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EXPLICIT-IMPLICIT SPLITTING SCHEMES FOR SOME SYSTEMS OF EVOLUTIONARY EQUATIONS

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This paper is dedicated to the 65-th birthday of Professor Francisco J. Lisbona

Abstract. In many applied problems, the individual components of the unknown vector are interconnected and therefore splitting schemes are applied in order to get a simple problem for evaluating unknowns at a new time level. On the basis of additive schemes (splitting schemes), there are constructed efficient computational algorithms for numerical solving the initial value problems for systems of time-dependent PDEs. The present paper deals with computational algorithms that are based on using explicit-implicit approximations in time. Typically, additive operator-difference schemes for systems of evolutionary equations are constructed for operators that are coupled in space. Here we investigate more general problems, where we have coupling of derivatives in time for components of the solution vector.

Key words. Evolutionary problem, splitting scheme, stability of operator-difference schemes, additive operator-difference schemes.

1. Introduction

In solving applied problems, we deal with boundary value problems for systems of time-dependent PDEs. To construct computational algorithms for such problems, we approximate the equations taking into account appropriate initial and boundary conditions. Approximation in space is based on finite difference schemes, finite element procedures or finite volume methods [8, 12, 19, 20]. Special requirements are applied to the approximation in time for numerical solving problems for systems of equations [1, 9, 13]. In addition to general requirements to satisfy the conditions of approximation and stability, it is necessary to keep in mind the issues of computational implementation of the constructed schemes, i.e., the issue how to solve of the corresponding discrete problem at a new time level. In this regard, the most impressive results are associated with the construction of special additive operator-difference schemes (splitting schemes) [15, 24].

Additive schemes (operator-splitting schemes) are used to solve various unsteady problems [15, 20, 24, 30]. They are designed for the efficient computational implementation of the corresponding discrete problem defining the approximate solution at a new time level. The transition to a chain of simpler problems allows us to construct efficient difference schemes. We speak of splitting with respect to spatial variables (locally one-dimensional schemes). In some cases, it is useful to separate subproblems of distinct nature — we have splitting into physical processes. Regionally additive schemes (domain decomposition methods) are focused on constructing computational algorithms for parallel computers.

The main theoretical results on stability and convergence of additive schemes have been obtained for scalar evolutionary first-order equations and, in some cases,

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for second-order equations [15, 20, 24, 30]. In computational practice, it is essential to construct splitting schemes for systems of evolutionary equations. For example, vector problems have individual components of the unknown vector that are interconnected with each other. In this case, the use of appropriate splitting schemes is intended to obtaining simple problems for the individual components of the solution at a new time level.

For standard parabolic and hyperbolic systems of equations with a self-adjoint elliptic operator, additive schemes have been constructed in [20] using the regularization principle for difference schemes. Splitting schemes for systems of equations can be constructed employing the triangular splitting of a problem operator into the sum of operators adjoint to each other, i.e., using the alternating triangle method developed by Samarskii. Additive schemes of this type were used in [14] for dynamic problems of elasticity. A similar approach [25, 29] was applied to problems of an incompressible fluid with a variable viscosity. Additive schemes for transient problems of electrodynamics were considered in [27].

The above-mentioned classes of additive operator-difference schemes for evolutionary equations are based on an additive splitting of the leading operator into several terms. For many problems of practical interest, it is interesting to investigate the problems that have an additive representation for the operator at the time derivative. In the first publication on this subject [28], there were proposed and examined vector additive operator-difference schemes, where the operator at the time derivative was split into the sum of self-adjoint and positive definite operators.

Among additive schemes, we highlight explicit-implicit schemes, where the different nature of terms of the problem operator is taken into account via inhomogeneous approximations in time. Explicit-implicit schemes are widely used for the numerical solution of convection-diffusion problems. Various variants of inhomogeneous discretization in time are given in [2]. One or another explicit approximation is applied to the convective transport operator, whereas the diffusive transport operator is approximated implicitly. Thus, the most severe restrictions on a time step due to diffusion are removed. In view of the subordination of the convective transport operator to the diffusive transport operator, we have already proved unconditional stability of the above-considered explicit-implicit schemes for time-dependent convection-diffusion problems. Similar techniques are used in the analysis of diffusion-reaction problems. In this case (see, e.g., [17]), the diffusive transport is treated implicitly, whereas for reactions (source terms), explicit approximations are used. Such explicit approximations demonstrate obvious advantages for problems with nonlinear terms describing reaction processes. Detailed consideration of the implicit-explicit (IMEX) algorithms is given in the book [11] containing references to other works in this field of research.

In this paper, we propose splitting schemes for additive representation of the leading operator of the problem, i.e., the operator at the time derivative. We separate the diagonal part of a problem operator matrix and employ explicit-implicit approximations in time. The paper is organized as follows. In Section 1, we formulate the initial-boundary value problem for a system of PDEs. After some discretization in time, we obtain the Cauchy problem for a system of evolutionary equations. The standard two-level operator-difference scheme is discussed in Section 2. Section 3 deals with the construction of the explicit-implicit scheme by means of separating the diagonal part of the leading operator of the problem. The general problem of splitting the operator at the time derivative is discussed in Section 4.

Explicit-implicit schemes of first-order approximation in time are constructed and investigated.

2. Boundary Value Problems for Systems of Equations

We consider the boundary value problem for the system of coupled parabolic equations in a bounded domain Ω. For the individual components $u_\alpha(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \Omega$, $\alpha = 1, 2, ..., p$, we have

$$\sum_{\beta=1}^{p} c_{\alpha\beta}(\boldsymbol{x}) \frac{\partial u_\beta}{\partial t} - \sum_{\beta=1}^{p} \operatorname{div}(k_{\alpha\beta}(\boldsymbol{x}) \operatorname{grad} u_\beta) = f_\alpha(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Omega. \tag{1}$$

System of equations (1) is supplemented with the following boundary and initial conditions, respectively:

$$u_\alpha(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \partial\Omega, \quad 0 < t \leq T, \tag{2}$$

$$u_\alpha(\boldsymbol{x}, 0) = u_\alpha^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \quad \alpha = 1, 2, \ldots, p. \tag{3}$$

We formulate the main restrictions on the coefficients for problem (1)–(3). The system of parabolic equations is considered under the restrictions

$$c_{\alpha\beta} = c_{\beta\alpha}, \quad \sum_{\alpha,\beta=1}^{p} c_{\alpha\beta} \xi_\alpha \xi_\beta \geq \delta \sum_{\alpha=1}^{p} \xi_\alpha^2, \quad \delta > 0,$$

$$k_{\alpha\beta} = k_{\beta\alpha}, \quad \sum_{\alpha,\beta=1}^{p} k_{\alpha\beta} \xi_\alpha \xi_\beta \geq \kappa \sum_{\alpha=1}^{p} \xi_\alpha^2, \quad \kappa > 0.$$

For real matrices $C = \{c_{\alpha\beta}\}$ and $K = \{k_{\alpha\beta}\}$, we have

$$C = C^* \geq \delta I, \quad K = K^* \geq \kappa I,$$

where I is the $p \times p$ identity matrix.

The above boundary value problems for systems of parabolic equations arise in many applied problems. The first example is the system of coupled parabolic equations of second order describing mass transfer in multicomponent media [7, 26]. The coefficients $k_{\alpha\beta}$ describe the diffusion phenomena: main-term diffusion at $\alpha = \beta$ and cross-term diffusion coefficients at $\alpha \neq \beta$. For multicomponent media, we have $c_{\alpha\beta} = \delta_{\alpha\beta} c_\alpha$, where $\delta_{\alpha\beta}$ is Kronecker's delta. The second example concerns fluid motion in porous media. The governing equations for a flow in fractured porous media employ the multiple porosity model (see, e.g., [3, 4]). In this case, $u_\alpha(\boldsymbol{x}, t)$ is the dynamic pore pressure in the p-porosity model. For these models, it is the principal moment that $c_{\alpha\beta} \neq \delta_{\alpha\beta} c_\alpha$. The third example of systems of coupled diffusion equations is the diffusion-chemotaxis-reaction processes. Such problems are studied numerically, for example, in [5, 6]. As in other applied problems, here the matrix K is non-symmetric and the equations themselves are nonlinear.

After approximation in space, from problem (1) with boundary conditions (2), we arrive at a system of ODEs. Let us formulate the corresponding Cauchy problem. In this paper, we restricted ourselves to the formulation of stability conditions for two- and three-level operator-difference schemes. Using the derived estimate for stability, we can consider the corresponding problem for error and obtain the estimate for convergence. Such an examination is appropriate to carry out taking into account the selected approximations in space, the smoothness of the input data and solution.

Let H_α, $\alpha = 1, 2, \ldots, p$ be finite-dimensional real Hilbert (Euclidean) spaces of grid functions, where the scalar product and the norm are denoted by $(\cdot, \cdot)_\alpha$ and

$\|\cdot\|_\alpha,\ \alpha = 1,2,\dots,p$, respectively. The individual components of the solution are denoted by $u_\alpha(t) \in H_\alpha,\ \alpha = 1,2,\dots,p$ for every t $(0 \le t \le T,\ T > 0)$. We search the solution for the system of evolutionary equations of first order:

$$\sum_{\beta=1}^{p} B_{\alpha\beta}\frac{du_\beta}{dt} + \sum_{\beta=1}^{p} A_{\alpha\beta}u_\beta = f_\alpha, \quad \alpha = 1,2,\dots,p. \tag{4}$$

Here $f_\alpha(t) \in L_2(0,T;H_\alpha),\ \alpha = 1,2,\dots,p$ are specified, and $B_{\alpha\beta},\ A_{\alpha\beta}$ are linear constant (independent of t) operators acting from H_β onto H_α $(A_{\alpha\beta} : H_\beta \to H_\alpha$, $B_{\alpha\beta} : H_\beta \to H_\alpha)$ for all $\alpha, \beta = 1,2,\dots,p$. System of equations (4) is supplemented with the initial data

$$u_\alpha(0) = v_\alpha^0, \quad \alpha = 1,2,\dots,p. \tag{5}$$

We treat system of equations (4) as a single evolutionary equation for vector $\boldsymbol{u} = \{u_1, u_2, \dots, u_p\}$:

$$\boldsymbol{B}\frac{d\boldsymbol{u}}{dt} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}(t), \quad 0 < t \le T, \tag{6}$$

where $\boldsymbol{f} = \{f_1, f_2, \dots, f_p\}$, and the elements of the operator matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are represented in the form

$$\boldsymbol{A} = \{A_{\alpha\beta}\}, \quad \boldsymbol{B} = \{B_{\alpha\beta}\}, \quad \alpha, \beta = 1,2,\dots,p.$$

On the direct sum of spaces [10] $\boldsymbol{H} = H_1 \oplus H_2 \oplus \dots \oplus H_p$, we put

$$(\boldsymbol{u}, \boldsymbol{v}) = \sum_{\alpha=1}^{p} (u_\alpha, v_\alpha)_\alpha, \quad \|\boldsymbol{u}\|^2 = \sum_{\alpha=1}^{p} \|u_\alpha\|_\alpha^2.$$

In view of (5), we define

$$\boldsymbol{u}(0) = \boldsymbol{v}^0, \tag{7}$$

where $\boldsymbol{v}^0 = \{v_1^0, v_2^0, \dots, v_p^0\}$.

Consider Cauchy problem (6), (7) under the condition that operators $\boldsymbol{A}$ and $\boldsymbol{B}$ are self-adjoint and positive definite in $\boldsymbol{H}$, i.e.,

$$\boldsymbol{A} = \boldsymbol{A}^* \ge \delta_A \boldsymbol{E}, \quad \delta_A > 0, \quad \boldsymbol{B} = \boldsymbol{B}^* \ge \delta_B \boldsymbol{E}, \quad \delta_B > 0, \tag{8}$$

where $\boldsymbol{E}$ is the identity operator in $\boldsymbol{H}$. The self-adjointness is associated with the fulfillment of the equalities

$$A_{\alpha\beta} = A_{\beta\alpha}^*, \quad B_{\alpha\beta} = B_{\beta\alpha}^*, \quad \alpha, \beta = 1,2,\dots,p$$

for the operators of the original system of equations (4).

Here is an elementary a priori estimate for the solution of Cauchy problem (6), (7). We will use it as a guide in investigating the corresponding operator-difference schemes. For $\boldsymbol{D} = \boldsymbol{D}^* > 0$, we use notation $\boldsymbol{H}_{\boldsymbol{D}}$ for a space $\boldsymbol{H}$ equipped with the scalar product $(\boldsymbol{y}, \boldsymbol{w})_{\boldsymbol{D}} = (\boldsymbol{D}\boldsymbol{y}, \boldsymbol{w})$ and the norm $\|\boldsymbol{y}\|_{\boldsymbol{D}} = (\boldsymbol{D}\boldsymbol{y}, \boldsymbol{y})^{1/2}$.

Multiplying both sides of equation (6) scalarly in $\boldsymbol{H}$ by $\dfrac{d\boldsymbol{u}}{dt}$, we obtain

$$\left(\boldsymbol{B}\frac{d\boldsymbol{u}}{dt}, \frac{d\boldsymbol{u}}{dt}\right) + \frac{1}{2}\frac{d}{dt}(\boldsymbol{A}\boldsymbol{u}, \boldsymbol{u}) = \left(\boldsymbol{f}, \frac{d\boldsymbol{u}}{dt}\right).$$

Taking into account (8) and using

$$\left(\boldsymbol{f}, \frac{d\boldsymbol{u}}{dt}\right) \le \left(\boldsymbol{B}\frac{d\boldsymbol{u}}{dt}, \frac{d\boldsymbol{u}}{dt}\right) + \frac{1}{4}\left(\boldsymbol{B}^{-1}\boldsymbol{f}, \boldsymbol{f}\right),$$

we derive the inequality

$$\frac{d}{dt}\|\boldsymbol{u}\|_{\boldsymbol{A}}^2 \leq \frac{1}{2}\|\boldsymbol{f}\|_{\boldsymbol{B}^{-1}}^2.$$

We get from it the following a priori estimate:

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{A}}^2 \leq \|\boldsymbol{v}^0\|_{\boldsymbol{A}}^2 + \frac{1}{2}\int_0^t \|\boldsymbol{f}(\theta)\|_{\boldsymbol{B}^{-1}}^2 d\theta, \tag{9}$$

which expresses the stability of the solution of problem (6), (7) with respect to the initial data and the right-hand side.

3. Scheme with Weights

To solve numerically the operator-differential problem (6), (7), we use the standard scheme with weights. We introduce a uniform grid in time

$$\overline{\omega}_\tau = \omega_\tau \cup \{T\} = \{t_n = n\tau, \quad n = 0, 1, ..., N_0, \quad \tau N_0 = T\}$$

and, we denote $\boldsymbol{y}^n = \boldsymbol{y}(t^n)$, $t^n = n\tau$. Let us approximate equation (6) by the two-level difference scheme

$$\boldsymbol{B}\frac{\boldsymbol{y}^{n+1} - \boldsymbol{y}^n}{\tau} + \boldsymbol{A}(\sigma \boldsymbol{y}^{n+1} + (1-\sigma)\boldsymbol{y}^n) = \boldsymbol{\varphi}^n, \tag{10}$$

where σ is a numerical parameter (weight) within $0 \leq \sigma \leq 1$, and, e.g., $\boldsymbol{\varphi}^n = \boldsymbol{f}(\sigma \boldsymbol{t}^{n+1} + (1-\sigma)\boldsymbol{t}^n)$. For simplicity, we restrict ourselves to the case of the same weight for all equations in system (4). In view of (7), we supplement (10) with the initial data

$$\boldsymbol{y}^0 = \boldsymbol{v}^0. \tag{11}$$

A detailed study of the scheme with weights (the necessary and sufficient condition for stability as well as the choice of a norm) was conducted in [20, 21]. Here we restrict ourselves to an elementary estimate for stability of operator-difference scheme (10), (11). Estimate (9) serves us as a guide in our study.

Theorem 1. *If $\sigma \geq 1/2$, then operator-difference scheme (10) is unconditionally stable in $\boldsymbol{H}_{\boldsymbol{A}}$, and the difference solution satisfies the levelwise estimate*

$$\|\boldsymbol{y}^{n+1}\|_{\boldsymbol{A}}^2 \leq \|\boldsymbol{y}^n\|_{\boldsymbol{A}}^2 + \frac{\tau}{2}\|\boldsymbol{\varphi}^n\|^2_{\left(\boldsymbol{B}+\left(\sigma-\frac{1}{2}\right)\tau\boldsymbol{A}\right)^{-1}}. \tag{12}$$

Proof. Scheme (10) can be written in the form

$$\left(\boldsymbol{B} + \left(\sigma - \frac{1}{2}\right)\tau\boldsymbol{A}\right)\frac{\boldsymbol{y}^{n+1} - \boldsymbol{y}^n}{\tau} + \boldsymbol{A}\frac{\boldsymbol{y}^{n+1} + \boldsymbol{y}^n}{2} = \boldsymbol{\varphi}^n.$$

Multiplying both sides of this equation scalarly in $\boldsymbol{H}$ by $2(\boldsymbol{y}^{n+1} - \boldsymbol{y}^n)$, we obtain the equality

$$\begin{aligned} 2\tau&\left(\left(\boldsymbol{B} + \left(\sigma - \frac{1}{2}\right)\tau\boldsymbol{A}\right)\frac{\boldsymbol{y}^{n+1} - \boldsymbol{y}^n}{\tau}, \frac{\boldsymbol{y}^{n+1} - \boldsymbol{y}^n}{\tau}\right) \\ &+ (\boldsymbol{A}\boldsymbol{y}^{n+1}, \boldsymbol{y}^{n+1}) - (\boldsymbol{A}\boldsymbol{y}^n, \boldsymbol{y}^n) = 2\tau\left(\boldsymbol{\varphi}^n, \frac{\boldsymbol{y}^{n+1} - \boldsymbol{y}^n}{\tau}\right). \end{aligned}$$

Using the inequality

$$\begin{aligned} \left(\boldsymbol{\varphi}^n, \frac{\boldsymbol{y}^{n+1} - \boldsymbol{y}^n}{\tau}\right) \leq &\left(\left(\boldsymbol{B} + \left(\sigma - \frac{1}{2}\right)\tau\boldsymbol{A}\right)\frac{\boldsymbol{y}^{n+1} - \boldsymbol{y}^n}{\tau}, \frac{\boldsymbol{y}^{n+1} - \boldsymbol{y}^n}{\tau}\right) \\ &+ \frac{1}{4}\left(\left(\boldsymbol{B} + \left(\sigma - \frac{1}{2}\right)\tau\boldsymbol{A}\right)^{-1}\boldsymbol{\varphi}^n, \boldsymbol{\varphi}^n\right), \end{aligned}$$

we derive the required estimate (12). □

Estimate (12) is just a discrete analog of estimate (9) and it ensures the unconditional stability of the difference scheme with weights (10), (11) under the natural condition $\sigma \geq 1/2$. Considering the corresponding problem for the error, we prove the convergence $\mathcal{O}((2\sigma - 1)\tau + \tau^2)$ of the solution of operator-difference problem (6), (7) to the solution of operator-differential problem (6), (7) in $\boldsymbol{H_A}$ under the restriction $\sigma \geq 1/2$. If $\sigma = 1/2$, then we have the second-order convergence rate with respect to τ.

Operator-difference scheme (10) may be written in the canonical form for the two-level schemes:

$$(\boldsymbol{B} + \sigma\tau\boldsymbol{A})\frac{\boldsymbol{y}^{n+1} - \boldsymbol{y}^n}{\tau} + \boldsymbol{A}\boldsymbol{y}^n = \boldsymbol{\varphi}^n. \tag{13}$$

The transition to a new time level requires to solve the problem

$$(\boldsymbol{B} + \sigma\tau\boldsymbol{A})\boldsymbol{y}^{n+1} = \boldsymbol{\psi}^n.$$

Concerning original problem (4), (5), we must solve the system of coupled equations

$$\sum_{\beta=1}^{p}(B_{\alpha\beta} + \sigma\tau A_{\alpha\beta})y_\beta^{n+1} = \psi_\alpha^n, \quad \alpha = 1, 2, \ldots, p.$$

Various iterative methods can be used for this procedure [18, 23].

Another opportunity is to take into account the specific features of the above unsteady problems and to construct splitting schemes, where the transition to a new time level involves the solution of simpler problems. For the problems of type (4), (5), it seems reasonable to employ the splitting schemes, where the transition to a new time level is performed solving the problems

$$(B_{\alpha\alpha} + \sigma\tau A_{\alpha\alpha})y_\alpha^{n+1} = \widetilde{\psi}_\alpha^n, \quad \alpha = 1, 2, \ldots, p.$$

This means that we have to invert only the diagonal part of the operator matrix $\boldsymbol{B} + \sigma\tau\boldsymbol{A}$ in our computations.

4. Schemes with a Diagonal Operator

We start with the case, where the problem of inversion of the operator $\boldsymbol{B}$ does not exist. Such a situation occurs if the operator matrix $\boldsymbol{B}$ at the time derivatives is diagonal, i.e.,

$$B_{\alpha\beta} = \delta_{\alpha\beta}B_\alpha, \quad \alpha = 1, 2, \ldots, p. \tag{14}$$

This class of problems appears in simulation of mass transfer in multicomponent media. In this case, the components of the solution vector are coupled due to the elements $A_{\alpha\beta},\ \alpha \neq \beta$ in the operator matrix $\boldsymbol{A}$.

Let us construct additive operator-difference schemes using the splitting of operator $\boldsymbol{A}$ with separation of the diagonal part. In this case, we obtain

$$\boldsymbol{A} = \boldsymbol{A}_0 + \boldsymbol{A}_1, \quad \boldsymbol{A}_0 = \operatorname{diag}(A_{11}, A_{22}, ..., A_{pp}). \tag{15}$$

In additive representation (15), we have

$$\boldsymbol{A}_0 = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{pp} \end{pmatrix}, \quad \boldsymbol{A}_1 = \begin{pmatrix} 0 & A_{12} & \cdots & A_{1p} \\ A_{21} & 0 & \cdots & A_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ A_{p1} & A_{p2} & \cdots & 0 \end{pmatrix}.$$

In our problem, in view of (8), we get

$$\boldsymbol{A}_0 + \boldsymbol{A}_1 \geq \delta_A \boldsymbol{E}, \quad \delta_A > 0. \tag{16}$$

Let us consider problem (4), (5) under the additional assumption:

$$\boldsymbol{A}_0 - \boldsymbol{A}_1 \geq 0. \tag{17}$$

In some cases, property (17) follows from (16).

The properties of operator $\boldsymbol{A}$ are associated with the properties of matrix K, i.e., with the coefficients $k_{\alpha\beta}, \alpha, \beta = 1, 2, ..., p$ in the boundary value problem (1)–(3). The positive definiteness of operator $\boldsymbol{A}$ follows from the positive definiteness of matrix K. In view of (15), the fulfillment of (17) may be associated with

$$\widetilde{K} = \widetilde{K}^* \geq 0, \quad \widetilde{K} = \{\widetilde{k}_{\alpha\beta}\},$$

where

$$\widetilde{k}_{\alpha\alpha} = k_{\alpha\alpha}, \quad \widetilde{k}_{\alpha\beta} = -k_{\alpha\beta}, \quad \alpha, \beta = 1, 2, ..., p.$$

For the system of two equations ($p = 2$), from $\boldsymbol{A}_0 + \boldsymbol{A}_1 \geq 0$, it follows immediately (17). For $p = 3$, we can highlight the case with non-positive off-diagonal coefficients:

$$k_{\alpha\beta} \leq 0, \quad \alpha, \beta = 1, 2, 3, \quad \alpha \neq \beta.$$

Under these restrictions, we have $\widetilde{K} = \{|k_{\alpha\beta}|\} \geq 0$ [16], which ensures the fulfillment of (17). For general systems, we emphasize the case of diagonal dominance of matrix K, where

$$k_{\alpha\alpha} \geq \sum_{\alpha \neq \beta = 1}^{p} |k_{\alpha\beta}|.$$

These examples demonstrate the fulfillment of conditions (16), (17) in a number of problems with the decomposition (15).

For numerical solving (6), (7) under constraints (14)–(16), we will use the two-level scheme:

$$\boldsymbol{B} \frac{\boldsymbol{y}^{n+1} - \boldsymbol{y}^n}{\tau} + \boldsymbol{A}_0 \boldsymbol{y}^{n+1} + \boldsymbol{A}_1 \boldsymbol{y}^n = \boldsymbol{\varphi}^n. \tag{18}$$

This scheme with inhomogeneous approximation in time belongs to the class of explicit-implicit schemes. Here only the diagonal part of the operator $\boldsymbol{A}$ is shifted to the upper time level. The computational implementation of the explicit-implicit scheme (18) is conducted by means of problems

$$(B_{\alpha\alpha} + \tau A_{\alpha\alpha}) y_\alpha^{n+1} = \psi_\alpha^n, \quad \alpha = 1, 2, ..., p$$

at the new time level. For these individual problems, it is possible to arrange independent (parallel) computing $y_\alpha^{n+1}, \ \alpha = 1, 2, ..., p$.

The main result on stability of the explicit-implicit scheme is formulated as the following statement.

Theorem 2. *If (17) holds, then explicit-implicit difference scheme satisfying (8), (15), (18) is unconditionally stable, and for the difference solution the following levelwise estimate is valid:*

$$\|\boldsymbol{y}^{n+1}\|_{\boldsymbol{A}}^2 \leq \|\boldsymbol{y}^n\|_{\boldsymbol{A}}^2 + \frac{\tau}{2} \|\boldsymbol{\varphi}^n\|^2_{\left(\boldsymbol{B} + \frac{\tau}{2}(\boldsymbol{A}_0 - \boldsymbol{A}_1)\right)^{-1}}. \tag{19}$$

Proof. For the proof, we write the explicit-implicit scheme in the form

$$\left(\boldsymbol{B} + \frac{\tau}{2}(\boldsymbol{A}_0 - \boldsymbol{A}_1)\right) \frac{\boldsymbol{y}^{n+1} - \boldsymbol{y}^n}{\tau} + \boldsymbol{A} \frac{\boldsymbol{y}^{n+1} + \boldsymbol{y}^n}{2} = \boldsymbol{\varphi}^n$$

and we multiply it scalarly in $\boldsymbol{H}$ by $2(\boldsymbol{y}^{n+1} - \boldsymbol{y}^n)$. Further arguments are similar to the proof of Theorem 1. □

Estimate (19) for stability with respect to the initial data and the right-hand side, proven for the explicit-implicit scheme (18), is not significantly different from the estimate corresponding to the standard scheme with weights (see (12)).

Explicit-implicit scheme (18) approximates equation (6) with the first order with respect to τ. It is possibile to construct unconditionally stable explicit-implicit schemes with second-order approximation in time. Instead of (18), we can use the three-level explicit-implicit scheme:

$$\boldsymbol{B}\frac{\boldsymbol{y}^{n+1} - \boldsymbol{y}^{n-1}}{2\tau} + \boldsymbol{A}_0(\sigma \boldsymbol{y}^{n+1} + (1-2\sigma)\boldsymbol{y}^n + \sigma \boldsymbol{y}^{n-1}) + \boldsymbol{A}_1 \boldsymbol{y}^n = \boldsymbol{\varphi}^n \tag{20}$$

with $\boldsymbol{\varphi}^n = \boldsymbol{f}(\boldsymbol{t}^n)$. To calculate the first step, we can apply, e.g, the two-level scheme

$$\boldsymbol{B}\frac{\boldsymbol{y}^1 - \boldsymbol{y}^0}{\tau} + (\boldsymbol{A}_0 + \boldsymbol{A}_1)\frac{\boldsymbol{y}^1 + \boldsymbol{y}^0}{2} = \frac{\boldsymbol{\varphi}^1 + \boldsymbol{\varphi}^0}{2}.$$

Investigation of stability is based on the following general statement from the theory of stability (correctness) for three-level operator-difference schemes [20, 21, 22].

Lemma 1. *Let in the three-level operator-difference scheme*

$$\boldsymbol{B}\frac{\boldsymbol{y}^{n+1} - \boldsymbol{y}^{n-1}}{2\tau} + \boldsymbol{D}\frac{\boldsymbol{y}^{n+1} - 2\boldsymbol{y}^n + \boldsymbol{y}^{n-1}}{\tau^2} + \boldsymbol{A}\boldsymbol{y}^n = \boldsymbol{\varphi}^n \tag{21}$$

operators $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{D}$ *are constant (independent of* n*) and*

$$\boldsymbol{A} = \boldsymbol{A}^* > 0, \quad \boldsymbol{B} = \boldsymbol{B}^* > 0, \quad \boldsymbol{D} = \boldsymbol{D}^* > 0. \tag{22}$$

If

$$\boldsymbol{D} > \frac{\tau^2}{4}\boldsymbol{A}, \tag{23}$$

then scheme (21), (22) is unconditionally stable and its solution satisfies the estimate

$$\mathcal{E}^{n+1} \le \mathcal{E}^n + \frac{\tau}{2}(\boldsymbol{B}^{-1}\boldsymbol{\varphi}^n, \boldsymbol{\varphi}^n), \tag{24}$$

where

$$\mathcal{E}^n = \left\| \frac{\boldsymbol{y}^n + \boldsymbol{y}^{n-1}}{2} \right\|^2_{\boldsymbol{A}} + \left\| \frac{\boldsymbol{y}^n - \boldsymbol{y}^{n-1}}{\tau} \right\|^2_{\boldsymbol{D} - \frac{\tau^2}{4}\boldsymbol{A}}.$$

Proof. Taking into account

$$\boldsymbol{y}^n = \frac{1}{4}(\boldsymbol{y}^{n+1} + 2\boldsymbol{y}^n + \boldsymbol{y}^{n-1}) - \frac{1}{4}(\boldsymbol{y}^{n+1} - 2\boldsymbol{y}^n + \boldsymbol{y}^{n-1}),$$

we write (21) as

$$\begin{aligned} \boldsymbol{B}\frac{\boldsymbol{y}^{n+1} - \boldsymbol{y}^{n-1}}{2\tau} &+ \left(\boldsymbol{D} - \frac{\tau^2}{4}\boldsymbol{A}\right)\frac{\boldsymbol{y}^{n+1} - 2\boldsymbol{y}^n + \boldsymbol{y}^{n-1}}{\tau^2} \\ &+ \boldsymbol{A}\frac{\boldsymbol{y}^{n+1} - 2\boldsymbol{y}^n + \boldsymbol{y}^{n-1}}{4} = \boldsymbol{\varphi}^n. \end{aligned} \tag{25}$$

Let

$$\boldsymbol{v}^n = \frac{1}{2}(\boldsymbol{y}^n + \boldsymbol{y}^{n-1}), \quad \boldsymbol{w}^n = \frac{\boldsymbol{y}^n - \boldsymbol{y}^{n-1}}{\tau}$$

and rewrite (25) in the form

$$\boldsymbol{B}\frac{\boldsymbol{w}^{n+1} + \boldsymbol{w}^n}{2} + \boldsymbol{R}\frac{\boldsymbol{w}^{n+1} - \boldsymbol{w}^n}{\tau} + \frac{1}{2}\boldsymbol{A}(\boldsymbol{v}^{n+1} + \boldsymbol{v}^n) = \boldsymbol{\varphi}^n, \tag{26}$$

where

$$\boldsymbol{R} = \boldsymbol{D} - \frac{\tau^2}{4}\boldsymbol{A}.$$

Multiplying scalarly both sides of (26) by

$$2(\boldsymbol{v}^{n+1} - \boldsymbol{v}^n) = \tau(\boldsymbol{w}^{n+1} + \boldsymbol{w}^n),$$

we get the equality

$$\begin{aligned}\frac{\tau}{2}(\boldsymbol{B}(\boldsymbol{w}^{n+1} + \boldsymbol{w}^n), \boldsymbol{w}^{n+1} + \boldsymbol{w}^n) + (\boldsymbol{R}(\boldsymbol{w}^{n+1} - \boldsymbol{w}^n), \boldsymbol{w}^{n+1} + \boldsymbol{w}^n) \\ + (\boldsymbol{A}(\boldsymbol{v}^{n+1} + \boldsymbol{v}^n), \boldsymbol{v}^{n+1} - \boldsymbol{v}^n) = \tau(\boldsymbol{\varphi}^n, \boldsymbol{w}^{n+1} + \boldsymbol{w}^n).\end{aligned} \tag{27}$$

For the right-hand side, we use the estimate

$$(\boldsymbol{\varphi}^n, \boldsymbol{w}^{n+1} + \boldsymbol{w}^n) \leq \frac{1}{2}(\boldsymbol{B}(\boldsymbol{w}^{n+1} + \boldsymbol{w}^n) + \frac{1}{2}(\boldsymbol{B}^{-1}\boldsymbol{\varphi}^n, \boldsymbol{\varphi}^n).$$

This makes it possible to get from (27) the inequality

$$\mathcal{E}^{n+1} \leq \mathcal{E}^n + \frac{\tau}{2}(\boldsymbol{B}^{-1}\boldsymbol{\varphi}^n, \boldsymbol{\varphi}^n), \tag{28}$$

where we use the notation

$$\mathcal{E}^n = (\boldsymbol{A}\boldsymbol{v}^n, \boldsymbol{v}^n) + (\boldsymbol{R}\boldsymbol{w}^n, \boldsymbol{w}^n).$$

Inequality (28) is the desired a priori estimate (24), if we show that $\mathcal{E}^n$ defines the squared norm of the difference solution. In view of the positivity of $\boldsymbol{A}$, it is sufficient to require the positivity for the operator $\boldsymbol{R}$ (see (23)). □

Scheme (20) may be written in the form (21) with

$$\boldsymbol{D} = \sigma\tau^2\boldsymbol{A}_0.$$

Taking this fact into account, stability condition (23) takes the form

$$0 < 4\sigma\boldsymbol{A}_0 - \boldsymbol{A} = 4\left(\sigma - \frac{1}{2}\right)\boldsymbol{A}_0 + \boldsymbol{A}_0 - \boldsymbol{A}_1.$$

Under assumptions (15), (17), this condition will be true for $\sigma > 1/2$. Thus, we can formulate the following statement.

Theorem 3. *Explicit-implicit scheme (20) satisfying (8), (15) and (17) is unconditionally stable for $\sigma > 1/2$, and the difference solution satisfies levelwise estimate (24), where*

$$\mathcal{E}^n = \left\|\frac{\boldsymbol{y}^n + \boldsymbol{y}^{n-1}}{2}\right\|^2_{\boldsymbol{A}} + \tau^2\left\|\frac{\boldsymbol{y}^n - \boldsymbol{y}^{n-1}}{\tau}\right\|^2_{\sigma\boldsymbol{A}_0 - \frac{1}{4}\boldsymbol{A}}.$$

5. General Case

For problem (6), (7) with a common (not diagonal) operator $\boldsymbol{B}$ ($B_{\alpha\beta} \neq \delta_{\alpha\beta}B_\alpha$), explicit-implicit difference schemes will be based on a decomposition of operator $\boldsymbol{B}$. Similarly to (15), we set

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1, \quad \boldsymbol{B}_0 = \mathrm{diag}(B_{11}, B_{22}, ..., B_{pp}). \tag{29}$$

In addition to the positive definiteness of operator $\boldsymbol{B}$, we assume that the inequality

$$\boldsymbol{B}_0 - \boldsymbol{B}_1 \geq 0 \tag{30}$$

holds. Thus, coefficients $c_{\alpha\beta}, \alpha, \beta = 1, 2, ..., p$ of the matrix C in problem (1)–(3) are considered under restrictions similar to the formulated above for the coefficients $k_{\alpha\beta}, \alpha, \beta = 1, 2, ..., p$ of matrix K.

To solve numerically (6), (7) under the constraints (8), (15), (17), (29), (30), we will use the three-level scheme:

$$
\begin{aligned}
(31) \qquad & \boldsymbol{B}_0 \frac{\boldsymbol{y}^{n+1} - \boldsymbol{y}^n}{\tau} + \boldsymbol{B}_1 \frac{\boldsymbol{y}^n - \boldsymbol{y}^{n-1}}{\tau} \\
& \qquad + \boldsymbol{A}_0(\sigma \boldsymbol{y}^{n+1} + (1-2\sigma)\boldsymbol{y}^n + \sigma \boldsymbol{y}^{n-1}) + \boldsymbol{A}_1 \boldsymbol{y}^n = \boldsymbol{\varphi}^n.
\end{aligned}
$$

Let us formulate the stability condition for this explicit-implicit scheme.

Theorem 4. *Explicit-implicit difference scheme (31) satisfying (8), (15), (17), (29), (30) is unconditionally stable for $\sigma > 1/2$, and difference solution satisfies the levelwise estimate (24), where*

$$
\mathcal{E}^n = \left\| \frac{\boldsymbol{y}^n + \boldsymbol{y}^{n-1}}{2} \right\|^2_{\boldsymbol{A}} + \tau^2 \left\| \frac{\boldsymbol{y}^n - \boldsymbol{y}^{n-1}}{\tau} \right\|^2_{\frac{1}{2\tau}(\boldsymbol{B}_0 - \boldsymbol{B}_1) + \sigma \boldsymbol{A}_0 - \frac{1}{4}\boldsymbol{A}}.
$$

Proof. We employ Lemma 1. Scheme (31) may be written in the form (21) with

$$
\boldsymbol{D} = \frac{\tau}{2}(\boldsymbol{B}_0 - \boldsymbol{B}_1) + \sigma \tau^2 \boldsymbol{A}_0.
$$

In the terms of this theorem, stability condition (23) will be valid for $\sigma > 1/2$. □

In many applied problems, conditions (17) and (30) are too strong. It is possible to consider problems with weaker restrictions

$$
\boldsymbol{A}_0 \geq \gamma_A \boldsymbol{A}_1, \quad \boldsymbol{B}_0 \geq \gamma_B \boldsymbol{B}_1,
$$

where $\gamma_A \geq 1$, $\gamma_B \geq 1$. Under these conditions it is possible to construct unconditionally stable explicit-implicit schemes, which are considered in the present paper in detail.

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