\omega}_0, \tag{3.6a}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(3.3)}$, $\overline{\omega}^{\,s}$ is a piecewise-uniform mesh, which is constructed as follows. The interval $[0,d]$ is divided into three intervals $[0,\sigma]$, $[\sigma, d-\sigma]$, $[d-\sigma, d]$, the step-sizes on these intervals are constant and they are equal to $h^{(1)} = 4\,\sigma\, N^{-1}$ on the intervals $[0,\sigma]$, $[d-\sigma, d]$, and to $h^{(2)} = 2(d-2\sigma)N^{-1}$ on the interval $h^{(2)} = 2(d-\sigma)N^{-1}$. The parameter σ is defined by the relation

$$\sigma = \sigma(\varepsilon,\, N,\, m) = \min\left[4^{-1}\, d,\ m^{-1}\, \varepsilon\, \ln N\right], \tag{3.6b}$$

† We denote by M (by m) sufficiently large (small) positive constants that do not depend on the value of the parameter ε. In the case of grid problems, these constants are also independent of the stencils of the difference schemes.

where m is an arbitrary nonnegative number; further, we use $m = 1$. The mesh $\overline{\omega}^{\,s}$ and the grid $\overline{G}_h^{\,s}$ are thus constructed.

For the solution of the difference scheme (3.2) on the grid (3.6), we obtain the following ε-dependent estimate:

$$(3.7)\qquad |u(x,t) - z(x,t)| \le M\,\left\{N^{-2}\,\min{}^{2}\left[\varepsilon^{-1},\,\ln N\right] + N_0^{-1}\right\},\quad (x,t) \in \overline{G}_h,$$

and also the ε-uniform estimate

$$(3.8)\qquad |u(x,t) - z(x,t)| \le M\,\left[N^{-2}\,\ln^2 N + N_0^{-1}\right],\quad (x,t) \in \overline{G}_h.$$

The scheme (3.2), (3.6) converges ε-uniformly with the first order in t, and with the second order, up to a logarithmic factor, in x; for fixed values of ε, this scheme converges with the first order in t and with the second order in x (see, e.g., [3, 5]).

We give some *a priori* bounds on the solution of the boundary value problem which guarantee validity of the estimates (3.7), (3.8). Write the problem solution as the sum of the functions

$$(3.9)\qquad u(x,t) = U(x,t) + V(x,t),\quad (x,t) \in \overline{G},$$

where $U(x,t)$ and $V(x,t)$ are the regular and singular components of the solution. When the data of the boundary value problem (2.2), (2.1) are sufficiently smooth, we obtain for $U(x,t)$ and $V(x,t)$ the following estimates (see, e.g., [6]):

$$(3.10a)\qquad \left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\,U(x,t)\right| \le M,$$

$$(3.10b)\qquad \left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\,V(x,t)\right| \le M\,\varepsilon^{-k}\,[\exp(-m\,\varepsilon^{-1}\,x) + \exp(-m\,\varepsilon^{-1}\,(d-x))],$$

$$(x,t) \in \overline{G},\quad k + 2k_0 \le 4,\quad m = m_{(3.6)}.$$

Let $z_U(x,t)$ and $z_V(x,t)$, $(x,t) \in \overline{G}_h$ be solutions of difference schemes corresponding to the solutions $u(x,t) = U(x,t)$ and $u(x,t) = V(x,t)$, $(x,t) \in \overline{G}$. Then for $z_U(x,t)$ and $z_V(x,t)$ we have the estimates

$$(3.11)\quad |U(x,t) - z_U(x,t)| \le M\,\left[N^{-2} + N_0^{-1}\right],\quad (x,t) \in \overline{G}_h,$$

$$(3.12)\quad |V(x,t) - z_V(x,t)| \le M\,\left\{N^{-2}\,\min{}^{2}\left[\varepsilon^{-1},\,\ln N\right] + N_0^{-1}\right\},\quad (x,t) \in \overline{G}_h.$$

4. Standard total and fractional errors

4.1. Some notations and definitions. We denote by $z_{i,n}^{\varepsilon,\,N,\,N_0} = z^{\varepsilon,\,N,\,N_0}(x_i,\,t_n)$ the discrete function $z(x,\,t)$, $(x,\,t) \in \overline{G}_h$, i.e., the solution of difference scheme (3.2), (3.6), considered in the node $(x,t) = (x_i,\,t_n)$ from the grid $\overline{G}_{h\,(3.6)} = \overline{G}^{\,\varepsilon,N,N_0}_{h\,(3.6)}$.

The standard actual error $\vartheta_{i,n}^{\varepsilon,\,N,\,N_0} = \vartheta^{\varepsilon,\,N,\,N_0}(x_i,\,t_n)$ of the solution computed in the nodes $(x_i,\,t_n) \in \overline{G}_{h\,(3.6)}$ for a prescribed value of ε is defined by

$$(4.1a)\qquad \vartheta_{i,n}^{\varepsilon,\,N,\,N_0} = \left|z_{i,n}^{\varepsilon,\,N,\,N_0} - u_{i,n}^{\varepsilon}\right|,\quad (x_i,\,t_n) \in \overline{G}_{h\,(3.6)}.$$

Here $u_{i,n}^{\varepsilon} = u^{\varepsilon}(x_i,\,t_n)$ is the exact problem solution $u(x,\,t)$ in the node $(x_i,\,t_n) \in \overline{G}_{h\,(3.6)}$.

The standard ε-dependent actual error $\vartheta^{\,\varepsilon,\,N,\,N_0}$ on $\overline{G}_h = \overline{G}_h^{\,\varepsilon,\,N,\,N_0}$ and the standard ε-uniform actual error $\vartheta^{\,N,\,N_0}$ are defined by

$$(4.1b)\qquad \vartheta^{\,\varepsilon,\,N,\,N_0} = \max_{i,\,n}\vartheta^{\,\varepsilon,\,N,\,N_0}(x_i,\,t_n),\quad \vartheta^{\,N,\,N_0} = \max_{\varepsilon}\vartheta^{\,\varepsilon,\,N,\,N_0}.$$

The theoretical errors given in (3.7) and (3.8) are relative to the values $\vartheta^{\,\varepsilon,\,N,\,N_0}$ and $\vartheta^{\,N,\,N_0}$.

First priority, we are interested in standard actual errors in the solution of the scheme approximating an initial boundary value problem. Moreover, we shall consider also errors of solutions to difference schemes approximated differential problems for regular and singular components. This allows us to reveal the behaviour of errors to the discrete solutions depending on the value of the parameter ε and the number of nodes used.

We shall denote by $\vartheta_u^{\,\varepsilon,\,N,\,N_0}$, $\vartheta_U^{\,\varepsilon,\,N,\,N_0}$ and $\vartheta_V^{\,\varepsilon,\,N,\,N_0}$ the standard actual errors $\vartheta^{\,\varepsilon,\,N,\,N_0}$ of the solutions $u(x,t)$, $U(x,t)$ and $V(x,t)$, respectively.

Because the function $u(x,\,t)$, as a rule, is unknown, we use "the fine grid technique" to analyze convergence of schemes. Let $z_{i,n}^{\varepsilon,\,\widehat{N},\,\widehat{N}_0} = z^{\,\varepsilon,\,\widehat{N},\,\widehat{N}_0}(x_i,\,t_n)$, $(x_i,\,t_n) \in \overline{G}_{h\,(3.6)}^{\varepsilon,\widehat{N},\widehat{N}_0}$ be the solution of the scheme on a sufficiently finest grid with the number of nodes $\widehat{N}+1$ and $\widehat{N}_0+1$ in x and t, respectively (see, e.g., ([2])). Using this discrete function, we construct an interpolant which is quadratic in x and linear in t. We shall use the function $\overline{z}^{\,\varepsilon,\,\widehat{N},\,\widehat{N}_0}(x,t)$, $(x,t) \in \overline{G}$, as an "exact solution" in analyzing convergence characteristics for difference schemes.

We define the standard total error $E_{i,n}^{\varepsilon,\,N,\,N_0}$ by

$$E_{i,n}^{\varepsilon,\,N,\,N_0} = \left| z_{i,n}^{\varepsilon,\,N,\,N_0} - \overline{z}_{i,n}^{\varepsilon,\,\widehat{N},\,\widehat{N}_0} \right|, \ (x_i,\,t_n) \in \overline{G}_h^{\,\varepsilon,\,N,\,N_0}, \tag{4.2a}$$

where $N < \widehat{N}$, $N_0 < \widehat{N}_0$ and $\widehat{N}$, $\widehat{N}_0$ are sufficiently large.

In a similar way, we define the standard fractional errors in space and in time, respectively, by

$$\widetilde{E}_{i,n;\,x}^{\varepsilon,\,N,\,\widehat{N}_0} = \left| z_{i,n}^{\varepsilon,\,N,\,\widehat{N}_0} - \overline{z}_{i,n}^{\varepsilon,\,\widehat{N},\,\widehat{N}_0} \right|, \ (x_i,\,t_n) \in \overline{G}_h^{\,\varepsilon,\,N,\,\widehat{N}_0}, \tag{4.2b}$$

$$\widetilde{E}_{i,n;\,t}^{\varepsilon,\,\widehat{N},\,N_0} = \left| z_{i,n}^{\varepsilon,\,\widehat{N},\,N_0} - \overline{z}_{i,n}^{\varepsilon,\,\widehat{N},\,\widehat{N}_0} \right|, \ (x_i,\,t_n) \in \overline{G}_h^{\,\varepsilon,\,\widehat{N},\,N_0}. \tag{4.2c}$$

We define the standard ε-dependent errors for a fixed value of the parameter ε

(4.3a)

$$E^{\varepsilon,N,N_0} = \max_{i,n} E_{i,n}^{\varepsilon,N,N_0}, \quad \widetilde{E}_x^{\,\varepsilon,N,\widehat{N}_0} = \max_{i,n} \widetilde{E}_{i,n;\,x}^{\varepsilon,N,\widehat{N}_0}, \quad \widetilde{E}_t^{\,\varepsilon,\,\widehat{N},\,N_0} = \max_{i,n} \widetilde{E}_{i,n;\,t}^{\varepsilon,\widehat{N},N_0},$$

and the standard ε-uniform errors

$$E^{N,N_0} = \max_{\varepsilon} E^{\,\varepsilon,N,N_0}, \quad \widetilde{E}_x^{\,N,\widehat{N}_0} = \max_{\varepsilon} \widetilde{E}_x^{\,\varepsilon,N,\widehat{N}_0}, \quad \widetilde{E}_t^{\,\widehat{N},\,N_0} = \max_{\varepsilon} \widetilde{E}_t^{\,\varepsilon,\widehat{N},N_0}. \tag{4.3b}$$

The standard convergence orders (total, fractional in space and in time) for a fixed ε are given by

$$\begin{aligned} r^{\,\varepsilon,N,N_0} &= \log\left(E^{\,\varepsilon,\,N,\,N_0}/E^{\varepsilon,2\,N,2\,N_0}\right)/\log\,2, \\ \widetilde{r}_x^{\,\varepsilon,N,\widehat{N}_0} &= \log\left(\widetilde{E}_x^{\,\varepsilon,\,N,\,\widehat{N}_0}/\widetilde{E}_x^{\,\varepsilon,2\,N,\,\widehat{N}_0}\right)/\log\,2, \\ \widetilde{r}_t^{\,\varepsilon,\widehat{N},N_0} &= \log\left(\widetilde{E}_t^{\,\varepsilon,\,\widehat{N},\,N_0}/\widetilde{E}_t^{\,\varepsilon,\widehat{N},\,2\,N_0}\right)/\log\,2. \end{aligned} \tag{4.4}$$

For the standard ε-uniform convergence orders we have

$$\begin{aligned} r^{\,N,N_0} &= \log\left(E^{\,N,\,N_0}/E^{\,2\,N,2\,N_0}\right)/\log\,2, \\ \widetilde{r}_x^{\,N,\widehat{N}_0} &= \log\left(\widetilde{E}_x^{\,N,\,\widehat{N}_0}/\widetilde{E}_x^{\,2\,N,\,\widehat{N}_0}\right)/\log 2, \\ \widetilde{r}_t^{\,\widehat{N},N_0} &= \log\left(\widetilde{E}_t^{\,\widehat{N},\,N_0}/\widetilde{E}_t^{\,\widehat{N},\,2\,N_0}\right)/\log 2. \end{aligned} \tag{4.5}$$

4.2. Computation of standard total and fractional errors. Now we give some details to compute the standard errors $E^{\varepsilon,\,N,\,N_0}_{i,n\,(4.2\mathrm{a})}$, $\widetilde{E}^{\,\varepsilon,\,N,\,\widehat{N}_0}_{i,n;\,x\,(4.2\mathrm{b})}$ and $\widetilde{E}^{\,\varepsilon,\,\widehat{N},\,N_0}_{i,n;\,t\,(4.2\mathrm{c})}$. Let the value N, which defines the number of mesh nodes in x, be prescribed by the relation $N^{(j)} = 2^{j-1} \times N^0$, $j = 1, 2, \ldots, J$, $\widehat{N} = N^{(J)} = 2^{J-1} \times N^0$, and let, in a similar way, the value N_0 be prescribed, N_0 is the number of mesh nodes in t.

Let we are interested in the standard total error $E^{\varepsilon,\,N,\,N_0}_{i,n\,(4.2\mathrm{a})}$. The discrete functions $z^{\,\varepsilon,\,N,\,N_0}(x,t)$ and $z^{\,\varepsilon,\,\widehat{N},\,\widehat{N}_0}(x,t)$ are defined on the piecewise-uniform grids $\overline{G}_h^{\,\varepsilon,\,N,\,N_0}$ and $\overline{G}_h^{\,\varepsilon,\,\widehat{N},\,\widehat{N}_0}$, respectively, moreover, $\overline{\omega}_0^{N_0} \subset \overline{\omega}_0^{\widehat{N}_0}$, because the meshes $\overline{\omega}_0^{N_0}$, $\overline{\omega}_0^{\widehat{N}_0}$ are uniform, and $\overline{\omega}^{\,\varepsilon,\,N} \subset \overline{\omega}^{\,\varepsilon,\,\widehat{N}}$ in the case when the mesh $\overline{\omega}^{\,\varepsilon,\,\widehat{N}}$ is uniform. In this case, the computation of $E^{\varepsilon,\,N,\,N_0}_{i,n\,(4.2\mathrm{a})}$ is significantly simplified. At the point $A = (x_i, t_n) \in \overline{G}_h^{\,\varepsilon,\,N,\,N_0}$, which is also a node $(x_j, t_k) \in \overline{G}_h^{\,\varepsilon,\,\widehat{N},\,\widehat{N}_0}$, we have

$$\overline{z}^{\,\varepsilon,\,\widehat{N},\,\widehat{N}_0}_{i,n} = \overline{z}^{\,\varepsilon,\,\widehat{N},\,\widehat{N}_0}(A) = z^{\,\varepsilon,\,\widehat{N},\,\widehat{N}_0}(x_j, t_k) = z^{\,\varepsilon,\,\widehat{N},\,\widehat{N}_0}_{j,k}$$

and, consequently,

$$E^{\varepsilon,\,N,\,N_0}_{i,n} = \left| z^{\,\varepsilon,\,N,\,N_0}_{i,n} - z^{\,\varepsilon,\,\widehat{N},\,\widehat{N}_0}_{j,k} \right|;$$

thus, an interpolation in x and t, when computing $E^{\varepsilon,\,N,\,N_0}_{i,n}$, is not required. But if the mesh $\overline{\omega}^{\,\varepsilon,\,\widehat{N}}$ is nonuniform then, by virtue of the condition $\sigma(\varepsilon,\,N) \neq \sigma(\varepsilon,\,\widehat{N})$,

the mesh $\omega^{\,\varepsilon,\,N}$, in general, does not have common points with the mesh $\omega^{\,\varepsilon,\,\widehat{N}}$.

In this case, for $t = t_n$, $t_n \in \omega_0$, using the function $z^{\,\varepsilon,\,\widehat{N},\,\widehat{N}_0}$ in the node $x_j \in \overline{\omega}^{\,\varepsilon,\,\widehat{N}}$, we find the value of the quadratic interpolant in x at the point $x_i \in \overline{\omega}^{\,\varepsilon,\,N}$. Here we use the following rule. When the point $x_i \in d^{\,l}$, where $d^{\,l} = [0, \sigma(\varepsilon,\,\widehat{N})] \cup [1 - \sigma(\varepsilon,\,\widehat{N}), 1]$ is the $\sigma(\varepsilon,\,\widehat{N})$-neighbourhood of ends to the segment $[0,1]$, we chose interpolation points according to the condition $x_j, x_{j+1}, x_{j+2} \in d^{\,l}$; when $x_i \in d^{\,e}$, where $d^{\,e} = [0,1] \setminus d^{\,l}$, we chose such points according to the condition $x_j, x_{j+1}, x_{j+2} \in d^{\,e}$. Such a choice of the nodes x_j, x_{j+1}, x_{j+2} allows us, using three equidistant nodes, to construct an interpolant on the set $[x_j, x_{j+2}]$ with good approximating properties. By virtue of the found value $\overline{z}^{\,\varepsilon,\,\widehat{N},\,\widehat{N}_0}_{i,n}$, we compute the standard total error $E^{\varepsilon,\,N,\,N_0}_{i,n}$ by formula (4.2a). In a similar way, we compute the standard fractional errors $\widetilde{E}^{\,\varepsilon,\,N,\,\widehat{N}_0}_{i,n;x\,(4.2\mathrm{b})}$, $\widetilde{E}^{\,\varepsilon,\,\widehat{N},\,N_0}_{i,n;t\,(4.2\mathrm{c})}$.

5. Analysis of standard errors for the solution of a model problem and for solution components

Here we consider a "standard approach" to study errors in solutions of difference schemes for a model initial-boundary value problem and their regular and singular components.

For numerical experiments, for simplicity, we consider the following model problem with constant coefficients:

$$(5.1)\quad Lu(x,t) \;\equiv\; \varepsilon^2 \frac{\partial^2}{\partial x^2} u(x,t) - \frac{\partial}{\partial t} u(x,t) - u(x,t) = f(x,t), \quad (x,t) \in G,$$

$$u(x,t) \;=\; \varphi(x,t), \quad (x,t) \in S.$$

For problem (5.1), we use the difference scheme

$$(5.2)\quad \Lambda z(x,t) \;\equiv\; \varepsilon^2 \delta_{\overline{x}\widehat{x}} z(x,t) - \delta_{\overline{t}} z(x,t) - z(x,t) = f(x,t), \quad (x,t) \in G_h,$$

$$z(x,t) \;=\; \varphi(x,t), \quad (x,t) \in S_h.$$

We consider problem (5.1) and difference scheme (5.2) under the condition

$$f(x,t) = -(x-0.2)^4 + 12\,\varepsilon^2(x-0.2)^2 - 3\,(t-0.4)^2 - (t-0.4)^3, \quad (x,t) \in G, \tag{5.3}$$
$$\varphi(x,t) = (x-0.2)^4 + (t-0.4)^3 + t^4, \quad (x,t) \in S.$$

The computed solution $z(x,t)$ of problem (5.2), (5.3) for $\varepsilon = 2^{-5}$, $N = N_0 = 64$ and its zoom in a neighbourhood of the boundary layer is given on Fig. 1

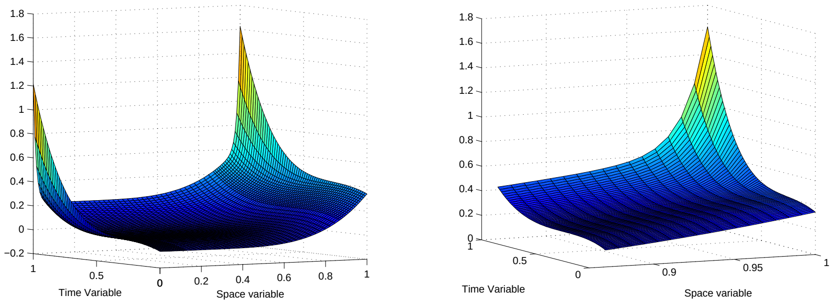

FIGURE 1. Computed solution $z(x,t)$ for $\varepsilon = 2^{-5}$, $N = N_0 = 64$ and its zoom in the right layer region $[1-\sigma, 1]$.

The solution $u(x,t)$ is represented as the sum of the functions (3.9), where the regular component is the solution of problem (5.1) under the condition

$$f(x,t) = -(x-0.2)^4 + 12\,\varepsilon^2(x-0.2)^2 - 3\,(t-0.4)^2 - (t-0.4)^3, \quad (x,t) \in G, \tag{5.4}$$
$$\varphi(x,t) = (x-0.2)^4 + (t-0.4)^3, \quad (x,t) \in S,$$

and the singular component $V(x,t)$ is the solution of problem (5.1) under the condition

$$f(x,t) = 0, \quad (x,t) \in G; \qquad \varphi(x,t) = t^4, \quad (x,t) \in S. \tag{5.5}$$

The regular component of the solution and its zoom in a neighbourhood of the right boundary layer are given on Fig. 2.

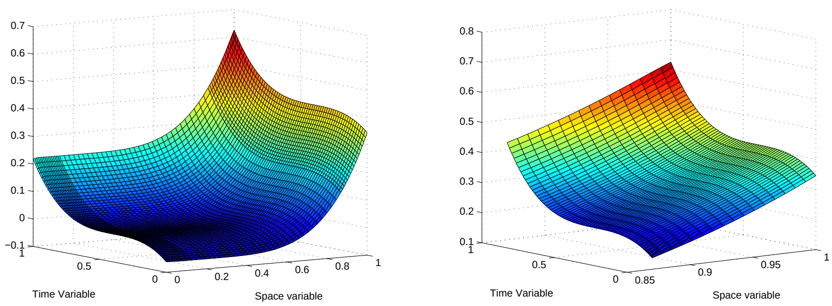

FIGURE 2. Regular component $U(x,t)$ and its layer region $[1-\sigma, 1]$ for $\varepsilon = 2^{-5}$ and $N = N_0 = 64$.

For the function $u(x,t)$, using the "exact" solution $\overline{z}_u^{\,\varepsilon,\,\widehat{N},\,\widehat{N}_0}$ with $\widehat{N} = \widehat{N}_0 = 1024$, we compute the standard total errors and orders of the convergence rate

$$E_u^{\varepsilon,\,N,\,N_0}, \quad r_u^{\varepsilon,\,N,\,N_0}; \quad E_u^{N,\,N_0}, \quad r_u^{N,\,N_0}; \tag{5.6}$$

the standard fractional errors in x and their convergence orders

$$\widetilde{E}_{u\,x}^{\varepsilon,\,N,\,\widehat{N}_0},\quad \widetilde{r}_{u\,x}^{\varepsilon,\,N,\,\widehat{N}_0};\quad \widetilde{E}_{u\,x}^{N,\,\widehat{N}_0},\quad \widetilde{r}_{u\,x}^{N,\,\widehat{N}_0}; \tag{5.7}$$

and also the standard fractional errors in t and their convergence orders

$$\widetilde{E}_{u\,t}^{\varepsilon,\,\widehat{N},\,N_0},\quad \widetilde{r}_{u\,t}^{\varepsilon,\,\widehat{N},\,N_0};\quad \widetilde{E}_{u\,t}^{\widehat{N},\,N_0},\quad \widetilde{r}_{u\,t}^{\widehat{N},\,N_0}. \tag{5.8}$$

The results of numerical experiments for (5.6), (5.7), (5.8) are given in Tables 1, 2, 3, respectively.

TABLE 1. Standard total errors $E_u^{\varepsilon,\,N,\,N_0}$, $E_u^{N,\,N_0}$ and the convergence orders $r_u^{\varepsilon,\,N,\,N_0}$, $r_u^{N,\,N_0}$ for $u = u_{(5.3)}$.

	N=32	N=64	N=128	N=256	N=512
	N_0=32	N_0=64	N_0=128	N_0=256	N_0=512
$\varepsilon = 2^{-0}$	1.766E-002	8.557E-003	3.996E-003	1.713E-003	5.711E-004
	1.045	1.099	1.222	1.585	
$\varepsilon = 2^{-1}$	2.321E-002	1.118E-002	5.204E-003	2.228E-003	7.421E-004
	1.054	1.103	1.224	1.586	
$\varepsilon = 2^{-2}$	1.894E-002	8.968E-003	4.132E-003	1.759E-003	5.845E-004
	1.078	1.118	1.232	1.590	
$\varepsilon = 2^{-3}$	2.175E-002	9.719E-003	4.310E-003	1.797E-003	5.906E-004
	1.162	1.173	1.262	1.605	
$\varepsilon = 2^{-4}$	2.917E-002	1.289E-002	5.150E-003	1.996E-003	6.305E-004
	1.178	1.324	1.367	1.663	
$\varepsilon = 2^{-5}$	2.913E-002	1.329E-002	5.658E-003	2.230E-003	7.008E-004
	1.132	1.232	1.343	1.670	
…	…	…	…	…	…
$\varepsilon = 2^{-15}$	2.913E-002	1.329E-002	5.658E-003	2.230E-003	7.008E-004
	1.132	1.232	1.343	1.670	
E_u^{N,N_0}	2.917E-002	1.329E-002	5.658E-003	2.230E-003	7.421E-004
r_u^{N,N_0}	1.134	1.232	1.343	1.587	

TABLE 2. Standard fractional errors $\widetilde{E}_{u\,x}^{\varepsilon,\,N,\,\widehat{N}_0}$, $\widetilde{E}_{u\,x}^{N,\,\widehat{N}_0}$ and the convergence orders $\widetilde{r}_{u\,x}^{\varepsilon,\,N,\widehat{N}_0}$ in space for $u = u_{(5.3)}$

	N=32	N=64	N=128	N=256	N=512
	$\widehat{N}_0 = 1024$	$\widehat{N}_0 = 1024$	$\widehat{N}_0 = 1024$	$\widehat{N}_0 = 1024$	$\widehat{N}_0 = 1024$
$\varepsilon = 2^{-0}$	3.297E-004	8.220E-005	2.031E-005	4.836E-006	9.671E-007
	2.004	2.017	2.070	2.322	
$\varepsilon = 2^{-1}$	5.721E-004	1.427E-004	3.527E-005	8.397E-006	1.680E-006
	2.003	2.017	2.070	2.322	
$\varepsilon = 2^{-2}$	1.256E-003	3.145E-004	7.779E-005	1.853E-005	3.706E-006
	1.998	2.015	2.070	2.322	
$\varepsilon = 2^{-3}$	4.776E-003	1.212E-003	3.008E-004	7.176E-005	1.436E-005
	1.978	2.011	2.067	2.321	
$\varepsilon = 2^{-4}$	1.361E-002	4.752E-003	1.195E-003	2.859E-004	5.730E-005
	1.518	1.991	2.064	2.319	
$\varepsilon = 2^{-5}$	1.357E-002	5.079E-003	1.728E-003	5.288E-004	1.312E-004
	1.417	1.555	1.709	2.011	
…	…	…	…	…	…
$\varepsilon = 2^{-15}$	1.357E-002	5.079E-003	1.728E-003	5.288E-004	1.312E-004
	1.417	1.555	1.709	2.011	
$\widetilde{E}_{u\,x}^{N,\,\widehat{N}_0}$	1.361E-002	5.079E-003	1.728E-003	5.288E-004	1.312E-004
$\widetilde{r}_{u\,x}^{N,\widehat{N}_0}$	1.422	1.555	1.709	2.011	

Discuss these results. It turns out that the standard total errors $E_u^{\varepsilon,\,N,\,N_0}$, $E_u^{N,\,N_0}$ given in Table 1 provide little information; they do not reveal the actual errors

TABLE 3. Standard fractional errors $\widetilde{E}_{u\,t}^{\varepsilon,\,\widehat{N},\,N_0}$, $\widetilde{E}_{u\,t}^{\widehat{N},\,N_0}$ and the convergence orders $\widetilde{r}_{u\,t}^{\varepsilon,\,\widehat{N},N_0}$ in time for $u = u_{(5.3)}$.

	$\widehat{N}=1024$	$\widehat{N}=1024$	$\widehat{N}=1024$	$\widehat{N}=1024$	$\widehat{N}=1024$
	N_0=32	N_0=64	N_0=128	N_0=256	N_0=512
$\varepsilon=2^{-0}$	1.734E-002	8.475E-003	3.976E-003	1.708E-003	5.702E-004
	1.032	1.092	1.219	1.583	
$\varepsilon=2^{-1}$	2.266E-002	1.104E-002	5.169E-003	2.219E-003	7.404E-004
	1.037	1.095	1.220	1.584	
$\varepsilon=2^{-2}$	1.787E-002	8.683E-003	4.061E-003	1.742E-003	5.811E-004
	1.041	1.096	1.221	1.584	
$\varepsilon=2^{-3}$	1.774E-002	8.626E-003	4.035E-003	1.731E-003	5.774E-004
	1.041	1.096	1.221	1.584	
$\varepsilon=2^{-4}$	1.774E-002	8.626E-003	4.035E-003	1.731E-003	5.774E-004
	1.041	1.096	1.221	1.584	
$\varepsilon=2^{-5}$	1.774E-002	8.625E-003	4.034E-003	1.731E-003	5.774E-004
	1.041	1.096	1.221	1.584	
...	...	...	...	...	...
$\varepsilon=2^{-15}$	1.774E-002	8.625E-003	4.034E-003	1.731E-003	5.774E-004
	1.041	1.096	1.221	1.584	
$\widetilde{E}_{u\,t}^{\widehat{N},\,N_0}$	2.266E-002	1.104E-002	5.169E-003	2.219E-003	7.404E-004
$\widetilde{r}_{u\,t}^{\widehat{N},N_0}$	1.037	1.095	1.220	1.584	

$\vartheta_{(4.1)}^{\varepsilon,\,N,\,N_0}$. Note that the standard total errors $E_u^{\varepsilon,\,N,\,N_0}$ in Table 1 are of order the sum of the standard fractional errors $\widetilde{E}_{u\,x}^{\varepsilon,\,N,\,\widehat{N}_0}$ and $\widetilde{E}_{u\,t}^{\varepsilon,\,\widehat{N},\,N_0}$ given in Table 2 and Table 3, respectively. But, the behaviour of these fractional errors is sufficiently strange. Indeed, the orders of the convergence rate $\widetilde{E}_{u\,x}^{\varepsilon,\,N,\,\widehat{N}_0}$ of the standard fractional errors in x are not less than 1.4, they increase with the growth of N, they are not less than 2.0 for $N_0 = 256$, and they are not less than 2.3 for $\varepsilon \geq 2^{-4}$. The orders of the convergence rate $\widetilde{E}_{u\,t}^{\varepsilon,\,\widehat{N},\,N_0}$ of the standard fractional errors in t are not less than 1.0, they increase as N becomes larger, and they are not less than 1.5 for $N_0 = 256$. The convergence orders of the standard fractional errors in x and t are not compatible with the theoretical results, i.e., with the estimates (3.7) and (3.8).

Thus, the results of Tables 1, 2 and 3 are unavailable to study the actual errors in discrete solutions of difference schemes on piecewise-uniform grids.

We now consider the standard fractional errors in x and in t and their corresponding orders of the convergence rate

- for the regular component $U(x,t)$

$$\widetilde{E}_{U\,x}^{\varepsilon,\,N,\,\widehat{N}_0},\quad \widetilde{r}_{U\,x}^{\varepsilon,\,N,\,\widehat{N}_0};\qquad \widetilde{E}_{U\,x}^{N,\,\widehat{N}_0},\quad \widetilde{r}_{U\,x}^{N,\,\widehat{N}_0};\tag{5.9}$$

$$\widetilde{E}_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0},\quad \widetilde{r}_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0};\qquad \widetilde{E}_{U\,t}^{\widehat{N},\,N_0},\quad \widetilde{r}_{U\,t}^{\widehat{N},\,N_0};\tag{5.10}$$

- and for the singular component $V(x,t)$

$$\widetilde{E}_{V\,x}^{\varepsilon,\,N,\,\widehat{N}_0},\ \widetilde{r}_{V\,x}^{\varepsilon,\,N,\,\widehat{N}_0};\quad \widetilde{E}_{V\,x}^{N,\,\widehat{N}_0},\ \widetilde{r}_{V\,x}^{N,\,\widehat{N}_0};\quad \widetilde{E}_{V\,t}^{\varepsilon,\,\widehat{N},\,N_0},\ \widetilde{r}_{V\,t}^{\varepsilon,\,\widehat{N},\,N_0};\quad \widetilde{E}_{V\,t}^{\widehat{N},\,N_0},\ \widetilde{r}_{V\,t}^{\widehat{N},\,N_0}.$$

The behaviour of these standard fractional errors and their convergence orders for the regular and singular solution components are also not compatible with the theoretical results, i.e., with the estimates (3.11), (3.12).

Thus, the standard fractional errors for the regular and singular solution components are also unavailable to study the actual errors in the discrete solutions of difference schemes on piecewise-uniform grids.

6. Regularized fractional and total errors

In this section we introduce errors of a new type, and we call them *regularized* errors. We introduce regularized fractional and total errors and show that the regularized errors reflect well the behaviour of the actual errors for the functions $u(x,t)$, $U(x,t)$ and $V(x,t)$.

6.1. Technics for computation of regularized fractional errors and orders of their convergence rate. On the basis of the standard fractional errors in x and t for the functions $u(x,t)$, $U(x,t)$ and $V(x,t)$, it is possible to construct regularized fractional errors in x and t. In this subsection we show how to compute the regularized fractional errors in t for the regular component $U(x,t)$ using the standard fractional errors $\widetilde{E}_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0}$ in (5.10).

Set $N_0^{(j)} = 32 \times 2^{j-1}$, $j = 1, \ldots, J$, and $N_0^{(J)} = \widehat{N}_0$ (in the tables $J = 6$). In a similar way, we introduce $N^{(j)}$, $j = 1, \ldots, J$.

Let, in the case of problem (5.1), (5.4), the function $v^{\varepsilon,\,\widehat{N}}(x,t) = v_U^{\varepsilon,\,\widehat{N}}(x,t)$, $(x,t) \in \overline{G}_{h\,x}$, be a solution of the method of lines in t

$$(6.1)\quad \begin{cases} \Lambda_x v^{\varepsilon,\,\widehat{N}}(x,t) \equiv \{\varepsilon^2 \delta_{\widehat{x}\overline{x}} - \dfrac{\partial}{\partial t} - 1\} v^{\varepsilon,\,\widehat{N}}(x,t) = f(x,t), & (x,t) \in G_{h\,x}, \\ v^{\varepsilon,\,\widehat{N}}(x,t) = \varphi(x,t), & (x,t) \in S_{h\,x}. \end{cases}$$

where $\overline{G}_{h\,x} = \overline{G}_{h\,x}^{\varepsilon,\,\widehat{N}} = \overline{\omega}_x^{\varepsilon,\,\widehat{N}} \times [0,1]$, $G_{h\,x} = \overline{G}_{h\,x} \cap G$, $S_{h\,x} = \overline{G}_{h\,x} \cap S$ and $\overline{\omega}_x^{\varepsilon,\,\widehat{N}}$ is a piecewise-uniform mesh in x. Introduce

$$(6.2a)\qquad e_t^{\varepsilon,\widehat{N},N_0} = e_{U\,t}^{\varepsilon,\widehat{N},N_0} = \max_{(x,t)\in\overline{G}_h^{\varepsilon,\widehat{N},N_0}} \big|\, e_t^{\varepsilon,\widehat{N},N_0}(x,t) \,\big|,$$

where

$$(6.2b)\qquad e_t^{\varepsilon,\widehat{N},N_0}(x,t) = z^{\varepsilon,\widehat{N},N_0}(x,t) - v^{\varepsilon,\widehat{N}}(x,t), \quad (x,t) \in \overline{G}_h^{\varepsilon,\widehat{N},N_0}.$$

We call the value $e_t^{\varepsilon,\widehat{N},N_0}$ the regularized fractional error in t. According to (6.2b), we have

$$(6.3)\qquad \widetilde{E}_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}} = \max_{(x,t)\in\overline{G}_h^{\varepsilon,\widehat{N},N_0^{(j)}}} |e_t^{\varepsilon,\,\widehat{N},N_0^{(j)}}(x,t) - \overline{e}_t^{\varepsilon,\,\widehat{N},N_0^{(J)}}(x,t)|.$$

Assume that the following condition holds:

$$(6.4)\qquad \max_{(x,t)\in\overline{G}_h^{\varepsilon,\widehat{N},N_0^{(j)}}} |e_t^{\varepsilon,\,\widehat{N},N_0^{(j)}}(x,t) - \overline{e}_t^{\varepsilon,\,\widehat{N},N_0^{(J)}}(x,t)| = e_t^{\varepsilon,\,\widehat{N},N_0^{(j)}} - \overline{e}_t^{\varepsilon,\,\widehat{N},N_0^{(J)}},$$

and the regularized fractional error in t has the following form:

$$(6.5)\qquad e_t^{\varepsilon,\,\widehat{N},N_0^{(j)}} = M\, N_0^{(j)-\overline{q}}, \quad j = 1, \ldots, J,$$

where $\overline{q} = \overline{q}_{(6.5)} = \overline{q}_t$ is the local fractional order of the convergence rate to the error in t; $M = M_t^{\varepsilon,\,\widehat{N},\,N_0^{(j)}} = M_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}}$ is a local constant, $\overline{q} = \overline{q}_t^{\varepsilon,\,\widehat{N},\,N_0^{(j)}} = \overline{q}_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}}$, which are depend on j rather weakly. Then for the error $\widetilde{E}_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}}$ we get the following representation:

$$(6.6)\qquad \widetilde{E}_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}} = M\,(N_0^{(j)-\overline{q}} - N_0^{(J)-\overline{q}}), \quad j = 1, \ldots, J-1.$$

From this we have

$$(6.7)\quad \alpha_j^{\varepsilon|\widehat{N}} \equiv \widetilde{E}_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}} \,/\, \widetilde{E}_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j+1)}} = 2^{\overline{q}} \times \frac{1-2^{-(J-j)\overline{q}}}{1-2^{-(J-1-j)\overline{q}}}, \quad j=1,\ldots,J-2.$$

In the case when the values $\widetilde{E}_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}}$ and $\widetilde{E}_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j+1)}}$ are known, the relation (6.7) allows us to find $\overline{q} = \overline{q}_{(6.5)}$, and, with this purpose, we solve with an iterative process the following recurrent relation:

$$(6.8)\qquad \overline{q}_{i+1} = (\ln 2)^{-1} \times \ln\left\{\alpha_j^{\varepsilon|\widehat{N}} \times \frac{1-2^{-(J-1-j)\overline{q}_i}}{1-2^{-(J-j)\overline{q}_i}}\right\}, \quad i=0,1,2,\ldots.$$

We set the value $\overline{q}_0$ by the relation

$$(6.9)\qquad \overline{q}_0 = (\ln 2)^{-1} \times \ln \alpha_j^{\varepsilon|\widehat{N}}.$$

The iterative process is stopped when $|\overline{q}_i - \overline{q}_{i+1}| \le tol$, where tol is a sufficiently small value (in our numerical experiments, we have taken $tol = 10^{-2}$). In this case we set

$$\overline{q} = \overline{q}_{(6.5)} = \overline{q}_{i+1}.$$

Once we dispose of the value $\overline{q} = \overline{q}(\varepsilon,\, j,\, J;\, \widehat{N}) = \overline{q}_t^{\varepsilon,\,\widehat{N},\,N_0^{(j)}}$, $j=1,2,\ldots,J-2$, by formula (6.6), we find the local constant $M = M_t^{\varepsilon,\,\widehat{N},\,N_0^{(j)}}$ in (6.5):

$$M_t^{\varepsilon,\,\widehat{N},\,N_0^{(j)}} = \widetilde{E}_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}} / (N_0^{(j)-\overline{q}} - N_0^{(J)-\overline{q}}), \quad j=1,\ldots,J-2.$$

Further, by formula (6.5), we find the value itself $e_t^{\varepsilon,\,\widehat{N},N_0^{(j)}}$, i.e., the regularized fractional error in t for a given value ε

$$e_t^{\varepsilon,\,\widehat{N},N_0^{(j)}} = e_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}} = M_t^{\varepsilon,\,\widehat{N},\,N_0^{(j)}}\; N_0^{(j)-\overline{q}}, \quad j=1,\ldots,J-2.$$

when we have computed the values $e_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}}$, $j=1,2,,...,J-2$, for each ε from 2^0 to 2^{-15}, we find the ε-uniform regularized fractional errors

$$e_{U\,t}^{\widehat{N},\,N_0^{(j)}} = \max_{\varepsilon} e_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}}, \quad j=1,2,,...,J-2,$$

and also the ε-dependent and ε-uniform orders of the convergence rate for the regularized fractional errors in t

$$(6.10)\quad q_{U\,t}^{\varepsilon,\widehat{N},N_0} = \log\left(\frac{e_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}}}{e_{U\,t}^{\varepsilon,\,\widehat{N},\,2\,N_0^{(j)}}}\right) / \log 2, \quad q_{U\,t}^{\widehat{N},N_0} = \log\left(\frac{e_{U\,t}^{\widehat{N},\,N_0^{(j)}}}{e_{U\,t}^{\widehat{N},\,2\,N_0^{(j)}}}\right) / \log 2.$$

Thus, for the regular component $U(x,t)$, we obtain the regularized fractional errors in t and the orders of the convergence rate

$$(6.11)\qquad e_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0},\;\; q_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0};\qquad e_{U\,t}^{\widehat{N},\,N_0},\;\; q_{U\,t}^{\widehat{N},\,N_0}.$$

In a similar way, the regularized fractional errors in t and their orders of the convergence rate are constructed for the singular components $V(x,t)$

$$(6.12)\qquad e_{V\,t}^{\varepsilon,\,\widehat{N},\,N_0},\;\; q_{V\,t}^{\varepsilon,\,\widehat{N},\,N_0};\qquad e_{V\,t}^{\widehat{N},\,N_0},\;\; q_{V\,t}^{\widehat{N},\,N_0}.$$

Using the data from (6.11) and (6.12), we find the regularized fractional errors in t for the solution $u(x,t)$ by the relations

$$(6.13)\qquad e_{u\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}} = e_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}} + e_{V\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}}, \quad e_{u\,t}^{\widehat{N},\,N_0^{(j)}} = \max_{\varepsilon} e_{u\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}}$$

and their orders of the convergence rate $q_{u\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(j)}}$, $q_{u\,t}^{\widehat{N},\,N_0^{(j)}}$ by formulas similar to (6.10). The results of the numerical experiments for (6.13) are given in Table 4.

TABLE 4. Regularized fractional errors $e_{u\,t}^{\varepsilon,\,\widehat{N},\,N_0}$, $e_{u\,t}^{\widehat{N},N_0}$ and the convergence orders $q_{u\,t}^{\varepsilon,\,\widehat{N},\,N_0}$, $q_{u\,t}^{\widehat{N},\,N_0}$ in time for $u = u_{(5.3)}$.

	$\widehat{N} = 1024$	$\widehat{N} = 1024$	$\widehat{N} = 1024$	$\widehat{N} = 1024$
	$N_0 = 32$	$N_0 = 64$	$N_0 = 128$	$N_0 = 256$
$\varepsilon = 2^{-0}$	1.793E-002 0.985	9.055E-003 0.991	4.547E-003 0.998	2.279E-003
$\varepsilon = 2^{-1}$	2.341E-002 0.990	1.179E-002 0.994	5.909E-003 0.999	2.959E-003
$\varepsilon = 2^{-2}$	1.846E-002 0.994	9.268E-003 0.997	4.640E-003 1.001	2.322E-003
$\varepsilon = 2^{-3}$	1.833E-002 0.993	9.207E-003 0.996	4.611E-003 1.000	2.308E-003
$\varepsilon = 2^{-4}$	1.833E-002 0.993	9.207E-003 0.996	4.611E-003 1.000	2.308E-003
$\varepsilon = 2^{-5}$	1.833E-002 0.993	9.206E-003 0.996	4.610E-003 1.000	2.307E-003
...	...	...	...	...
$\varepsilon = 2^{-15}$	1.833E-002 0.993	9.206E-003 0.996	4.610E-003 1.000	2.307E-003
$e_{u\,t}^{\widehat{N},N_0}$ $q_{u\,t}^{\widehat{N},\,N_0}$	2.341E-002 0.990	1.179E-002 0.996	5.909E-003 0.998	2.959E-003

As it follows from the experimental results, the regularized fractional errors in t for the function $u(x,t)$ are the sum of the errors for the components $U(x,t)$ and $V(x,t)$, moreover, the fractional errors for $U(x,t)$ and $V(x,t)$ are of the same order. In Table 4, the orders of the convergence rate for the regularized fractional errors in t are close to one. This agrees with the theoretical results, i.e., with the estimates (3.7), (3.8) for the solution of the initial-boundary value problem and with the estimates (3.11), (3.12) for the regular and singular components.

6.2. Regularized fractional errors and their orders of the convergence rate in x. Similarly to the construction of the regularized fractional errors and the convergence orders in t for the functions $U(x,t)$, $V(x,t)$ and $u(x,t)$, we construct the regularized fractional errors in x

$$e_{U\,x}^{\varepsilon,\,N,\,\widehat{N}_0}, \quad q_{U\,x}^{\varepsilon,\,N,\,\widehat{N}_0}; \quad e_{U\,x}^{N,\,\widehat{N}_0}, \quad q_{U\,x}^{N,\,\widehat{N}_0}; \tag{6.14a}$$

$$e_{V\,x}^{\varepsilon,\,N,\,\widehat{N}_0}, \quad q_{V\,x}^{\varepsilon,\,N,\,\widehat{N}_0}; \quad e_{V\,x}^{N,\,\widehat{N}_0}, \quad q_{V\,x}^{N,\,\widehat{N}_0}; \tag{6.14b}$$

$$e_{u\,x}^{\varepsilon,\,N,\,\widehat{N}_0}, \quad q_{u\,x}^{\varepsilon,\,N,\,\widehat{N}_0}; \quad e_{u\,x}^{N,\,\widehat{N}_0}, \quad q_{u\,x}^{N,\,\widehat{N}_0}. \tag{6.14c}$$

The results of numerical experiments for (6.14c) are given for $u = u_{(5.3)}$ in Table 5.

From Table 5 it follows that the regularized fractional error in x for the function $u(x,t)$ converges with order higher than one, and the order increases with growth of N for $\varepsilon \le 2^{-5}$, moreover, for $\varepsilon > 2^{-5}$ the order of the convergence rate is close to two. These results agree well with the theoretical results, i.e., with the estimates (3.7), (3.8) for the solution of the problem and with the estimates (3.11), (3.12) for the regular and singular components.

TABLE 5. Regularized fractional errors $e_{u\,x}^{\varepsilon,\,N,\,\widehat{N}_0}$, $e_{u\,x}^{N,\widehat{N}_0}$ and the convergence orders $q_{u\,x}^{\varepsilon,\,N,\,\widehat{N}_0}$, $q_{u\,x}^{N,\,\widehat{N}_0}$ in space for $u = u_{(5.3)}$.

	N=32	N=64	N=128	N=256
	$\widehat{N}_0 = 1024$	$\widehat{N}_0 = 1024$	$\widehat{N}_0 = 1024$	$\widehat{N}_0 = 1024$
$\varepsilon = 2^{-0}$	3.301E-004 2.000	8.252E-005 2.000	2.063E-005 2.000	5.157E-006
$\varepsilon = 2^{-1}$	5.727E-004 1.999	1.433E-004 2.000	3.583E-005 2.000	8.955E-006
$\varepsilon = 2^{-2}$	1.257E-003 1.994	3.158E-004 1.998	7.902E-005 2.000	1.976E-005
$\varepsilon = 2^{-3}$	4.781E-003 1.974	1.217E-003 1.993	3.056E-004 1.997	7.653E-005
$\varepsilon = 2^{-4}$	1.368E-002 1.519	4.773E-003 1.973	1.215E-003 1.993	3.050E-004
$\varepsilon = 2^{-5}$	1.367E-002 1.407	5.157E-003 1.514	1.793E-003 1.595	5.933E-004
...	...	...	...	...
$\varepsilon = 2^{-15}$	1.367E-002 1.407	5.157E-003 1.514	1.793E-003 1.595	5.932E-004
$e_{u_x}^{N,\widehat{N}_0}$ $q_{u_x}^{N,\,\widehat{N}_0}$	1.368E-002 1.408	5.157E-003 1.524	1.793E-003 1.596	5.933E-004

6.3. Regularized total errors. For the regular and singular components $U(x,t)$ and $V(x,t)$, using the regularized fractional errors in x and t from (6.14a), (6.14b), we introduce the ε-dependent regularized total errors

$$e_{U\,x,t}^{\varepsilon,\,N^{(jx)},\,N_0^{(jt)}} = e_{U\,x}^{\varepsilon,\,N^{(jx)},\,\widehat{N}_0} + e_{U\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(jt)}}, \tag{6.15a}$$

$$e_{V\,x,t}^{\varepsilon,\,N^{(jx)},\,N_0^{(jt)}} = e_{V\,x}^{\varepsilon,\,N^{(jx)},\,\widehat{N}_0} + e_{V\,t}^{\varepsilon,\,\widehat{N},\,N_0^{(jt)}}; \tag{6.15b}$$

ε-uniform regularized total errors

$$e_{U\,x,t}^{N^{(jx)},\,N_0^{(jt)}} = \max_{\varepsilon} e_{U\,x,t}^{\varepsilon,\,N^{(jx)},\,N_0^{(jt)}}, \quad e_{V\,x,t}^{N^{(jx)},\,N_0^{(jt)}} = \max_{\varepsilon} e_{V\,x,t}^{\varepsilon,\,N^{(jx)},\,N_0^{(jt)}}; \tag{6.15c}$$

and the orders of the convergence rate. Thus, we have

$$e_{U\,x,t}^{\varepsilon,\,N^{(j)},\,N_0^{(j)}}, \quad q_{U\,x,t}^{\varepsilon,\,N^{(j)},\,N_0^{(j)}}; \quad e_{U\,x,t}^{N^{(j)},\,N_0^{(j)}}, \quad q_{U\,x,t}^{N^{(j)},\,N_0^{(j)}}; \tag{6.16a}$$

$$e_{V\,x,t}^{\varepsilon,\,N^{(j)},\,N_0^{(j)}}, \quad q_{V\,x,t}^{\varepsilon,\,N^{(j)},\,N_0^{(j)}}; \quad e_{V\,x,t}^{N^{(j)},\,N_0^{(j)}}, \quad q_{V\,x,t}^{N^{(j)},\,N_0^{(j)}}. \tag{6.16b}$$

For the solution $u(x,t)$ of the initial-boundary value problem, taking into account (6.16), we define the regularized total errors by the relations similar to (6.13):

$$e_{u\,x,t}^{\varepsilon,\,N^{(j)},\,N_0^{(j)}} = e_{U\,x,t}^{\varepsilon,\,N^{(j)},\,N_0^{(j)}} + e_{V\,x,t}^{\varepsilon,\,N^{(j)},\,N_0^{(j)}}, \quad e_{u\,x,t}^{N^{(j)},\,N_0^{(j)}} = \max_{\varepsilon} e_{u\,x,t}^{\varepsilon,\,N^{(j)},\,N_0^{(j)}}, \tag{6.17}$$

and, by formulas similar to (6.10), we find their orders of the convergence rate $q_{u\,x,t}^{\varepsilon,\,N^{(j)},\,N_0^{(j)}}$, $q_{u\,x,t}^{N^{(j)},\,N_0^{(j)}}$. Thus, we get

$$e_{u\,x,t}^{\varepsilon,\,N^{(j)},\,N_0^{(j)}}, \quad q_{u\,x,t}^{\varepsilon,\,N^{(j)},\,N_0^{(j)}}; \quad e_{u\,x,t}^{N^{(j)},\,N_0^{(j)}}, \quad q_{u\,x,t}^{N^{(j)},\,N_0^{(j)}}. \tag{6.18}$$

The results of numerical experiments to (6.17), (6.18) for $u(x,t)$ are given in Table 6.

According to Table 6, the regularized total error has the convergence order close to one or a few larger than one, and it agrees well with the theoretical results, i.e., with the estimates (3.7), (3.8) for the solution of the initial-boundary value problem and with the estimates (3.11) and (3.12) for the regular and singular components.

TABLE 6. Regularized total errors $e_{u\,x,t}^{\varepsilon,\,N,\,N_0}$, $e_{u\,x,t}^{N,N_0}$ and the convergence orders $q_{u\,x,t}^{\varepsilon,\,N,\,N_0}$, $q_{u\,x,t}^{N,\,N_0}$ for $u = u_{(5.3)}$.

	N=32	N=64	N=128	N=256
	$N_0 = 32$	$N_0 = 64$	$N_0 = 128$	$N_0 = 256$
$\varepsilon = 2^{-0}$	1.826e-002 0.999	9.137e-003 1.000	4.567e-003 1.000	2.284e-003
$\varepsilon = 2^{-1}$	2.399E-002 1.008	1.193E-002 1.005	5.945E-003 1.002	2.968E-003
$\varepsilon = 2^{-2}$	1.972E-002 1.041	9.584E-003 1.022	4.719E-003 1.011	2.342E-003
$\varepsilon = 2^{-3}$	2.311E-002 1.149	1.042E-002 1.084	4.916E-003 1.044	2.384E-003
$\varepsilon = 2^{-4}$	3.201E-002 1.195	1.398E-002 1.263	5.825E-003 1.157	2.613E-003
$\varepsilon = 2^{-5}$	3.200E-002 1.156	1.436E-002 1.165	6.403E-003 1.143	2.901E-003
...	...	...	...	...
$\varepsilon = 2^{-15}$	3.200E-002 1.156	1.436E-002 1.165	6.403E-003 1.143	2.901E-003
$e_{u,\,x,\,t}^{N,\,N_0}$ $q_{u,\,x,\,t}^{N,\,N_0}$	3.201E-002 1.156	1.436E-002 1.165	6.403E-003 1.110	2.968E-003

6.4. Comparison of the numerical results for standard and regularized errors. Finally, we compare the results of the numerical experiments for the standard fractional and total errors $\widetilde{E}_{u\,x}^{\varepsilon,\,N,\,\widehat{N}_0}$, $\widetilde{E}_{u\,t}^{\varepsilon,\,\widehat{N},\,N_0}$, $E_{u}^{\varepsilon,\,N,\,N_0}$ from Tables 1, 2, 3 and the regularized fractional and total errors $e_{u\,x}^{\varepsilon,\,N,\,\widehat{N}_0}$, $e_{u\,t}^{\varepsilon,\,\widehat{N},\,N_0}$, $e_{u\,x,t}^{\varepsilon,\,N,\,N_0}$ from Table 4, 5, 6, respectively.

Regularized fractional and total errors and their orders of the convergence rate agree well with the theoretical results **for all values** of ε and $N < \widehat{N}$, $N_0 < \widehat{N}_0$. Standard fractional and total errors and their orders of the convergence rate agree with the theoretical results for all values of ε only **under rather restrictive conditions** prescribed to N, N_0, namely, under the condition $N \ll \widehat{N}$, $N_0 \ll \widehat{N}_0$ for standard total errors and their convergence orders, under the conditions $N \ll \widehat{N}$ and $N_0 \ll \widehat{N}_0$ for standard fractional errors in x and t, respectively.

7. Conclusions

1. For singularly perturbed parabolic reaction-diffusion problems, a new approach is elaborated to study numerically special ε-uniformly convergent difference schemes on piecewise-uniform grids. This approach is based on analysis of the regularized errors for discrete solutions using a "standard fine grid technique" for numerical study of the difference schemes.

1.a. The new approach allows us to find the regularized errors of discrete solutions as well as the total errors (in both variables x, t) and the fractional errors (in x and t) generated in the approximation of differential derivatives by grid derivatives.

1.b. The regularized total and fractional errors agree well with the theoretical estimates for actual total and fractional errors and their orders of the convergence rate.

1.c. From the numerical experiments, one can see that the regularized fractional errors of the discrete solution converge with order close to one in t and with order higher than one in x, moreover, the order of the convergence rate in x increases (but does not exceed 2) as the number of nodes in the spatial mesh grows. It should be noted that the regularized total errors are essentially less informative than the fractional errors.

2. It is also shown that the "standard" technique turns out inefficient for numerical study of the difference schemes. This technique brings to large errors already when estimating the total actual error.

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