INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 11, Number 2, Pages 255–270



# EXPANDED MIXED FINITE ELEMENT DOMAIN DECOMPOSITION METHODS ON TRIANGULAR GRIDS

ANDRÉS ARRARÁS AND LAURA PORTERO

(Communicated by F.J. Gaspar)

*This paper is dedicated to Professor Francisco J. Lisbona on the occasion of his 65th birthday*

**Abstract.** In this work, we present a cell-centered time-splitting technique for solving evolutionary diffusion equations on triangular grids. To this end, we consider three variables (namely the pressure, the flux and a weighted gradient) and construct a so-called expanded mixed finite element method. This method introduces a suitable quadrature rule which permits to eliminate both fluxes and gradients, thus yielding a cell-centered semidiscrete scheme for the pressure with a local 10-point stencil. As for the time integration, we use a domain decomposition operator splitting based on a partition of unity function. Combining this splitting with a multiterm fractional step formula, we obtain a collection of uncoupled subdomain problems that can be efficiently solved in parallel. A priori error estimates for both the semidiscrete and fully discrete schemes are derived on smooth triangular meshes with six triangles per internal vertex.

**Key words.** Cell-centered finite difference, domain decomposition, error estimates, fractional step, mixed finite element, operator splitting.

## 1. Introduction

We consider a parabolic initial-boundary value problem that models single phase flow in porous media. The problem can be written as a system of two first-order equations of the form

$$p_t + \nabla \cdot \mathbf{u} = f \qquad \text{in } \Omega \times (0,T], \tag{1a}$$

$$\mathbf{u} = -K\nabla p \qquad \text{in } \Omega \times (0,T], \tag{1b}$$

$$p = p_0 \qquad \text{in } \Omega \times \{0\}, \tag{1c}$$

$$p = g \qquad \text{on } \Gamma_D \times (0,T], \tag{1d}$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma_N \times (0,T], \tag{1e}$$

where $\Omega \subset \mathbb{R}^2$ is a convex polygonal domain with Lipschitz continuous boundary $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ such that $\Gamma_D \cap \Gamma_N = \emptyset$. In this formulation, $K \equiv K(\mathbf{x}) \in \mathbb{R}^{2\times 2}$ is a symmetric and positive definite tensor satisfying, for some $0 < \kappa_* \le \kappa^* < \infty$,

$$\kappa_* \xi^T \xi \le \xi^T K\, \xi \le \kappa^* \xi^T \xi \qquad \forall\, \xi \ne \mathbf{0} \in \mathbb{R}^2, \tag{2}$$

and $\mathbf{n}$ is the outward unit vector normal to $\partial\Omega$. Typically, $p$ represents the fluid pressure, $\mathbf{u}$ is the Darcy velocity and $K$ denotes the hydraulic conductivity tensor.

In this work, we propose and analyze a family of mixed finite element (MFE) time-splitting methods for the solution of problem (1). Via the method of lines approach, the original problem is first reduced to a system of ordinary differential equations using a spatial semidiscretization technique. More precisely, we consider a variant of the standard mixed formulation called the expanded MFE method (cf.

Received by the editors November 12, 2012 and, in revised form, July 7, 2013.
2000 *Mathematics Subject Classification.* 65M06, 65M12, 65M15, 65M20, 65M55, 65M60, 76S05.

[1, 2, 3, 10, 12]). Besides the pressure $p$ and the flux $\mathbf{u}$, this method introduces an additional explicit unknown, namely the adjusted gradient $\lambda$. The newly defined variable avoids inverting tensor $K$, thus allowing for the presence of non-negative conductivities in the flow domain $\Omega$ (as a difference, $K$ is assumed to be strictly positive in the standard mixed method). Following [1, 2], we consider the lowest order Raviart–Thomas ($RT_0$) finite element spaces on triangles (cf. [21]), and subsequently define a suitable quadrature rule that permits to eliminate both $\mathbf{u}$ and $\lambda$. As a result, the expanded MFE formulation is reduced to a cell-centered finite difference scheme for the pressure with a local 10-point stencil. In the context of elliptic problems, this idea has been already studied in [6, 8, 16] for the standard mixed method on triangular grids. Similar strategies have been also investigated in the case of rectangular elements (cf. [3, 22, 28]).

The stiff initial value problem resulting from the previous stage is integrated in time by using a domain decomposition splitting technique. This kind of splitting was first introduced in [25, 26] for the construction of regionally-additive schemes and has been subsequently used in [14, 15, 19] for solving linear parabolic problems. In combination with this splitting, we define a family of time integrators belonging to the class of $m$-part fractional step Runge–Kutta (FSRK$_m$) methods (cf. [9]). Such methods are composed by merging together $m$ diagonally implicit Runge–Kutta schemes into a single composite formula. In particular, we consider the so-called Yanenko's method (cf. [30]), which has been proved to be unconditionally contractive for different splitting functions (see [17, 27]). The fully discrete scheme is thus a collection of uncoupled subdomain problems that can be solved in parallel without the need for Schwarz-type iteration procedures.

The design and analysis of expanded MFE fractional step methods for parabolic problems have been addressed in the earlier works [4, 5]. In both cases, though, the problems were discretized on rectangular meshes using an alternating direction implicit (ADI) technique. In the present paper, we extend the results from these works to the case of domain decomposition splitting methods on triangular grids, thus yielding added flexibility to the resulting algorithms.

The rest of the paper is outlined as follows. In Section 2, we introduce the expanded MFE method and subsequently derive a cell-centered finite difference scheme for the pressure. The convergence analysis of the semidiscrete scheme is described in the next section. Section 4 further presents the family of fractional step time integrators based on a domain decomposition splitting technique. Finally, a priori error estimates for the fully discrete scheme are obtained in Section 5.

## 2. The expanded mixed finite element method

In order to define an expanded formulation, we need to introduce the additional unknown $\lambda \equiv \lambda(\mathbf{x}, t) = -G^{-1}\nabla p$. This variable is referred to as the adjusted gradient and involves a symmetric and positive definite tensor $G \equiv G(\mathbf{x}) \in \mathbb{R}^{2\times 2}$, to be defined below. In this context, the equation (1b) can be rewritten as

$$G\lambda = -\nabla p \qquad \text{in } \Omega \times (0,T], \tag{3a}$$

$$\mathbf{u} = KG\lambda \qquad \text{in } \Omega \times (0,T]. \tag{3b}$$

These two equations, together with (1a) and the corresponding initial and boundary data, represent the so-called expanded mixed formulation in the triple $(\mathbf{u}, \lambda, p)$.

**2.1. The weak formulation.** For a domain $R \subset \mathbb{R}^2$, let $W^{k,p}(R)$ be the standard Sobolev space, with $k \in \mathbb{R}$ and $1 \le p \le \infty$, endowed with the norm and seminorm $\|\cdot\|_{k,p,R}$ and $|\cdot|_{k,p,R}$, respectively. Let $H^k(R)$ be the Hilbert space $W^{k,2}(R)$,

endowed with the norm and seminorm $\|\cdot\|_{k,R}$ and $|\cdot|_{k,R}$, respectively. We further denote by $(\cdot,\cdot)_R$ and $\|\cdot\|_R$ the inner product and norm, respectively, in either $L^2(R)$ or $(L^2(R))^2$. The subscript $R$ will be omitted whenever $R \equiv \Omega$. For a section $S$ of the domain boundary, $\langle\cdot,\cdot\rangle_S$ and $\|\cdot,\cdot\|_S$ represent the $L^2(S)$-inner product (or duality pairing) and norm, respectively. We shall also use the space

$$H(\mathrm{div}; R) = \{\mathbf{v} \in (L^2(R))^2 : \nabla\cdot\mathbf{v} \in L^2(R)\},$$

with corresponding norm

$$\|\mathbf{v}\|_{\mathrm{div};\,R} = (\|\mathbf{v}\|_R^2 + \|\nabla\cdot\mathbf{v}\|_R^2)^{1/2}.$$

Finally, if $\chi \equiv \chi(R)$ denotes any of the above normed spaces on $R$, with associated norm $\|\cdot\|_\chi$, we shall consider $L^q([0,T];\chi)$ as the space of $\chi$-valued functions $\varphi : [0,T] \to \chi(R)$, equipped with the norm

$$\|\varphi\|_{L^q(\chi)} \equiv \|\varphi\|_{L^q([0,T];\,\chi)} = \begin{cases} \left(\displaystyle\int_0^T \|\varphi(t)\|_\chi^q\, dt\right)^{1/q} & \text{if } 1 \le q < \infty, \\ \text{ess sup}_{t\in[0,T]}\, \|\varphi(t)\|_\chi & \text{if } q = \infty. \end{cases}$$

In this framework, the weak form of the expanded mixed formulation reads: *Find* $(\mathbf{u},\lambda,p) : [0,T] \to H_0(\mathrm{div};\Omega)\times(L^2(\Omega))^2 \times L^2(\Omega)$ *such that*

$$(p_t, w) + (\nabla\cdot\mathbf{u}, w) = (f, w) \qquad \forall\, w \in L^2(\Omega), \tag{4a}$$

$$(G\lambda, \mathbf{v}) = (p, \nabla\cdot\mathbf{v}) - \langle g, \mathbf{v}\cdot\mathbf{n}\rangle_{\Gamma_D} \qquad \forall\, \mathbf{v} \in H_0(\mathrm{div};\Omega), \tag{4b}$$

$$(G\mathbf{u}, \mu) = (GKG\lambda, \mu) \qquad \forall\, \mu \in (L^2(\Omega))^2, \tag{4c}$$

$$p(0) = p_0, \tag{4d}$$

where

$$H_0(\mathrm{div};\Omega) = \{\mathbf{v} \in H(\mathrm{div};\Omega) : \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma_N\}.$$

Note that, if $G = K^{-1}$, (3b) implies $\lambda = \mathbf{u}$ and the standard mixed formulation is obtained. Instead, if $G$ is considered to be the identity matrix, we derive the expanded mixed formulation proposed in [10]. In this paper, we shall locally define $G$ in terms of an affine mapping $F_T$, as specified below.

**2.2. Mixed finite element spaces.** Let $\mathcal{T}_h$ be a conforming, shape-regular and quasi-uniform partition of $\Omega$, where $h = \max_{T\in\mathcal{T}_h} \mathrm{diam}(T)$. We assume that this partition involves six triangles per interior vertex. Let $\hat{T}$ be the reference equilateral triangle with vertices $\hat{\mathbf{r}}_1 = (-1,0)^T$, $\hat{\mathbf{r}}_2 = (1,0)^T$ and $\hat{\mathbf{r}}_3 = (0,\sqrt{3})^T$, and introduce a family of bijective affine mappings $\{F_T\}_{T\in\mathcal{T}_h}$ such that $F_T(\hat{T}) = T$. We further define, for each mapping $F_T$, the Jacobian matrix $B_T$ and its determinant $J_T = |\det(B_T)|$. The corresponding vertices of $T$ are denoted by $\mathbf{r}_i = (x_i, y_i)^T$, while the outward unit vectors normal to the edges of $\hat{T}$ and $T$ are represented by $\hat{\mathbf{n}}_i$ and $\mathbf{n}_i$, respectively, for $i = 1,2,3$ (cf. Figure 1). For later use, we also introduce the notations $\hat{\mathbf{n}}_{\hat{e}}$ and $\mathbf{n}_e$ to define the outward unit vectors normal to the edges $\hat{e} \subset \partial\hat{T}$ and $e \subset \partial T$, respectively.

Let $\hat{V}(\hat{T})$ and $\hat{W}(\hat{T})$ be the $RT_0$ finite element spaces on the reference element $\hat{T}$, i.e.,

$$\hat{V}(\hat{T}) = (\mathbb{P}_0(\hat{T}))^2 \oplus \hat{\mathbf{x}}\,\mathbb{P}_0(\hat{T}), \qquad \hat{W}(\hat{T}) = \mathbb{P}_0(\hat{T}),$$

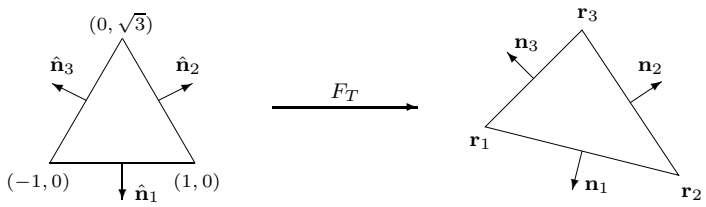


FIGURE 1. Affine mapping $F_T$ from the reference element $\hat{T}$ onto a generic triangle $T \in \mathcal{T}_h$.

where $\mathbb{P}_0(\hat{T})$ denotes the set of constant functions defined on $\hat{T}$. The corresponding spaces on a generic element $T \in \mathcal{T}_h$ are defined via the transformations

$$\mathbf{v} \leftrightarrow \hat{\mathbf{v}} : \mathbf{v} = \left( \frac{1}{J_T} B_T \, \hat{\mathbf{v}} \right) \circ F_T^{-1},$$
$$w \leftrightarrow \hat{w} : w = \hat{w} \circ F_T^{-1}.$$

The former is known as the Piola transformation (cf. [24]) and it is defined to preserve the continuity of the normal components of velocity vectors across interelement edges. This is a necessary condition that must be fulfilled when building approximations to $H(\mathrm{div};\Omega)$. The Piola transformation further satisfies the following properties (cf. [7]):

$$(\nabla \cdot \mathbf{v}, w)_T = (\hat{\nabla} \cdot \hat{\mathbf{v}}, \hat{w})_{\hat{T}}, \qquad \langle \mathbf{v} \cdot \mathbf{n}_e, w \rangle_e = \langle \hat{\mathbf{v}} \cdot \hat{\mathbf{n}}_{\hat{e}}, \hat{w} \rangle_{\hat{e}}.$$

In this context, the MFE spaces $V_h \times W_h \subset H_0(\mathrm{div};\Omega) \times L^2(\Omega)$ on $\mathcal{T}_h$ are given by

$$V_h = \left\{ \mathbf{v} \in H_0(\mathrm{div};\Omega) : \mathbf{v}|_T \leftrightarrow \hat{\mathbf{v}},\ \hat{\mathbf{v}} \in \hat{V}(\hat{T}) \ \forall\, T \in \mathcal{T}_h \right\},$$
$$W_h = \{ w \in L^2(\Omega) : w|_T \leftrightarrow \hat{w},\ \hat{w} \in \hat{W}(\hat{T}) \ \forall\, T \in \mathcal{T}_h \}.$$

Note that $V_h$ is also a subspace of $(L^2(\Omega))^2$, since $H_0(\mathrm{div};\Omega) \subset (L^2(\Omega))^2$. For later use, let $\hat{\mathcal{P}} : L^2(\hat{T}) \to \hat{W}(\hat{T})$ be the $L^2(\hat{T})$-projection operator satisfying the orthogonality condition $(\hat{\varphi} - \hat{\mathcal{P}}\hat{\varphi}, \hat{w})_{\hat{T}} = 0$, for any $\hat{\varphi} \in L^2(\hat{T})$ and $\hat{w} \in \hat{W}(\hat{T})$. Accordingly, let $\mathcal{P}_h : L^2(\Omega) \to W_h$ be the $L^2(\Omega)$-projection operator, which is locally defined on each element $T$ as $\mathcal{P}_h\varphi|_T = \hat{\mathcal{P}}\hat{\varphi} \circ F_T^{-1}$, for any $\varphi \in L^2(\Omega)$. Using a scaling argument and the Bramble–Hilbert lemma, it can be shown that

$$\|\varphi - \mathcal{P}_h\varphi\| \le Ch\,|\varphi|_1, \tag{5}$$

where $C$ is a positive constant, defined to be independent of $h$.

**2.3. The semidiscrete scheme.** The spatial discretization of the variational formulation (4) requires computing two integrals of the form $(G\mathbf{q}, \mathbf{v})$, for $\mathbf{q}$, $\mathbf{v} \in V_h$, which approximate the left-hand sides of the equations (4b) and (4c). In doing so, we shall consider a suitable quadrature rule that permits to reduce the semidiscrete scheme to a cell-centered finite difference method for the pressure. The integration on each element $T \in \mathcal{T}_h$ is accomplished by mapping to the reference element $\hat{T}$, where the quadrature rule is defined. Using the Piola transformation, for any $\mathbf{q}$, $\mathbf{v} \in V_h$ and $\hat{\mathbf{q}}$, $\hat{\mathbf{v}} \in \hat{V}(\hat{T})$, we have

$$(G\mathbf{q}, \mathbf{v})_T = \left( \frac{1}{J_T} B_T^T G B_T \, \hat{\mathbf{q}}, \hat{\mathbf{v}} \right)_{\hat{T}}.$$

If we choose $G|_T = J_T B_T^{-T} B_T^{-1}$ on each element $T \in \mathcal{T}_h$, we simplify the interaction of the functions on $\hat{T}$, thus obtaining $(G\mathbf{q}, \mathbf{v})_T = (\hat{\mathbf{q}}, \hat{\mathbf{v}})_{\hat{T}}$. Note that $G$ is indeed symmetric and positive definite on $T \in \mathcal{T}_h$. In virtue of the previous result, the quadrature rule on $T$ is defined as (cf. [1, 2])

$$(G\mathbf{q}, \mathbf{v})_{Q,T} \equiv (\hat{\mathbf{q}}, \hat{\mathbf{v}})_{\hat{Q},\hat{T}} = \frac{|\hat{T}|}{6} \left( \sum_{i=1}^{3} \hat{\mathbf{q}}(\hat{\mathbf{r}}_i) \cdot \hat{\mathbf{v}}(\hat{\mathbf{r}}_i) + 3\hat{\mathbf{q}}(\hat{\mathbf{x}}_c) \cdot \hat{\mathbf{v}}(\hat{\mathbf{x}}_c) \right),$$

where $|\hat{T}|$ is the area of $\hat{T}$ and $\hat{\mathbf{x}}_c = \frac{1}{3}(\hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_2 + \hat{\mathbf{r}}_3)$. This quadrature rule is exact for polynomials of degree 1. Furthermore, if we denote by $\hat{\mathbf{v}}_i$ the basis function of $\hat{V}(\hat{T})$ associated to the $i$-th edge $\hat{e}_i$, for $i = 1, 2, 3$, the following orthogonality condition is satisfied:

$$(\hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j)_{\hat{Q},\hat{T}} = \begin{cases} 0, & \text{if } i \neq j, \\ \frac{1}{6} |\hat{T}| \, |e_i|^2, & \text{if } i = j, \end{cases} \tag{6}$$

where $|e_i|$ is the length of the corresponding $i$-th edge $e_i$ of $T$, for $i = 1, 2, 3$. The global quadrature rule is thus given by $(G\mathbf{q}, \mathbf{v})_Q = \sum_{T \in \mathcal{T}_h} (G\mathbf{q}, \mathbf{v})_{Q,T}$.

As a result, the expanded MFE approximation to (4) reads: *Find* $(\mathbf{u}_h, \lambda_h, p_h) : [0, T] \to V_h \times V_h \times W_h$ *such that*

$$(p_{h,t}, w) + (\nabla \cdot \mathbf{u}_h, w) = (f, w) \qquad \forall\, w \in W_h, \tag{7a}$$

$$(G\lambda_h, \mathbf{v})_Q = (p_h, \nabla \cdot \mathbf{v}) - \langle g, \mathbf{v} \cdot \mathbf{n} \rangle_{\Gamma_D} \qquad \forall\, \mathbf{v} \in V_h, \tag{7b}$$

$$(G\mathbf{u}_h, \mu)_Q = (GKG\lambda_h, \mu) \qquad \forall\, \mu \in V_h, \tag{7c}$$

$$p_h(0) = S_h p(0), \tag{7d}$$

where $S_h p(0)$ denotes the MFE elliptic projection of $p(0)$ (to be defined below). Note that the initial condition $p_h(0)$ determines $\lambda_h(0)$ through (7b); in turn, $\mathbf{u}_h(0)$ is determined by $\lambda_h(0)$ through (7c).

**2.4. Reduction to a cell-centered finite difference scheme for the pressure.** In this subsection, we describe how to obtain a cell-centered finite difference discretization in the pressure variable from the expanded MFE formulation (7). Let $N_e$ and $N_T$ be the number of edges and elements in $\mathcal{T}_h$, respectively. If we denote by $\{\mathbf{v}_i\}_{i=1}^{N_e}$ and $\{w_j\}_{j=1}^{N_T}$ the respective basis functions of $V_h$ and $W_h$, the semidiscrete solution $(\mathbf{u}_h, \lambda_h, p_h)$ can be expressed as

$$\begin{aligned} \mathbf{u}_h(\mathbf{x}, t) &= \textstyle\sum_{i=1}^{N_e} U_{h,i}(t)\, \mathbf{v}_i(\mathbf{x}), \\ \lambda_h(\mathbf{x}, t) &= \textstyle\sum_{i=1}^{N_e} \Lambda_{h,i}(t)\, \mathbf{v}_i(\mathbf{x}), \\ p_h(\mathbf{x}, t) &= \textstyle\sum_{i=1}^{N_T} P_{h,i}(t)\, w_i(\mathbf{x}). \end{aligned}$$

Defining the vectors $U_h = (U_{h,1}, U_{h,2}, \ldots, U_{h,N_e})^T$, $\Lambda_h = (\Lambda_{h,1}, \Lambda_{h,2}, \ldots, \Lambda_{h,N_e})^T$ and $P_h = (P_{h,1}, P_{h,2}, \ldots, P_{h,N_T})^T$, the differential system stemming from (7a)–(7c) can be written in matrix form as

$$\begin{pmatrix} P_h' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & -D^{-1}B & 0 \\ B^T & 0 & S \\ 0 & -S & C \end{pmatrix} \begin{pmatrix} P_h \\ U_h \\ \Lambda_h \end{pmatrix} = \begin{pmatrix} F_h \\ G_h \\ 0 \end{pmatrix}, \tag{8}$$

where the matrices $B \in \mathbb{R}^{N_T \times N_e}$, $C \in \mathbb{R}^{N_e \times N_e}$ and $S \in \mathbb{R}^{N_e \times N_e}$ are given by

$$\begin{aligned} (B)_{ij} &= -(\nabla \cdot \mathbf{v}_j, w_i), && \text{for } i = 1, 2, \ldots, N_T,\ j = 1, 2, \ldots, N_e, \\ (C)_{ij} &= (GKG\mathbf{v}_j, \mathbf{v}_i), && \text{for } i, j \in \{1, 2, \ldots, N_e\}, \\ (S)_{ij} &= (G\mathbf{v}_j, \mathbf{v}_i)_Q, && \text{for } i, j \in \{1, 2, \ldots, N_e\}. \end{aligned}$$

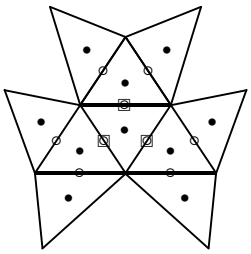

FIGURE 2. Stencil for the discrete diffusion $A_h P_h$ on a generic element $T \in \mathcal{T}_h$. Symbols: $\square$, $U_h$; $\circ$, $\Lambda_h$; $\bullet$, $P_h$.

Note that $C$ is symmetric, positive definite and sparse, and $S$ is diagonal with positive diagonal entries (see (6)). In addition, $D \in \mathbb{R}^{N_T \times N_T}$ is given by $D = \mathrm{diag}(|T_1|, |T_2|, \ldots, |T_{N_T}|)$, where $|T_i|$ denotes the area of $T_i$, for $i = 1, 2, \ldots, N_T$. Finally, the vectors $F_h \in \mathbb{R}^{N_T}$ and $G_h \in \mathbb{R}^{N_e}$ satisfy

$$(F_h)_i = \tfrac{1}{|T_i|}\,(f, w_i), \qquad \text{for } i = 1, 2, \ldots, N_T,$$
$$(G_h)_i = -\langle g, \mathbf{v}_i \cdot \mathbf{n}\rangle_{\Gamma_D}, \qquad \text{for } i = 1, 2, \ldots, N_e.$$

In order to derive a cell-centered scheme for the pressure, we need to express $U_h$ in terms of $P_h$ and substitute this expression into the first equation of (8). To this end, we first obtain $\Lambda_h = S^{-1}(G_h - B^T P_h)$ from the second equation, and subsequently derive $U_h = S^{-1} C \Lambda_h$ from the third equation. Inserting these expressions back into the first equation, we get

$$P_h' + D^{-1} B S^{-1} C S^{-1} B^T P_h = F_h + D^{-1} B S^{-1} C S^{-1} G_h.$$

Hence, if we denote $A_h = D^{-1} B S^{-1} C S^{-1} B^T$ and $M_h = D^{-1} B S^{-1} C S^{-1} G_h$, we can rewrite the expanded MFE method (7) as a stiff initial value problem of the form[1]: *Find* $P_h : [0, T] \to \mathcal{H}_p$ *such that*

$$P_h'(t) + A_h P_h(t) = F_h(t) + M_h(t) \qquad t \in (0, T], \tag{9a}$$
$$P_h(0) = P_h^0, \tag{9b}$$

where $P_h^0 \in \mathbb{R}^{N_T}$ is given by $(P_h^0)_i = \frac{1}{|T_i|} \int_{T_i} p_0(\mathbf{x})\, d\mathbf{x}$, for $i = 1, 2, \ldots, N_T$. The discrete diffusion term $A_h P_h$ shows a local 10-point stencil on a generic element $T \in \mathcal{T}_h$ (cf. Figure 2).

## 3. Error analysis of the semidiscrete scheme

In this section, we derive a priori error estimates for the expanded MFE formulation (7). The results are based on the convergence behaviour of a so-called MFE elliptic projection, first proposed in [29]. Recalling (4), such a projection can be expressed in the form: *Find* $(\mathcal{R}_h \mathbf{u}, \mathcal{L}_h \lambda, \mathcal{S}_h p) : [0, T] \to V_h \times V_h \times W_h$ *such that*

$$(\nabla \cdot \mathcal{R}_h \mathbf{u}, w) = (f - p_t, w) \qquad \forall\, w \in W_h, \tag{10a}$$
$$(G\mathcal{L}_h \lambda, \mathbf{v})_Q = (\mathcal{S}_h p, \nabla \cdot \mathbf{v}) - \langle g, \mathbf{v} \cdot \mathbf{n}\rangle_{\Gamma_D} \qquad \forall\, \mathbf{v} \in V_h, \tag{10b}$$
$$(G\mathcal{R}_h \mathbf{u}, \mu)_Q = (GKG\mathcal{L}_h \lambda, \mu) \qquad \forall\, \mu \in V_h, \tag{10c}$$
$$(\mathcal{S}_h p(0), w) = (p_0, w) \qquad \forall\, w \in W_h. \tag{10d}$$

[1]For a fixed $t \in [0, T]$, the vector space $\mathcal{H}_p$ contains the pressure functions $P_h(t)$ whose degrees of freedom are located at the centroids of the triangles.

Note that the triple $(\mathcal{R}_h\mathbf{u}, \mathcal{L}_h\lambda, \mathcal{S}_h p)$ is precisely the solution of the expanded MFE approximation to a continuous elliptic problem whose exact solution is $(\mathbf{u}, \lambda, p)$. This formulation shows a dual–dual mixed structure and can be proved to satisfy a discrete inf–sup condition (cf. [12]). Subtracting (10) from (4), we get the error equations

$$(\nabla \cdot (\mathbf{u} - \mathcal{R}_h\mathbf{u}), w) = 0 \qquad \forall\, w \in W_h, \tag{11a}$$

$$(G\lambda, \mathbf{v}) - (G\mathcal{L}_h\lambda, \mathbf{v})_Q = (p - \mathcal{S}_h p, \nabla \cdot \mathbf{v}) \qquad \forall\, \mathbf{v} \in V_h, \tag{11b}$$

$$(G\mathbf{u}, \mu) - (G\mathcal{R}_h\mathbf{u}, \mu)_Q = (GKG(\lambda - \mathcal{L}_h\lambda), \mu) \qquad \forall\, \mu \in V_h, \tag{11c}$$

$$(p(0) - \mathcal{S}_h p(0), w) = 0 \qquad \forall\, w \in W_h. \tag{11d}$$

The convergence analysis of this formulation has been developed in [2]. The main result is provided in the next lemma, which assumes the existence of a global mapping $F : \hat{\Omega} \to \Omega$; here, $\hat{\Omega}$ denotes a reference domain that can be discretized using a mesh of equilateral triangles.

**Lemma 1** (Arbogast-Dawson-Keenan-Wheeler-Yotov [2, Theorem 8.4])**.** *Let the triple $(\mathcal{R}_h\mathbf{u}, \mathcal{L}_h\lambda, \mathcal{S}_h p)$ be the mixed finite element elliptic projection of $(\mathbf{u}, \lambda, p)$ on $\mathcal{T}_h$ as given by (10). If there exists a $\mathcal{C}^3(\hat{\Omega} \cup \partial\hat{\Omega})$ mapping $F : \hat{\Omega} \to \Omega$, then it holds, for all $t \in [0, T]$,*

$$\|\mathbf{u}(t) - \mathcal{R}_h\mathbf{u}(t)\| + \|\lambda(t) - \mathcal{L}_h\lambda(t)\| \le Ch, \tag{12}$$

$$\|p(t) - S_h p(t)\| \le Ch, \tag{13}$$

*where $C$ is a positive constant, defined to be independent of $h$.*

**Remark 1.** *Experimentally, for smooth problems, $S_h p(t)$ has been observed to be $\mathcal{O}(h^2)$ superconvergent to $p(t)$ at the centroids of the triangles, for all $t \in [0, T]$.*

Based on (12) and (13), the error estimates for the semidiscrete scheme (7) are derived in the following theorem.

**Theorem 1.** *Let the triple $(\mathbf{u}_h, \lambda_h, p_h)$ be the expanded mixed finite element approximation to $(\mathbf{u}, \lambda, p)$ on $\mathcal{T}_h$ as given by (7). Under the hypotheses of Lemma 1, it holds*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(L^2)} + \|\lambda - \lambda_h\|_{L^\infty(L^2)} \le Ch, \tag{14}$$

$$\|p - p_h\|_{L^\infty(L^2)} \le Ch, \tag{15}$$

*where $C$ is a positive constant, defined to be independent of $h$.*

*Proof.* Let us begin with the analysis of both the adjusted gradient and pressure. For all $t \in [0, T]$, the triangle inequality implies

$$\|\lambda - \lambda_h\| \le \|\lambda - \mathcal{L}_h\lambda\| + \|\mathcal{L}_h\lambda - \lambda_h\|, \tag{16}$$

$$\|p - p_h\| \le \|p - \mathcal{S}_h p\| + \|\mathcal{S}_h p - p_h\|. \tag{17}$$

Note that, in both equations, the first term on the right-hand side is bounded by Lemma 1. In order to derive the corresponding bounds for the second terms, we subtract (7) from (4) to obtain

$$\begin{aligned}
(p_t - p_{h,t}, w) + (\nabla \cdot (\mathbf{u} - \mathbf{u}_h), w) &= 0 && \forall\, w \in W_h,\\
(G\lambda, \mathbf{v}) - (G\lambda_h, \mathbf{v})_Q &= (p - p_h, \nabla \cdot \mathbf{v}) && \forall\, \mathbf{v} \in V_h,\\
(G\mathbf{u}, \mu) - (G\mathbf{u}_h, \mu)_Q &= (GKG(\lambda - \lambda_h), \mu) && \forall\, \mu \in V_h,\\
(p(0) - p_h(0), w) &= 0 && \forall\, w \in W_h.
\end{aligned}$$

If we take into account (11) and further denote $\xi_h = \mathcal{S}_h p - p_h$, $\gamma_h = p - \mathcal{S}_h p$, $\zeta_h = \mathcal{R}_h \mathbf{u} - \mathbf{u}_h$ and $\eta_h = \mathcal{L}_h \lambda - \lambda_h$, the preceding equations can be written in the form

$$(\xi_{h,t} + \gamma_{h,t}, w) + (\nabla \cdot \zeta_h, w) = 0 \qquad \forall\, w \in W_h, \tag{18a}$$

$$(G\eta_h, \mathbf{v})_Q = (\xi_h, \nabla \cdot \mathbf{v}) \qquad \forall\, \mathbf{v} \in V_h, \tag{18b}$$

$$(G\zeta_h, \mu)_Q = (GKG\eta_h, \mu) \qquad \forall\, \mu \in V_h, \tag{18c}$$

$$\xi_h(0) = 0. \tag{18d}$$

In this derivation, we use the easily established fact that the projection operator $\mathcal{S}_h$ commutes with time differentiation. At this point, we take two different approaches to analyze the convergence of either $\xi_h$ or $\eta_h$.

In the former case, we choose $\mathbf{v} = \zeta_h$, $\mu = \eta_h$ and $w = \xi_h$. Then, if we add (18a) and (18b), and subsequently subtract (18c), we obtain

$$(\xi_{h,t}, \xi_h) + (GKG\eta_h, \eta_h) = -(\gamma_{h,t}, \xi_h).$$

Using the Cauchy–Schwarz and Young's inequalities, together with (2), we get

$$\frac{1}{2}\frac{d}{dt}\|\xi_h\|^2 + C_0\|\eta_h\|^2 \leq \frac{1}{2}\left(\|\gamma_{h,t}\|^2 + \|\xi_h\|^2\right),$$

where $C_0$ is a positive constant, defined to be independent of $h$. Integration with respect to the time variable from 0 to $t$ yields

$$\|\xi_h(t)\|^2 + 2C_0 \int_0^t \|\eta_h\|^2 d\tau \leq \int_0^t \left(\|\gamma_{h,t}\|^2 + \|\xi_h\|^2\right) d\tau,$$

for all $t \in [0, T]$, since $\xi_h(0) = 0$ (see (18d)). The subsequent application of Gronwall's lemma leads to

$$\|\xi_h(t)\|^2 + 2C_0 \int_0^t \|\eta_h\|^2 d\tau \leq C_1 \int_0^t \|\gamma_{h,t}\|^2 \, d\tau,$$

where $C_1$ is also a positive constant not depending on $h$. The right-hand side of this expression is bounded by (13). Since the second term on the left-hand side is non-negative, it follows that $\|\xi_h(t)\| \leq Ch$, for all $t \in [0, T]$. Hence, (15) follows by inserting this bound and (13) into (17), and taking the supremum over all $t$.

To study the convergence of $\eta_h$, we differentiate (18b) with respect to $t$ and choose $\mathbf{v} = \zeta_h$, $\mu = \eta_{h,t}$ and $w = \xi_{h,t}$. This yields

$$(\xi_{h,t}, \xi_{h,t}) + (GKG\eta_h, \eta_{h,t}) = -(\gamma_{h,t}, \xi_{h,t}).$$

In this case, we have

$$\|\xi_{h,t}\|^2 + \frac{1}{2}\frac{d}{dt}(GKG\eta_h, \eta_h) \leq \frac{1}{2}\left(\|\gamma_{h,t}\|^2 + \|\xi_{h,t}\|^2\right).$$

Integrating this expression with respect to $t$ and taking into account (2), we obtain

$$\|\eta_h(t)\|^2 \leq C \int_0^t \|\gamma_{h,t}\|^2 \, d\tau,$$

for all $t \in [0, T]$. Note that $\eta_h(0) = 0$, as can be derived from (18b) by choosing $\mathbf{v} = \eta_h(0)$ (provided that $\xi_h(0) = 0$). Once again, (13) can be used to bound the right-hand side of this inequality. Thus, recalling (12) and (16) and taking the supremum over all $t$, we obtain the bound in (14) for the adjusted gradient.

Finally, the velocity error $\mathbf{u} - \mathbf{u}_h$ is analyzed via a different approach. We define the discrete $(L^2(\Omega))^2$-projection operator $\mathcal{Q}_V : (L^2(\Omega))^2 \to V_h$ as given by

$$((\mathcal{Q}_V \mathbf{q} - \mathbf{q}), \mathbf{v})_Q = 0 \qquad \forall\, \mathbf{v} \in V_h.$$

This operator satisfies the approximation property (cf. [2])

$$\|\mathcal{Q}_V \mathbf{q} - \mathbf{q}\| \leq Ch\,\|\mathbf{q}\|_1. \tag{19}$$

For all $t \in [0,T]$, we split the error term in the $L^2(\Omega)$-norm as

$$\|\mathbf{u} - \mathbf{u}_h\| \leq \|\mathbf{u} - \mathcal{Q}_V \mathbf{u}\| + \|\mathcal{Q}_V \mathbf{u} - \mathbf{u}_h\|. \tag{20}$$

The first term on the right-hand side is bounded by (19), whilst the second one can be proved to satisfy (cf. [2, Theorem 8.4])

$$\|\mathcal{Q}_V \mathbf{u} - \mathbf{u}_h\| \leq C(h\|\mathbf{u}\|_2 + \|\lambda - \lambda_h\|). \tag{21}$$

As derived above, the error term $\|\lambda - \lambda_h\|$ is $\mathcal{O}(h)$. Hence, inserting (19) and (21) into (20) and taking the supremum over all $t$, we obtain the bound in (14) for the velocity variable and complete the proof. □

## 4. The domain decomposition splitting method

In this section, we construct an efficient time integrator for solving the initial value problem (9). To this end, we first introduce a domain decomposition operator splitting for the discrete diffusion and right-hand side. This splitting is subsequently combined with a fractional step formula, which reduces the system of ordinary differential equations (9) to a collection of algebraic linear systems (one per internal stage).

**4.1. A domain decomposition operator splitting.** Let $\Omega_1^*, \Omega_2^*, \ldots, \Omega_m^*$ form a non-overlapping decomposition of $\Omega$ into $m$ subdomains. This decomposition fulfills the conditions $\overline{\Omega} = \cup_{k=1}^m \overline{\Omega}_k^*$ and $\Omega_k^* \cap \Omega_l^* = \emptyset$, for $k \neq l$. In turn, each $\Omega_k^* \subset \Omega$ is considered to be an open disconnected set involving $m_k$ connected components, i.e., $\Omega_k^* = \cup_{l=1}^{m_k} \Omega_{kl}^*$, for $k = 1, 2, \ldots, m$. Such components are pairwise disjoint (that is, $\Omega_{ki}^* \cap \Omega_{kj}^* = \emptyset$, for $i \neq j$) and chosen to be shape regular of diameter $H$. Typically, the components $\Omega_{kl}^*$ correspond to the elements in a coarse partition $\mathcal{T}_H$ of $\Omega$ with mesh size $H$.

Let $\Omega_{kl}$ be the extension of $\Omega_{kl}^*$ obtained by translating its internal boundaries, $\partial\Omega_{kl}^* \cap \Omega$, within a distance $\gamma \equiv \beta H$ in $\Omega$. The parameter $\beta > 0$ is usually referred to as the overlapping factor and its value must be chosen in such a way that the extended components are also pairwise disjoint (i.e., $\Omega_{ki} \cap \Omega_{kj} = \emptyset$, for $i \neq j$). If we denote by $\Omega_k \subset \Omega$ the open disconnected set defined as $\Omega_k = \cup_{l=1}^{m_k} \Omega_{kl}$, for $k = 1, 2, \ldots, m$, then the collection $\Omega_1, \Omega_2, \ldots, \Omega_m$ form an overlapping decomposition of $\Omega$ into $m$ subdomains. Such a decomposition satisfies $\Omega = \cup_{k=1}^m \Omega_k$.

Next, we construct a smooth partition of unity consisting of a family of $m$ non-negative and $\mathcal{C}^\infty(\Omega)$ functions $\{\rho_k(\mathbf{x})\}_{k=1}^m$. Each function $\rho_k : \overline{\Omega} \to [0,1]$ is

$$\rho_k(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{x} \in \overline{\Omega} \setminus \overline{\Omega}_k, \\ h_k(\mathbf{x}), & \text{if } \mathbf{x} \in \cup_{l=1;\, l\neq k}^m (\overline{\Omega}_k \cap \overline{\Omega}_l), \\ 1, & \text{if } \mathbf{x} \in \overline{\Omega}_k \setminus \cup_{l=1;\, l\neq k}^m (\overline{\Omega}_k \cap \overline{\Omega}_l), \end{cases} \tag{22}$$

where $h_k(\mathbf{x})$ is $\mathcal{C}^\infty(\Omega)$ and satisfies $0 \leq h_k(\mathbf{x}) \leq 1$ and $\sum_{k=1}^m h_k(\mathbf{x}) = 1$. Therefore, the family of functions $\{\rho_k(\mathbf{x})\}_{k=1}^m$ fulfills, for any $\mathbf{x} \in \overline{\Omega}$,

$$\text{supp}(\rho_k(\mathbf{x})) \subset \overline{\Omega}_k, \qquad 0 \leq \rho_k(\mathbf{x}) \leq 1, \qquad \sum_{k=1}^m \rho_k(\mathbf{x}) = 1. \tag{23}$$

In this framework, we introduce the decompositions

$$A_h = \textstyle\sum_{k=1}^m A_{h,k}, \qquad M_h = \sum_{k=1}^m M_{h,k}, \qquad F_h = \sum_{k=1}^m F_{h,k},$$

where $A_{h,k} \in \mathbb{R}^{N_T \times N_T}$ and $M_{h,k} \in \mathbb{R}^{N_T}$ are given by

$$A_{h,k} = D^{-1} B S^{-1} C_k S^{-1} B^T, \qquad M_{h,k} = D^{-1} B S^{-1} C_k S^{-1} G_h, \tag{24}$$

for $k = 1, 2, \ldots, m$. Here, we define $C_k \in \mathbb{R}^{N_e \times N_e}$ as $(C_k)_{ij} = (G\rho_k K G \mathbf{v}_j, \mathbf{v}_i)$, for $i, j \in \{1, 2, \ldots, N_e\}$. On the other hand, each vector component of $F_{h,k} \in \mathbb{R}^{N_T}$ fulfills

$$(F_{h,k})_i = \tfrac{1}{|T_i|} (\rho_k f, w_i), \quad \text{for } i = 1, 2, \ldots, N_T,\ k = 1, 2, \ldots, m.$$

As a result, the system of ordinary differential equations (9a) can be expressed as a split system of the form

$$P_h' + \sum_{k=1}^{m} A_{h,k} P_h = \sum_{k=1}^{m} L_{h,k}, \tag{25}$$

where $L_{h,k} = F_{h,k} + M_{h,k}$.

**Lemma 2.** *The matrices $\{A_{h,k}\}_{k=1}^m$ defined in (24) satisfy*

$$\xi^T A_{h,k} \xi \geq 0 \qquad \forall\, \xi \neq \mathbf{0} \in \mathbb{R}^{N_T}, \tag{26}$$

*for $k = 1, 2, \ldots, m$.*

*Proof.* Since $C$ is symmetric and positive definite, $C_k$ will be symmetric and non-negative definite, due to the presence of $\rho_k$ in its definition (see (23)). As a result, the matrix $N_k = B S^{-1} C_k S^{-1} B^T$ is also symmetric and non-negative definite. Hence, for any $\xi \neq \mathbf{0} \in \mathbb{R}^{N_T}$, we consider $\theta = D^{-1/2}\xi$, thus obtaining

$$\xi^T D^{-1} N_k \xi = \theta^T D^{-1/2} N_k D^{1/2} \theta \geq 0,$$

which follows from the Rayleigh quotient, since $D^{-1/2} N_k D^{1/2}$ is similar to $N_k$. This result implies (26) and completes the proof. □

A natural way to solve (25) is the use of a fractional step time integrator which takes advantage of the multiterm partitioning. Since the matrices $\{A_{h,k}\}_{k=1}^m$ do not commute pairwise, we further require a method whose stability is not affected by the lack of commutativity of the split terms. In the sequel, we present a family of time-splitting formulae defined to be unconditionally stable even in the non-commuting case.

**4.2. The fully discrete scheme.** Let us consider a family of FSRK$_m$ methods involving an arbitrary number $m$ of implicit parts. When applied to the split system (25), it gives rise to the fully discrete scheme

$$\left\{ \begin{array}{l} \text{For } n = 0, 1, \ldots, N: \\ \quad P_h^{n,0} = P_h^n, \\ \quad \text{For } k = 1, 2, \ldots, m: \\ \qquad P_h^{n,k} = P_h^{n,k-1} + \tau\,(-A_{h,k} P_h^{n,k} + L_{h,k}(t_{n+1})), \\ \quad P_h^{n+1} = P_h^{n,m}. \end{array} \right. \tag{27}$$

For the sake of simplicity, the time step $\tau$ is considered to be constant, $t_n = n\tau$ and $N = [T/\tau] - 1$. The fully discrete solution $P_h^n$ is an approximation to the solution of (25) at $t = t_n$. This family of time integrators, first proposed in [30], is typically referred to as the fractional implicit Euler method, since one integration step of (27) may be seen as $m$ consecutive implicit Euler steps.

The linear system to solve at the $k$-th internal stage of (27) is given by

$$(I + \tau A_{h,k}) P_h^{n,k} = T_h^{n,k}, \tag{28}$$

for $k = 1, 2, \ldots, m$, where $I$ denotes the identity matrix and $T_h^{n,k}$ is defined to be $T_h^{n,k} = P_h^{n,k-1} + \tau L_{h,k}(t_{n+1})$. Since $A_{h,k}$ contains the function $\rho_k(\mathbf{x})$, with $\mathrm{supp}(\rho_k(\mathbf{x})) \subset \overline{\Omega}_k$ (cf. (23)), the previous system is restricted to subdomain $\Omega_k$. Further, as $\Omega_k$ involves $m_k$ disjoint connected components, namely $\Omega_{kl}$, the linear system (28) is indeed a collection of $m_k$ uncoupled linear subsystems of the form

$$(I_{kl} + \tau A_{h,kl}) \mathcal{R}_{kl} P_h^{n,k} = \mathcal{R}_{kl} T_h^{n,k}, \tag{29}$$

where $I_{kl} = \mathcal{R}_{kl} I \mathcal{R}_{kl}^T$ and $A_{h,kl} = \mathcal{R}_{kl} A_{h,k} \mathcal{R}_{kl}^T$, $\mathcal{R}_{kl}$ being a restriction matrix from $\Omega$ to $\Omega_{kl}$ (see [18, Remark 5.1] for details). Unlike most classical domain decomposition algorithms (cf. [20]), the solution of (29) does not require any Schwarz iteration procedure, since the internal stages in (27) are sequentially solved (i.e., interface conditions need not be imposed on subdomains during the solution process).

## 5. Error analysis of the fully discrete scheme

In this section, we describe the convergence analysis of the fully discrete scheme. For that purpose, let us first define the full discretization error at $t_{n+1}$ as the difference $\bar{p}_h(t_{n+1}) - P_h^{n+1}$, where $\bar{p}_h(t)$ stands for $r_h p(\mathbf{x}, t)$ and $r_h : W_h \to \mathcal{H}_p$ denotes the restriction operator of the scalar functions in $W_h$ to the cell centers of $\mathcal{T}_h$. Then, the global error can be decomposed in the form

$$\bar{p}_h(t_{n+1}) - P_h^{n+1} = (\bar{p}_h(t_{n+1}) - \hat{P}_h^{n+1}) + (\hat{P}_h^{n+1} - P_h^{n+1}), \tag{30}$$

where $\hat{P}_h^{n+1}$ is the numerical solution obtained when applying (27) with a time step $\tau$ and the initial value $P_h^n = \bar{p}_h(t_n)$. The difference $\bar{p}_h(t_{n+1}) - \hat{P}_h^{n+1}$ is commonly known as the full truncation error at $t_{n+1}$ and will be denoted by $\beta_h^{n+1}$. In the sequel, we shall describe in detail how to derive suitable bounds for (30).

Throughout this section, $(\cdot,\cdot)_{\ell^2}$ stands for the discrete $L^2$-inner product in $\mathcal{H}_p$, and $\|\cdot\|_{\ell^2} = (\cdot,\cdot)_{\ell^2}^{1/2}$ represents the induced discrete $L^2$-norm.

**5.1. Stability.** In order to study the stability of the fully discrete scheme (27), we consider the perturbed scheme

$$\left\{\begin{array}{l} Q_h^0 = P_h^0 + \varepsilon_h^0, \\ \text{For } n = 0, 1, \ldots, N: \\ \quad Q_h^{n,0} = Q_h^n, \\ \quad \text{For } k = 1, 2, \ldots, m: \\ \qquad Q_h^{n,k} = Q_h^{n,k-1} + \tau\left(-A_{h,k} Q_h^{n,k} + L_{h,k}(t_{n+1})\right) + \tau \delta_h^{n,k}, \\ \quad Q_h^{n+1} = Q_h^{n,m}, \end{array}\right. \tag{31}$$

where $\varepsilon_h^0$ denotes the error in the initial data, and $\delta_h^{n,k}$ may stand for round-off errors, errors due to non-exactly solving the implicit relations or discretization errors. Let us next define, for $n = 0, 1, \ldots, N_T$,

$$\varepsilon_h^{n+1} = Q_h^{n+1} - P_h^{n+1}. \tag{32}$$

Subtracting (27) from (31), we may write

$$\varepsilon_h^{n+1} = R_h\, \varepsilon_h^n + \tau \sum_{k=1}^{m} S_h^k \delta_h^{n,k}, \tag{33}$$

where

$$R_h = (I + \tau A_{h,m})^{-1} (I + \tau A_{h,m-1})^{-1} \cdots (I + \tau A_{h,1})^{-1}, \tag{34}$$

$$S_h^k = (I + \tau A_{h,m})^{-1} (I + \tau A_{h,m-1})^{-1} \cdots (I + \tau A_{h,k})^{-1}, \tag{35}$$

for $k = 1, 2, \ldots, m$. Observe that $R_h = S_h^1$. In this framework, the stability of the fully discrete scheme (27) will be guaranteed if $\varepsilon_h^{n+1}$ can be bounded in terms of the perturbations. To obtain this bound, we first quote an auxiliary lemma, which can be derived using well-known stability results (cf. [13]).

**Lemma 3.** *Let $\Lambda \in \mathbb{R}^{s\times s}$ satisfy $\xi^T \Lambda \xi \geq 0$, for any $\xi \neq \mathbf{0} \in \mathbb{R}^s$, with $s \in \mathbb{N}$. Then, it holds*

$$\|(I + \mu\Lambda)^{-1}\|_2 \leq 1, \tag{36}$$

*where $\mu$ is a positive constant and $\|\cdot\|_2$ denotes the spectral norm.*

The stability of the fully discrete scheme (27) now follows.

**Theorem 2.** *Let $\varepsilon_h^{n+1}$ be defined by (32). If the split matrices $\{A_{h,k}\}_{k=1}^m$ are given by (24), then it holds, for $n = 0, 1, \ldots, N$,*

$$\|\varepsilon_h^{n+1}\|_{\ell^2} \leq \|\varepsilon_h^0\|_{\ell^2} + C \max_{\substack{0\leq j\leq n\\ 1\leq k\leq m}} \|\delta_h^{j,k}\|_{\ell^2}. \tag{37}$$

*Proof.* The expression (33) can be rewritten in the form

$$\varepsilon_h^{n+1} = (R_h)^{n+1}\varepsilon_h^0 + \tau \sum_{j=0}^{n} \sum_{k=1}^{m} (R_h)^{n-j} S_h^k \delta_h^{j,k}.$$

Recall that the split matrices $\{A_{h,k}\}_{k=1}^m$ are involved in the definition of $R_h$ and $\{S_h^k\}_{k=1}^m$ through (34) and (35), respectively. Since they satisfy (26), the bound (36) applies. Hence, (37) follows from the triangle inequality in the norm $\|\cdot\|_{\ell^2}$. □

Note that (37) shows unconditional stability of the method with respect to the initial error $\varepsilon_h^0$ and the perturbations $\delta_h^{j,k}$.

**5.2. Consistency.** Let us consider the perturbed scheme (31), with $Q_h^n = \bar{p}_h(t_n)$ and $Q_h^{n,k} = \bar{p}_h(t_{n+1})$, for $k = 1, 2, \ldots, m$. As a result, $Q_h^{n+1} = \bar{p}_h(t_{n+1})$ and (32) represents the full discretization error at $t_{n+1}$, i.e., $\varepsilon_h^{n+1} = \bar{p}_h(t_{n+1}) - P_h^{n+1}$. Since $\hat{P}_h^{n+1}$ is the numerical solution obtained in one single step of (27), starting at $P_h^n = \bar{p}_h(t_n)$, we have that $P_h^{n+1} - \hat{P}_h^{n+1} = R_h \varepsilon_h^n$ and the relation (30) can be expressed as

$$\varepsilon_h^{n+1} = R_h \varepsilon_h^n + \beta_h^{n+1}. \tag{38}$$

A term-by-term comparison of (33) and (38) yields the following expression for the full truncation error

$$\beta_h^{n+1} = \tau \sum_{k=1}^{m} S_h^k \delta_h^{n,k}. \tag{39}$$

For later use, we define the spatial truncation error $\alpha_h(t)$ as given by

$$\alpha_h(t) = \bar{p}_h'(t) + A_h \bar{p}_h(t) - L_h(t), \qquad t \in [0, T], \tag{40}$$

where $\bar{p}_h'(t)$ is $r_h p_t(\mathbf{x}, t)$ and $L_h(t) = F_h(t) + M_h(t)$. The full truncation error can be bounded as follows.

**Theorem 3.** *Let $\bar{p}_h(t)$ fulfill*

$$\|\bar{p}_h''(t)\|_{\ell^2} \leq C, \tag{41a}$$

$$\|A_{h,j}\bar{p}_{h,k}'(t)\|_{\ell^2} \leq C, \tag{41b}$$

*where $\bar{p}_{h,k}'(t) = L_{h,k}(t) - A_{h,k}\bar{p}_h(t)$, for $j,\, k \in \{1, 2, \ldots, m\}$ and $t \in [t_n, t_{n+1}]$. Then, the full truncation error $\beta_h^{n+1}$ satisfies, for $n = 0, 1, \ldots, N_T$,*

$$\|\beta_h^{n+1}\|_{\ell^2} \leq C(\tau^2 + \tau\,\|\alpha_h(t_{n+1})\|_{\ell^2}) \qquad \forall\, \tau \in (0, \tau_0], \tag{42}$$

*where $C$ is a positive constant, defined to be independent of $h$ and $\tau$.*

*Proof.* Let $Q_h^n = \bar{p}_h(t_n)$ and $Q_h^{n,k} = \bar{p}_h(t_{n+1})$, for $k = 1, 2, \ldots, m$, in (31). A suitable Taylor expansion of $\bar{p}_h$ around $t = t_{n+1}$, together with the relation

$$\alpha_h(t_{n+1}) = \bar{p}_h'(t_{n+1}) + \sum_{k=1}^{m}(A_{h,k}\bar{p}_h(t_{n+1}) - L_{h,k}(t_{n+1})),$$

yields the following expressions for the perturbation terms

$$\delta_h^{n,1} = \alpha_h(t_{n+1}) + \sum_{k=2}^{m}(L_{h,k}(t_{n+1}) - A_{h,k}\bar{p}_h(t_{n+1})) - \tfrac{1}{2}\tau\bar{p}_h''(\xi_n),$$

$$\delta_h^{n,k} = A_{h,k}\bar{p}_h(t_{n+1}) - L_{h,k}(t_{n+1}), \qquad \text{for } k = 2, 3, \ldots, m,$$

where $\xi_n \in [t_n, t_{n+1}]$. Inserting these expressions into the definition (39) of the full truncation error, we get

$$\begin{aligned}\beta_h^{n+1} = {} & R_h\left(\tau\alpha_h(t_{n+1}) - \tfrac{1}{2}\tau^2\bar{p}_h''(\xi_n)\right) \\ & + \sum_{j=1}^{m-1} S_h^j\tau^2 A_{h,j}\left(\sum_{k=j+1}^{m} A_{h,k}\bar{p}_h(t_{n+1}) - L_{h,k}(t_{n+1})\right).\end{aligned} \tag{44}$$

Finally, (42) is obtained by using (36), with $\Lambda = A_{h,k}$, together with the bounds (41). □

**Remark 2.** *Conditions of type (41b) are derived in* [15]. *In this work, the authors prove similar bounds for a standard 5-point finite difference approximation to the diffusion term $-\nabla\cdot(\rho_k K\nabla p)$. In their analysis, they consider a two-dimensional parabolic problem with homogeneous Dirichlet boundary data. These results may be extended to the expanded mixed finite element method presented above by using the following idea. Recall that the split discrete diffusion term is given by $D^{-1}BS^{-1}C_kS^{-1}B^TP_h$. Since $B^TP_h$ involves first-order differences in the pressure variable, $S^{-1}C_kS^{-1}B^TP_h$ can be viewed as a linear combination of pressure differences, whose coefficients depend on the elements of tensor $K$. This term represents an approximation to the negative partitioned flux, $\rho_k K\nabla p$. Since $B$ provides an extra level of first-order differences, $BS^{-1}C_kS^{-1}B^TP_h$ becomes a 10-point stencil approximation to $-\nabla\cdot(\rho_k K\nabla p)$ on three-line triangular meshes. Note that pre-multiplication by the inverse of $D$ introduces a scaling of the stencil coefficients, but does not modify the stencil structure. In this framework, taking into account that the smooth partition of unity $\{\rho_k(\mathbf{x})\}_{k=1}^m$ defined in (22) satisfies*

$$\|\rho_k\|_{C^q(\Omega)} \leq C\gamma^{-q},$$

*where the constant $C > 0$ is independent of $k$ and $\gamma$ (cf.* [15]*), the upper bound in (41b) can be proved to satisfy*

$$\|A_{h,j}\bar{p}_{h,k}'(t)\|_{\ell^2} \leq C\left(\frac{1}{\gamma^2}\,\|f\|_{C^2(\Omega)} + \frac{1}{\gamma^3}\,\|p\|_{C^4(\Omega)}\right). \tag{45}$$

*Here, we make use of the Hölder norms*

$$\|u\|_{C^q(\Omega)} = \sup_{|\alpha|\leq q} \max_{\overline{\Omega}} |\partial^\alpha u|,$$

*where $q$ is a non-negative integer and $\partial^\alpha$ represents the multi-index notation*

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}, \qquad \alpha = (\alpha_1, \alpha_2).$$

*Further considering (44) in combination with (45), we may derive a bound for the full truncation error $\beta_h^{n+1}$ in terms of the overlap $\gamma$. However, as noted in* [15]*, the resulting estimate is too pessimistic when compared to actual numerical results.*

**5.3. Convergence.** The convergence of the fully discrete scheme (27) follows from the preceding results on consistency and stability. In this subsection, with an abuse of notation, we shall introduce the discrete norm

$$\|\bar{p}_h - P_h\|_{\ell^\infty(\ell^2)} = \max_{0\leq n\leq N_T} \|\bar{p}_h(t_{n+1}) - P_h^{n+1}\|_{\ell^2} \tag{46}$$

to compute the full discretization error of the method.

**Theorem 4.** *Under the hypotheses of Theorem 3, the fully discrete solution $P_h^{n+1}$ of the method (27), with $\{A_{h,k}\}_{k=1}^m$ as given by (24), satisfies*

$$\|\bar{p}_h - P_h\|_{\ell^\infty(\ell^2)} \leq C(\tau + h^2 + \max_{0\leq t\leq T} \|\alpha_h(t)\|_{\ell^2}), \tag{47}$$

*where $C$ is a positive constant, defined to be independent of $h$ and $\tau$.*

*Proof.* Expanding the recurrence relation (38) for the full discretization error, we get

$$\varepsilon_h^{n+1} = (R_h)^{n+1}\varepsilon_h^0 + \sum_{j=1}^{n+1} (R_h)^{n+1-j}\beta_h^j, \tag{48}$$

where $\varepsilon_h^0 = \bar{p}_h(0) - P_h^0$. Since $P_h^0 = r_h(\mathcal{P}_h p(0))$, and $\mathcal{P}_h p$ is $\mathcal{O}(h^2)$-close to $p$ at the center of mass of each element, then $\|\varepsilon_h^0\|_{\ell^2} \leq Ch^2$. Recalling (36), with $\Lambda = A_{h,k}$, together with the bounds (41) and the consistency result (42), we obtain

$$\|\varepsilon_h^{n+1}\|_{\ell^2} \leq Ch^2 + \sum_{j=1}^{n+1} C(\tau^2 + \tau\,\|\alpha_h(t_j)\|_{\ell^2}) \leq C(\tau + h^2 + \max_{0\leq t\leq T} \|\alpha_h(t)\|_{\ell^2}).$$

Taking the maximum over $n$ implies (47) and completes the proof. □

**Remark 3.** *Suitable Taylor expansions of the coefficients of the local 10-point stencil associated to the spatial discretization permit us to prove first-order convergence of its local truncation error $\alpha_h(t)$. The combination of this bound with the thesis of Theorem 4 yields the convergence result*

$$\|r_h p - P_h\|_{\ell^\infty(\ell^2)} \leq C(\tau + h). \tag{49}$$

*In addition, if we define $p_h^{n+1}$ to be the element of $W_h$ satisfying $r_h p_h^{n+1} = P_h^{n+1}$, it can be shown that*

$$\|p(t_{n+1}) - p_h^{n+1}\| \leq \|p(t_{n+1}) - \mathcal{P}_h p(t_{n+1})\| + \|r_h(\mathcal{P}_h p(t_{n+1})) - P_h^{n+1}\|_{\ell^2},$$

*where we use the condition $\|w\| = \|r_h w\|_{\ell^2}$, for any $w \in W_h$. The former term on the right-hand side is $\mathcal{O}(h)$ due to (5), while the latter can be split in the form*

$$\|r_h(\mathcal{P}_h p(t_{n+1})) - P_h^{n+1}\|_{\ell^2} \leq \|r_h(\mathcal{P}_h p(t_{n+1}) - p(t_{n+1}))\|_{\ell^2} + \|r_h p(t_{n+1}) - P_h^{n+1}\|_{\ell^2}.$$

*As mentioned above, the first term is, in this case,* $\mathcal{O}(h^2)$*, and the second is* $\mathcal{O}(\tau+h)$ *due to (49). Thus, if we take the maximum over* $n$*, we obtain*

$$\|p - p_h\|_{\ell^\infty(L^2)} = \max_{0\le n\le N_T} \|p(t_{n+1}) - p_h^{n+1}\| \le C(\tau + h).$$

**Remark 4.** *Similar domain decomposition splitting methods of first and second order in time are studied in* [14, 15] *for a finite difference spatial discretization on rectangular grids. In particular, the authors propose and analyze a first-order fractional step method based on the approximate matrix factorization technique (cf.* [13]*), and a second-order ADI scheme inspired by the method of Douglas and Gunn (cf.* [11]*). In the latter case, the second-order convergence is achieved at the price of losing the unconditional stability without additional assumptions. More specifically, the scheme can be proved to be unconditionally stable if the split matrices* $\{A_{h,k}\}_{k=1}^m$ *are required to commute pairwise. In the non-commuting case, the unconditional stability is preserved for* $m = 2$ *in the norm* $\|(I + \alpha\tau A_{h,2}) \cdot \|_{\ell^2}$*, with* $\alpha > 0$*, but the method turns to be conditionally stable for* $m \ge 3$*. In* [14]*, an alternative method that provides second-order convergence with unconditional stability is also discussed. The method is based on the so-called Strang splitting (cf.* [23]*) in combination with the Crank–Nicolson scheme. In this case, although the unconditional stability is preserved, the method requires more linear systems to be solved than the preceding schemes.*

## Acknowledgments

This research is partially supported by MICINN grant MTM2010-21037.

## References

[1] Arbogast, T., Dawson, C.N. and Keenan, P.T., Mixed finite elements as finite differences for elliptic equations on triangular elements, Tech. Report CRPC-TR94392, Rice University, 1994.

[2] Arbogast, T., Dawson, C.N., Keenan, P.T., Wheeler, M.F. and Yotov, I., Enhanced cell-centered finite differences for elliptic equations on general geometry, SIAM J. Sci. Comput. 19 (1998), pp. 404–425.

[3] Arbogast, T., Wheeler, M.F. and Yotov, I., Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences, SIAM J. Numer. Anal. 34 (1997), pp. 828–852.

[4] Arrarás, A., Mimetic fractional step methods for parabolic problems, Ph.D. thesis, Universidad Carlos III de Madrid, 2011.

[5] Arrarás, A., Portero, L. and Jorge, J.C., A combined mixed finite element ADI scheme for solving Richards' equation with mixed derivatives on irregular grids, Appl. Numer. Math. 59 (2009), pp. 454–467.

[6] Baranger, J., Maître, J.-F. and Oudin, F., Connection between finite volume and mixed finite element methods, RAIRO Modél. Math. Anal. Numér. 30 (1996), pp. 445–465.

[7] Brezzi, F. and Fortin, M., Mixed and Hybrid Finite Element Methods, Springer Ser. Comput. Math. 15, Springer, Berlin, 1991.

[8] Brezzi, F., Fortin, M. and Marini, L.D., Error analysis of piecewise constant pressure approximations of Darcy's law, Comput. Methods Appl. Mech. Engrg. 195 (2006), pp. 1547–1559.

[9] Bujanda, B. and Jorge, J.C., Fractional step Runge–Kutta methods for time dependent coefficient parabolic problems, Appl. Numer. Math. 45 (2003), pp. 99–122.

[10] Chen, Z., Expanded mixed finite element methods for linear second-order elliptic problems, I, RAIRO Modél. Math. Anal. Numér. 32 (1998), pp. 479–499.

[11] Douglas, Jr., J. and Gunn, J.E., A general formulation of alternating direction method: Part I. Parabolic and hyperbolic problems, Numer. Math. 6 (1964), pp. 428–453.

[12] Gatica, G.N. and Heuer, N., An expanded mixed finite element approach via a dual–dual formulation and the minimum residual method, J. Comput. Appl. Math. 132 (2001), pp. 371–385.

[13] Hundsdorfer, W. and Verwer, J.G., Numerical Solution of Time-Dependent Advection–Diffusion–Reaction Equations, Springer Ser. Comput. Math. 33, Springer, Berlin, 2003.

[14] Mathew, T.P.A., Domain Decomposition Methods for the Numerical Solution of Partial Differential Equations, Lect. Notes Comput. Sci. Eng. 61, Springer, Berlin, 2008.
[15] Mathew, T.P., Polyakov, P.L., Russo, G. and Wang, J., Domain decomposition operator splittings for the solution of parabolic equations, SIAM J. Sci. Comput. 19 (1998), pp. 912–932.
[16] Micheletti, S., Sacco, R. and Saleri, F., On some mixed finite element methods with numerical integration, SIAM J. Sci. Comput. 23 (2001), pp. 245–270.
[17] Portero, L., Arrarás, A. and Jorge, J.C., Contractivity of domain decomposition splitting methods for nonlinear parabolic problems, J. Comput. Appl. Math. 234 (2010), pp. 1078–1087.
[18] Portero, L., Arrarás, A. and Jorge, J.C., Domain decomposition operator splitting for mimetic finite difference discretizations of non-stationary problems, Comput. & Fluids 46 (2011), pp. 398–403.
[19] Portero, L., Bujanda, B. and Jorge, J.C., A combined fractional step domain decomposition method for the numerical integration of parabolic problems, in Parallel Processing and Applied Mathematics, R. Wyrzykowski et al., eds., Lecture Notes in Comput. Sci. 3019, Springer-Verlag, Berlin, 2004, pp. 1034–1041.
[20] Quarteroni, A. and Valli, A., Domain Decomposition Methods for Partial Differential Equations, Numer. Math. Sci. Comput., Oxford University Press, New York, 1999.
[21] Raviart, R.A. and Thomas, J.M., A mixed finite element method for 2nd order elliptic problems, in Mathematical Aspects of Finite Element Methods, I. Galligani and E. Magenes, eds., Lecture Notes in Math. 606, Springer, Berlin, 1977, pp. 292–315.
[22] Russell, T.F. and Wheeler, M.F., Finite element and finite difference methods for continuous flows in porous media, in The Mathematics of Reservoir Simulation, R.E. Ewing, ed., Frontiers Appl. Math. 1, SIAM, Philadelphia, 1983, pp. 35–106.
[23] Strang, G., On the construction and comparison of difference schemes, SIAM J. Numer. Anal. 5 (1968), pp. 506–517.
[24] Thomas, J.M., Sur l'analyse numérique des méthods d'éléments finis hybrides et mixtes, Ph.D. thesis, Université Pierre et Marie Curie, 1977.
[25] Vabishchevich, P.N., Difference schemes with domain decomposition for solving nonstationary problems, U.S.S.R. Comput. Math. Math. Phys. 29 (1989), pp. 155–160.
[26] Vabishchevich, P.N., Parallel domain decomposition algorithms for time-dependent problems of mathematical physics, in Advances in Numerical Methods and Applications, I.T. Dimov et al., eds., World Scientific, River Edge, 1994, pp. 293–299.
[27] Verwer, J.G., Contractivity of locally one-dimensional splitting methods, Numer. Math 44 (1984), pp. 247–259.
[28] Weiser, A. and Wheeler, M.F., On convergence of block-centered finite differences for elliptic problems, SIAM J. Numer. Anal. 25 (1988), pp. 351–375.
[29] Wheeler, M.F., A priori $L_2$ error estimates for Galerkin approximations to parabolic partial differential equations, SIAM J. Numer. Anal. 10 (1973), pp. 723–759.
[30] Yanenko, N.N., The Method of Fractional Steps. The Solution of Problems of Mathematical Physics in Several Variables, Springer, Berlin, 1971.

Departamento de Ingeniería Matemática e Informática, Universidad Pública de Navarra, Edificio de Las Encinas, Campus de Arrosadía, 31006 Pamplona (Spain)
*E-mail*: andres.arraras@unavarra.es and laura.portero@unavarra.es