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A FULLY DISCRETE CALDERÓN CALCULUS FOR TWO DIMENSIONAL TIME HARMONIC WAVES

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Dedicated to Francisco 'Paco' Lisbona on the occasion of his 65th birthday

Abstract. In this paper, we present a fully discretized Calderón Calculus for the two dimensional Helmholtz equation. This full discretization can be understood as highly non-conforming Petrov-Galerkin methods, based on two staggered grids of mesh size h, Dirac delta distributions substituting acoustic charge densities and piecewise constant functions for approximating acoustic dipole densities. The resulting numerical schemes from this calculus are all of order h^2 provided that the continuous equations are well posed. We finish by presenting some numerical experiments illustrating the performance of this discrete calculus.

Key words. Calderón calculus, Boundary Element Methods, Dirac deltas distributions, Nyström methods.

1. Introduction

In this paper we present a very simple and compatible Nyström discretization of all boundary integral operators for the Helmholtz equation in a smooth parametrizable curve in the plane. The discretization uses a naïve quadrature method for logarithmic integral equations, based on two staggered grids, and due to Jukka Saranen and Liisa Schroderus [13] (see also [15] and [2]). This is combined with an equally simple staggered grid discretization of the hypersingular operator, recently discovered in [8]. If the displaced grids used for the discretization of these two operators are mutually reversed, then it is possible to combine these two discretizations with a simple minded Nyström method for the double layer operator and its adjoint. The complete set of operators is complemented with a fully discrete version of the single and double layer potentials. We will explain the construction of the discrete set and reinterpret it as a non-conforming Petrov- Galerkin discretization of the operators (using Dirac deltas and piecewise constant functions) to which we apply midpoint integration in every element integral.

Once the semivariational form has been reached we will show inf-sup conditions for all discrete operators involved and consistency error estimates based on asymptotic expansions of the error in the style of [2, 5, 6]. We will finally state and sketch the proof of some convergence error estimates. While some of the results, for individual equations (mainly based on indirect boundary integral formulations) had already appeared in previous papers, this is the first time that the entire Calderón Calculus is presented in its entirety. Let it be emphasized, that this is probably the simplest form of *discretizing simultaneously all the potentials and integral operators* for the Helmholtz equation in the plane and that the methods we obtain are of

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order two. Barring the conceptual difficulty of understanding the boundary integral operators, the methods have the simplicity of basic Finite Difference Methods and require no effort in their implementation: all discrete elements are described in full, natural data structures can be easily figured out from the way the geometry is sampled, and no additional discretization step (quadrature, assembly by element, mapping to a reference element) is required. The methods will be presented for the case of a single curve, but we will hint at its immediate extension to the case of multiple scatterers.

In a final section devoted to numerical experiments, we will show how to use the methods for transmission problems and how to construct combined field integral representations.

2. Calderón calculus for exterior Helmholtz boundary problems

2.1. Potentials and operators. Let Γ be a smooth simple closed curve given by a regular 1-periodic positively oriented parametrization $\mathbf{x} = (x_1, x_2) : \mathbb{R} \to \Gamma \subset \mathbb{R}^2$. Let $\mathbf{n}(t) := (x_2'(t), -x_1'(t))$ be a non-normalized outward pointing normal vector at $\mathbf{x}(t) \in \Gamma$. The domain exterior to Γ will be denoted Ω^+. As a reminder of the fact that we are taking limits from this exterior domain, the superscript $+$ will be used in trace and normal derivative operators.

Let us introduce the exterior Helmholtz equation

$$\Delta U + k^2 U = 0 \text{ in } \Omega^+, \quad \nabla U(\mathbf{z}) \cdot (\tfrac{1}{|\mathbf{z}|}\mathbf{z}) - \imath k U(\mathbf{z}) = o(\tfrac{1}{\sqrt{|\mathbf{z}|}}), \quad \text{as } |\mathbf{z}| \to \infty, \tag{1}$$

where $k > 0$ is the wavenumber. Given 1-periodic complex-valued functions η and ψ, the (parametrized) single and double layer potentials for the Helmholtz equation (1) are defined, respectively, with the formulas

$$\begin{aligned}
\big(\mathrm{S}\,\eta\big)(\mathbf{z}) &:= \frac{\imath}{4}\int_0^1 H_0^{(1)}(k|\mathbf{z} - \mathbf{x}(t)|)\eta(t)\,\mathrm{d}t,\\
\big(\mathrm{D}\,\psi\big)(\mathbf{z}) &:= \frac{\imath k}{4}\int_0^1 H_1^{(1)}(k|\mathbf{z} - \mathbf{x}(t)|)\frac{(\mathbf{z} - \mathbf{x}(t)) \cdot \mathbf{n}(t)}{|\mathbf{z} - \mathbf{x}(t)|}\psi(t)\,\mathrm{d}t
\end{aligned}$$

for arbitrary $\mathbf{z} \in \mathbb{R}^2 \setminus \Gamma$. (Here $H_n^{(1)}$ is the Hankel function of the first kind and order n.) The single and double layer potentials define radiating solutions of the Helmholtz equation for any η, ψ. Moreover, if U is a $\mathcal{C}^1(\overline{\Omega^+})$ solution of (1) and we define

$$\varphi = \gamma^+ U := U|_\Gamma \circ \mathbf{x}, \qquad \lambda = \partial_{\mathbf{n}}^+ U := ((\nabla U)|_\Gamma \circ \mathbf{x}) \cdot \mathbf{n}, \tag{2}$$

then [9, 14]

$$U(\mathbf{z}) = (\mathrm{D}\,\varphi)(\mathbf{z}) - (\mathrm{S}\,\lambda)(\mathbf{z}), \quad \mathbf{z} \in \Omega^+. \tag{3}$$

We note that the representation formula (3), depending on parametrized Cauchy data (2), can be extended to any locally H^1 solution of (1). In this work we will restrict our attention to smooth solutions though.

Associated to the layer potentials we have three integral operators.

$$(\mathrm{V}\eta)(s) := \frac{\imath}{4}\int_0^1 H_0^{(1)}(k|\mathbf{x}(s) - \mathbf{x}(t)|)\eta(t)\,\mathrm{d}t, \tag{4a}$$

$$(\mathrm{K}\psi)(s) := \frac{\imath k}{4}\int_0^1 H_1^{(1)}(k|\mathbf{x}(s) - \mathbf{x}(t)|)\frac{(\mathbf{x}(s) - \mathbf{x}(t)) \cdot \mathbf{n}(t)}{|\mathbf{x}(s) - \mathbf{x}(t)|}\psi(t)\,\mathrm{d}t, \tag{4b}$$

$$(\mathrm{J}\eta)(s) := \frac{\imath k}{4}\int_0^1 H_1^{(1)}(k|\mathbf{x}(s) - \mathbf{x}(t)|)\frac{(\mathbf{x}(t) - \mathbf{x}(s)) \cdot \mathbf{n}(s)}{|\mathbf{x}(s) - \mathbf{x}(t)|}\eta(t)\,\mathrm{d}t, \tag{4c}$$

as well as the integrodifferential operator

$$\mathrm{W}\psi := -(\mathrm{V}\psi')' - k^2 \mathrm{V}_{\mathbf{n}}\psi, \tag{4d}$$

where

$$(\mathrm{V}_{\mathbf{n}}\psi)(s) := \frac{\imath}{4}\int_0^1 H_0^{(1)}(k|\mathbf{x}(s)-\mathbf{x}(t)|)\big(\mathbf{n}(t)\cdot\mathbf{n}(s)\big)\psi(t)\,\mathrm{d}t.$$

The operators in (4) are respectively called single layer, double layer, adjoint double layer, and hypersingular operator. The operator W admits a different expression in terms of finite parts integrals (see [14, Lemma 2.5.6]), which is where its name comes from.

Layer operators and potentials are related via the so-called jump relations [9, 11, 14], namely, the exterior parametrized boundary values of the layer operators are given by the formulas

$$\begin{array}{rclcrcl} \gamma^+\mathrm{S}\,\eta &=& \mathrm{V}\eta, & & \gamma^+\mathrm{D}\,\psi &=& \frac{1}{2}\psi + \mathrm{K}\psi,\\ \partial_{\mathbf{n}}^+\mathrm{S}\,\eta &=& -\frac{1}{2}\eta + \mathrm{J}\eta, & & \partial_{\mathbf{n}}^+\mathrm{D}\,\psi &=& -\mathrm{W}\psi. \end{array} \tag{5}$$

The matrix of operators

$$\mathcal{C}^+ := \begin{bmatrix} \frac{1}{2}\mathrm{I}+\mathrm{K} & -\mathrm{V}\\ -\mathrm{W} & \frac{1}{2}\mathrm{I}-\mathrm{J}\end{bmatrix}$$

is the exterior Calderón projector. It follows from (3) and (5), that if (φ,λ) are the parametrized Cauchy data (2) for a solution of (1), then $\mathcal{C}^+(\varphi,\lambda)^\top = (\varphi,\lambda)^\top$ or, equivalently

$$\mathcal{D}^+\begin{bmatrix}\varphi\\ \lambda\end{bmatrix} := \begin{bmatrix} \frac{1}{2}\mathrm{I}-\mathrm{K} & \mathrm{V}\\ \mathrm{W} & \frac{1}{2}\mathrm{I}+\mathrm{J}\end{bmatrix}\begin{bmatrix}\varphi\\ \lambda\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}. \tag{6}$$

Note that K and J are transposed of each other, while V and W are symmetric.

2.2. Boundary integral equations for exterior problems. We next summarize a collection of boundary integral equations leading to the solution of (1) with a given boundary condition:

$$\gamma^+ U = \beta_0 \qquad \text{or} \qquad \partial_{\mathbf{n}}^+ U = \beta_1. \tag{7}$$

The data functions in the right-hand side of (7) are 1-periodic functions and the boundary operators are those of (2). Recall that the Dirichet or Neumann exterior problem for the Helmholtz equation with Sommerfeld radiation condition at infinity are uniquely solvable.

A *direct method* for solving the exterior Dirichlet problem starts in the representation formula (3), equates $\varphi = \beta_0$, and then uses one of the two identities in (6) to set up an integral equation in order to find λ. Similarly, for the Neumann problem, we impose $\lambda = \beta_1$, and then use one of the equations in (6) in search of φ. The resulting integral equations are collected in Table 1.

An *indirect method* based on the single layer potential representation looks for $U = \mathrm{S}\eta$ and then uses the expressions in the first column of (5) to set up an integral equation depending on which boundary data is known. Similarly, we can look for $U = \mathrm{D}\psi$ and use the boundary integral operators that appear in the right column of (5) to build an equation. These equations are gathered in Table 2.

Proposition 2.1 (See [12, Section 3.2])**.** *Let Ω be the domain interior to Γ.*

(a) *Equations* (dN01)*,* (iN01)*,* (dD01)*, and* (iD01) *are uniquely solvable if and only if $-k^2$ is not a Dirichlet eigenvalue of the Laplace operator in Ω.*
(b) *Equations* (dN02)*,* (iN02)*,* (dD02)*, and* (iD02) *are uniquely solvable if and only if $-k^2$ is not a Neumann eigenvalue of the Laplace operator in Ω.*

Table 1. BIEs for direct formulations. The representation formula is (3). All these equations are solvable. Uniqueness is discussed in Proposition 2.1.

Dirichlet	$\mathrm{V}\lambda = -\frac{1}{2}\varphi + \mathrm{K}\varphi$	$\varphi = \beta_0$	(dD01)
	$\frac{1}{2}\lambda + \mathrm{J}\lambda = -\mathrm{W}\varphi,$	$\varphi = \beta_0$	(dD02)
Neumann	$-\frac{1}{2}\varphi + \mathrm{K}\varphi = \mathrm{V}\lambda,$	$\lambda = \beta_1$	(dN01)
	$-\mathrm{W}\varphi = \frac{1}{2}\lambda + \mathrm{J}\lambda,$	$\lambda = \beta_1$	(dN02)

Table 2. BIEs for indirect formulations. The potential representation is given next to the boundary integral equation. Unique solvability of these equations is discussed in Proposition 2.1.

Dirichlet	$\mathrm{V}\eta = \beta_0,$	$U = \mathrm{S}\,\eta$	(iD01)
	$\frac{1}{2}\psi + \mathrm{K}\psi = \beta_0,$	$U = \mathrm{D}\,\psi$	(iD02)
Neumann	$-\frac{1}{2}\eta + \mathrm{J}\eta = \beta_1,$	$U = \mathrm{S}\,\eta$	(iN01)
	$\mathrm{W}\psi = -\beta_1,$	$U = \mathrm{D}\,\psi$	(iN02)

The equations of Tables 1 and 2 involve the four operators of the matrix $\mathcal{D}^+$ in (6) and their transposes. The operators in the first row of $\mathcal{D}^+$ are invertible when $-k^2$ is not an interior Dirichlet eigenvalue. The operators in the second row of $\mathcal{D}^+$ are invertible when $-k^2$ is not an interior Neumann eigenvalue. The precise Sobolev space setting where these equations are well posed will be explained in Section 4.1. In addition to these equations, the Calderón Calculus, given by the jump relations (5) and the identities (6), can be used to construct combined integral equations and several other associated boundary integral equations, some of which are invertible for all values of k.

3. The fully discrete calculus

3.1. Matrix representation. Let N be a positive integer, $h := 1/N$, and let us consider the uniform grid in parametric space

$$s_i := (i - \tfrac{1}{2})h, \quad t_i := ih, \qquad i \in \mathbb{Z},$$

thus defined so that t_i is the midpoint of the interval (s_i, s_{i+1}). The following quantities will be *all the geometric elements of* Γ that will be used in the discrete

Calculus:

$$\begin{array}{llllllll} \mathbf{m}_i & := & \mathbf{x}(t_i), & \mathbf{b}_i & := & \mathbf{x}(s_i), & & \\ \mathbf{n}_i & := & h\mathbf{n}(t_i), & \ell_i & := & |\mathbf{n}_i| = h|\mathbf{x}'(t_i)|, & & \mathbf{s}_i = h^2\mathbf{x}''(t_i). \end{array} \tag{8}$$

These quantities make up the *main discretization grid.* Note that they are defined for $i \in \mathbb{Z}$, modulo N. For practical reasons, we will need a discrete function $n(i)$, that gives the next index in a rotating (modulo N) form, so that $n(i) = i + 1$ for $i \le N - 1$ and $n(N) = 1$. We now take $\varepsilon \in (-1/2, 1/2) \setminus \{0\}$ and repeat the same construction with the displaced grid in parametric space:

$$t_i^\varepsilon = (i + \varepsilon)h, \qquad s_i^\varepsilon = (i + \varepsilon - \tfrac{1}{2})h.$$

The quantities $\mathbf{m}_i^\varepsilon$, $\mathbf{b}_i^\varepsilon$, $\mathbf{n}_i^\varepsilon$, ℓ_i^ε, and $\mathbf{s}_i^\varepsilon$ are defined accordingly. They constitute the *companion grid.*

Given column vectors $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_N)^\top \in \mathbb{C}^N$, $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_N)^\top \in \mathbb{C}^N$, we consider the discrete single and double layer potentials:

$$\mathrm{S}_h(\mathbf{z})\,\boldsymbol{\eta} \;:=\; \sum_{j=1}^{N} \frac{\imath}{4} H_0^{(1)}(k|\mathbf{z} - \mathbf{m}_j^\varepsilon|)\eta_j, \tag{9a}$$

$$\mathrm{D}_h(\mathbf{z})\,\boldsymbol{\psi} \;:=\; \sum_{j=1}^{N} \frac{\imath k}{4} H_1^{(1)}(k|\mathbf{z} - \mathbf{m}_j|)\frac{(\mathbf{z} - \mathbf{m}_j)\cdot \mathbf{n}_j}{|\mathbf{z} - \mathbf{m}_j|}\psi_j. \tag{9b}$$

We also consider four $N \times N$ matrices V_h, K_h, J_h and W_h, given by

$$\mathrm{V}_{ij} \;=\; \frac{\imath}{4} H_0^{(1)}(k|\mathbf{m}_i - \mathbf{m}_j^\varepsilon|), \tag{10a}$$

$$\mathrm{K}_{ij} \;:=\; \begin{cases} \dfrac{\mathbf{s}_i \cdot \mathbf{n}_i}{4\pi \ell_i^2}, & i = j, \\[2mm] \dfrac{\imath k}{4} H_1^{(1)}(k|\mathbf{m}_i - \mathbf{m}_j|)\dfrac{(\mathbf{m}_i - \mathbf{m}_j)\cdot \mathbf{n}_j}{|\mathbf{m}_i - \mathbf{m}_j|}, & i \neq j, \end{cases} \tag{10b}$$

$$\mathrm{J}_{ij} \;:=\; \begin{cases} \dfrac{\mathbf{s}_i^\varepsilon \cdot \mathbf{n}_i^\varepsilon}{4\pi (\ell_i^\varepsilon)^2}, & i = j, \\[2mm] \dfrac{\imath k}{4} H_1^{(1)}(k|\mathbf{m}_i^\varepsilon - \mathbf{m}_j^\varepsilon|)\dfrac{(\mathbf{m}_j^\varepsilon - \mathbf{m}_i^\varepsilon)\cdot \mathbf{n}_i^\varepsilon}{|\mathbf{m}_i^\varepsilon - \mathbf{m}_j^\varepsilon|}, & i \neq j, \end{cases} \tag{10c}$$

$$\mathrm{W}_{ij} \;:=\; \widetilde{\mathrm{V}}_{n(i),n(j)} + \widetilde{\mathrm{V}}_{ij} - \widetilde{\mathrm{V}}_{n(i),j} - \widetilde{\mathrm{V}}_{i,n(j)} - k^2(\mathbf{n}_i^\varepsilon \cdot \mathbf{n}_j)\mathrm{V}_{ji}, \tag{10d}$$

where

$$\widetilde{\mathrm{V}}_{ij} = \frac{\imath}{4} H_0^{(1)}(k|\mathbf{b}_i^\varepsilon - \mathbf{b}_j|).$$

Note that the diagonal values in K_h and J_h are defined using the limit values in the kernels of the integral operators K and J as $|s - t| \to 0$.

Remark 3.1. *As can be seen from* (9) *and* (10)*, the structure of the matrices and operators does not remember where the discrete geometric data come from. The formulas* (9) *and* (10) *use discrete data* $\{\mathbf{m}_i, \mathbf{n}_i, \mathbf{b}_i, \ell_i, \mathbf{s}_i\}$ *and* $\{\mathbf{m}_i^\varepsilon, \mathbf{n}_i^\varepsilon, \mathbf{b}_i^\varepsilon, \ell_i^\varepsilon, \mathbf{s}_i^\varepsilon\}$*, sampled from the curve. It is immaterial whether these data have been sampled from a simple curve or several simple non-intersecting curves. The next-index function* $n(i)$ *used in* W_h *has to be adapted to contain cycles of nodes showing the different connected components of the collection of curves.*

Discretization of the integral equations in Tables 1 and 2 is almost straightforward based on these matrices and potentials. The Dirichlet and Neumann data in (7) are discretized by vectors of samples:

$$\boldsymbol{\beta}_0 := (\beta_0(t_1), \ldots, \beta_0(t_N))^\top \qquad \boldsymbol{\beta}_1 := h\,(\beta_1(t_1^\varepsilon), \ldots, \beta_1(t_N^\varepsilon))^\top. \tag{11}$$

The different scaling of these vectors will be clear from the interpretation of these methods that we will give in Section 3.2. At this stage, it can be justified with some arguments of dimensional analysis, given the fact that β_1 corresponds to data of a derivative of the function. By the definition of the parametrized boundary operators (2), of the Cauchy data (7) and of the discrete quantities (8), we can similarly write

$$\boldsymbol{\beta}_0 = (U(\mathbf{m}_1), \dots, U(\mathbf{m}_N))^\top \qquad \boldsymbol{\beta}_1 = (\nabla U(\mathbf{m}_1^\varepsilon) \cdot \mathbf{n}_1^\varepsilon, \dots, \nabla U(\mathbf{m}_N^\varepsilon) \cdot \mathbf{n}_N^\varepsilon)^\top.$$

The *discrete direct methods* use a representation formula

$$U_h(\mathbf{z}) = \mathrm{D}_h(\mathbf{z})\boldsymbol{\varphi} - \mathrm{S}_h(\mathbf{z})\boldsymbol{\lambda} \tag{12}$$

and one of the linear systems of Table 3. The *discrete indirect methods* appear collected in Table 4, including the corresponding potential representation.

TABLE 3. Discrete direct methods, with representation formula (12).

Dirichlet	$\mathrm{V}_h\boldsymbol{\lambda} = -\frac{1}{2}\boldsymbol{\varphi} + \mathrm{K}_h\boldsymbol{\varphi}$	$\boldsymbol{\varphi} = \boldsymbol{\beta}_0$	(dD01h)
	$\frac{1}{2}\boldsymbol{\lambda} + \mathrm{J}_h\boldsymbol{\lambda} = -\mathrm{W}_h\boldsymbol{\varphi},$	$\boldsymbol{\varphi} = \boldsymbol{\beta}_0$	(dD02h)
Neumann	$-\frac{1}{2}\boldsymbol{\varphi} + \mathrm{K}_h\boldsymbol{\varphi} = \mathrm{V}_h\boldsymbol{\lambda},$	$\boldsymbol{\lambda} = \boldsymbol{\beta}_1$	(dN01h)
	$-\mathrm{W}_h\boldsymbol{\varphi} = \frac{1}{2}\boldsymbol{\lambda} + \mathrm{J}_h\boldsymbol{\lambda},$	$\boldsymbol{\lambda} = \boldsymbol{\beta}_1$	(dN02h)

TABLE 4. Discrete indirect methods.

Dirichlet	$\mathrm{V}_h\boldsymbol{\eta} = \boldsymbol{\beta}_0,$	$U_h = \mathrm{S}_h\,\boldsymbol{\eta}$	(iD01h)
	$\frac{1}{2}\boldsymbol{\psi} + \mathrm{K}_h\boldsymbol{\psi} = \boldsymbol{\beta}_0,$	$U_h = \mathrm{D}_h\,\boldsymbol{\psi}$	(iD02h)
Neumann	$-\frac{1}{2}\boldsymbol{\eta} + \mathrm{J}_h\boldsymbol{\eta} = \boldsymbol{\beta}_1,$	$U_h = \mathrm{S}_h\,\boldsymbol{\eta}$	(iN01h)
	$\mathrm{W}_h\boldsymbol{\psi} = -\boldsymbol{\beta}_1,$	$U_h = \mathrm{D}_h\,\boldsymbol{\psi}$	(iN02h)

3.2. Reinterpretation as non-conforming Petrov-Galerkin methods. Our method can be understood as a collection of non-conforming Petrov-Galerkin methods with a very simple quadrature rule for approximating any integral appearing in the scheme. The basic idea is the following: the input of D (and therefore W and K) will be approximated with a piecewise constant function on the main grid; the input of S (and therefore V and J) will be approximated with a linear combination of Dirac deltas on the companion grid; tests related to Dirichlet problems will be carried out by Dirac deltas on the main grid; tests related to Neumann problems will be done with piecewise constants on the companion grid; finally, all integrals will be broken into subintervals of the grid and approximated with a midpoint rule.

In order to write the methods of Section 3.1 in the form where we will develop their convergence analysis, we need to define some new discrete elements. First of all, we consider the (periodic) Dirac delta distribution δ_z at a point z. Its action on any periodic function that is continuous around z will be denoted $\{\delta_z, \rho\} = \{\rho, \delta_z\} := \rho(z)$. Given an interval $I \subset \mathbb{R}$, we will denote by χ_I the periodized characteristic function of I, i.e., the characteristic function of the set $I + \mathbb{Z}$. We then consider four discrete spaces

$$\begin{aligned} S_h &:= \mathrm{span}\{\chi_{(s_{i-1},s_i)} \,:\, i=1,\ldots,N\}, & S_{h,\varepsilon} &:= \mathrm{span}\{\chi_{(s^\varepsilon_{i-1},s^\varepsilon_i)} \,:\, i=1,\ldots,N\},\\ S_h^{-1} &:= \mathrm{span}\{\delta_{t_i} \,:\, i=1,\ldots,N\}, & S_{h,\varepsilon}^{-1} &:= \mathrm{span}\{\delta_{t_i^\varepsilon} \,:\, i=1,\ldots,N\}. \end{aligned}$$

For elements of these spaces we will identify the vector of their coefficients –with respect to the basis that has been used to define the space–, using the same letter in boldface font. For example,

$$S_h^{-1} \ni \mu_h = \sum_{j=1}^N \mu_j \delta_{t_j} \longleftrightarrow \boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)^\top \in \mathbb{C}^N.$$

The two discrete operators

$$Q_h^{-1}\rho := h\sum_{j=1}^N \rho(t_j)\delta_{t_j} \qquad Q_{h,\varepsilon}^{-1}\rho := h\sum_{j=1}^N \rho(t_j^\varepsilon)\delta_{t_j^\varepsilon}$$

complete the collection of elements needed for a variational description of the discrete Calderón Calculus. They will be used to denote midpoint quadrature approximations. For example,

$$\{Q_h^{-1}\rho, \phi\} = h\sum_{j=1}^N \rho(t_j)\phi(t_j) \approx \int_0^1 \rho(t)\phi(t)\mathrm{d}t.$$

The discrete potentials (9) can be easily described in this language:

$$S_{h,\varepsilon}^{-1} \ni \eta_h \mapsto \mathrm{S}_h(\cdot)\boldsymbol{\eta} = \mathrm{S}\eta_h, \qquad S_h \ni \psi_h \mapsto \mathrm{D}_h(\cdot)\boldsymbol{\psi} = \mathrm{D}Q_h^{-1}\psi_h.$$

Observe how in the double layer potential we are just applying the midpoint rule to approximate $\mathrm{D}\psi_h$, while no additional integration is needed in the already fully discrete expression for $\mathrm{S}\eta_h$.

The matrices (10) have their variational counterparts as bilinear forms:

$$\begin{aligned} S_h^{-1} \times S_{h,\varepsilon}^{-1} \ni (\mu_h, \eta_h) &\longmapsto \mathrm{v}(\mu_h,\eta_h) := \{\mu_h, \mathrm{V}\eta_h\} = \boldsymbol{\mu}^\top \mathrm{V}_h \boldsymbol{\eta},\\ S_h^{-1} \times S_h \ni (\mu_h, \psi_h) &\longmapsto \mathrm{k}(\mu_h,\psi_h) := \{\mu_h, \mathrm{K}Q_h^{-1}\psi_h\} = \boldsymbol{\mu}^\top \mathrm{K}_h \boldsymbol{\psi},\\ S_{h,\varepsilon} \times S_{h,\varepsilon}^{-1} \ni (\phi_h, \eta_h) &\longmapsto \mathrm{j}(\phi_h,\eta_h) := \{Q_{h,\varepsilon}^{-1}\phi_h, \mathrm{J}\eta_h\} = \boldsymbol{\phi}^\top \mathrm{J}_h \boldsymbol{\eta},\\ S_{h,\varepsilon} \times S_h \ni (\phi_h, \psi_h) &\longrightarrow \mathrm{w}(\phi_h,\psi_h) := \boldsymbol{\phi}^\top \mathrm{W}_h \boldsymbol{\psi}. \end{aligned}$$

The bilinear form w can be understood as follows

$$\mathrm{w}(\phi_h,\psi_h) = \{\phi_h', \mathrm{V}\psi_h'\} - k^2\{Q_{h,\varepsilon}^{-1}\phi_h, \mathrm{V}_{\mathbf{n}} Q_h^{-1}\psi_h\},$$

just by noticing that $\chi'_{(s_{i-1},s_i)} = \delta_{s_{i-1}} - \delta_{s_i}$ and that a change of sign has to be applied to the leading integrodifferential part of W (see (4d)) when changing the differentiation to the test function. The rationale behind this choice of spaces can be observed in the matrix of operators $\mathcal{D}^+$ in (6). As trial spaces we are considering $S_h \times S_{h,\varepsilon}^{-1}$, while the rows of $\mathcal{D}^+$ are respectively tested with S_h^{-1} and $S_{h,\varepsilon}$. This means that the operators of the second kind ($\pm\frac{1}{2}\mathrm{I}+\mathrm{K}$ and $\pm\frac{1}{2}\mathrm{I}+\mathrm{J}$) are discretized on a single grid (each of them on a different grid though), while the operators of the first kind (V and W) use two grids. This is actually a requirement due to the fact that the kernels of V and $\mathrm{V}_{\mathbf{n}}$ cannot be evaluated in the diagonal $s=t$, where they have a logarithmic singularity. Once this choice of trial and test spaces has been taken as a first step in the discretization of the four operators in (6), midpoint integration is applied to all remaining integrals. The operators Q_h^{-1} and $Q_{h,\varepsilon}^{-1}$ are used as a way of enforcing full discretization of every operator acting on a piecewise constant function.

To describe variationally the equations in Tables 3 and 4 we first cast the data function (β_0 for the Dirichlet problem and β_1 for the Neumann problem) in the discrete spaces

$$\beta_0^h := \sum_{j=1}^N \beta_0(t_j)\chi_{(s_{j-1},s_j)} \in S_h, \qquad \beta_1^h := Q_{h,\varepsilon}^{-1}\beta_1 = h\sum_{j=1}^N \beta_1(t_j^\varepsilon)\delta_{t_j^\varepsilon} \in S_{h,\varepsilon}^{-1},$$

so that their coefficients coincide with the sample vectors (11). The equations (dN01h) correspond then to writing $\lambda_h=\beta_1^h$, solving

$$\varphi_h \in S_h \quad \text{s.t.} \quad -\tfrac{1}{2}\{\mu_h,\varphi_h\}+\mathrm{k}(\mu_h,\varphi_h)=\mathrm{v}(\mu_h,\lambda_h) \quad \forall \mu_h\in S_h^{-1}, \tag{13}$$

and finally using $U_h = \mathrm{D}Q_h^{-1}\varphi_h - \mathrm{S}\lambda_h$ as discrete representation formula. The indirect method (iN02h) corresponds to solving

$$\psi_h\in S_h \quad \text{s.t} \quad \mathrm{w}(\phi_h,\psi_h) = -\{\beta_1^h,\phi_h\} = -\{Q_{h,\varepsilon}^{-1}\phi_h,\beta_1\} \quad \forall \phi_h \in S_{h,\varepsilon},$$

for a potential representation $U_h = \mathrm{D}Q_h^{-1}\psi_h$. The indirect method (iD01h) is equivalent to solving

$$\eta_h \in S_{h,\varepsilon}^{-1} \quad \text{s.t.} \quad \mathrm{v}(\mu_h,\eta_h)=\{\mu_h,\beta_0^h\}=\{\mu_h,\beta_0\} \quad \forall\mu_h\in S_h^{-1}.$$

The remaining five discrete equations in Tables 3 and 4 can be easily rewritten using these same elements.

4. Numerical analysis

4.1. Stability. Analysis of the methods in Section 3.1 is carried out in the form given in Section 3.2, in the frame of periodic Sobolev spaces. For $s\in\mathbb{R}$ we define the space H^s as the completion of the space of trigonometric polynomials $\mathrm{span}\,\{\exp(2\pi\imath m\,\cdot\,) : m\in\mathbb{Z}\}$ with respect to the norm

$$\|\rho\|_s^2 = |\widehat{\rho}(0)|^2 + \sum_{m\neq 0}|m|^{2s}|\widehat{\rho}(m)|^2, \quad \widehat{\rho}(m) := \int_0^1 \rho(t)\exp(-2\pi\imath m t)\,\mathrm{d}t.$$

An extensive treatment of these spaces can be found in [14]. The operators (4) can be extended to act on all Sobolev spaces H^s. In particular, the following result holds (see [9, Table 2.1.1] and [12, Section 3.2]).

Proposition 4.1. *The operators*

$$\pm\tfrac{1}{2}+\mathrm{K},\ \pm\tfrac{1}{2}+\mathrm{J}: H^s\to H^s, \quad \mathrm{V}:H^s\to H^{s+1}, \quad \mathrm{W}:H^s\to H^{s-1} \tag{14}$$

are bounded for all s. If, in addition, $-k^2$ is neither a Dirichlet nor a Neumann eigenvalue of the Laplacian in Ω (cf. Proposition 2.1), then all of them are invertible.

Proposition 4.2. *Assume that $-k^2$ is neither a Dirichlet nor a Neumann eigenvalue of the Laplacian in Ω and let $\varepsilon \in (-1/2, 1/2) \setminus \{0\}$. Then there exist positive numbers $c_{\mathrm{V}}, c_{\mathrm{K}}, c_{\mathrm{J}}, c_{\mathrm{W}} > 0$ so that for all h small enough*

$$\inf_{0\neq\eta_h\in S^{-1}_{h,\varepsilon}} \sup_{0\neq\mu_h\in S^{-1}_h} \frac{|\mathrm{v}(\mu_h,\eta_h)|}{\|\mu_h\|_{-1}\|\eta_h\|_{-1}} \geq c_{\mathrm{V}}, \tag{15}$$

$$\inf_{0\neq\psi_h\in S_h} \sup_{0\neq\mu_h\in S^{-1}_h} \frac{|\pm\frac{1}{2}\{\mu_h,\psi_h\}+\mathrm{k}(\mu_h,\psi_h)|}{\|\mu_h\|_{-1}\|\psi_h\|_{0}} \geq c_{\mathrm{K}}, \tag{16}$$

$$\inf_{0\neq\eta_h\in S^{-1}_{h,\varepsilon}} \sup_{0\neq\phi_h\in S_{h,\varepsilon}} \frac{|\pm\frac{1}{2}\{\phi_h,\eta_h\}+\mathrm{j}(\phi_h,\eta_h)|}{\|\phi_h\|_{0}\|\eta_h\|_{-1}} \geq c_{\mathrm{J}}, \tag{17}$$

$$\inf_{0\neq\psi_h\in S_h} \sup_{0\neq\phi_h\in S_{h,\varepsilon}} \frac{|\mathrm{w}(\phi_h,\psi_h)|}{\|\phi_h\|_{0}\|\psi_h\|_{0}} \geq c_{\mathrm{W}}. \tag{18}$$

The constants can depend on ε.

Proof. Condition (15) was proved in [2, Proposition 8], although it is based on a stability result (phrased in different terms) given in [13]. Condition (18) has been proven in [8, Theorem 1]. With minor modifications, the proof of [7, Theorem 2] can be used to prove (17). It is then easy to note that this result would also hold for the spaces S_h^{-1} and S_h (it all amounts to displacing the grid for both test and trial functions). Then, by an easy transposition argument, (16) holds. □

The value $\varepsilon = 0$ is not a practicable option for the choice of the grids: in this case both grids coincide and we are obliged to evaluate the singular kernels in their diagonal. The choices $\varepsilon = \pm 1/2$ lead to a discretization of V (they give the same one) that is not stable, i.e., the inf-sup condition does not hold. The proof of the inf-sup condition for the discretization of W in [8] requires also that $\varepsilon \neq \pm 1/2$, because it is based on the result for V, although numerical evidence points to this being just a technical restriction, which is not in the case of V. Note finally that dependence of the methods on ε is 1-periodic.

4.2. Consistency analysis via asymptotic expansions. We next study the consistency of the approximation of the bilinear forms associated to the four operators (4) by their discrete counterparts, as well as the approximation of the identity operators that appear in the equations of Tables 1 and 2. The consistency error analysis is carried out by comparison with a *quasioptimal projection* of the corresponding unknown (the input of the integral operator) in the discrete space. These projections are defined by matching the central Fourier coefficients:

$$\begin{aligned} S^{-1}_{h,\varepsilon} &\ni D^{-1}_{h,\varepsilon}\eta, \quad \widehat{D^{-1}_{h,\varepsilon}\eta}(m) = \widehat{\eta}(m), \quad -N/2 < m \leq N/2, \\ S_h &\ni D_h\psi, \quad \widehat{D_h\psi}(m) = \widehat{\psi}(m), \quad -N/2 < m \leq N/2. \end{aligned}$$

The operator D_h was studied in [1], while $D^{-1}_{h,\varepsilon}$ proceeds from [2]. It is proved in those references that

$$\|D_h\psi - \psi\|_s \leq C_{s,r} h^{r-s}\|\psi\|_r \qquad s \leq r \leq 1,\ s < 1/2, \tag{19a}$$

$$\|D^{-1}_{h,\varepsilon}\eta - \eta\|_s \leq C h^{r-s}\|\eta\|_r \qquad s \leq r \leq 0,\ s < -1/2. \tag{19b}$$

Proposition 4.3. *For all $\eta \in H^3$ and $\psi \in H^4$ it holds*

$$\begin{aligned} |\{\phi_h, D^{-1}_{h,\varepsilon}\eta\} - \{\phi_h, Q^{-1}_{h,\varepsilon}\eta\}| &\leq Ch^3\|\eta\|_3\|\phi_h\|_0, \quad \forall \phi_h \in S_{h,\varepsilon}, \\ |\{\mu_h, D_h\psi\} - \{\mu_h,\psi\} + h^2\tfrac{1}{24}\{\mu_h,\psi''\}| &\leq Ch^3\|\psi\|_4\|\mu_h\|_{-1}, \quad \forall \mu_h \in S^{-1}_h. \end{aligned}$$

The constants in the bounds are independent of ε.

Proof. The second expansion follows from [2, Theorem 7]. To prove the first one, note that by [2, Lemma 5]

$$D_{h,\varepsilon}^{-1}\eta - Q_{h,\varepsilon}^{-1}\eta = Q_{h,\varepsilon}^{-1}E_h\eta, \text{ where } E_h\eta := \sum_{-\frac{N}{2}<m\le\frac{N}{2}} \widehat{\eta}(m)\exp(2\pi\imath m\cdot) - \eta. \tag{20}$$

A direct computation (see also [8, Lemma 9]) shows then that

$$\begin{aligned}|\{\phi_h, Q_{h,\varepsilon}^{-1}E_h\eta\}| &\le \|\phi_h\|_0\|E_h\eta\|_0 + \left|\{\phi_h, Q_{h,\varepsilon}^{-1}E_h\eta\} - \int_0^1 \phi_h(t)(E_h\eta)(t)\mathrm{d}t\right| \\ &\le \|\phi_h\|_0(\|E_h\eta\|_0 + \pi h\|E_h\eta\|_1) \le Ch^3\|\phi_h\|_0\|\eta\|_3,\end{aligned}$$

where the last inequality follows from the fact that $\|E_h\eta\|_s \le Ch^{t-s}\|\eta\|_t$ for all $t \ge s$ [14, Theorem 8.2.1]. □

For simplicity, in what follows we will write $\mathrm{P} \in \mathcal{E}(n)$ when P is a periodic pseudodifferential operator of order n, i.e., $\mathrm{P} : H^s \to H^{s-n}$ is bounded for all s.

Proposition 4.4. *There exists $\mathrm{P_k} \in \mathcal{E}(1)$ so that for all $\eta \in H^3$ and $\psi \in H^4$,*

$$\begin{aligned}|\mathrm{j}(\phi_h, D_{h,\varepsilon}^{-1}\eta) - \{Q_{h,\varepsilon}^{-1}\phi_h, \mathrm{J}\eta\}| &\le Ch^3\|\eta\|_3\|\phi_h\|_0, \quad \forall \phi_h \in S_{h,\varepsilon}, \\ |\mathrm{k}(\mu_h, D_h\psi) - \{\mu_h, \mathrm{K}\psi\} - h^2\{\mu_h, \mathrm{P_k}\psi\}| &\le Ch^3\|\psi\|_4\|\mu_h\|_{-1}, \quad \forall \mu_h \in S_h^{-1}.\end{aligned}$$

The coefficient $\mathrm{P_k}$ and the constants in the bounds do not depend on ε.

Proof. We refer to [7], where similar expansions are derived. □

The study of the approximation properties of V and W is strongly influenced by the parameter ε. We write

$$C_1(\varepsilon) := \frac{1}{2\pi\imath}\log(4\sin^2(\pi\varepsilon)) \qquad C_2(\varepsilon) := \frac{1}{2}\int_0^\varepsilon C_1(t)\mathrm{d}t,$$

and note that $C_1(\pm 1/6) = 0$.

Proposition 4.5. *There exists a smooth function a_v and operators $\mathrm{L_v} \in \mathcal{E}(1)$, $\mathrm{L}_\mathrm{w}^1 \in \mathcal{E}(2)$, $\mathrm{L}_\mathrm{w}^{2a}, \mathrm{L}_\mathrm{w}^{2b} \in \mathcal{E}(3)$ such that for all $\eta \in H^3$ and $\psi \in H^4$,*

$$\begin{aligned}&|\mathrm{v}(\mu_h, D_{h,\varepsilon}^{-1}\eta) - \{\mu_h, \mathrm{V}\eta\} - hC_1(\varepsilon)\{\mu_h, a_\mathrm{v}\eta\} \\ &\qquad -h^2C_2(\varepsilon)\{\mu_h, \mathrm{L_v}\eta\}| \le Ch^3\|\eta\|_3\|\mu_h\|_{-1}, \quad \forall \mu_h \in S_h^{-1}, \\ &|\mathrm{w}(\phi_h, D_h\psi) - \{Q_{h,\varepsilon}^{-1}\phi_h, \mathrm{W}\psi\} - hC_1(\varepsilon)\{Q_{h,\varepsilon}^{-1}\phi_h, \mathrm{L}_\mathrm{w}^1\psi\} \\ &\qquad -h^2\{Q_{h,\varepsilon}^{-1}\phi_h, (C_2(\varepsilon)\mathrm{L}_\mathrm{w}^{2a} + \mathrm{L}_\mathrm{w}^{2b})\psi\}\big| \le Ch^3\|\psi\|_4\|\phi_h\|_0, \quad \forall \phi_h \in S_{h,\varepsilon}.\end{aligned}$$

Proof. The first expansion is given in [2, Theorem 7], while the second one is proved in [8, Proposition A.4]. □

The key fact at this point is that by letting $\varepsilon = \pm 1/6$ all the expansions start at h^2. This will be crucial since, as we will see in the next subsection, we can identify the order of the method with the first power of h appearing in the consistency expansion. The relevance of identifying the h^2 term of the asymptotic expansion of the consistency error in Propositions 4.3, 4.4 and 4.5 is related to the possibility of moving from the norms given by the inf-sup conditions in Proposition 4.2 to stronger norms when producing estimates of the convergence error. (See Theorem 4.7 below.)

Remark 4.6. *If in Propositions 4.3, 4.4 and 4.5 we only assume that* $\eta \in H^2$ *and* $\psi \in H^3$, *and we eliminate the* h^2 *term from the left-hand side of the bounds, then the result holds with a bound of the form* $Ch^2\|\eta\|_2$ *or* $Ch^2\|\psi\|_3$.

4.3. Convergence estimates. We collect in this subsection the convergence results for all the numerical schemes presented in this paper.

Theorem 4.7. *Assume that k satisfies the hypothesis of Proposition 4.2 and* $\varepsilon = \pm 1/6$. *Let* $(\lambda_h, \varphi_h) \in S^{-1}_{h,\varepsilon} \times S_h$ *be the pair associated to the solution* $(\boldsymbol{\lambda}, \boldsymbol{\varphi})$ *of any of* (dD01h), (dD02h), (dN01h) *or* (dN02h). *Then*

$$\|\varphi_h - D_h\varphi\|_0 + \|\lambda_h - \lambda\|_{-1} \le Ch^2(\|\varphi\|_3 + \|\lambda\|_2).$$

Moreover,

$$\max_j |\varphi_j - \beta_0(t_j)| + \max_j |h^{-1}\lambda_j - \beta_1(t_j)| \le Ch^2(\|\varphi\|_4 + \|\lambda\|_4).$$

Proof. We will only show the case (dN01h), all others being very similar. Using the variational representation of (dN01h) in (13), we can write

$$-\tfrac{1}{2}\{\mu_h, \varphi_h\} + \mathrm{k}(\mu_h, \varphi_h) = \{\mu_h, -\tfrac{1}{2}\varphi + \mathrm{K}\varphi\} + \{\mu_h, \mathrm{V}(Q^{-1}_{h,\varepsilon}\lambda - \lambda)\} \quad \forall \mu_h \in S^{-1}_h.$$

Using now Propositions 4.3 (second bound), 4.4 (second bound) and 4.5 (first bound) –see also Remark 4.6– and (20), it follows that

$$\begin{aligned} |-\tfrac{1}{2}\{\mu_h, \varphi_h - D_h\varphi\} + \mathrm{k}(\mu_h, \varphi_h - D_h\varphi)| \quad &\le \quad Ch^2\|\mu_h\|_{-1}(\|\varphi\|_3 + \|\lambda\|_2) \\ &\quad + |\mathrm{v}(\mu_h, Q^{-1}_{h,\varepsilon}E_h\lambda)| \quad \forall \mu_h \in S^{-1}_h. \end{aligned}$$

Using [2, Lemma 13] and the fact that

$$\|Q^{-1}_{h,\varepsilon}\eta\|_{-1} \le C(\|\eta\|_0 + h\|\eta\|_1),$$

(see [2, Lemma 6]) we can prove that

$$|\mathrm{v}(\mu_h, Q^{-1}_{h,\varepsilon}E_h\lambda)| \le C\|\mu_h\|_{-1}(\|E_h\lambda\|_0 + h\|E_h\lambda\|_1) \le Ch^2\|\mu_h\|_{-1}\|\lambda\|_2 \quad \forall \mu_h \in S^{-1}_h.$$

Therefore, by Proposition 4.2, the bound for $\|\varphi_h - D_h\varphi\|_0$ follows. The bound for

$$\|\lambda - \lambda_h\|_{-1} = \|\lambda - Q^{-1}_{h,\varepsilon}\lambda\|_{-1} \le \|\lambda - D^{-1}_{h,\varepsilon}\lambda\|_{-1} + \|D^{-1}_{h,\varepsilon}\lambda - Q^{-1}_{h,\varepsilon}\lambda\|_{-1}$$

follows from (19b) and (20). The uniform estimates require including the h^2 term of the consistency error expansion: see [2, Corollary 11] and [8, Theorem 6.4] for very similar arguments. □

Theorem 4.8. *Assume that k satisfies the hypothesis of Proposition 4.2 and* $\varepsilon = \pm 1/6$. *Let* $\psi_h \in S_h$ *be associated to the solution* $\boldsymbol{\psi}$ *of* (iD02h) *or* (iN02h) *and let* $\eta_h \in S^{-1}_{h,\varepsilon}$ *be associated to the solution* $\boldsymbol{\eta}$ *of* (iD01h) *or* (iN01h). *Then*

$$\|D_h\psi - \psi_h\|_0 \le Ch^2\|\psi\|_3 \qquad \|\eta - \eta_h\|_{-1} \le Ch^2\|\eta\|_2.$$

Proof. The proof is very similar to the one of Theorem 4.7. The absence of integral operators in the right hand side makes the arguments slightly simpler. □

In all cases it is possible to prove that the estimates can be transferred to the computation of potential, with the direct representation (12) in the case of direct methods, or the associated layer potential in the case of indirect methods. In all cases, we can prove $|U(\mathbf{z}) - U_h(\mathbf{z})| \le C(\mathbf{z})h^2$.

Remark 4.9. *If we take* $\varepsilon \neq \pm 1/6$, *the methods involving* V_h *or* W_h *are of order one.*

5. Experiments

In the following experiments we consider a single elliptical obstacle with boundary

$$\tfrac{1}{4}(x-0.1)^2 + (y-0.2)^2 = 1.$$

We will check solutions in two observation points inside the ellipse $\mathbf{x}_1 = (0.2, 0.4)$ and $\mathbf{x}_2 = (-0.2, -0.4)$. The examples will use more complicated integral equations than those explained in the previous sections, in order to put the discrete Calderón Calculus to a more demanding test.

5.1. A transmission problem. Consider the coupling of the exterior Helmholtz equation (1) with an interior equation with different wave number

$$\Delta V + (k/c)^2 V = 0 \text{ in } \Omega$$

(here $c > 0$) and transmission conditions

$$\gamma^+ U + \beta_0 = \gamma^- V, \qquad \partial_{\mathbf{n}}^+ U + \beta_1 = \alpha \partial_{\mathbf{n}}^- V$$

(with $\alpha > 0$). Data are taken so that the exact solution is

$$U(\mathbf{z}) = H_0^{(1)}(k|\mathbf{z} - \mathbf{x}_0|), \qquad V(\mathbf{z}) = \exp(\imath (k/c)\mathbf{z}\cdot\mathbf{d}) \qquad \mathbf{x}_0 \in \Omega, \qquad |\mathbf{d}| = 1.$$

We use the symmetric formulation of Martin Costabel and Ernst Stephan [4] (see also [10]). The main unknowns are $\varphi^- = \gamma^- V$ and $\lambda^- = \alpha \partial_{\mathbf{n}}^- V$. The system they satisfy is

$$\begin{bmatrix} \mathrm{W}_k + \alpha \mathrm{W}_{k/c} & \mathrm{J}_k + \mathrm{J}_{k/c} \\ -\mathrm{K}_k - \mathrm{K}_{k/c} & \mathrm{V}_k + \frac{1}{\alpha}\mathrm{V}_{k/c} \end{bmatrix} \begin{bmatrix} \varphi^- \\ \lambda^- \end{bmatrix} = \begin{bmatrix} \mathrm{W}_k & \frac{1}{2}\mathrm{I} + \mathrm{J}_k \\ \frac{1}{2}\mathrm{I} - \mathrm{K}_k & \mathrm{V}_k \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \tag{21}$$

where we have tagged the integral operators with the corresponding wave number. The potential representation for the exterior and interior fields is

$$U = -\mathrm{S}_k(\lambda^- - \beta_1) + \mathrm{D}_k(\varphi^- - \beta_0), \qquad V = \alpha^{-1}\mathrm{S}_{k/c}\lambda^- - \mathrm{D}_{k/c}\varphi^-. \tag{22}$$

Discretization is carried out by simply substituting the elements of (21) and (22) by their discrete counterparts: the data functions are sampled with (11), the integral operators are build with (10) and the potentials with (9). We solve and tabulate the following errors:

$$\mathrm{E}_h^\lambda := \max_j |h^{-1}\lambda_j^- - \alpha\partial_{\mathbf{n}}^+ V(t_j^\varepsilon)| \qquad \mathrm{E}_h^\varphi := \max_j |\varphi_j^- - \gamma^- V(t_j)|$$

$$\mathrm{E}_h^V := \max_{\ell=1,2} |V_h(\mathbf{x}_\ell) - V(\mathbf{x}_\ell)|$$

These experiments are reported in Tables 5 and 6. The parameters are $k = 3$, $c = 2/3$ and $\alpha = 3/2$.

5.2. Burton-Miller integral equation. Consider now the exterior Helmholtz equation (1) with boundary condition $\gamma^+ U + \gamma U_{\mathrm{inc}} = 0$, where $\Delta U_{\mathrm{inc}} + k^2 U_{\mathrm{inc}} = 0$ in a neighborhood of the interior domain $\overline{\Omega}$. The well known Burton-Miller integral equation [3, Section 3.9] is

$$\tfrac{1}{2}\xi + \mathrm{J}\xi + c\mathrm{V}\xi = \partial_{\mathbf{n}} U_{\mathrm{inc}} + c\gamma U_{\mathrm{inc}}. \tag{23}$$

The exterior normal derivative can be computed after solving this equation and there are two potential representations of the solution

$$\lambda = \xi - \partial_{\mathbf{n}} U_{\mathrm{inc}} \qquad U = -\mathrm{S}\xi = -\mathrm{S}\lambda - \mathrm{D}\gamma U_{\mathrm{inc}}.$$

TABLE 5. Errors E_h^{λ} (left column) and E_h^{φ} (right column) for the Transmission Problem in Experiment 1.

N	error	e.c.r
10	$4.6842E(+000)$	
20	$1.2470E(+000)$	1.9093
40	$3.7207E(-001)$	1.7448
80	$9.4663E(-002)$	1.9747
160	$2.3768E(-002)$	1.9938
320	$5.9518E(-003)$	1.9976
640	$1.4886E(-003)$	1.9994

N	error	e.c.r
10	$5.8671E(-001)$	
20	$1.9979E(-001)$	1.5542
40	$4.9104E(-002)$	2.0246
80	$1.2376E(-002)$	1.9883
160	$3.1081E(-003)$	1.9934
320	$7.7699E(-004)$	2.0001
640	$1.9423E(-004)$	2.0001

TABLE 6. Error E_h^{V} (potential solution V at two interior observation points) for the Transmission Problem in Experiment 1.

N	error	e.c.r
10	$1.8729E(-001)$	
20	$2.0779E(-002)$	3.1721
40	$4.0885E(-003)$	2.3455
80	$9.6559E(-004)$	2.0821
160	$2.4527E(-004)$	1.9770
320	$6.1837E(-005)$	1.9878
640	$1.5527E(-005)$	1.9937

The value $c = -\imath k$ is the usual choice in (23). For this value, the equation (23) is uniquely solvable independently of the frequency. Since $\mathrm{S}\xi = U_{\mathrm{inc}}$ in the interior domain, we compare errors

$$\mathrm{E}_h^{U} := \max_{\ell=1,2} |\mathrm{S}_h(\mathbf{x}_\ell)\boldsymbol{\xi} - U_{\mathrm{inc}}(\mathbf{x}_\ell)|$$

We also compare the density $\boldsymbol{\xi}$ with the solution of Problem (dD01h) (Table 3) computing the compared error

$$\mathrm{E}_h^{\xi} := \max_j |\underbrace{h^{-1}\lambda_j}_{\text{(dD01h)}} - \underbrace{(h^{-1}\xi_j - \partial_{\mathbf{n}} U_{\mathrm{inc}}(t_j^{\varepsilon}))}_{\text{Burton-Miller I.E.}}|.$$

In our numerical experiments we have taken $U_{\mathrm{inc}}(\mathbf{x}) = \exp(\imath k\mathbf{d}\cdot\mathbf{x})$, i.e. an acoustic plane wave, with direction given by the unit vector $\mathbf{d} = (1,1)/\sqrt{2}$ and wave number $k = 2$. The results are gathered in Table 7.

5.3. Conclusions. We have presented a collection of compatible discretizations of the two potentials and four boundary integral operators associated to the Helmholtz equation on smooth parametrizable curves in the plane. We have shown discrete stability of the discrete versions for all the operators in absence of resonances. We have also given convergence estimates for eight integral equations that solve the exterior Dirichlet and Neumann problems, with direct and indirect boundary integral equations. Finally, we have tested the methods in more complicated cases, such as systems of boundary integral equations arising from transmission problems and combined field integral equations.

TABLE 7. Errors E_h^U (left columns) and E_h^ξ (right columns) for the Burton-Miller integral equation in Experiment 2.

N	error	e.c.r	N	error	e.c.r
10	$1.7205E(-001)$		10	$7.6790E(+000)$	
20	$3.6082E(-002)$	2.2535	20	$1.8790E(+000)$	2.0310
40	$1.1990E(-002)$	1.5894	40	$4.1656E(-001)$	2.1734
80	$3.7936E(-003)$	1.6602	80	$8.5219E(-002)$	2.2893
160	$1.0571E(-003)$	1.8435	160	$1.4703E(-002)$	2.5351
320	$2.7581E(-004)$	1.9384	320	$2.2452E(-003)$	2.7112
640	$7.2185E(-005)$	1.9339	640	$7.1749E(-004)$	1.6458

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