INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 11, Number 2, Pages 385–399

PARAMETER-UNIFORM CONVERGENCE FOR A FINITE DIFFERENCE METHOD FOR A SINGULARLY PERTURBED LINEAR REACTION-DIFFUSION SYSTEM WITH DISCONTINUOUS SOURCE TERMS

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(Communicated by C. Rodrigo)

This paper is dedicated to Professor Francisco J. Lisbona on his 65th birthday

Abstract. A singularly perturbed linear system of second order ordinary differential equations of reaction-diffusion type with discontinuous source terms is considered. A small positive parameter multiplies the leading term of each equation. These singular perturbation parameters are assumed to be distinct. The components of the solution exhibit overlapping boundary and interior layers. A numerical method is constructed that uses a classical finite difference scheme on a piecewise uniform Shishkin mesh. It is proved that the numerical approximations obtained by this method are essentially first order convergent uniformly with respect to all of the perturbation parameters. Numerical illustrations are presented in support of the theory.

Key words. Singular perturbation problems, system of differential equations, reaction - diffusion equations, discontinuous source terms, overlapping boundary and interior layers, classical finite difference scheme, Shishkin mesh, parameter - uniform convergence.

1. Introduction

A singularly perturbed linear system of second order ordinary differential equations of reaction - diffusion type with discontinuous source terms is considered in the interval $\Omega = \{x : 0 < x < 1\}$. A single discontinuity in the source terms is assumed to occur at a point $d \in \Omega$. Introduce the notation $\Omega^- = (0, d)$, $\overline{\Omega^-} = [0, d]$, $\Omega^+ = (d, 1)$, $\overline{\Omega^+} = [d, 1]$ and denote the jump at d in any function $\vec{\omega}$ by $[\vec{\omega}](d) = \vec{\omega}(d+) - \vec{\omega}(d-)$. The corresponding self-adjoint two point boundary value problem is

$$(1)\quad -E\vec{u}''(x) + A(x)\vec{u}(x) = \vec{f}(x) \text{ on } \Omega^- \cup \Omega^+,\ \vec{u} \text{ given on } \Gamma \text{ and } \vec{f}(d-) \neq \vec{f}(d+)$$

where $\Gamma = \{0, 1\}$, $\overline{\Omega} = \Omega \cup \Gamma$. The norms $\| \vec{V} \| = \max_{1 \le k \le n} |V_k|$ for any n-vector $\vec{V}$, $\| y \| = \sup_{0 \le x \le 1} |y(x)|$ for any scalar-valued function y and $\| \vec{y} \| = \max_{1 \le k \le n} \| y_k \|$ for any vector-valued function $\vec{y}$ are introduced. Here $\vec{u}$ is a column $n-$vector, E and $A(x)$ are $n \times n$ matrices, $E = \mathrm{diag}(\vec{\varepsilon})$, $\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n)$ with $0 < \varepsilon_i \le 1$ for all $i = 1, \ldots, n$. The ε_i are assumed to be distinct and, for convenience, to have the ordering

$$\varepsilon_1 < \cdots < \varepsilon_n.$$

For simplicity, cases with some of the parameters coincident are not considered here. In these cases the number of layer functions is reduced and, consequently, the number of transition parameters in the Shishkin mesh defined in Section 4 is reduced. The methods of proof are essentially the same.

Received by the editors October 31, 2012 and, in revised form, August 4, 2013.
2000 *Mathematics Subject Classification.* 65L10, 65L12, 65L20, 65L70.

The problem can also be written in the operator form

$$\vec{L}\vec{u} \;=\; \vec{f} \text{ on } \Omega^- \cup \Omega^+,\; \vec{u} \text{ given on } \Gamma \text{ and } \vec{f}(d-) \neq \vec{f}(d+)$$

where the operator $\vec{L}$ is defined by

$$\vec{L} \;=\; -ED^2 + A, \quad D^2 = \frac{d^2}{dx^2}.$$

For all $x \in \overline{\Omega}$, it is assumed that the components $a_{ij}(x)$ of $A(x)$ satisfy the inequalities

$$a_{ii}(x) > \sum_{\substack{j \neq i \\ j=1}}^{n} |a_{ij}(x)| \text{ for } 1 \leq i \leq n \text{ and } a_{ij}(x) \leq 0 \text{ for } i \neq j \tag{2}$$

and, for some α,

$$0 < \alpha < \min_{\substack{x \in [0,1] \\ 1 \leq i \leq n}} (\sum_{j=1}^{n} a_{ij}(x)). \tag{3}$$

It is assumed that $a_{ij} \in C^{(2)}(\overline{\Omega})$, $f_i \in C^{(2)}(\Omega^- \cup \Omega^+)$ for $i, j = 1, \ldots, n$. Then (1) has a solution $\vec{u} \in C(\overline{\Omega}) \cap C^{(1)}(\Omega) \cap C^{(4)}(\Omega^- \cup \Omega^+)$. Because $\vec{f}$ is discontinuous at d, the solution $\vec{u}(x)$ does not necessarily have a continuous second order derivative at the point d. Thus $\vec{u}(x) \notin C^2(\Omega)$, but the first derivative of the solution exists and is continuous. In Section 2, for the construction of the solution in Theorem 2.1 and in the definition of the singular component, we need the Schur product of two $n-$ vectors, which is defined by

$$\vec{\mu} \cdot \vec{\eta} \;=\; (\mu_1\eta_1,\, \mu_2\eta_2,\, \ldots,\, \mu_n\eta_n) \in \mathbb{R}^n \tag{4}$$

for $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\vec{\eta} = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{R}^n$. It is also assumed that

$$\sqrt{\varepsilon}_n \leq \frac{\sqrt{\alpha}}{6}. \tag{5}$$

Throughout the paper C denotes a generic positive constant, which is independent of x and of all singular perturbation and discretization parameters. Furthermore, inequalities between vectors are understood in the componentwise sense.

For a general introduction to parameter-uniform numerical methods for singular perturbation problems, see for example [1], [2] and [3]. Parameter-uniform numerical methods for scalar problems with discontinuous data are reported in [4], [5], [6] and [7]. The present paper extends the results in [4] for a single equation to a general system of equations.

The plan of the paper is as follows. In the next two sections, the analytical results of the continuous problem are given. In Section 4 piecewise-uniform Shishkin meshes, which are fitted to resolve the interior and boundary layers, are introduced. In Section 5 the discrete problem is defined and the corresponding maximum principle and stability result are established. In Section 6 the statement and proof of the parameter-uniform error estimate are given. Section 7 contains numerical illustrations.

2. Standard analytical results

Theorem 2.1. *Problem* (1) *has a solution* $\vec{u} \in C(\overline{\Omega}) \cap C^{(1)}(\Omega) \cap C^{(4)}(\Omega^- \cup \Omega^+)$.

Proof. The proof is by construction. Let $\vec{y}_1, \vec{y}_2$ be particular solutions of the differential equations

$$-E\vec{y}_1''(x) + A(x)\vec{y}_1(x) = \vec{f}(x),\ x \in \Omega^-$$

and

$$-E\vec{y}_2''(x) + A(x)\vec{y}_2(x) = \vec{f}(x),\ x \in \Omega^+.$$

Consider the function

$$\vec{y}(x) = \begin{cases} \vec{y}_1(x) + (\vec{u}(0) - \vec{y}_1(0)) \cdot \vec{\phi}_1(x) + \vec{A}_1 \cdot \vec{\phi}_2(x), & x \in \Omega^- \\ \vec{y}_2(x) + \vec{B}_1 \cdot \vec{\phi}_1(x) + (\vec{u}(1) - \vec{y}_2(1)) \cdot \vec{\phi}_2(x), & x \in \Omega^+ \end{cases} \tag{6}$$

where $\vec{\phi}_1(x),\ \vec{\phi}_2(x)$ are the solutions of the boundary value problems

$$-E\vec{\phi}_1''(x) + A(x)\vec{\phi}_1(x) = \vec{0},\ x \in \Omega,\ \vec{\phi}_1(0) = \vec{1},\ \vec{\phi}_1(1) = \vec{0}$$

$$-E\vec{\phi}_2''(x) + A(x)\vec{\phi}_2(x) = \vec{0},\ x \in \Omega,\ \vec{\phi}_2(0) = \vec{0},\ \vec{\phi}_2(1) = \vec{1}$$

and $\vec{A}_1,\ \vec{B}_1$ are constant vectors to be chosen so that $\vec{y} \in C^{(1)}(\Omega)$. In fact, the constants $\vec{A}_1$ and $\vec{B}_1$ are derived from the conditions

$$\vec{y}(d-) = \vec{y}(d+) \text{ and } \vec{y}\,'(d-) = \vec{y}\,'(d+).$$

These conditions lead to a system of $2n$ equations in the unknowns $A_{1,i},\ B_{1,i}$, $i = 1, 2, \ldots, n$, given by

$$\begin{pmatrix} \vec{\phi}_2(d) & -\vec{\phi}_1(d) \\ \vec{\phi}_2'(d) & -\vec{\phi}_1'(d) \end{pmatrix} \begin{pmatrix} \vec{A}_1 \\ \vec{B}_1 \end{pmatrix} = \begin{pmatrix} \vec{k}_1 \\ \vec{k}_2 \end{pmatrix}$$

where

$$\vec{k}_1 = [\vec{u}(1) - \vec{y}_2(1)] \cdot \vec{\phi}_2(d) - [\vec{u}(0) - \vec{y}_1(0)] \cdot \vec{\phi}_1(d)$$

$$\vec{k}_2 = [\vec{u}(1) - \vec{y}_2(1)] \cdot \vec{\phi}_2'(d) - [\vec{u}(0) - \vec{y}_1(0)] \cdot \vec{\phi}_1'(d).$$

Using the definition given in (4), the above could be rewritten as

$$\left.\begin{array}{l} \phi_{2,i}(d)A_{1,i} - \phi_{1,i}(d)B_{1,i} = k_{1,i}, \\ \phi_{2,i}'(d)A_{1,i} - \phi_{1,i}'(d)B_{1,i} = k_{2,i}, \end{array}\right\} \text{ for } i = 1, 2, \ldots, n.$$

Note that on the open interval $(0, 1),\ 0 < \vec{\phi}_1,\ \vec{\phi}_2 < 1$. Thus $\vec{\phi}_1,\ \vec{\phi}_2$ cannot have an internal maximum or minimum and also

$$\vec{\phi}_1' < \vec{0},\ \vec{\phi}_2' > \vec{0},\ x \in (0, 1).$$

Hence $\phi_{1,i}(d)\phi_{2,i}'(d) - \phi_{2,i}(d)\phi_{1,i}'(d) > 0, i = 1, 2, \ldots, n,$ ensures the existence of $\vec{A}_1$ and $\vec{B}_1$. □

The operator $\vec{L}$ satisfies the following maximum principle.

Lemma 2.2. *Let* $A(x)$ *satisfy (2) and (3). Let* $\vec{\psi}$ *be any vector-valued function in the domain of* $\vec{L}$ *such that* $\vec{\psi} \geq \vec{0}$ *on* Γ, $\vec{L}\vec{\psi} \geq \vec{0}$ *on* $\Omega^- \cup \Omega^+$ *and* $[\vec{\psi}](d) = \vec{0}$, $[\vec{\psi}'](d) \leq \vec{0}$, *then* $\vec{\psi} \geq \vec{0}$ *on* $\overline{\Omega}$.

Proof. Let i^*, x^* be such that $\psi_{i^*}(x^*) = \min\limits_{i,x} \psi_i(x)$. If $\psi_{i^*}(x^*) \geq 0$, there is nothing to prove. Suppose therefore that $\psi_{i^*}(x^*) < 0$, then the proof is completed by showing that this leads to a contradiction. With the above assumption on the boundary values, either $x^* \in \Omega^- \cup \Omega^+$ or $x^* = d$. In the first case $\psi''_{i^*}(x^*) \geq 0$ and so

$$(\vec{L}\vec{\psi})_{i^*}(x^*) = -\varepsilon_{i^*}\psi''_{i^*}(x^*) + \sum_{j=1}^{n} a_{i^*,j}(x^*)\psi_j(x^*) < 0,$$

which is false. In the second case the argument depends on whether or not ψ_{i^*} is differentiable at d. If $\psi'_{i^*}(d)$ does not exist, then $[\psi'_{i^*}](d) \neq 0$ and because $\psi'_{i^*}(d-) \leq 0$, $\psi'_{i^*}(d+) \geq 0$ it is clear that $[\psi'_{i^*}](d) > 0$, which is a contradiction. On the other hand, let ψ_{i^*} be differentiable at d. As $(A\vec{\psi})_{i^*}(d) < 0$ and all entries of A and all ψ_j are in $C(\overline{\Omega})$, there exists an interval $[d_1, d]$ on which $(A\vec{\psi})_{i^*}(x) < 0$. If $\psi''_{i^*}(\hat{x}) \geq 0$ at any point $\hat{x} \in [d_1, d)$, then $(\vec{L}\vec{\psi})_{i^*}(\hat{x}) < 0$, contradicting the hypotheses of the lemma. Thus we can assume that $\psi''_{i^*}(x) < 0$ on $[d_1, d)$. But this implies that $\psi'_{i^*}(x)$ is strictly decreasing on $[d_1, d)$ and we know already that $\psi'_{i^*}(d) = 0$ and $\psi'_{i^*} \in C(\overline{\Omega})$, so $\psi'_{i^*}(x) > 0$ on $[d_1, d)$. Consequently the continuous function $\psi_{i^*}(x)$ cannot have a minimum at $x = d$, which contradicts our earlier assumption that $x^* = d$. This completes the proof. □

Let $\tilde{A}(x)$ be any principal sub-matrix of $A(x)$ and $\tilde{\vec{L}}$ the corresponding operator. To see that any $\tilde{\vec{L}}$ satisfies the same maximum principle as $\vec{L}$, it suffices to observe that the elements of $\tilde{A}(x)$ satisfy *a fortiori* the same inequalities as those of $A(x)$.

As a consequence of the maximum principle, there is established the stability result for the problem (1) in the following

Lemma 2.3. *Let $A(x)$ satisfy (2) and (3). If $\vec{\psi}$ is any vector-valued function in the domain of $\vec{L}$, then for each i, $1 \leq i \leq n$ and $x \in \overline{\Omega}$,*

$$|\psi_i(x)| \leq \max\left\{\| \vec{\psi} \|_\Gamma, \frac{1}{\alpha} \| \vec{f} \|_{\Omega^- \cup \Omega^+}\right\}.$$

Proof. Define the two functions

$$\vec{\theta}^{\pm}(x) = \max\left\{\| \vec{\psi} \|_\Gamma, \frac{1}{\alpha} \| \vec{f} \|_{\Omega^- \cup \Omega^+}\right\}\vec{e} \pm \vec{\psi}(x)$$

where $\vec{e} = (1, \ldots, 1)^T$ is the unit column $n-$vector. Using the properties of A it is not hard to verify that $\vec{\theta}^{\pm} \geq \vec{0}$ on Γ and $\vec{L}\vec{\theta}^{\pm} \geq \vec{0}$ on $\Omega^- \cup \Omega^+$. Furthermore, since $\vec{u} \in C^{(1)}(\Omega)$,

$$[\vec{\theta}^{\pm}](d) = \pm[\vec{\psi}](d) = \vec{0} \text{ and } [\vec{\theta}^{\pm,\prime}](d) = \pm[\vec{\psi}'](d) = \vec{0}.$$

It follows from Lemma 2.2 that $\vec{\theta}^{\pm} \geq \vec{0}$ on $\overline{\Omega}$. □

Standard estimates of the exact solution and its derivatives are contained in the following lemma.

Lemma 2.4. *Let $A(x)$ satisfy (2) and (3) and let $\vec{u}$ be the exact solution of (1). Then, for each $i = 1, \ldots, n$, $x \in \Omega^- \cup \Omega^+$ and $k = 0, 1, 2$,*

$$|u_i^{(k)}(x)| \leq C\varepsilon_i^{-\frac{k}{2}}(||\vec{u}||_\Gamma + ||\vec{f}||_{\Omega^- \cup \Omega^+}),$$

$$|u_i^{(3)}(x)| \leq C\varepsilon_1^{-\frac{1}{2}}\varepsilon_i^{-1}(||\vec{u}||_\Gamma + ||\vec{f}||_{\Omega^- \cup \Omega^+} + \sqrt{\varepsilon_1}||\vec{f}'||_{\Omega^- \cup \Omega^+})$$

and

$$|u_i^{(4)}(x)| \leq C\varepsilon_1^{-1}\varepsilon_i^{-1}(||\vec{u}||_\Gamma + ||\vec{f}||_{\Omega^-\cup\Omega^+} + \varepsilon_1||\vec{f}''||_{\Omega^-\cup\Omega^+}).$$

Proof. The proof is similar to the proof of Lemma 3 in [10]. □

The reduced solution $\vec{u}_0$ of (1) is the solution of the reduced equation $A\vec{u}_0 = \vec{f}$. The Shishkin decomposition of the exact solution $\vec{u}$ of (1) is $\vec{u} = \vec{v} + \vec{w}$ where the smooth component $\vec{v}$ is the solution of $\vec{L}\vec{v} = \vec{f}$ on $\Omega^- \cup \Omega^+$ with $\vec{v} = \vec{u}_0$ on Γ, $\vec{v}(d-) = (A(d))^{-1}\vec{f}(d-)$, $\vec{v}(d+) = (A(d))^{-1}\vec{f}(d+)$ and the singular component $\vec{w}$ is the solution of $\vec{L}\vec{w} = \vec{0}$ on $\Omega^- \cup \Omega^+$ with $[\vec{w}](d) = -[\vec{v}](d)$, $[\vec{w}'](d) = -[\vec{v}'](d)$, $\vec{w} = \vec{u} - \vec{v}$ on Γ.

For convenience, the singular component is given a further decomposition

$$\vec{w}(x) = \begin{cases} \vec{w}_1^L(x) + \vec{w}_1^R(x) & \text{on } \Omega^- \\ \vec{w}_2^L(x) + \vec{w}_2^R(x) & \text{on } \Omega^+ \end{cases}$$

where $\vec{w}_1^L(x) = \vec{w}(0)\cdot\vec{\psi}_1(x)$, $\vec{w}_1^R(x) = \vec{A}_2\cdot\vec{\psi}_2(x)$, $\vec{w}_2^L(x) = \vec{B}_2\cdot\vec{\psi}_3(x)$, $\vec{w}_2^R(x) = \vec{w}(1)\cdot\vec{\psi}_4(x)$ with

$$-E\vec{\psi}_1''(x) + A(x)\vec{\psi}_1(x) = \vec{0} \quad \text{on } \Omega^-, \quad \vec{\psi}_1(0) = \vec{1},\ \vec{\psi}_1(d) = \vec{0}$$

$$-E\vec{\psi}_2''(x) + A(x)\vec{\psi}_2(x) = \vec{0} \quad \text{on } \Omega^-, \quad \vec{\psi}_2(0) = \vec{0},\ \vec{\psi}_2(d) = \vec{1}$$

$$-E\vec{\psi}_3''(x) + A(x)\vec{\psi}_3(x) = \vec{0} \quad \text{on } \Omega^+, \quad \vec{\psi}_3(d) = \vec{1},\ \vec{\psi}_3(1) = \vec{0}$$

$$-E\vec{\psi}_4''(x) + A(x)\vec{\psi}_4(x) = \vec{0} \quad \text{on } \Omega^+, \quad \vec{\psi}_4(d) = \vec{0},\ \vec{\psi}_4(1) = \vec{1}.$$

Here too, $\vec{A}_2$ and $\vec{B}_2$ are constants, independent of x and $\vec{\varepsilon}$, to be chosen in a way similar to that used in determining $\vec{A}_1$ and $\vec{B}_1$ in Theorem 2.1.

Bounds on the smooth component and its derivatives are contained in the following lemma.

Lemma 2.5. *Let $A(x)$ satisfy (2) and (3). Then the smooth component $\vec{v}$ and its derivatives satisfy, for all $x \in \Omega^- \cup \Omega^+$, $i = 1, \ldots, n$ and $k = 0, \ldots, 4$,*

$$|v_i^{(k)}(x)| \leq C(1 + \varepsilon_i^{1-\frac{k}{2}}).$$

Proof. Using the techniques given in [10] on the intervals Ω^- and Ω^+ separately, it is not hard to see that the above estimate holds. □

3. Improved estimates

The layer functions $B_{1,i}^L$, $B_{1,i}^R$, $B_{2,i}^L$, $B_{2,i}^R$, $B_{1,i}$, $B_{2,i}$, $i = 1, \ldots, n$, , associated with the solution $\vec{u}$, are defined by

$$B_{1,i}^L(x) = e^{-x\sqrt{\alpha}/\sqrt{\varepsilon_i}},\ B_{1,i}^R(x) = e^{-(d-x)\sqrt{\alpha}/\sqrt{\varepsilon_i}},\ B_{1,i} = B_{1,i}^L + B_{1,i}^R \text{ on } \overline{\Omega^-},$$
$$B_{2,i}^L(x) = e^{-(x-d)\sqrt{\alpha}/\sqrt{\varepsilon_i}},\ B_{2,i}^R(x) = e^{-(1-x)\sqrt{\alpha}/\sqrt{\varepsilon_i}},\ B_{2,i} = B_{2,i}^L + B_{2,i}^R \text{ on } \overline{\Omega^+}.$$

The following elementary properties of the layer functions $B_{1,i}^L$, $B_{1,i}^R$, for all $1 \leq i < j \leq n$ and $0 \leq x < y \leq d$, should be noted:
(a) $B_{1,i}^L(x) < B_{1,j}^L(x)$, $B_{1,i}^L(x) > B_{1,i}^L(y)$, $0 < B_{1,i}^L(x) \leq 1$.
(b) $B_{1,i}^R(x) < B_{1,j}^R(x)$, $B_{1,i}^R(x) < B_{1,i}^R(y)$, $0 < B_{1,i}^R(x) \leq 1$.
(c) $B_{1,i}^L(x)$ ($B_{1,i}^R(x)$) is monotone decreasing (increasing) for increasing $x \in [0, \frac{d}{2}]$

$([\frac{d}{2}, d])$.
(d) $B_{1,i}(x) \leq 2B^L_{1,i}(x)$ for $x \in [0, \frac{d}{2}]$ and $B_{1,i}(x) \leq 2B^R_{1,i}(x)$ for $x \in [\frac{d}{2}, d]$.
Similar properties for $B^L_{2,i},\ B^R_{2,i},$ for all $1 \leq i < j \leq n$ and $d \leq x < y \leq 1$, hold good.

Remark: In [10], a sequence of points $x^{(s)}_{i,j}$ are introduced, which lead to novel estimates for the derivatives of the singular components. These are used to prove second order convergence of the method. In [8], Linss and Madden use a single point of this kind with $s = 1/2$.

Here too, if $x^{(s)}_{i,j}$ and $1 - x^{(s)}_{i,j}$ are those points of the boundary layers then $d - x^{(s)}_{i,j}$ and $d + x^{(s)}_{i,j}$ play the same role in case of the interior layers. In this paper, result of first order convergence is established for the method. But, using these points, finding a second order convergent method could be done in future. For the interested readers, the properties of these points are stated below.

Definition 3.1. *For $B^L_{1,i},\ B^L_{1,j}$, each i, j, $1 \leq i \neq j \leq n$ and each $s, s > 0$, the point $x^{(s)}_{i,j}$ is defined by*

$$\frac{B^L_{1,i}(x^{(s)}_{i,j})}{\varepsilon_i^s} = \frac{B^L_{1,j}(x^{(s)}_{i,j})}{\varepsilon_j^s}. \tag{7}$$

It is remarked that

$$\begin{gathered}\frac{B^R_{1,i}(d - x^{(s)}_{i,j})}{\varepsilon_i^s} = \frac{B^R_{1,j}(d - x^{(s)}_{i,j})}{\varepsilon_j^s}, \quad \frac{B^L_{2,i}(d + x^{(s)}_{i,j})}{\varepsilon_i^s} = \frac{B^L_{2,j}(d + x^{(s)}_{i,j})}{\varepsilon_j^s}, \\ \frac{B^R_{2,i}(1 - x^{(s)}_{i,j})}{\varepsilon_i^s} = \frac{B^R_{2,j}(1 - x^{(s)}_{i,j})}{\varepsilon_j^s}.\end{gathered} \tag{8}$$

In the next lemma the existence and uniqueness of the points $x^{(s)}_{i,j}$ are shown. Various properties are also established.

Lemma 3.2. *For all i, j such that $1 \leq i < j \leq n$ and $0 < s \leq 3/2$, the points $x^{(s)}_{i,j}$ exist, are uniquely defined and satisfy the following inequalities*

$$\frac{B^L_{1,i}(x)}{\varepsilon_i^s} > \frac{B^L_{1,j}(x)}{\varepsilon_j^s}, \ x \in [0, x^{(s)}_{i,j}), \quad \frac{B^L_{1,i}(x)}{\varepsilon_i^s} < \frac{B^L_{1,j}(x)}{\varepsilon_j^s}, \ x \in (x^{(s)}_{i,j}, d]. \tag{9}$$

Moreover

$$x^{(s)}_{i,j} < x^{(s)}_{i+1,j}, \ \text{if } i+1 < j \ \text{ and } \ x^{(s)}_{i,j} < x^{(s)}_{i,j+1}, \ \text{ if } i < j. \tag{10}$$

Also

$$x^{(s)}_{i,j} < 2s\frac{\sqrt{\varepsilon_j}}{\sqrt{\alpha}} \ \text{ and } \ x^{(s)}_{i,j} \in (0, \frac{d}{2}) \ \text{ if } i < j. \tag{11}$$

Analogous results hold for $B^R_{1,i}$, $B^L_{2,i}$ and $B^R_{2,i}$ and the points $d - x^{(s)}_{i,j}$, $d + x^{(s)}_{i,j}$, $1 - x^{(s)}_{i,j}$.

Proof. The proof is similar to the proof of Lemma 5 in [10]. □

Bounds on the singular component $\vec{w}$ of $\vec{u}$ and its derivatives are contained in

Lemma 3.3. *Let $A(x)$ satisfy (2) and (3).Then there exists a constant C, such that, for $i = 1, \ldots, n$ and $x \in \Omega^-$,*

$$\left|w^L_{1,i}(x)\right| \leq CB^L_{1,n}(x), \ \left|w^{L,\prime}_{1,i}(x)\right| \leq C\sum_{q=i}^{n} \frac{B^L_{1,q}(x)}{\sqrt{\varepsilon}_q}, \ \left|w^{L,\prime\prime}_{1,i}(x)\right| \leq C\sum_{q=i}^{n} \frac{B^L_{1,q}(x)}{\varepsilon_q},$$

$$\left|w^{L,(3)}_{1,i}(x)\right| \leq C\sum_{q=1}^{n} \frac{B^L_{1,q}(x)}{\varepsilon_q^{3/2}}, \ \left|\varepsilon_i w^{L,(4)}_{1,i}(x)\right| \leq C\sum_{q=1}^{n} \frac{B^L_{1,q}(x)}{\varepsilon_q}.$$

Analogous results hold for $w^R_{1,i}$, $w^L_{2,i}$ and $w^R_{2,i}$ and their derivatives.

Proof. The argument used in Lemma 7 of [10] are applied on Ω^- and Ω^+ separately to obtain the required bounds of $\vec{w}$. □

Using the bounds of $\vec{w}$ and Lemma 2.5, it is not hard to see that the smooth component and its derivatives can be bounded by the sharper estimates given in the following

Lemma 3.4. *Let $A(x)$ satisfy (2) and (3). Then the smooth component $\vec{v}$ of the solution $\vec{u}$ of* (1) *satisfies for $i = 1, \ldots, n$, $k = 0, 1, 2, 3$,*

$$|v_i^{(k)}(x)| \leq C \begin{cases} 1 + \displaystyle\sum_{q=i}^{n} \frac{B_{1,q}(x)}{\varepsilon_q^{\frac{k}{2}-1}} & on\ \Omega^- \\ \\ 1 + \displaystyle\sum_{q=i}^{n} \frac{B_{2,q}(x)}{\varepsilon_q^{\frac{k}{2}-1}} & on\ \Omega^+. \end{cases}$$

4. The Shishkin mesh

A piecewise uniform Shishkin mesh with N mesh-intervals is now constructed on $\Omega^- \cup \Omega^+$ as follows. Let $\Omega^N = \Omega^{-N} \cup \Omega^{+N}$ where $\Omega^{-N} = \{x_j\}_{j=1}^{\frac{N}{2}-1}$, $\Omega^{+N} = \{x_j\}_{j=\frac{N}{2}+1}^{N-1}$ and $x_{\frac{N}{2}} = d$. Then $\overline{\Omega^-}^N = \{x_j\}_{j=0}^{\frac{N}{2}}$, $\overline{\Omega^+}^N = \{x_j\}_{j=\frac{N}{2}}^{N}$, $\overline{\Omega^-}^N \cup \overline{\Omega^+}^N = \overline{\Omega}^N = \{x_j\}_{j=0}^{N}$ and $\Gamma^N = \Gamma$. The interval $[0, d]$ is subdivided into $2n+1$ sub-intervals

$$[0, \tau_1] \cup \cdots \cup (\tau_{n-1}, \tau_n] \cup (\tau_n, d - \tau_n] \cup (d - \tau_n, d - \tau_{n-1}] \cup \cdots \cup (d - \tau_1, d].$$

The n parameters τ_r, which determine the points separating the uniform meshes, are defined by $\tau_0 = 0$, $\tau_{n+1} = \frac{d}{2}$,

$$\tau_n = \min\left\{\frac{d}{4}, 2\frac{\sqrt{\varepsilon_n}}{\sqrt{\alpha}} \ln N\right\} \tag{12}$$

and, for $r = n-1, \ldots, 1$,

$$\tau_r = \min\left\{\frac{r\tau_{r+1}}{r+1}, 2\frac{\sqrt{\varepsilon_r}}{\sqrt{\alpha}} \ln N\right\}. \tag{13}$$

Clearly

$$0 < \tau_1 < \ldots < \tau_n \leq \frac{d}{4}.$$

Then, on the sub-interval $(\tau_n, d-\tau_n]$ a uniform mesh with $\frac{N}{4}$ mesh points is placed and on each of the sub-intervals $(\tau_r, \tau_{r+1}]$ and $(d-\tau_{r+1}, d-\tau_r]$, $r = 0, 1, \ldots, n-1$, a uniform mesh of $\frac{N}{8n}$ mesh points is placed. In particular, when all the parameters τ_r, $r = 1, \ldots, n$, take their left-hand value, the Shishkin mesh $\overline{\Omega^-}^N$ becomes a

classical uniform mesh throughout from 0 to d.
Similarly the interval $[d, 1]$ is subdivided into $2n+1$ sub-intervals

$$[d, d+\sigma_1] \cup \cdots \cup (d+\sigma_{n-1}, d+\sigma_n] \cup (d+\sigma_n, 1-\sigma_n] \cup (1-\sigma_n, 1-\sigma_{n-1}] \cup \cdots \cup (1-\sigma_1, 1].$$

The n parameters σ_r, which determine the points separating the uniform meshes, are defined by $\sigma_0 = 0$, $\sigma_{n+1} = \frac{1-d}{2}$,

$$\sigma_n = \min\left\{\frac{1-d}{4}, 2\frac{\sqrt{\varepsilon_n}}{\sqrt{\alpha}} \ln N\right\}$$

and, for $r = n-1, \ldots, 1$,

$$\sigma_r = \min\left\{\frac{r\sigma_{r+1}}{r+1}, 2\frac{\sqrt{\varepsilon_r}}{\sqrt{\alpha}} \ln N\right\}.$$

Clearly

$$0 < \sigma_1 < \ldots < \sigma_n \leq \frac{1-d}{4}.$$

Then, on the sub-interval $(d+\sigma_n, 1-\sigma_n]$ a uniform mesh with $\frac{N}{4}$ mesh points is placed and on each of the sub-intervals $(d+\sigma_r, d+\sigma_{r+1}]$ and $(1-\sigma_{r+1}, 1-\sigma_r]$, $r = 0, 1, \ldots, n-1$, a uniform mesh of $\frac{N}{8n}$ mesh points is placed. In particular, when all the parameters σ_r, $r = 1, \ldots, n$, take their left-hand value, the Shishkin mesh $\overline{\Omega^+}^N$ becomes a classical uniform mesh throughout from d to 1.
When $d = 1/2$ and when all the transition parameters τ_r and σ_r, $r = 1, \ldots, n$, take their left-hand value then the mesh $\overline{\Omega}^N$ is the classical uniform mesh with step size N^{-1} throughout from 0 to 1. In practice, it is convenient to take

$$N = 8nk, \ k \geq 2, \tag{14}$$

where n is the number of distinct singular perturbation parameters involved in (1). This construction leads to a class of 2^{n+1} piecewise uniform Shishkin meshes $\overline{\Omega}^N$. From the above construction of $\overline{\Omega^-}^N$, it is clear that the transition points $\{\tau_r, d-\tau_r\}_{r=1}^n$ are the only points at which the mesh-size can change and that it does not necessarily change at each of these points. The following notation is introduced: if $x_j = \tau_r$, then $h_r^- = x_j - x_{j-1}, h_r^+ = x_{j+1} - x_j$, $J = \{\tau_r : h_r^+ \neq h_r^-\}$. In general, for each point x_j in the mesh-interval $(\tau_{r-1}, \tau_r]$,

$$x_j - x_{j-1} = 8n\, N^{-1}(\tau_r - \tau_{r-1}). \tag{15}$$

Also, for $x_j \in (\tau_n, \frac{d}{2}]$, $x_j - x_{j-1} = 4N^{-1}(d - 2\tau_n)$ and for $x_j \in (0, \tau_1]$, $x_j - x_{j-1} = 8nN^{-1}\tau_1$. Thus, for $1 \leq r \leq n-1$, the change in the step-size at the point $x_j = \tau_r$ is

$$h_r^+ - h_r^- = 8nN^{-1}((r+1)d_r - rd_{r-1}), \tag{16}$$

where

$$d_r = \frac{r\tau_{r+1}}{r+1} - \tau_r \tag{17}$$

with the convention $d_0 = 0$. Notice that $d_r \geq 0$, that Ω^{-N} is the classical uniform mesh when $d_r = 0$ for all $r = 1, \ldots, n$ and, from (12) and (13), that

$$\tau_r \leq C\sqrt{\varepsilon_r} \ln N, \quad 1 \leq r \leq n. \tag{18}$$

It follows from (15) and (18) that for $r = 1, \ldots, n-1$,

$$h_r^- + h_r^+ \leq C\sqrt{\varepsilon_{r+1}} N^{-1} \ln N. \tag{19}$$

Also

$$\tau_r = \frac{r}{s}\tau_s \text{ when } d_r = \cdots = d_s = 0,\ 1 \le r \le s \le n. \tag{20}$$

Similar results hold good for $\overline{\Omega^+}^N$ and $\sigma_r,\ r = 1,\ \ldots\ , n.$
The results in the following lemma are used later.

Lemma 4.1. *Assume that* $d_r > 0$ *for some* $r, 1 \le r \le n$. *Then the following inequalities hold*

$$B^L_{1,r}(d-\tau_r) \le B^L_{1,r}(\tau_r) = N^{-2}, \tag{21}$$

$$x^{(s)}_{r-1,r} \ \le\ \tau_r - h^-_r \text{ for } 0 < s \le 3/2, 1 < r \le n, \tag{22}$$

$$B^L_{1,q}(\tau_r - h^-_r) \le CB^L_{1,q}(\tau_r) \ \text{ for } \ 1 \le r \le q \le n, \tag{23}$$

$$\frac{B^L_{1,q}(\tau_r)}{\sqrt{\varepsilon_q}} \le C\frac{1}{\sqrt{\varepsilon_r}\ln N} \ \text{ for } \ 1 \le q \le n,\ \ 1 \le r \le n. \tag{24}$$

Analogous results hold for $B^R_{1,r},\ B^L_{2,r},\ B^R_{2,r}$.

Proof. Using the definitions of $B^L_{1,r}(x)$ and τ_r, (21) follows.
By Lemma 3.2,

$$x^{(s)}_{r-1,r} < 2s\frac{\sqrt{\varepsilon_r}}{\sqrt{\alpha}} = \frac{s\tau_r}{\ln N} \le \frac{\tau_r}{2}.$$

Also, by (14) and (15),

$$h^-_r = 8nN^{-1}(\tau_r - \tau_{r-1}) = \frac{(\tau_r - \tau_{r-1})}{k} < \frac{\tau_r}{2}.$$

It follows that $x^{(s)}_{r-1,r} + h^-_r \le \tau_r$ as required.
To verify (23) note, from (15), that

$$h^-_r = 8nN^{-1}(\tau_r - \tau_{r-1}) \le 8nN^{-1}\tau_r = 2^4 nN^{-1}\frac{\sqrt{\varepsilon_r}}{\sqrt{\alpha}}\ln N.$$

But

$$e^{2^4 nN^{-1}\frac{\sqrt{\varepsilon_r}}{\sqrt{\alpha}}\ln N} \le (N^{\frac{1}{N}})^{16n} \le C.$$

Since $r \le q$,

$$\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_q}}h^-_r \le \frac{\sqrt{\varepsilon_r}}{\sqrt{\varepsilon_q}}8nN^{-1}\tau_r \le 16nN^{-1}\ln N\frac{\sqrt{\varepsilon_r}}{\sqrt{\alpha}}.$$

It follows that

$$B^L_{1,q}(\tau_r - h^-_r) = B^L_{1,q}(\tau_r)e^{\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_q}}h^-_r} \le CB^L_{1,q}(\tau_r)$$

as required.
To verify (24), if $q \ge r$ the result is trivial. On the other hand, if $q < r$,

$$B^L_{1,q}(\tau_r) = e^{-\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_q}}\tau_r} = e^{-2\frac{\sqrt{\varepsilon_r}}{\sqrt{\varepsilon_q}}\ln N} \le \frac{C}{\ln N}\frac{\sqrt{\varepsilon_q}}{\sqrt{\varepsilon_r}},$$

where the inequality is obtained by using the result $e^{-t} \le \frac{1}{t}$ for all $t \ge 0$. □

5. The discrete problem

In this section a classical finite difference operator with an appropriate Shishkin mesh is used to construct a numerical method for (1), which is shown later to be essentially first order parameter-uniform convergent.
The discrete two-point boundary value problem is now defined to be the finite difference method

$$\begin{aligned} &-E\delta^2\vec{U}(x) + A(x)\vec{U}(x) = \vec{f}(x) \text{ on } \Omega^N, \\ &\vec{U} = \vec{u} \text{ on } \Gamma^N,\ D^-\vec{U}(x_{N/2}) = D^+\vec{U}(x_{N/2}). \end{aligned} \tag{25}$$

This is used to compute numerical approximations to the exact solution of (1). Note that (25) can also be written in the operator form

$$\vec{L}^N\vec{U} \ = \ \vec{f} \text{ on } \Omega^N,\ \vec{U} = \vec{u} \text{ on } \Gamma^N,\ D^-\vec{U}(x_{N/2}) = D^+\vec{U}(x_{N/2})$$

where

$$\vec{L}^N \ = \ -E\delta^2 + A$$

and $\delta^2,\ D^+$ and D^- are the classical finite difference operators as in [10].
The following discrete results are analogous to those for the continuous case.

Lemma 5.1. *Let $A(x)$ satisfy (2) and (3). Then, for any vector-valued mesh function $\vec{\Psi}$, the inequalities $\vec{\Psi} \ \geq \ \vec{0}$ on Γ^N, $\vec{L}^N\vec{\Psi} \ \geq \ \vec{0}$ on Ω^N and $D^+\vec{\Psi}(x_{N/2}) - D^-\vec{\Psi}(x_{N/2}) \leq \vec{0}$ imply that $\vec{\Psi} \geq \vec{0}$ on $\overline{\Omega}^N$.*

Proof. Let i^*, j^* be such that $\Psi_{i^*}(x_{j^*}) = \min_{i,j} \Psi_i(x_j)$ and assume that the lemma is false. Then $\Psi_{i^*}(x_{j^*}) < 0$. From the hypotheses we have $j^* \neq 0,\ N$ and $\Psi_{i^*}(x_{j^*}) - \Psi_{i^*}(x_{j^*-1}) \leq 0,\ \Psi_{i^*}(x_{j^*+1}) - \Psi_{i^*}(x_{j^*}) \geq 0$, so $\delta^2\Psi_{i^*}(x_{j^*}) \ \geq \ 0$. It follows that

$$\left(\vec{L}^N\vec{\Psi}\right)_{i^*}(x_{j^*}) \ = \ -\varepsilon_{i^*}\delta^2\Psi_{i^*}(x_{j^*}) + \sum_{k=1}^{n} a_{i^*,\,k}(x_{j^*})\Psi_k(x_{j^*}) \ < \ 0.$$

If $x_{j^*} \in \Omega^N$, this leads to a contradiction. Because of the boundary values, the only other possibility is that $x_{j^*} = x_{N/2}$. Then

$$D^-\Psi_{i^*}(x_{N/2}) \leq 0 \leq D^+\Psi_{i^*}(x_{N/2}) \leq D^-\Psi_{i^*}(x_{N/2})$$

and so

$$\Psi_{i^*}(x_{\frac{N}{2}-1}) = \Psi_{i^*}(x_{N/2}) = \Psi_{i^*}(x_{\frac{N}{2}+1}) < 0.$$

Then $\left(\vec{L}^N\vec{\Psi}\right)_{i^*}(x_{\frac{N}{2}-1}) < 0$, which provides the desired contradiction. □

An immediate consequence of this is the following discrete stability result.

Lemma 5.2. *Let $A(x)$ satisfy (2) and (3). Then, for any vector-valued mesh function $\vec{\Psi}$ defined on $\overline{\Omega}^N$ such that $D^+\vec{\Psi} = D^-\vec{\Psi}$ at $x_{N/2}$,*

$$|\vec{\Psi}(x_j)| \ \leq \ \max\left\{||\vec{\Psi}||_{\Gamma^N}, \frac{1}{\alpha}||\vec{L}^N\vec{\Psi}||_{\Omega^{-N}\cup\Omega^{+N}}\right\},\ 0 \leq j \leq N.$$

Proof. Define the two functions

$$\vec{\Theta}^{\pm}(x_j) = \max\left\{||\vec{\Psi}||_{\Gamma^N}, \frac{1}{\alpha}||\vec{L}^N\vec{\Psi}||_{\Omega^{-N}\cup\Omega^{+N}}\right\}\vec{e} \pm \vec{\Psi}(x_j)$$

where $\vec{e} = (1, \ \dots \ , 1)^T$ is the unit column $n-$vector. Using the properties of A it is not hard to verify that $\vec{\Theta}^{\pm} \geq \vec{0}$ on Γ^N, for $x_j \neq x_{N/2}$, $\vec{L}^N \vec{\Theta}^{\pm} \geq \vec{0}$ on Ω^N and at $x_j = x_{N/2}$,

$$(D^+ - D^-)\vec{\Theta}^{\pm}(x_j) = \pm(D^+ - D^-)\vec{\Psi}(x_j) = 0.$$

It follows from Lemma 5.1 that $\vec{\Theta}^{\pm} \geq \vec{0}$ on $\overline{\Omega}^N$. □

6. Error estimate

Analogously to the continuous case, the discrete solution $\vec{U}$ can be decomposed into $\vec{V}_1$ and $\vec{W}_1$ on Ω^{-N} and $\vec{V}_2$ and $\vec{W}_2$ on Ω^{+N} which are defined to be the solutions of the following discrete problems

$$\begin{aligned}(\vec{L}^N \vec{V}_1)(x_j) &= \vec{f}(x_j), \ x_j \in \Omega^{-N}, \ \vec{V}_1(0) = \vec{v}(0), \ \vec{V}_1(x_{N/2}) = \vec{v}(d-), \\ (\vec{L}^N \vec{V}_2)(x_j) &= \vec{f}(x_j), \ x_j \in \Omega^{+N}, \ \vec{V}_2(1) = \vec{v}(1), \ \vec{V}_2(x_{N/2}) = \vec{v}(d+)\end{aligned}$$

and

$$(\vec{L}^N \vec{W}_1)(x_j) = \vec{0}, \ x_j \in \Omega^{-N}, \ \vec{W}_1(0) = \vec{w}(0),$$

$$(\vec{L}^N \vec{W}_2)(x_j) = \vec{0}, \ x_j \in \Omega^{+N}, \ \vec{W}_2(1) = \vec{w}(1),$$

$$\vec{W}_1(x_{N/2}) + \vec{V}_1(x_{N/2}) = \vec{W}_2(x_{N/2}) + \vec{V}_2(x_{N/2}),$$

$$D^- \vec{W}_1(x_{N/2}) + D^- \vec{V}_1(x_{N/2}) = D^+ \vec{W}_2(x_{N/2}) + D^+ \vec{V}_2(x_{N/2}).$$

The error at each point $x_j \in \overline{\Omega}^N$ is denoted by $\vec{e}(x_j) = \vec{U}(x_j) - \vec{u}(x_j)$. Then the local truncation error $\vec{L}^N \vec{e}(x_j)$, for $j \neq N/2$, has the decomposition

$$\vec{L}^N \vec{e}(x_j) \quad = \quad \vec{L}^N (\vec{V} - \vec{v})(x_j) + \vec{L}^N (\vec{W} - \vec{w})(x_j).$$

The smooth and singular error components are bounded in the following theorems.

Theorem 6.1. *Let $A(x)$ satisfy (2) and (3). If $\vec{v}$ denotes the smooth component of the exact solution of (1) and $\vec{V}$ the smooth component of the solution of the discrete problem (25), then, for $j \neq N/2$,*

$$|(\vec{L}^N (\vec{V} - \vec{v}))_i(x_j)| \leq C \, (N^{-1} \ln N)^2. \tag{26}$$

Proof. Following the techniques in [10] and using Lemma 3.4, it is not hard to see that (26) holds. □

Theorem 6.2. *Let $A(x)$ satisfy (2) and (3). If $\vec{w}$ denotes the singular component of the exact solution of (1) and $\vec{W}$ the singular component of the solution of the discrete problem (25), then, for $j \neq N/2$,*

$$|(\vec{L}^N (\vec{W} - \vec{w}))_i(x_j)| \leq C \, (N^{-1} \ln N)^2. \tag{27}$$

Proof. Following the techniques in [10] and using Lemmas 3.3 and 4.1, it is not hard to see that (27) holds. □

At the point $x_j = x_{N/2}$, for $i = 1, \dots, n$,

$$\begin{aligned}(D^+ - D^-)e_i(x_{\frac{N}{2}}) &= (D^+ - D^-)(U_i - u_i)(x_{\frac{N}{2}}) \\ &= (D^+ - D^-)U_i(x_{\frac{N}{2}}) - (D^+ - D^-)u_i(x_{\frac{N}{2}}).\end{aligned}$$

Recall that $(D^+ - D^-)U_i(x_{\frac{N}{2}}) = 0$. Let $h^* = \max\{h^-_{N/2}, h^+_{N/2}\}$.
Then

$$\begin{aligned}
|(D^+ - D^-)e_i(x_{\frac{N}{2}})| &= |(D^+ - D^-)u_i(x_{\frac{N}{2}})| \\
&\le |(D^+ - \frac{d}{dx})u_i(x_{\frac{N}{2}})| + |(D^- - \frac{d}{dx})u_i(x_{\frac{N}{2}})| \\
&\le \frac{1}{2}h^+_{N/2}\,|u_i''(\eta)|_{\eta\in\Omega^+} + \frac{1}{2}h^-_{N/2}\,|u_i''(\xi)|_{\xi\in\Omega^-} \\
&\le C\,h^* \max_{x\in\Omega^-\cup\Omega^+} |u_i''(x)|\,.
\end{aligned}$$

Therefore,

$$|(D^+ - D^-)e_i(x_{\frac{N}{2}})| \;\le\; C\,\frac{h^*}{\varepsilon_i}. \tag{28}$$

From now on, we have the general setting $h_k = x_k - x_{k-1}$ and $h_{k+1} = x_{k+1} - x_k$ for any $x_k \in \overline{\Omega}^N = \{x_k\}_{k=0}^N$.

Define, for $i = 1, \ldots, n$, a set of discrete barrier functions on $[0,1]$ by

$$\omega_i(x_j) = \begin{cases} \dfrac{\Pi_{k=1}^{j}(1+\sqrt{\alpha}h_k/\sqrt{2\varepsilon_i})}{\Pi_{k=1}^{N/2}(1+\sqrt{\alpha}h_k/\sqrt{2\varepsilon_i})}, & 0 \le j \le N/2 \\[2ex] \dfrac{\Pi_{k=j}^{N-1}(1+\sqrt{\alpha}h_{k+1}/\sqrt{2\varepsilon_i})}{\Pi_{k=N/2}^{N-1}(1+\sqrt{\alpha}h_{k+1}/\sqrt{2\varepsilon_i})}, & N/2 \le j \le N. \end{cases}$$

Note that

$$\omega_i(0) = 0,\ \ \omega_i(d) = 1,\ \ \omega_i(1) = 0 \tag{29}$$

and, for $0 \le j \le N$,

$$0 \le \omega_i(x_j) \le 1. \tag{30}$$

It is not hard to see that, for $x_j \in \overline{\Omega^-}^N$,

$$D^+\omega_i(x_j) = \frac{\sqrt{\alpha}}{\sqrt{2\varepsilon_i}}\omega_i(x_j),\ \ D^-\omega_i(x_j) = \frac{\sqrt{\alpha}}{\sqrt{2\varepsilon_i}(1+\sqrt{\alpha}h_j/\sqrt{2\varepsilon_i})}\omega_i(x_j) \tag{31}$$

and, for $x_j \in \overline{\Omega^+}^N$,

$$D^+\omega_i(x_j) = -\frac{\sqrt{\alpha}}{\sqrt{2\varepsilon_i}(1+\sqrt{\alpha}h_{j+1}/\sqrt{2\varepsilon_i})}\omega_i(x_j),\ \ D^-\omega_i(x_j) = -\frac{\sqrt{\alpha}}{\sqrt{2\varepsilon_i}}\omega_i(x_j). \tag{32}$$

In particular, at $x_j = x_{N/2}$,

$$(D^+ - D^-)\omega_i(x_j) \le -\frac{C}{\sqrt{\varepsilon_i}}. \tag{33}$$

We now state and prove the main theoretical result of the paper.

Theorem 6.3. *Let $\vec{u}(x_j)$ be the solution of the continuous problem* (1) *and $\vec{U}(x_j)$ be the solution of the discrete problem* (25). *Then,*

$$\parallel \vec{U}(x_j) - \vec{u}(x_j) \parallel \le C\,N^{-1}\ln N.$$

Proof. Consider the mesh function $\vec{\Psi}$ given by

$$\Psi_i(x_j) = C_1 N^{-1}\ln N + C_2 \frac{h^*}{\sqrt{\varepsilon_i}}\omega_i(x_j) \pm e_i(x_j),\ 1 \le i \le n,\ 0 \le j \le N,$$

where C_1 and C_2 are constants. Then, for appropriate choices of C_1 and C_2, using Theorems 6.1, 6.2 and the fact that $h^* \le C\sqrt{\varepsilon_1}\, N^{-1}\ln N$,

$$\begin{aligned}(\vec{L}^N\vec{\Psi})_i(x_j) &= C_1\sum_{j=1}^{n} a_{ij}(x)N^{-1}\ln N + C_2\frac{h^*}{\sqrt{\varepsilon_i}}(\vec{L}^N\vec{\omega})_i(x_j) \pm (\vec{L}^N\vec{e})_i(x_j)\\ &\ge 0,\ \text{for } j \ne N/2,\ \text{using (31) and (32)}\end{aligned}$$

and

$$\begin{aligned}D^+\Psi_i(d) - D^-\Psi_i(d) &\le -C_2\frac{Ch^*}{\varepsilon_i} \pm C\frac{h^*}{\varepsilon_i},\ \text{using (28) and (33)}\\ &\le 0.\end{aligned}$$

Also, using (29), $\Psi_i(0) = C_1N^{-1}\ln N \ge 0$, $\Psi_i(1) = C_1N^{-1}\ln N \ge 0$.
Therefore, using Lemma 5.1 for $\vec{\Psi}$, it follows that $\Psi_i(x_j) \ge 0$ for all $i = 1,\dots,n,\ 0 \le j \le N$. As, from (30), $\omega_i(x_j) \le 1$ for $1 \le i \le n,\ 0 \le j \le N$, for N sufficiently large,

$$\| \vec{U} - \vec{u} \| \le CN^{-1}\ln N,$$

which completes the proof. □

Remark: It may be conjectured that a more accurate finite difference operator could lead to an improvement in the parameter-uniform error estimate. However, the authors know of no results in this direction, even for the simplest system with one equation.

7. Numerical results

The above numerical method is applied to the following singularly perturbed boundary value problems.

Example 7.1. *Consider*

$$-E\vec{u}''(x) + A(x)\vec{u}(x) = \vec{f}(x)\ \textit{for}\ x \in (0,0.5)\cup(0.5,1),\ \vec{u}(0) = \vec{0},\ \vec{u}(1) = \vec{0}$$

where $E = \mathrm{diag}(\varepsilon_1,\ \varepsilon_2)$, $A = \begin{pmatrix} 6+x & -1 \\ x-1 & 6 \end{pmatrix}$, $\vec{f} = (2,\,2)^T$ for $0 < x < 0.5$ and $\vec{f} = (1,\,1)^T$ for $0.5 < x < 1$. It is seen that both components of the source function $\vec{f}$ have a discontinuity at $x = 0.5$. For various values of $\varepsilon_1,\ \varepsilon_2,\ N = 16k,\ k = 2^r,\ r = 1,\cdots,7$, and $\alpha = 3.9$, the $\vec{\varepsilon}$–uniform order of convergence and the $\vec{\varepsilon}$–uniform error constant are computed using the general methodology from [3]. The results are presented in Table 1.

TABLE 1. *Values of* D_ε^N, D^N, p^N, p^* *and* $C_{p^*}^N$ *for* $\varepsilon_1 = \dfrac{\eta}{8}$, $\varepsilon_2 = \eta$.

η	Number of mesh points N					
	32	64	128	256	512	1024
2^0	0.251E-02	0.122E-02	0.602E-03	0.299E-03	0.149E-03	0.743E-04
2^{-4}	0.257E-02	0.684E-03	0.175E-03	0.441E-04	0.134E-04	0.660E-05
2^{-8}	0.102E-01	0.761E-02	0.259E-02	0.687E-03	0.176E-03	0.443E-04
2^{-12}	0.319E-02	0.204E-02	0.119E-02	0.835E-03	0.619E-03	0.289E-03
2^{-16}	0.319E-02	0.204E-02	0.118E-02	0.833E-03	0.619E-03	0.289E-03
2^{-20}	0.319E-02	0.204E-02	0.118E-02	0.833E-03	0.618E-03	0.289E-03
2^{-24}	0.318E-02	0.204E-02	0.118E-02	0.833E-03	0.618E-03	0.289E-03
2^{-28}	0.318E-02	0.204E-02	0.118E-02	0.833E-03	0.618E-03	0.289E-03
D^N	0.102E-01	0.761E-02	0.259E-02	0.835E-03	0.619E-03	0.289E-03
p^N	0.418E+00	0.156E+01	0.163E+01	0.431E+00	0.110E+01	
C_p^N	0.172E+00	0.172E+00	0.782E-01	0.337E-01	0.334E-01	0.209E-01
The order of $\vec{\varepsilon}$-uniform convergence $p^* = 0.418$						
Computed $\vec{\varepsilon}$-uniform error constant, $C_{p^*}^N = 0.172$						

Example 7.2. *Consider*

$$-E\vec{u}''(x) + A(x)\vec{u}(x) = \vec{f}(x) \textit{ for } x \in (0, 0.5) \cup (0.5, 1),\ \vec{u}(0) = \vec{2},\ \vec{u}(1) = \vec{2}$$

where $E = \text{diag}(\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3)$, $A = \begin{pmatrix} 8 & -1 & -2 \\ -x & 6+x & -1 \\ -1 & -(1+x^2) & 7+x \end{pmatrix}$, $\vec{f} = (1+2x,\ 2,\ 1+x)^T$ for $0 < x < 0.5$ and $\vec{f} = (2x+3,\ 0,\ 2-x^2)^T$ for $0.5 < x < 1$. It is seen that all the three components of the source function $\vec{f}$ have a discontinuity at $x = 0.5$. For various values of $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$, $N = 24k$, $k = 2^r$, $r = 2, \dots, 7$, and $\alpha = 4.9$, the $\vec{\varepsilon}$–uniform order of convergence and the $\vec{\varepsilon}$–uniform error constant are computed using the general methodology from [3]. The results are presented in Table 2.

Acknowledgment

The Authors wish to acknowledge the valuable suggestions made by the unknown referees.

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TABLE 2. *Values of* D_{ε}^{N}, D^{N}, p^{N}, p^{*} *and* $C_{p^*}^{N}$ *for* $\varepsilon_1 = \frac{\eta}{16}$, $\varepsilon_2 = \frac{\eta}{4}$, $\varepsilon_3 = \eta$.

η	Number of mesh points N				
	96	192	384	768	1536
2^{0}	0.515E-02	0.255E-02	0.127E-02	0.631E-03	0.315E-03
2^{-4}	0.274E-02	0.698E-03	0.176E-03	0.440E-04	0.110E-04
2^{-8}	0.281E-01	0.102E-01	0.274E-02	0.699E-03	0.176E-03
2^{-12}	0.700E-02	0.501E-02	0.294E-02	0.201E-02	0.102E-02
2^{-16}	0.699E-02	0.501E-02	0.295E-02	0.201E-02	0.102E-02
2^{-20}	0.699E-02	0.501E-02	0.295E-02	0.201E-02	0.102E-02
2^{-24}	0.699E-02	0.501E-02	0.295E-02	0.201E-02	0.102E-02
2^{-28}	0.699E-02	0.501E-02	0.295E-02	0.201E-02	0.102E-02
D^{N}	0.281E-01	0.102E-01	0.295E-02	0.201E-02	0.102E-02
p^{N}	0.146E+01	0.180E+01	0.553E+00	0.980E+00	
C_{p}^{N}	0.110E+01	0.588E+00	0.248E+00	0.248E+00	0.185E+00
Computed order of $\vec{\varepsilon}$-uniform convergence, $p^{*} = 0.553$					
Computed $\vec{\varepsilon}$-uniform error constant, $C_{p^*}^{N} = 1.101$					

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