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ANALYSIS AND FINITE ELEMENT APPROXIMATION OF BIOCONVECTION FLOWS WITH CONCENTRATION DEPENDENT VISCOSITY

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Abstract. The problem of a stationary generalized convective flow modelling bioconvection is considered. The viscosity is assumed to be a function of the concentration of the micro-organisms. As a result the PDE system describing the bioconvection model is quasilinear. The existence and uniqueness of the weak solution of the PDE system is obtained under minimum regularity assumption on the viscosity. Numerical approximations based on the finite element method are constructed and error estimates are obtained. Numerical experiments are conducted to demonstrate the accuracy of the numerical method as well as to simulate bioconvection pattern formations based on realistic model parameters.

Key words. bio-convection, nonlinear partial differential equations, finite element method

1. Introduction

Bio-convection occurs due to on average upwardly swimming micro-organisms which are slightly denser than water in suspensions. A fluid dynamical model treating the micro-organisms as collections of particles was first derived independently by M.Levandowsky, W. S. Hunter and E. A. Spiegel [16], and Y. Moribe [22] which we describe as follows. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. At point $x \in \Omega$, let $\mathbf{u}(x) = \{\mathbf{u}_j(x)\}_{j=1}^3$ and $p(x)$ respectively denote the velocity and pressure of the culture fluid while $c(x)$ refers to the concentration of the micro-organisms. The steady state system for $(\mathbf{u}, c, p)$ takes the form

$$
\begin{aligned}
-\,\mathbf{div}\,(\nu(c)D(\mathbf{u})) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= -g(1+\gamma c)i_3 + \mathbf{f}\,, \quad \text{in } \Omega\,,\\
\mathbf{div}\,\mathbf{u} &= 0\,, \quad \text{in } \Omega\,,\\
-\theta\Delta c + \mathbf{u}\cdot\nabla c + U\frac{\partial c}{\partial x_3} &= 0\,, \quad \text{in } \Omega\,.
\end{aligned}
\tag{1.1}
$$

Here $\nu(\cdot) > 0$, as a function of the concentration c, denotes the kinematic viscosity of the culture fluid, $D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ denotes the stress tensor, $\mathbf{f}$ refers to the volume-distributed external force, g is the acceleration of gravity, θ and U are the diffusion rate and the mean velocity of upward swimming of the micro-organisms respectively, $i_3 = (0, 0, 1)$ is the vertical unitary vector, and the constant $\gamma > 0$ is given by $\gamma = \rho_0/\rho_m - 1$, where ρ_0 is the density of the micro-organisms and ρ_m is the density of the culture fluid.

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The bioconvection model (1.1) is a special case of a more general equation describing the diffusion and transformation of an admixture in a region [1]. The first equation is a Navier-Stokes type equation describing the motion of the viscous micro-organisms while the second equation describes the incompressibility of the culture fluid. The last equation of (1.1) describes the mass conservation:

$$\frac{d}{dt}c + \textbf{div } q = 0\,, \quad \text{in } \ \Omega\,,$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + (u, \nabla)$ is the material derivative along the fluid particle and $q = -\theta\nabla c + Uci_3$ represents the flux of micro-organisms. We prescribe the boundary conditions for $\mathbf{u}$ and c as

$$\begin{aligned} \mathbf{u} = 0\,, &\quad \text{on } \partial\Omega\,, \\ \theta\frac{\partial c}{\partial \mathbf{n}} - Ucn_3 = 0\,, &\quad \text{on } \partial\Omega\,. \end{aligned} \tag{1.2}$$

The second equation of (1.2) refers to zero flux on the boundary where $\mathbf{n} = (n_1, n_2, n_3)$ is the exterior unitary normal vector on $\partial\Omega$. We further assume the fixed total mass for the micro-organisms:

$$\frac{1}{|\Omega|}\int_\Omega c(x)dx = \alpha\,, \tag{1.3}$$

for some constant α. Condition (1.3) assures that no micro-organisms are allowed to leave or enter the container. Now the complete system describing the motion of micro-organisms takes the form

$$\begin{cases} -\,\textbf{div }(\nu(c)D(\mathbf{u})) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = -g(1+\gamma c)i_3 + \mathbf{f}\,, \quad \text{in } \Omega\,, \\ \textbf{div } \mathbf{u} = 0\,, \quad \text{in } \Omega\,, \\ -\theta\Delta c + \mathbf{u}\cdot\nabla c + U\frac{\partial c}{\partial x_3} = 0\,, \quad \text{in } \Omega\,, \\ \mathbf{u} = 0\,, \quad \theta\frac{\partial c}{\partial \mathbf{n}} - Ucn_3 = 0\,, \quad \text{on } \partial\Omega\,, \\ \frac{1}{|\Omega|}\int_\Omega c(x)dx = \alpha\,. \end{cases} \tag{1.4}$$

In an ideal Newtonian fluid, the viscosity ν is a constant. In this case, the existence of the solution as well as the positivity of the concentration are proved in [14] where the authors considered both the stationary and evolutionary cases. The evolutionary case of system (1.1) with constant viscosity ν is studied numerically in [12]. The numerical study of slightly different bioconvection models can be found in [4], [8], [9], [7] and [13].

In general, for particle models, the viscosity is related to the concentration of the solute. Albert Einstein showed in his Ph.D thesis [6] that

$$\frac{\nu}{\nu_0} = 1 + \xi c \tag{1.5}$$

when the concentration c is small, where ν is the viscosity of the suspension, ν_0 is the viscosity of the pure solution and ξ is a proportionality coefficient, often chosen to be 2.5. This model was later extended by adding a quadratic term of c by Batchelor [2] for larger c ($\geq 10\%$). When the concentration is much higher, the relative viscosity $\frac{\nu}{\nu_0}$ varies as an exponential function of concentration c ([17], [15] and [3]). A recent work [5] showed the existence and uniqueness of a periodic solution of the time dependent case of (1.1) under the assumption that $\nu(\cdot)$ is a C^1 function, and, for some positive constants ν_* and ν^*

$$\nu_* < \nu(x) < \nu^*\,, \quad \forall x \in R\,, \quad \text{and} \quad \sup_{x\in R} \nu'(x) < \infty\,.$$

In this paper, we first improve the existence result of [5] by allowing ν only to be continuous and bounded. Then we focus our study on numerical simulations of (1.4). Specifically, we shall construct numerical approximations for the exact solution $(\mathbf{u}, c, p)$ of (1.4) using the finite element method with rigorous error analysis. we also conduct numerical experiments to first verify the efficiency and accuracy of our numerical algorithms and then study bioconvection pattern formations using realistic lab data. Though the numerical method is the standard finite element approximation, it still represents one of the first attempts of studying such a bioconvection model through numerical simulations. We plan to consider more sophisticated and efficient numerical methods in future work.

The paper is organized as follows. In the rest of this section, we introduce notations and assumptions that will be used throughout the rest of the paper. In Section 2, we first prove the existence of a weak solution of (1.4) under an assumption on ν which is weaker than (1.6). In Section 3, we consider the finite element approximation of (1.4) and derive error estimates through rigorous error analysis. In the last section we first present a numerical experiment to demonstrate the efficiency and accuracy of our numerical method. Then we conduct a numerical experiment to show the effect of nonlinear viscosity based on the data from lab experiments.

Notations and Assumptions Denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions with compact support in Ω, by $L^2(\Omega)$ the space of square integrable functions on Ω, and by $W^{k,p}(\Omega)$ the Sobolev space consisting of functions in $L^p(\Omega)$ with each of their partial derivatives through order k also in $L^p(\Omega)$. In particular we use $H^k(\Omega)$ to denote the Hilbert space $W^{k,2}(\Omega)$. Let $\mathbf{H}^k(\Omega) = (H^k(\Omega))^3$ and $\mathbf{L}^2(\Omega) = (L^2(\Omega))^3$. The space $\mathbf{H}_0^1(\Omega)$ is the closure of $(C_0^\infty(\Omega))^3$ in $\mathbf{H}^1(\Omega)$. Without confusion, we use $\|\cdot\|_k$ to denote the norms of $H^k(\Omega)$ and $\mathbf{H}^k(\Omega)$. Similarly $\|\cdot\|$ denotes the norm of $\mathbf{L}^2(\Omega)$ and $L^2(\Omega)$. We shall use $(\cdot,\cdot)$ to denote both the L^2 and $\mathbf{L}^2$ inner product. Throughout the paper, C refers to a general constant whose value varies at different appearances.

We assume that the kinematic viscosity $\nu(\cdot): \mathbb{R} \to \mathbb{R}$ is continuous and there exist constants ν_*, ν^* such that

$$0 < \nu_* \le \nu(x) \le \nu^*\,, \quad \forall x \in \mathbb{R}\,. \tag{1.6}$$

2. Existence and uniqueness of a weak solution

2.1. The weak formulation. First note that p is uniquely determined by (1.4) subject to difference of a constant. Denote by $L_0^2(\Omega)$ the closed subspace of $L^2(\Omega)$ orthogonal to constants, i.e.,

$$L_0^2(\Omega) = \{p \in L^2(\Omega);\ \int_\Omega p\ dx = 0\}.$$

Define the following bilinear and trilinear forms

$$a(c,r) = (\nabla c, \nabla r)\,, \quad \forall c, r \in H^1(\Omega)\,,$$
$$B_0(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega \mathbf{u}\cdot\nabla\mathbf{v}\ \mathbf{w}\ dx\,, \quad \forall \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}_0^1(\Omega)\,,$$
$$B(\mathbf{u},c,r) = \int_\Omega \mathbf{u}\cdot\nabla c\ r\ dx\,, \quad \forall \mathbf{u} \in \mathbf{H}_0^1(\Omega), \quad c, r \in H^1(\Omega)\,,$$
$$b(q,\mathbf{v}) = -(q,\ \mathbf{div}\ \mathbf{v})\,, \quad \forall q \in L_0^2(\Omega)\,, \quad \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,,$$

and set

$$\tilde{H} = H^1(\Omega) \cap L_0^2(\Omega) = \{c \in H^1(\Omega)\ :\ \int_\Omega c\ dx = 0\}\,.$$

We observe that the trilinear form $B_0(\cdot,\cdot,\cdot)$ and $B(\cdot,\cdot,\cdot)$ are continuous on $\mathbf{H}_0^1(\Omega)$. In fact, from Holder's inequality and the Sobolev imbedding theorem, we have that

$$B_0(\mathbf{u},\mathbf{v},\mathbf{w}) \le C_{B_0}\|\mathbf{u}\|_{\mathbf{L}^4(\Omega)}\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\Omega)}\|\mathbf{w}\|_{\mathbf{L}^4(\Omega)} \le C\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1 \tag{2.1}$$

where $C_{B_0} > 0$ is a constant. Similarly

$$B(\mathbf{u},c,r) \le C_B\|\mathbf{u}\|_1\|c\|_1\|r\|_1 \tag{2.2}$$

where $C_B > 0$ is a constant. Define

$$\mathbf{V} = \{\mathbf{u} \in \mathbf{H}_0^1(\Omega)\ :\ \ \mathbf{div}\ \mathbf{u} = 0 \text{ in } \Omega\}\,. \tag{2.3}$$

For $\mathbf{u} \in \mathbf{V}$, integrating by parts gives

$$\begin{cases} B_0(\mathbf{u},\mathbf{v},\mathbf{w}) + B_0(\mathbf{u},\mathbf{w},\mathbf{v}) = 0\,, \\ \quad B(\mathbf{u},c,r) + B(\mathbf{u},r,c) = 0\,, \end{cases} \tag{2.4}$$

or equivalently

$$B_0(\mathbf{u},\mathbf{v},\mathbf{v}) = 0\,, \quad B(\mathbf{u},r,r) = 0\,. \tag{2.5}$$

Then condition (1.3) is equivalent to requiring $c - \alpha \in \tilde{H}$. Define an auxiliary concentration $c_\alpha = c - \alpha$ with $\mathbf{f}_\alpha = \mathbf{f} - g\gamma\alpha i_3$. Then the weak formulation of (1.4) is derived by multiplying (1.4) by test functions and integrating by parts (without confussion, we write $c = c_\alpha$ and $\mathbf{f} = \mathbf{f}_\alpha$).

Definition 2.1. Given $\mathbf{f}$ in $\mathbf{L}^2(\Omega)$. $(\mathbf{u}, p, c) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \tilde{H}$ is said to be a weak solution of system (1.4) if

$$\begin{cases} (\nu(c+\alpha)D(\mathbf{u}), D(\mathbf{v})) + B_0(\mathbf{u},\mathbf{u},\mathbf{v}) + b(p,\mathbf{v}) \\ \qquad = -(g(1+\gamma c)i_3, \mathbf{v}) + (\mathbf{f},\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,, \\ \qquad b(q,\mathbf{u}) = 0\,, \quad \forall q \in L_0^2(\Omega)\,, \\ \theta a(c,r) + B(\mathbf{u},c,r) - U(c, \dfrac{\partial r}{\partial x_3}) = U\alpha(\dfrac{\partial r}{\partial x_3}, 1)\,, \quad \forall r \in \tilde{H}\,. \end{cases} \tag{2.6}$$

To solve system (2.6), it suffices to solve the associated problem: find a pair $(\mathbf{u}, c) \in \mathbf{V} \times \tilde{H}$ such that

$$\begin{cases} (\nu(c+\alpha)D(\mathbf{u}), D(\mathbf{v})) + B_0(\mathbf{u},\mathbf{u},\mathbf{v}) = -(g(1+\gamma c)i_3 + \mathbf{f}, \mathbf{v}),\ \forall \mathbf{v} \in \mathbf{V}\,, \\ \qquad \theta a(c,r) + B(\mathbf{u},c,r) - U(c, \dfrac{\partial r}{\partial x_3}) = U\alpha(\dfrac{\partial r}{\partial x_3}, 1)\,,\ \forall r \in \tilde{H}\,. \end{cases} \tag{2.7}$$

Remark 2.2. It is easy to verify that if $(\mathbf{u}, c, p)$ is a solution of system (2.6), then $(\mathbf{u}, c)$ must be a solution of (2.7). The converse is also true since the bilinear form $b(\cdot,\cdot)$ defined above satiesfies the inf-sup condition (see [10]), i.e., for some $\beta > 0$

$$\sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)} \frac{b(q,\mathbf{v})}{\|\mathbf{v}\|_1} \geq \beta\|q\|\,, \quad \forall q \in L_0^2(\Omega)\,.$$

2.2. Existence. To prove the existence of a weak solution of (2.7), we construct a sequence of approximate weak solutions using the Galerkin method, which will also be helpful in our later discussion about the finite element method. First we notice that

$$\begin{cases} \|\mathbf{v}\|_1 \leq C_\Omega \|\nabla \mathbf{v}\|\,, & \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,, \\ \|r\|_1 \leq C_\Omega \|\nabla r\|\,, & \forall r \in \tilde{H}\,, \end{cases} \tag{2.8}$$

for some constant C_Ω independent of $\mathbf{v}$ and r. (2.8) is the Poincaré inequality where the first inequality holds because $\mathbf{v} = 0$ on the boundary while the second one is due to the fact that $\int_\Omega r dx = 0$. We also need the following lemma on Nemytskii operators (see [18]).

Lemma 2.3. *Assume that a function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ satisfies the Carathéodory conditions:*

(i) $f(x,u)$ is a continuous function of u for almost all $x \in \Omega$;

(ii) $f(x,u)$ is a measurable function of x for all $u \in \mathbb{R}^m$.

Furthermore, assume that, for some constant C and $g \in L^q(\Omega)$

$$|f(x,u)| \leq C|u|^{p-1} + g(x), \quad x \in \Omega,\ u \in \mathbb{R}^m$$

where $1 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the Nemytskii operator $F(u) : \Omega \to \mathbb{R}$ defined by

$$F(u)(x) = f(x, u(x))$$

is a bounded and continuous map from $L^p(\Omega;\mathbb{R}^m)$ into $L^q(\Omega;\mathbb{R})$.

It is obvious that the viscosity $\nu(\cdot)$ satisfying condition (1.6) is a Nemytskii operator.

Since $\mathbf{V}$ and $\tilde{H}$ are both separable Hilbert spaces, there exist sequences $\{\mathbf{v}_j\}_{j=1}^\infty$ and $\{r_j\}_{j=1}^\infty$ such that $\{\mathbf{v}_j\}_{j=1}^\infty$ and $\{r_j\}_{j=1}^\infty$ are orthonormal basis of $\mathbf{V}$ and $\tilde{H}$, respectively. Let $\mathbf{V}_m$, $\tilde{H}_m$ be the finite dimensional subspaces of $\mathbf{V}$, $\tilde{H}$ generated by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ and $\{r_1, r_2, \ldots, r_m\}$, respectively. The first step of the Galerkin method is to seek $(\mathbf{u}^m, c^m) \in \mathbf{V}_m \times \tilde{H}_m$ such that

$$\begin{cases} (\nu(c^m+\alpha)D(\mathbf{u}^m), D(\mathbf{v})) + B_0(\mathbf{u}^m,\mathbf{u}^m,\mathbf{v}) = -((g+\gamma c^m)i_3,\mathbf{v}) \\ \qquad\qquad\qquad\qquad\qquad\qquad + (\mathbf{f},\mathbf{v})\,,\ \forall \mathbf{v} \in \mathbf{V}_m\,, \\ \theta a(c^m, r) + B(\mathbf{u}^m, c^m, r) - U(c^m, \dfrac{\partial r}{\partial x_3}) = U\alpha(\dfrac{\partial r}{\partial x_3}, 1)\,,\ \forall r \in R_m\,. \end{cases} \tag{2.9}$$

The existence of a solution of (2.9) is guaranteed for any integar $m > 0$ either by a direct corollary of Brouwer fixed point theorem or using Riesz' theorem.

We next show that $\{\mathbf{u}^m\}_{m=1}^\infty$ and $\{c^m\}_{m=1}^\infty$ are uniformly bounded in $\mathbf{V}$ and $\tilde{H}$, respectively.

Lemma 2.4. *Assume that*

$$\frac{\theta}{C_\Omega^2} > U\,. \tag{2.10}$$

Then there exists a constant C independent of m such that

$$\|c^m\|_1 + \|\mathbf{u}^m\|_1 < C. \tag{2.11}$$

Proof. Let $\mathbf{v} = \mathbf{u}^m$, $r = c^m$ in (2.9). From (2.5) we have

$$(\nu(c^m + \alpha)D(\mathbf{u}^m), D(\mathbf{u}^m)) = -((g + \gamma c^m)i_3, \mathbf{u}^m) + (\mathbf{f}, \mathbf{u}^m)\,,$$

$$\theta a(c^m, c^m) - U(c^m, \frac{\partial c^m}{\partial x_3}) = U\alpha(\frac{\partial c^m}{\partial x_3}, 1)\,.$$

Thus it follows from (1.6) and Young's inequality that

$$\begin{aligned} \nu_*\|\nabla \mathbf{u}^m\|^2 &\le (\nu(c^m + \alpha)D(\mathbf{u}^m), D(\mathbf{u}^m)) \\ &\le |-((g + \gamma c^m)i_3, \mathbf{u}^m) + (\mathbf{f}, \mathbf{u}^m)| \\ &\le (\gamma\|c^m\| + \|\mathbf{f} - g i_3\|)\,\|\mathbf{u}^m\|\,. \end{aligned} \tag{2.12}$$

Also

$$\begin{aligned} \theta\|\nabla c^m\|^2 &\le \theta a(c^m, c^m) \\ &\le |U(c^m, \frac{\partial c^m}{\partial x_3}) + U\alpha|\Omega|^{\frac{1}{2}}(\frac{\partial c^m}{\partial x_3}, 1)| \\ &\le U\|c^m\|_1^2 + U\alpha|\Omega|^{\frac{1}{2}}\|c^m\|_1\,. \end{aligned} \tag{2.13}$$

Using the above inequality, (2.8), and assumption (2.10) we obtain

$$\|c^m\|_1 \le \left(\frac{\theta}{C_\Omega^2} - U\right)^{-1} U\alpha|\Omega|^{\frac{1}{2}},. \tag{2.14}$$

Substituting c^m in (2.12) with the right hand side of (2.14) gives

$$\|\mathbf{u}_m\|_1 \le \frac{C_\Omega^2}{\nu_*}\left((\frac{\theta}{C_\Omega^2} - U)^{-1}U\alpha\right) + \|\mathbf{f} - g i_3\|\,.$$

□

We are now ready to show the existence of a solution of (2.7).

Theorem 2.5. *Assume that* (1.6) *and* (2.10) *hold, and* $\mathbf{f} \in \mathbf{L}^2(\Omega)$. *Then system* (2.7) *has a weak solution.*

Proof. Consider sequences $\{\mathbf{u}^m\}_{m=1}^\infty$, $\{c^m\}_{m=1}^\infty$ defined by (2.9). From Lemma 2.4, there exist $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and $c \in \tilde{H}$ (via subsequences if necessary) such that

$$\mathbf{u}^m \rightharpoonup \mathbf{u} \text{ in } \mathbf{H}_0^1(\Omega) \quad \textit{and} \quad c^m \rightharpoonup c \text{ in } \tilde{H}\,, \quad \textit{as } m \to \infty\,. \tag{2.15}$$

Due to the Sobolev compact embedding theorem, we know that

$$\mathbf{u}^m \to \mathbf{u} \textit{ in } \mathbf{L}^2(\Omega) \quad \textit{and} \quad c^m \to c \textit{ in } L^2(\Omega)\,, \quad \textit{as } m \to \infty\,. \tag{2.16}$$

We now show that the weak limit $(\mathbf{u}, c)$ is a solution of (2.7). Let $\mathbf{v}$ and r be test functions such that

$$\mathbf{v} \in \mathbf{V} \cap (C_0^\infty(\Omega))^3\,, \quad r \in C^\infty(\Omega) \cap \tilde{H}\,. \tag{2.17}$$

First notice that

$$\begin{aligned} &(\nu(c + \alpha)D(\mathbf{u}), D(\mathbf{v})) - (\nu(c^m + \alpha)D(\mathbf{u}^m), D(\mathbf{v})) \\ &\quad = (\nu(c + \alpha)D(\mathbf{u} - \mathbf{u}^m), D(\mathbf{v})) + ((\nu(c + \alpha) - \nu(c^m + \alpha))D(\mathbf{u}^m), D(\mathbf{v})) \\ &\quad := \mathrm{I} + \mathrm{II}\,. \end{aligned}$$

From (1.6), (2.15) we have that

$$|\mathrm{I}| = |(D(\mathbf{u}^m - \mathbf{u}), \nu(c + \alpha)D(\mathbf{v}))| \to 0\,, \quad \textit{as } m \to \infty\,.$$

Property (2.16) and the fact that ν is Nemytskii operator implies that

$$\nu(c^m+\alpha)\to\nu(c+\alpha)\ in\ L^2(\Omega)\,,\quad as\ m\to\infty\,. \tag{2.18}$$

Thus from (2.17), Lemma 2.4 and Holder's inequality, we have

$$|\mathrm{II}|\le C\|\nu(c^m+\alpha)-\nu(c+\alpha)\|\|\mathbf{u}^m\|_1\to 0\,,\quad as\ m\to\infty\,.$$

Combining the above estimates we obtain

$$(\nu(c^m+\alpha)D(\mathbf{u}^m),D(\mathbf{v}))\to(\nu(c+\alpha)D(\mathbf{u}),D(\mathbf{v}))\,,\quad as\ m\to\infty\,. \tag{2.19}$$

Next by Green's formula

$$B_0(\mathbf{u}^m,\mathbf{u}^m,\mathbf{v})=\sum_{i,j=1}^{3}\int_\Omega \mathbf{u}_j^m(\frac{\partial \mathbf{u}_i^m}{\partial x_j}\mathbf{v}_i)\ dx=-\sum_{i,j=1}^{3}\int_\Omega \mathbf{u}_i^m\mathbf{u}_j^m(\frac{\partial \mathbf{v}_i}{\partial x_j})\ dx\,.$$

By assumption (2.17), we know that $\frac{\partial \mathbf{v}}{\partial x_j}$ is uniformly bounded while $\mathbf{u}^m\to\mathbf{u}$ in $\mathbf{L}^2(\Omega)$ implies that $\mathbf{u}_i^m\mathbf{u}_j^m\to\mathbf{u}_i\mathbf{u}_j$ in $\mathbf{L}^1(\Omega)$ as $m\to\infty$. Thus

$$\lim_{m\to\infty}B_0(\mathbf{u}^m,\mathbf{u}^m,\mathbf{v})=-\sum_{i,j=1}^{3}\int_\Omega \mathbf{u}_i\mathbf{u}_j(\frac{\partial \mathbf{v}_i}{\partial x_j})\ dx=-B_0(\mathbf{u},\mathbf{v},\mathbf{u})=B_0(\mathbf{u},\mathbf{u},\mathbf{v})\,.$$

Following the same argument, we have that

$$B_0(\mathbf{u}^m,\mathbf{u}^m,\mathbf{v})\to B_0(\mathbf{u},\mathbf{u},\mathbf{v})\,,\quad B(\mathbf{u}^m,c^m,r)\to B(\mathbf{u},c,r)\,,\quad as\ m\to\infty\,. \tag{2.20}$$

Again from (2.15), we have that

$$\begin{aligned}(g(1+\gamma c^m)i_3,\mathbf{v})&\to(g(1+\gamma c)i_3,\mathbf{v})\,,\\ \theta a(c^m,r)&\to\theta a(c,r)\,,\\ U(c^m,\frac{\partial r}{\partial x_3})&\to U(c,\frac{\partial r}{\partial x_3})\,,\quad as\ m\to\infty\,.\end{aligned} \tag{2.21}$$

As the test functions $\mathbf{v}$, r defined in (2.17) are dense in $\mathbf{V}$ and $\tilde{H}$, conclusions (2.19), (2.20) and (2.21) hold for $\forall\mathbf{v}\in\mathbf{V}$ and $\forall r\in\tilde{H}$. Letting $m\to\infty$ in (2.9) and using the above results we obtain

$$(\nu(c+\alpha)D(\mathbf{u}),D(\mathbf{v}))+B_0(\mathbf{u},\mathbf{u},\mathbf{v})=-((g+\gamma c)i_3,\mathbf{v})+(\mathbf{f},\mathbf{v})\,,\quad\forall\mathbf{v}\in\mathbf{V}_m\,,$$

$$\theta a(c,r)+B(\mathbf{u},c,r)-U(c,\frac{\partial r}{\partial x_3})=U\alpha(\frac{\partial r}{\partial x_3},1)\,,\quad\forall r\in R_m\,.$$

Again since $\mathbf{v}$ and r are dense in $\mathbf{V}$ and $\tilde{H}$, we conclude that $(\mathbf{u},c)$ is a solution of (2.7). □

2.3. Uniqueness. First we notice that the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition (see Remark 2.2). Therefore for each solution $(\mathbf{u},c)\in\mathbf{V}\times\tilde{H}$ of system (2.7), there exists a unique $p\in L_0^2(\Omega)$ satisfying system (2.6) (see [10]). Hence to prove the uniqueness of solution for (2.6), it suffices to prove that system (2.7) has a unique solution.

Following the proof of Lemma 2.4 we can obtain the following estimates for $\mathbf{u}$ and c.

$$\|\mathbf{u}\|_1\le C_3\quad and\quad \|c\|_1\le C_4\,. \tag{2.22}$$

where

$$C_3=\frac{C_\Omega^2}{\nu_*}(\gamma C_4+\|gi_3+\mathbf{f}\|)\,,\quad C_4=\frac{U\alpha}{|\Omega|^{\frac{1}{2}}(\frac{\theta}{C_\Omega^2}-U)}\,.$$

Theorem 2.6. *Assume that*

(H1) *The hypothesis of Theorem 2.5 holds;*
(H2) *The viscosity* $\nu(\cdot)$ *is Lipschitz continuous, i.e., there exists a constant* $\nu_L > 0$ *such that*

$$|\nu(x_1) - \nu(x_2)| \leq \nu_L |x_1 - x_2| \,, \quad \forall x_1, x_2 \in \mathbb{R}\,;$$

(H3) *There exists a constant* C_0 *such that* $\|D(\mathbf{u})\|_\infty \leq C_0$*;*
(H4) *The following inequality*

$$\frac{\nu_*}{C_\Omega^2} - \left(\frac{C_B C_4}{\frac{\theta}{C_\Omega^2} - U} (\nu_L C_0 + g\gamma) + C_{B_0} C_3 \right) > 0$$

holds.

Then the solution $(\mathbf{u}, c)$ *of system* (2.7) *is unique.*

Proof. Let $(\mathbf{u}, c)$ and $(\bar{\mathbf{u}}, \bar{c})$ be two different solutions of (2.7). Substituting both solutions into (2.7) with $\mathbf{v} = \mathbf{u} - \bar{\mathbf{u}}$ and $r = c - \bar{c}$, and substracting the equation for $(\mathbf{u}, c)$ from the equation for $(\bar{\mathbf{u}}, \bar{c})$, we have that

$$\begin{aligned} &(\nu(c+\alpha)D(\mathbf{u}), D(\mathbf{u}-\bar{\mathbf{u}})) - (\nu(\bar{c}+\alpha)D(\bar{\mathbf{u}}), D(\mathbf{u}-\bar{\mathbf{u}}))+ \\ &B_0(\mathbf{u},\mathbf{u},\mathbf{u}-\bar{\mathbf{u}}) - B_0(\bar{\mathbf{u}},\bar{\mathbf{u}},\mathbf{u}-\bar{\mathbf{u}}) = -g\gamma(c-\bar{c},\mathbf{u}-\bar{\mathbf{u}})\,, \end{aligned} \tag{2.23}$$

and

$$\theta a(c-\bar{c}, c-\bar{c}) + B(\mathbf{u}, c, c-\bar{c}) - B(\bar{\mathbf{u}}, \bar{c}, c-\bar{c}) - U(c-\bar{c}, \frac{\partial(c-\bar{c})}{\partial x_3}) = 0\,. \tag{2.24}$$

According to property (2.5), we have the identity

$$\begin{cases} B_0(\mathbf{u},\mathbf{u},\mathbf{u}-\bar{\mathbf{u}}) - B_0(\bar{\mathbf{u}},\bar{\mathbf{u}},\mathbf{u}-\bar{\mathbf{u}}) = B_0(\mathbf{u}-\bar{\mathbf{u}},\mathbf{u},\mathbf{u}-\bar{\mathbf{u}})\,, \\ B(\mathbf{u},c,c-\bar{c}) - B(\bar{\mathbf{u}},\bar{c},c-\bar{c}) = B(\mathbf{u}-\bar{\mathbf{u}},c,c-\bar{c})\,. \end{cases} \tag{2.25}$$

Thus it follows from (2.24), (2.8) and (2.2) that

$$\begin{aligned} \frac{\theta}{C_\Omega^2}\|c-\bar{c}\|_1^2 &\leq |B(\mathbf{u}-\bar{\mathbf{u}}, c, c-\bar{c})| + U(c-\bar{c}, \frac{\partial(c-\bar{c})}{\partial x_3}) \\ &\leq C_B C_4 \|c-\bar{c}\|_1 \|\mathbf{u}-\bar{\mathbf{u}}\|_1 + U\|c-\bar{c}\|_1^2\,. \end{aligned}$$

From (2.10) we obtain

$$\|c-\bar{c}\|_1 \leq \frac{C_B C_4}{\frac{\theta}{C_\Omega^2} - U} \|\mathbf{u}-\bar{\mathbf{u}}\|_1\,. \tag{2.26}$$

Substituting the above estimate into (2.23) and combining (1.6), (2.8), (2.1) and (2.25), we have that

$$\begin{aligned} &\frac{\nu_*}{C_\Omega^2}\|\mathbf{u}-\bar{\mathbf{u}}\|_1^2 \leq (\nu(c+\alpha)D(\mathbf{u}-\bar{\mathbf{u}}), D(\mathbf{u}-\bar{\mathbf{u}})) \\ &\leq |((\nu(c+\alpha)-\nu(\bar{c}+\alpha))D(\bar{\mathbf{u}}), D(\mathbf{u}-\bar{\mathbf{u}}))| + |B_0(\mathbf{u}-\bar{\mathbf{u}},\mathbf{u},\mathbf{u}-\bar{\mathbf{u}})| + g\gamma|(c-\bar{c},\mathbf{u}-\bar{\mathbf{u}})| \\ &\leq \nu_L C_0\|c-\bar{c}\|_1\|\mathbf{u}-\bar{\mathbf{u}}\|_1 + C_{B_0}\|\mathbf{u}-\bar{\mathbf{u}}\|_1^2\|\mathbf{u}\|_1 + g\gamma\|c-\bar{c}\|_1\|\mathbf{u}-\bar{\mathbf{u}}\|_1 \\ &\leq \left(\frac{C_B C_4}{\frac{\theta}{C_\Omega^2} - U}(\nu_L C_0 + g\gamma) + C_{B_0} C_3 \right) \|\mathbf{u}-\bar{\mathbf{u}}\|_1^2. \end{aligned}$$

By assumption (H4) we conclude that

$$\|\mathbf{u}-\bar{\mathbf{u}}\|_1 = \|c-\bar{c}\|_1 = 0\,.$$

□

Remark 2.7. In practice, we need to verify condition (2.10) and (H4). First notice that since the micro-organisms are slightly denser than water, $\gamma = \rho_0/\rho_m - 1$ is small. Therefore to verify (2.10) and (H4), we only need ν_*, θ to be sufficiently large while U, C_Ω are sufficiently small, i.e., a suspension with highly viscous culture fluid, large diffusion rate, and slowly upswimming micro-organisms in a small container.

3. Numerical approximations with the finite element method

In this section, we construct and analyze finite element approximations for the weak solution of (2.6). Throughout this section, we assume that the hypothesis of Theorem 2.6 holds.

Let τ_h be a regular triangulation of Ω ([21]) and $\mathbf{X}_h$, M_h and S_h be finite element subspaces of $\mathbf{H}_0^1(\Omega)$, $L_0^2(\Omega)$ and $\tilde{H}$, respectively. Assume that the following discrete inf-sup condition holds.

$$\sup_{\mathbf{v}\in\mathbf{X}_h} \frac{b(q,\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{X}_h}} \geq \beta\|q\|_{M_h}\,, \quad \forall q \in M_h \tag{3.1}$$

where $\beta > 0$ is a constant. Furthermore we assume that $\mathbf{X}_h$, M_h and S_h satisfy the following approximation properties.

$$\inf_{\mathbf{v_h}\in\mathbf{X}_h} \|\mathbf{v}-\mathbf{v}_h\|_1 \leq Ch^s\|\mathbf{v}\|_{s+1}, \qquad \forall \mathbf{v}\in\mathbf{H}^{s+1}(\Omega),\ 0<s\leq k, \tag{3.2}$$

$$\inf_{q_h\in M^h} \|q-q_h\| \leq Ch^s\|q\|_s, \qquad \forall q\in H^s(\Omega),\ 0<s\leq k, \tag{3.3}$$

$$\inf_{t_h\in S_h} \|t-t_h\|_1 \leq Ch^s\|t\|_{s+1}, \qquad \forall t\in\mathbf{H}^{s+1}(\Omega),\ 0<s\leq k. \tag{3.4}$$

See [11, 10, 20] for constructions of these spaces satisfying (3.1)–(3.4). Next we define the discrete divergence free space

$$\mathbf{V}_h = \{\mathbf{v}\in\mathbf{X}_h, (\ \mathbf{div}\ \mathbf{v}, q_h) = 0\,, \quad \forall q_h \in M_h\}\,.$$

Notice that in general, $\mathbf{V}_h$ is not a subspace of $\mathbf{V}$. Thus in general the identity (2.5) does not hold. To obtain a property similar to (2.5) on $\mathbf{V}_h$, we define auxiliary forms $\hat{B}_0$ and $\hat{B}$ by

$$\hat{B}_0(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}B_0(\mathbf{u},\mathbf{v},\mathbf{w}) - \frac{1}{2}B_0(\mathbf{u},\mathbf{w},\mathbf{v})\,,$$
$$\hat{B}(\mathbf{u},c,r) = \frac{1}{2}B(\mathbf{u},c,r) - \frac{1}{2}B(\mathbf{u},r,c)\,.$$

It is easy to verify that

$$\begin{aligned} \hat{B}_0(\mathbf{u},\mathbf{v},\mathbf{w}) = B_0(\mathbf{u},\mathbf{v},\mathbf{w})\,, \quad \hat{B}(\mathbf{u},c,r) = B(\mathbf{u},c,r)\,, \\ \forall \mathbf{u}\in\mathbf{V}\,, \mathbf{v},\mathbf{w}\in\mathbf{H}_0^1(\Omega)\,, c,r\in\tilde{H}\,. \end{aligned} \tag{3.5}$$

In addition, we have

$$\hat{B}_0(\mathbf{u},\mathbf{v},\mathbf{v}) = 0\,, \quad \hat{B}(\mathbf{u},c,c) = 0\,, \quad \forall \mathbf{u},\mathbf{v}\in\mathbf{H}_0^1(\Omega)\,, c\in\mathbf{H}_0^1(\Omega) \tag{3.6}$$

and

$$\begin{cases} \hat{B}_0(\mathbf{u},\mathbf{v},\mathbf{w}) \leq C_{B_0}\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1\,, \quad \forall \mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{H}_0^1(\Omega)\,, \\ \hat{B}(\mathbf{u},c,r) \leq C_B\|\mathbf{u}\|_1\|c\|_1\|r\|_1\,, \quad \forall \mathbf{u}\in\mathbf{H}_0^1(\Omega)\,, c,r\in\tilde{H}\,, \end{cases} \tag{3.7}$$

where C_B and C_{B_0} are the same as in (2.1) and (2.2).

We define the finite element approximation of (2.7) as follows.

Definition 3.1. The finite element approximation of (2.7) is to find $(\mathbf{u}_h, p_h, c_h) \in \mathbf{X}_h \times M_h \times S_h$, such that

$$(3.8)\quad \begin{cases} (\nu(c_h+\alpha)D(\mathbf{u}_h), D(\mathbf{v})) + \hat{B}_0(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) - (p_h,\ \mathbf{div}\ \mathbf{v}) \\ \qquad = -(g(1+\gamma c_h)i_3, \mathbf{v}) + (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{X}_h\,, \\ (\ \mathbf{div}\ \mathbf{u}_h, q) = 0\,, \quad \forall q \in M_h\,, \\ \theta a(c_h, r) + \hat{B}(\mathbf{u}_h, c_h, r) - U(c_h, \dfrac{\partial r}{\partial x_3}) = U\alpha(\dfrac{\partial r}{\partial x_3}, 1)\,, \quad \forall r \in S_h\,. \end{cases}$$

Analogous to the continuous case, we definite an auxiliary system as follows. Find $(\mathbf{u}_h, c_h) \in V_h \times S_h$ such that

$$(3.9)\quad \begin{cases} (\nu(c_h+\alpha)D(\mathbf{u}_h), D(\mathbf{v})) + \hat{B}_0(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) \\ \qquad = -(g(1+\gamma c_h)i_3, \mathbf{v}) + (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h\,, \\ \theta a(c_h, r) + \hat{B}(\mathbf{u}_h, c_h, r) - U(c_h, \dfrac{\partial r}{\partial x_3}) = U\alpha(\dfrac{\partial r}{\partial x_3}, 1)\,, \quad \forall r \in S_h\,. \end{cases}$$

Because of properties (3.6) and (3.7), we can prove the existence of a weak solution of (3.9) following the same approach as in the continuous case. Then we obtain a solution $(\mathbf{u}_h, p_h, c_h)$ of (3.8) (see [10]) by solving

$$(3.10)\quad \begin{aligned} (p_h,\ \mathbf{div}\ \mathbf{v}) &= (\nu(c_h+\alpha)D(\mathbf{u}_h), D(\mathbf{v})) \\ &+ \hat{B}_0(\mathbf{u}_h, \mathbf{u}_h^n, \mathbf{v}) + ((g+\gamma c_h)i_2, \mathbf{v}) - (\mathbf{f}^n, \mathbf{v})\,, \quad \forall \mathbf{v} \in \mathbf{X}_h\,. \end{aligned}$$

According to (3.1), p_h is uniquely solvable in the quotient space Q_h/N_h where $N_h = \{q_h \in Q_h\,, (q_h,\ \mathbf{div}\ \mathbf{v}) = 0\,, \forall \mathbf{v} \in \mathbf{X}_h\}$.

Following a similar argument as in the continuous case we can show that $\|\mathbf{u}_h\|_1$ and $\|c_h\|_1$ are uniformly bounded, i.e., there exist constants C_3 and C_4 independent of h such that

$$(3.11)\quad \|\mathbf{u}_h\|_1 \le C_3\,, \quad \|c_h\|_1 \le C_4.$$

To carry out the error estimate, we introduce the Ritz Galerkin projections $r_h : \mathbf{H}_0^1(\Omega) \to \mathbf{V}_h$, $s_h : \tilde{H} \to S_h$, and L^2 projection $\pi_h : L_0^2(\Omega) \to M_h$ and split the errors into two parts:

$$(3.12)\quad \begin{cases} \mathbf{u} - \mathbf{u}_h = \mathbf{u} - r_h\mathbf{u} + r_h\mathbf{u} - \mathbf{u}_h := \rho_{\mathbf{u}}^h + \theta_{\mathbf{u}}^h\,, \\ p - p_h = p - \pi_h p + \pi_h p - p_h := \rho_p^h + \theta_p^h\,, \\ c - c_h = c - s_h c + s_h c - c_h := \rho_c^h + \theta_c^h\,. \end{cases}$$

From the approximation property (3.2)– (3.4) we known that (see [21])

$$(3.13)\quad \begin{cases} \|r_h\mathbf{u}\|_1 \le C(\mathbf{u}), & \|\rho_{\mathbf{u}}^h\|_1 \le Ch^s\|\mathbf{v}\|_{s+1}, & \mathbf{u} \in \mathbf{H}^{s+1}(\Omega),\ 0 < s \le k, \\ \|s_h c\|_1 \le C(c), & \|\rho_c^h\|_1 \le Ch^s\|c\|_{s+1}, & c \in \mathbf{H}^{s+1}(\Omega),\ 0 < s \le k, \\ \|\pi_h p\| \le C(p), & \|\rho_p^h\| \le Ch^s\|p\|_s, & p \in \mathbf{H}^s(\Omega),\ 0 < s \le k. \end{cases}$$

Theorem 3.2. *Assume that the hypothesis of Theorem 2.5 and Theorem 2.6 hold. Then for* $\mathbf{u} \in H^{s+1}(\Omega)$, $p \in H^s(\Omega)$ *and* $c \in H^{s+1}(\Omega)$, *there exists a constant* C *independent of* h *such that*

$$(3.14)\quad \|\mathbf{u} - \mathbf{u}_h\|_1 + \|c - c_h\|_1 + \|p - p_h\| \le Ch^s\,, \quad 0 < s \le k\,.$$

Proof. By (3.13), it surffies to estimate $\theta_{\mathbf{u}}^h$, θ_p^h and θ_c^h. Subtracting (3.9) from (2.6) with $\mathbf{v} = \theta_{\mathbf{u}}^h$, $r = \theta_c^h$ and using (3.5) we have that

$$(3.15)\quad \begin{aligned} (\nu(c+\alpha)D(\mathbf{u}), D(\theta_{\mathbf{u}}^h)) - (\nu(c_h+\alpha)D(\mathbf{u}_h), D(\theta_{\mathbf{u}}^h)) + \hat{B}_0(\mathbf{u}, \mathbf{u}, \theta_{\mathbf{u}}^h) \\ - \hat{B}_0(\mathbf{u}_h, \mathbf{u}_h, \theta_{\mathbf{u}}^h) + b(p, \theta_{\mathbf{u}}^h) = -g\gamma((c - c_h))i_3, \theta_{\mathbf{u}}^h) \end{aligned}$$

and

$$\theta a(c-c_h,\theta_c^h)+\hat{B}(\mathbf{u},c,\theta_c^h)-\hat{B}(\mathbf{u}_h,c_h,\theta_c^h)-U(c-c_h,\frac{\partial\theta_c^h}{\partial x_3})=0\,. \tag{3.16}$$

It follows from (1.6), (3.6) and (3.7) that

$$\begin{aligned}\frac{\theta}{C_\Omega^2}\|\theta_c^h\|_1^2\le \theta a(\theta_c^h,\theta_c^h)&=-\theta a(\rho_c^h,\theta_c^h)-\hat{B}(\theta_{\mathbf{u}}^h,c_h,\theta_c^h)-\hat{B}(r_h\mathbf{u},\rho_c^h,\theta_c^h)\\&\quad-\hat{B}(\rho_{\mathbf{u}}^h,c,\theta_c^h)+U(\theta_c^h,\frac{\partial\theta_c^h}{\partial x_3})+U(\rho_c^h,\frac{\partial\theta_c^h}{\partial x_3})\\&\le\theta\|\theta_c^h\|_1\|\rho_c^h\|_1+C_B\|\theta_c^h\|_1(\|\theta_{\mathbf{u}}^h\|_1\|c_h\|_1+\|r_h\mathbf{u}\|_1\|\rho_c^h\|_1\\&\quad+\|\rho_{\mathbf{u}}^h\|_1\|c\|_1)+U\|\theta_c^h\|_1^2+U\|\rho_c^h\|_1\|\theta_c^h\|_1\,.\end{aligned}$$

Moving the term $U\|\theta_c^h\|_1^2$ to the left and dividing by $\|\theta_c^h\|_1$ and using (2.22) and (3.11) we have

$$\|\theta_c^h\|_1\le\frac{1}{\frac{\theta}{C_\Omega^2}-U}\left((\theta+U+C_B\|r_h\mathbf{u}\|_1)\|\rho_c^h\|_1+C_BC_4(\|\theta_{\mathbf{u}}^h\|_1+\|\rho_{\mathbf{u}}^h\|_1)\right)\,. \tag{3.17}$$

Similarly for (3.15), from (1.6), (H2), (H3), (2.22), (3.7) and (3.11), we have that

$$\begin{aligned}\frac{\nu_*}{C_\Omega^2}\|\theta_{\mathbf{u}}^h\|_1^2&\le(\nu(c_h+\frac{\alpha}{|\Omega|})D(\theta_{\mathbf{u}}^h),D(\theta_{\mathbf{u}}^h))\\&=\ -(\nu(c_h+\alpha)D(\rho_{\mathbf{u}}^h),D(\theta_{\mathbf{u}}^h))+((\nu(c_h+\alpha)\\&\quad-\nu(c+\alpha))D(\mathbf{u}),D(\theta_{\mathbf{u}}^h))-\hat{B}_0(\theta_{\mathbf{u}}^h,\mathbf{u}_h,\theta_{\mathbf{u}}^h)-B_0(r_h\mathbf{u},\rho_{\mathbf{u}}^h,\theta_{\mathbf{u}}^h)\\&\quad-\hat{B}_0(\rho_{\mathbf{u}}^h,\mathbf{u},\theta_{\mathbf{u}}^h)-b(\rho_p,\theta_{\mathbf{u}}^h)-g\gamma((c-c_h))i_3,\theta_{\mathbf{u}}^h)\\&\le\nu^*\|\rho_{\mathbf{u}}^h\|_1\|\theta_{\mathbf{u}}^h\|_1+(\nu_LC_0+g\gamma)\|c_h-c\|_1\|\theta_{\mathbf{u}}^h\|_1\\&\quad+C_{B_0}\|\theta_{\mathbf{u}}^h\|_1(C_3\|\theta_{\mathbf{u}}^h\|_1+\|r_h\mathbf{u}\|_1\|\rho_{\mathbf{u}}^h\|_1+C_3\|\rho_{\mathbf{u}}^h\|_1)+\|\rho_p\|\|\theta_{\mathbf{u}}^h\|_1\,.\end{aligned}$$

Notice that by (3.17)

$$\begin{aligned}\|c_h-c\|_1&\le\|\rho_c^h\|_1+\|\theta_c^h\|_1\\&\le\|\rho_c^h\|_1+\frac{1}{\frac{\theta}{C_\Omega^2}-U}\left((\theta+U+C_B\|r_h\mathbf{u}\|_1)\|\rho_c^h\|_1+C_BC_4(\|\theta_{\mathbf{u}}^h\|_1+\|\rho_{\mathbf{u}}^h\|_1)\right)\,.\end{aligned}$$

This implies that

$$\begin{aligned}&(\frac{\nu_*}{C_\Omega^2}-C_{B_0}C_3-(\nu_LC_0+g\gamma)\frac{C_BC_4}{\frac{\theta}{C_\Omega^2}-U})\|\theta_{\mathbf{u}}^h\|_1\le(\nu^*+C_{B_0}\|r_h\mathbf{u}\|_1+C_{B_0}C_3\\&\quad+\frac{C_BC_4(\nu_LC_0+g\gamma)}{\frac{\theta}{C_\Omega^2}-U})\|\rho_{\mathbf{u}}^h\|_1+(\nu_LC_0+g\gamma)\left(1+\frac{\theta+U+C_B\|r_h\mathbf{u}\|_1}{\frac{\theta}{C_\Omega^2}-U}\right)\|\rho_c^h\|_1+\|\rho_p\|\,.\end{aligned}$$

By assumption (H4) and (3.13) we obtain

$$\|\theta_{\mathbf{u}}^h\|_1\le C(\|\rho_{\mathbf{u}}^h\|_1+\|\rho_c^h\|_1+\|\rho_p\|)\,, \tag{3.18}$$

and by (3.17) and (3.13) we have

$$\|\theta_c^h\|_1\le C(\|\rho_{\mathbf{u}}^h\|_1+\|\rho_c^h\|_1+\|\rho_p\|)\,. \tag{3.19}$$

From (3.12), (3.18) and (3.19) we obtain

$$\|\mathbf{u}-\mathbf{u}_h\|_1+\|c-c_h\|_1\le C\left(\|\rho_{\mathbf{u}}^h\|_1+\|\rho_c^h\|_1+\|\rho_p^h\|\right)\,. \tag{3.20}$$

It remains to estimate $\|p - p_h\|$. Subtracting (3.8) from (2.6) gives

$$-b(\mathbf{v}, \theta_p^h) = (\nu(c+\alpha)D(\mathbf{u}), D(\mathbf{v})) - (\nu(c_h+\alpha)D(\mathbf{u}_h), D(\mathbf{v}))$$
$$+\hat{B}_0(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \hat{B}_0(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) + b(\rho_p^h, \mathbf{v}) + g\gamma((c-c_h))i_3, \mathbf{v})\,.$$

By (3.1), (3.7), (2.22) and (3.11) we have that

$$\begin{aligned}\|\theta_p^h\| \le & \frac{1}{\beta} \sup_{\mathbf{v}\in\mathbf{X}_h} \frac{1}{\|\mathbf{v}\|_1}(-(\nu(c_h+\alpha)D(\mathbf{u}-\mathbf{u}_h), D(\mathbf{v})) \\ & - ((\nu(c+\alpha) - \nu(c_h+\alpha))D(\mathbf{u}), D(\mathbf{v})) - \hat{B}_0(\mathbf{u}-\mathbf{u}_h, \mathbf{u}, \mathbf{v}) \\ & + \hat{B}_0(\mathbf{u}_h, \mathbf{u}-\mathbf{u}_h, \mathbf{v}) - b(\rho_p^h, \mathbf{v}) - g\gamma((c-c_h)i_3, \mathbf{v})) \\ \le & \frac{1}{\beta} \sup_{\mathbf{v}\in\mathbf{X}_h} \frac{1}{\|\mathbf{v}\|_1}((\nu^* + 2C_{B_0}C_3)\|\mathbf{u}-\mathbf{u}_h\|_1\|\mathbf{v}\|_1 \\ & + (\nu_L C_0 + g\gamma)\|c-c_h\|_1\|\mathbf{v}\|_1 + \|\rho_p^h\|\|\mathbf{v}\|_1) \\ \le & C(\|\mathbf{u}-\mathbf{u}_h\|_1 + \|c-c_h\|_1 + \|\rho_p^h\|)\,.\end{aligned}$$

Combing the above estimate with (3.18) and (3.19), we obtain

$$\|p - p_h\|_1 \le C(\|\rho_{\mathbf{u}}^h\|_1 + \|\rho_c^h\|_1 + \|\rho_p^h\|)\,. \tag{3.21}$$

The result of the theorem then follows from (3.13). □

4. Numerical experiments

In this section we shall conduct two numerical experiments. The first one uses artificial data to verify the error estimates while the second one uses data obtained from lab experiments. We shall use Taylor-Hood finite element spaces ([19]) for $\mathbf{V}_h$ and Q_h and continuous piecewise quadratic function spaces for S_h.

Example 1 In this example we choose the domain $\Omega = [-1,1]\times[-1,1]$, and γ, U, θ and ν as

$$\gamma = 0.1,\, U = 0.1,\, \theta = 1\,,$$

and

$$\nu(x) = \sin^2 x + 1, \quad x \in \Omega\,.$$

The forcing terms are chosen so that the exact solution is given by

$$\begin{aligned}\mathbf{u} &= (\sin\pi x \sin\pi y, \sin\pi x \sin\pi y)^T\,, \\ p &= \sin\pi x \sin\pi y\,, \\ c &= \sin\pi x \sin\pi y.\end{aligned}$$

The numerical errors for different mesh sizes are listed in Table 1. The convergence rates listed in the table are consistent with our theoretical result.

Example 2 In this example we consider a 10 cm × 10 cm container filled with micro-organisms suspensions under zero external force, i.e., $\mathbf{f} \equiv 0$. For computation simplicity, we study the domain on x-z plane. The parameters of the model, obtained from lab experiments (see [12]), are given in Table 2.

As a volume concentration, c is given by

$$c = nv_0\,,$$

Table 1. Convergence rate

h	$\|p-p_h\|$	$\|u-u_h\|$	$\|c-c_h\|$	$\|p-p_h\|_1$	$\|u-u_h\|_1$	$\|c-c_h\|_1$
1/2	0.2520	0.0078	0.0049	0.9846	0.0854	0.0460
1/4	0.0323	0.0010	6.8E-04	0.3847	0.0207	0.0118
1/8	0.0055	1.31E-04	8.88E-05	0.1786	0.0050	0.030
1/16	0.0011	1.65E-05	1.13E-05	0.0877	0.0012	7.45E-04
1/32	2.28E-04	2.07E-06	1.42E-06	0.0436	3.09E-04	1.86E-04
1/64	4.90E-05	2.60E-07	1.80E-07	0.0216	7.68E-05	4.7E-05
conv. rate	2.22	2.99	2.97	1.02	2.01	2.00

Table 2. Parameter values

ν_0 cm^2/sec	g m/sec^2	γ	θ cm^2/sec	U cm/sec
0.01	9.81	0.1	0.0025	0.01

where n is the number of organisms per unit volume and v_0 is the volume of an individual organism. Define

$$\nu(c) = \begin{cases} \nu_0\,, & c < 0\,, \\ \nu_0(1 + 2.5\ c + 5.3\ c^2)\,, & 0 < c < 10\%\,, \\ \nu_0 \exp(\dfrac{2.5\ c}{1 - 1.4\ c})\,, & 10\% < c < 60\%\,, \\ \nu_0 \exp(9.375)\,, & c > 60\%\,, \end{cases} \tag{4.1}$$

where ν_0 is the viscosity of the culture fluid. (4.1) combines the work of Batchelor's [2] for low concentration and Mooney's [17] for high concentration. Note that $\exp(\frac{2.5\ c}{1-1.4\ c})$ has a low limit $\nu_* = \nu_0$ but tends to infinity when the maximum concentration $\varphi_m = \frac{1}{1.4}$ is reached since the suspension is acting like a solid, where no movement of neighboring particles are allowed. Therefore we set the upper bound $\nu^* = \nu_0 \exp(9.375)$ such that the viscosity defined in (4.1) satisfies property (1.6). In what follows, we consider four different cases with various values for α.

Case 1: $\alpha = 1\%$. The velocity and concentration are given in Figure 1. We can see that a bioconvection pattern can not be formed and the concentration has a homogeneous horizontal distribution. This is because the right hand side of the first equation in (1.1) almost equal to $-g$. As a result, $\mathbf{u} \approx 0$ while p is almost linear with $\nabla p \approx -g$ and $\frac{\partial c}{\partial x} \approx 0$ because of zero velocity $\mathbf{u}$. The micro-organisms do not move and the concentration stays linear in the vertical direction with zero horizontal gradient. From observed experiments, for a shallow container with low concentration of micro-organisms, the micro-organisms will stay at the surface of the suspension due to the upswimming since the effect of gravity can be neglected. In fact, bioconvection only occurs for sufficiently deep container. The higher the concentration, the shallower the container will be. In this case, 1% concentration is not large enough to form a bioconvection pattern in a 10 cm deep container.

Case 2: $\alpha = 20\%$. Figure 2 shows the distribution of concentration and the velocity filed with streamlines. Here the color denotes the magnitude of the velocity. The figure shows that a bioconvection pattern can be formed for sufficiently large concentration. Our simulation result is consistent with the results obtained in [12]. From the figure we also observe that two convections, separated from the center, flow steadily in opposite directions. The highest velocity happens in the

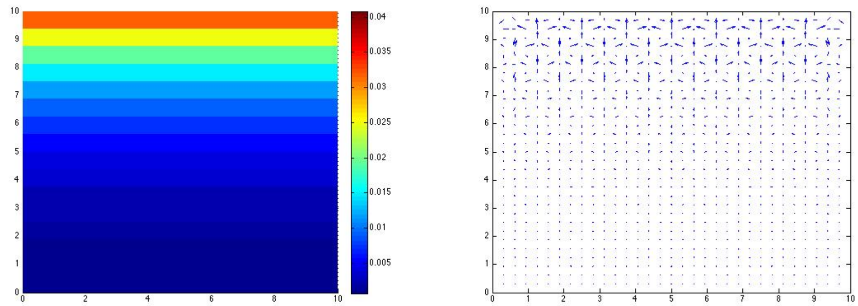

FIGURE 1. Concentration and velocity field for $\alpha = 1\%$

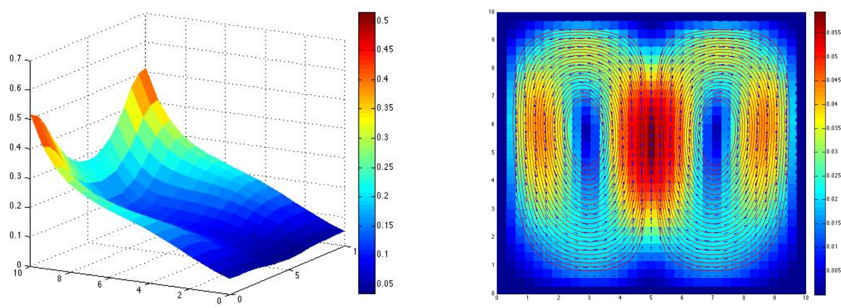

FIGURE 2. Concentration and velocity field for $\alpha = 20\%$

middle, where the concentration is low, due to the upswimming under small effect of the gravity. Another high speed motion is observed on the left and right side of the container, which is caused mostly by gravity due to high concentration in the left and right up corners. In this way, randomly upswimming micro-organisms form steady convections because of drag force generated by the motion. Only a few micro-organisms remain at the bottom while most of the micro-organisms stay close to the surface.

Case 3: $\alpha = 20\%$, however the constant viscosity $\nu(c) \equiv 0.01$ is used in this case to compare with Case 2 where concentration dependent viscosity (4.1) is used. The result is shown in Figure 3. From the graph, we can see that both models capture the motion of the bioconvection but the concentration distribution and velocity field are slightly different. The velocity of the nonlinear case are slower and smoother due to a relatively higher viscosity, which involves the concentration. The difference is more notable where the concentration is high. The nonlinear viscosity case reflects higher concentrated micro-organisms at the top corners since more micro-organisms are washed up by the drag force and stays there due to a high viscosity. One can see that the introduction of nonhomogeneous viscosity captures the feature of the bioconvection better in the simulation.

Case 4: $\alpha = 30\%$. The velocity field and concentration distribution are given in Figure 4. From the figure we observe that as concentration increases, the effect of the gravity become more significant, which leads to a faster convection. However, once the pattern is formed, the distribution of the contraction stays the same.

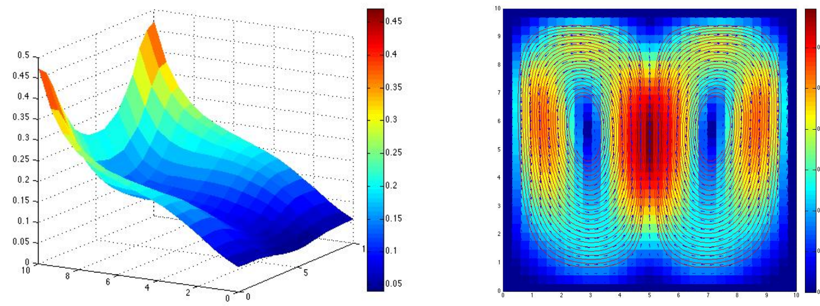

FIGURE 3. Concentration and velocity field for $\alpha = 20\%$ with constant viscosity

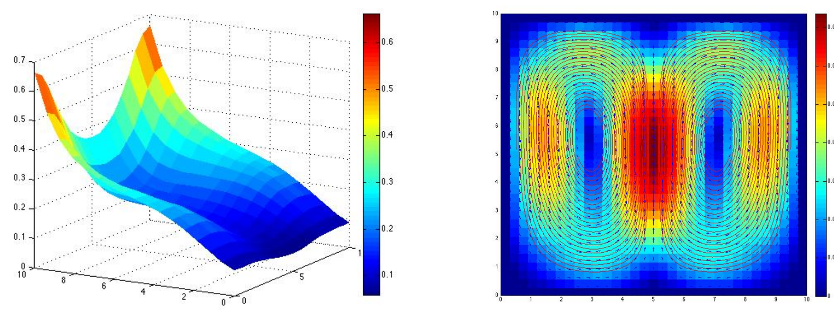

FIGURE 4. Concentration and velocity field for $\alpha = 30\%$

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