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ON THE TIME APPROXIMATION OF THE STOKES EQUATIONS WITH NONLINEAR SLIP BOUNDARY CONDITIONS

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Abstract. This work is concerned with the numerical approximation of the unsteady Stokes flow of a viscous incompressible fluid driven by a threshold slip boundary condition of friction type. The continuous problem is formulated as variational inequality, which is next discretize in time based on backward Euler's scheme. We prove existence and uniqueness of the solution of the time discrete problem by means of a regularization approach. Finally, we derive error estimates that justify the convergence property of the discretization proposed.

Key words. eywords: Stokes equations, slip boundary condition, variational inequality, regularization, convexity, monotone

1. Introduction

We consider unsteady flows of incompressible viscous fluids modeled by the Stokes system

$$\boldsymbol{u}_t - 2\nu \operatorname{div} \varepsilon(\boldsymbol{u}) + \nabla p = \boldsymbol{f} \quad \text{in } Q = \Omega \times (0,T), \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } Q, \tag{1.2}$$

where Ω is the flow region, a bounded domain in $\mathbb{R}^2$, while $\varepsilon(\boldsymbol{u}) = \frac{1}{2}[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T]$. The motion of the incompressible fluid is described by the velocity $\boldsymbol{u}(\boldsymbol{x},t)$ and pressure $p(\boldsymbol{x},t)$. In (1.1) $\boldsymbol{f}(\boldsymbol{x},t)$ is the external body force per unit volume, while ν is the kinematic viscosity. Equations (1.1) and (1.2) are supplemented by boundary and initial conditions. We first assume that

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0 \quad \text{on } \overline{\Omega}, \tag{1.3}$$

where $\boldsymbol{u}_0$ is a given function, for which precise assumptions will be introduced below, and $\overline{\Omega}$ is the closure of Ω. Next in order to describe the motion of the fluid at the boundary, we assume that the boundary of Ω, say, $\partial\Omega$ is made of two components S (say the outer wall) and Γ (the inner wall), and we require that $\overline{\partial\Omega} = \overline{S \cup \Gamma}$, with $S \cap \Gamma = \emptyset$. We assume the homogeneous Dirichlet condition on Γ, that is

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma \times (0,T). \tag{1.4}$$

We have chosen to work with homogeneous condition on the velocity in order to avoid the technical arguments linked to the Hopf lemma (see [1], Chapter 4, Lemma 2.3). On S, we first assume the impermeability condition

$$u_N = \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } S \times (0,T), \tag{1.5}$$

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where $\boldsymbol{n}$ is the outward unit normal on the boundary $\partial\Omega$, and u_N is the normal component of the velocity,while $\boldsymbol{u_\tau} = \boldsymbol{u} - u_N\boldsymbol{n}$ is its tangential component. In addition to (1.5) we also impose on S, a threshold slip condition [2, 3, 4], which is the main ingredient of this work. The threshold slip condition can be formulated with the knowledge of a positive function $g : S \longrightarrow (0, \infty)$ which is called the barrier of threshold function and the tangential part of $\boldsymbol{Tn}$ as follows:

(1.6)
$$\left.\begin{array}{ll} \text{if} \quad |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| < g \quad \text{then} \quad \boldsymbol{u_\tau} = \boldsymbol{0}, \\ \text{if} \quad |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| = g \quad \text{then} \quad \boldsymbol{u_\tau} \neq \boldsymbol{0}\ ,\ \textit{and}\ -(\boldsymbol{Tn})_{\boldsymbol{\tau}} = g\dfrac{\boldsymbol{u_\tau}}{|\boldsymbol{u_\tau}|} \end{array}\right\} \text{ on } \quad S \times (0, T).$$

Of course in (1.6), $\boldsymbol{T} = 2\nu\varepsilon(\boldsymbol{u}) - p\boldsymbol{I}$ is the Cauchy stress tensor with $\boldsymbol{I}$ being the identity tensor. It should quickly be mentioned that (1.6) is equivalent to [5]

$$(\boldsymbol{Tn})_{\boldsymbol{\tau}} \cdot \boldsymbol{u_\tau} + g|\boldsymbol{u_\tau}| = 0 \text{ on } \quad S \times (0, T), \tag{1.7}$$

which is re-written with the use of sub-differential as

$$-(\boldsymbol{Tn})_{\boldsymbol{\tau}} \in \ g\partial|\boldsymbol{u_\tau}| \quad \text{on } S \times (0, T), \tag{1.8}$$

where $\partial|\cdot|$ is the sub-differential of the real valued function $|\cdot|$, with $|\boldsymbol{w}|^2 = \boldsymbol{w} \cdot \boldsymbol{w}$. We recall that if $\boldsymbol{X}$ is a Hilbert space equipped with the inner product denoted as $\cdot$, and $x_0 \in \boldsymbol{X}$, then

$$y \in \partial\Psi(x_0) \text{ if and only if } \Psi(x) - \Psi(x_0) \geq y \cdot (x - x_0) \quad \forall x \in \boldsymbol{X}. \tag{1.9}$$

The slip boundary conditions of friction type (1.6) can be justified by the fact that frictional effects of the fluid at the pores of the solid can be very important, and this can be seen in fiber spinning. Hence one observes that different boundary conditions describe different physical phenomena. The boundary condition (1.6) has also been applied successfully to some flow phenomena in concrete situations such as oil flow over or beneath sand layers [6, 7]. In [8], a generalization of the boundary condition (1.6) is formulated and analyzed for the steady Stokes flow, while the case of Navier-Stokes equations has been examined in [9]. We should re-iterate that for fluids with moderate velocities and stresses, the no-slip condition is well suited and describe the fact that the fluid adheres to the boundary of the flow domain.

The subject of the present work is to approximate the two-dimensional problem (1.1)$\cdots$(1.6) in time, using the implicit Euler scheme, and establish its well-posedness, stability and measure the difference between the exact and discrete solution. We also want our scheme to have some properties observed at the continuous level. The existence theory of (1.1)$\cdots$(1.6) provided in [2, 3, 4] used semi-group approach, so no estimates of the solution are available in that research. Hence for completeness, we revisit the existence and uniqueness question by adopting the Galerkin's approach together with the energy method. By doing so, we have some a priori estimates that we would like our numerical scheme to have. Similar studies have been presented for the case of Navier-Stokes equations with Dirichlet or periodic boundary conditions in [10, 11, 12, 13, 14], reaction diffusion equations and parabolic p-Laplacian in [15, 16]. To better understand the analysis of the time discrete problem, it is important to present the main steps of the existence result of (1.1)$\cdots$ (1.6) which is done by regularization approach, Faedo-Galerkin approximation and using some compactness arguments [5]. The regularization is important because we have a non differentiable term which brings the inequality into the system. But also from the numerical analysis viewpoint, the regularization itself is worth considering (as we are going to see, the solution of the regularized and

non regularized problems are very close). The regularization procedure consists of replacing the original problem by a sequence of "better behaved approximate problems" indexed by a small positive parameter ϵ. This first phase also transforms the problem of variational inequality into a variational equation. We then solve the regularized problem by the Faedo-Galerkin method. Next, some a priori estimates of the regularized solution are obtained; and finally, using some compactness properties, the solution of the original problem is recovered when ϵ goes to zero. One of the difficulties in the algorithm just described is to obtain the pressure despite the fact that one has a nonlinear problem to solve. Indeed, in order to take advantage of the incompressibility condition, the problem in its weak form is written as a variational inequality with only one unknown (the velocity). But because of the inequality sign in our formulation, the pressure will not be obtained in the usual way (see for example the Navier-Stokes equations with Dirichlet boundary condition in [17]). Instead one constructs a regularized pressure by using the classical approach and then by compactness properties we recovered the pressure when ϵ goes to zero. Our focus in the analysis of the continuous problem is rather to derive a priori estimates that will be discussed later at the discrete level. We propose a time semi-discrete problem that relies on Backward Euler's scheme. The use of an implicit scheme seems necessary in order to avoid Courant-Friedrichs-Levy conditions. We prove that the discrete problem has a unique solution, mimic the a priori estimates of the continuous problem, and finally derive the error estimate. The method of proof follows our presentation for the continuous problem. It should be mentioned that the analysis of the time discrete problem associated with $(1.1)\cdots(1.6)$ is nontrivial because of the presence of; the non-differentiable and nonlinear term $|\boldsymbol{u_\tau}|$, and the inequality structure of the resulting variational formulation due to the boundary condition (1.7). The numerical analysis (error estimates, convergence) of flow phenomena driven by boundary conditions of friction type are from the author's knowledge infrequent in the literature, except the works of [18, 19, 20], but the literature on numerical analysis of variational inequalities is well documented [21, 22, 23, 24] just to mentioned a few. This work proposes some insights in this direction by considering only the discretization in time, and analyzing a discrete scheme capable of replicating some essential features of the continuous model. The analysis we propose here takes its roots in [10, 12, 13, 16] in the sense that those researchers answered similar questions for Navier-Stokes equations with Dirichlet or periodic boundary conditions. But in the context of variational inequalities, such questions have not yet been considered in the literature to the best of our knowledge. We are therefore interested in a priori error analysis for the backward Euler method with fixed time step assuming smoothness of the weak solution. R. Nochetto, G. Savare and C. Verdi in [25] have developed a posteriori error estimates for a class of nonlinear evolution equations of parabolic type with nonclassical boundary conditions taking into account the exact regularity of the weak solution. In [18, 19, 20], finite element method are considered and convergence analysis is obtained. But it is important to observe that the regularity assumption considered in those works has not yet been proved in the literature. In [26] we studied the convergence of finite element/discontinuous Galerkin approximation of $(1.1)\cdots(1.6)$ with low regularity. The rest of the paper is organized as follows. First we formulate the problem $(1.1)\cdots(1.6)$ in terms of variational inequalities and solve it via a regularization and compactness method (Section 2). The time discrete problem is formulated in Section 3 using the implicit Euler scheme. Then, existence and uniqueness result

are obtained by combining regularization and monotonicity approaches. Some additional a priori estimates of the discrete solution are obtained. Error estimates are discussed in Section 4.

2. Slip boundary condition/ variational formulations

2.1. Preliminaries. We denote by $H^k(\Omega)$ and $\|\cdot\|_k$ the standard Sobolev space and its norm, while $H^{-k}(\Omega)$ stands for the dual space of $H_0^k(\Omega)$. We denote by (a,b) the L^2 inner product of a and b over Ω. For any separable Banach space E equipped with the norm $\|\cdot\|_E$, we denote by $\mathcal{C}(0,T;E)$ the space of continuous functions on $[0,T]$ with values in E and by $\mathcal{D}'(0,T;E)$ the space of distributions with values in E. For each real number s with $s \geq 0$, we define the space $H^s(0,T;E)$ in the following way: when $s \in \mathbb{N}$, it is the space of functions on $(0,T)\times E$ such that the mappings $\phi \longmapsto \|\partial_t^m \phi\|_E$, $0 \leq m \leq s$, are square-integrable on $(0,T)$; otherwise, it is defined by interpolation between $H^{[s]+1}(0,T;E)$ and $H^{[s]}(\Omega,T;E)$, where $[s]$ stands for the integer part of s.
We introduce the following functions spaces

$$\boldsymbol{V} := \left\{ \boldsymbol{v} \in H^1(\Omega)^2,\ \boldsymbol{v}|_\Gamma = 0\ ,\ \boldsymbol{v}\cdot\boldsymbol{n}|_S = 0\ \right\},$$
$$M := L_0^2(\Omega) = \left\{ q \in L^2(\Omega)\ ,\ (q,1) = 0\ \right\}.$$

Throughout the paper we assume that Ω is a bounded, convex planar domain with polygonal boundary $\partial\Omega$, and C denotes a generic positive constants which may take different values even in the same calculation. The entities on which may depend, are given in brackets, e.g. $C(\Omega)$ denotes a constant which depends at most on Ω. As usual, $\phi(t)$ stands for the function $\boldsymbol{x} \in \Omega \mapsto \phi(\boldsymbol{x},t)$.

2.2. Variational inequality and Solvability. Let us first explain how the mixed formulation as well as the existence and uniqueness of the weak solution of (1.1)... (1.6) are obtained.
We assume that $\boldsymbol{f}$ is an element of $L^\infty(0,T;L^2(\Omega)^2)$ for any $T \in \mathbb{R}_+$. We also assume that the datum $\boldsymbol{u}_0 \in L^2(\Omega)^2$ and satisfies the following compatibility condition

$$\operatorname{div} \boldsymbol{u}_0 = 0 \text{ in } \Omega. \tag{2.1}$$

This condition is not necessary for all the results that will follow,but since it is not restrictive,we shall assume it from now on. We then introduce the following variational formulation for (1.1)... (1.6): Find $(\boldsymbol{u}(t),p(t)) \in \boldsymbol{V}\times M$ such that

$$\boldsymbol{u}(0) = \boldsymbol{u}_0\ ,\ \text{in } \Omega, \tag{2.2}$$

and, for a.e. t, with $0 \leq t \leq T$,

$$\begin{cases} \text{for all } (\boldsymbol{v},q) \in \boldsymbol{V}\times M, \\ \langle \boldsymbol{u}'(t), \boldsymbol{v}-\boldsymbol{u}(t)\rangle + a(\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t)) - b(p(t),\boldsymbol{v}-\boldsymbol{u}(t)) \\ +J(\boldsymbol{v}) - J(\boldsymbol{u}(t)) \geq \ell(\boldsymbol{v}-\boldsymbol{u}(t)), \\ b(q,\boldsymbol{u}(t)) = 0, \end{cases} \tag{2.3}$$

where;

$$a(\boldsymbol{u},\boldsymbol{v}) = \nu(\varepsilon(\boldsymbol{u}),\varepsilon(\boldsymbol{v}))\ ,\ b(q,\boldsymbol{v}) = (q,\operatorname{div}\boldsymbol{v}),\ \ell(\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v})\ ,\ J(\boldsymbol{v}) = (g,|\boldsymbol{v}_{\boldsymbol{\tau}}|)_S.$$

It is immediate that for (2.2) to make sense, we should require at least continuity in time for $\boldsymbol{u}(\boldsymbol{x},t)$. One can readily verify (see [5] where many similar examples are treated) that any solution of (2.2) and (2.3) is a solution of (1.1) ... (1.6) in the sense of distributions, and vise versa provided that additional regularity

assumptions on the solutions hold true. The principal result about the problem (2.2) and (2.3) can be stated as follows

Theorem 2.1. *If Assume* $(\boldsymbol{f}, g)$ *belong to* $L^\infty\left(0,T;L^2(\Omega)^2\right) \times L^\infty(S)$*, with* $\boldsymbol{u}_0 \in H^1(\Omega)^2$*, such that (2.1) hold true. Then, the problem (2.2), and (2.3) admits only one solution* $(\boldsymbol{u}, p)$*, with*

$$\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{V}) \ , \ \boldsymbol{u}' \in L^2(0,T;\boldsymbol{V}') \text{ and } p \in L^2(0,T;L^2_0(\Omega)).$$

Moreover, this solution satisfy the a priori estimates

$$\begin{aligned}
\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega)^2)} &\leq \|\boldsymbol{u}_0\| \exp(-Ct) + C\|\boldsymbol{f}\|_{L^\infty(0,T;L^2(\Omega)^2)}, \\
\|\boldsymbol{u}'\|_{L^2(0,T;L^2(\Omega)^2)} &\leq \|\boldsymbol{f}\|_{L^2(0,T;L^2(\Omega)^2)} + C(\Omega,\nu)\|\boldsymbol{u}_0\|_1 + C\|g\|_{L^\infty(S)} \\
&+ C\|g\|^{1/2}_{L^\infty(S)} \\
\nu\|\nabla \boldsymbol{u}\|^2_{L^\infty(0,T;L^2(\Omega)^2)} &\leq \|\boldsymbol{f}\|^2_{L^2(0,T;L^2(\Omega)^2)} + C(\Omega,\nu)\|\boldsymbol{u}_0\|^2_1 + C\|g\|^2_{L^\infty(S)} \\
&+ C\|g\|_{L^\infty(S)}.
\end{aligned} \tag{2.4}$$

Proof. The proof is done in many steps.
step 1:regularization. We first introduce some preliminaries tools for the analysis of the variational problem (2.2) and (2.3). We recall that the kernel of the continuous bilinear form $b(\cdot,\cdot)$ on $L^2(\Omega) \times V$ defined by

$$Z(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{V} \ , \ b(q,\boldsymbol{v}) = 0 \ , \ q \in L^2(\Omega) \ \right\},$$

is characterized by

$$Z(\Omega) = \{ \boldsymbol{v} \in \boldsymbol{V} \ , \ \operatorname{div} \boldsymbol{v} = 0 \ \}.$$

Moreover, $b(\cdot,\cdot)$ satisfies the following inf-sup condition: there exists a constant $C(\Omega)$ such that

$$\text{for all } q \in M \ , \ \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{b(q,\boldsymbol{v})}{\|\boldsymbol{v}\|_1} \geq C(\Omega)\|q\| \ . \tag{2.5}$$

Next, arguing as in [8] (see theorem 3-1), the variational problem (2.2) and (2.3) is equivalent to: Find $\boldsymbol{u}(t))$ satisfying (2.2), (2.1) such that

$$\begin{cases}
\boldsymbol{u} \in L^\infty(0,T;Z(\Omega)) \ , \ \boldsymbol{u}' \in L^2(0,T;Z(\Omega)') \text{ for almost all } t \\
\text{ and for all } \boldsymbol{v} \in Z(\Omega), \\
\langle \boldsymbol{u}'(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle + a(\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) + J(\boldsymbol{v}) - J(\boldsymbol{u}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)).
\end{cases} \tag{2.6}$$

For the well posedness of (2.2) and (2.6), we regularize [5] the variational problem (2.6) by introducing the functional

$$\boldsymbol{v} \in \boldsymbol{V} \longmapsto J_\epsilon(\boldsymbol{v}) = \int_S g\sqrt{|\boldsymbol{v}_{\boldsymbol{\tau}}|^2 + \epsilon^2}\ ds, \ 0 < \epsilon << 1.$$

We observe that

(a) J_ϵ is convex and differentiable, with Gateaux-derivative $K_\epsilon : \boldsymbol{V} \mapsto \boldsymbol{V}'$ given by

$$\langle K_\epsilon(\boldsymbol{u}), \boldsymbol{v}\rangle = \int_S g \frac{\boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{v}_{\boldsymbol{\tau}}}{\sqrt{|\boldsymbol{u}_{\boldsymbol{\tau}}|^2 + \epsilon^2}} ds$$

(b) K_ϵ is monotone, that is

$$\text{for all } \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}\ ,\ \langle K_\epsilon(\boldsymbol{u}) - K_\epsilon(\boldsymbol{v}), \boldsymbol{u} - \boldsymbol{v}\rangle \geq 0. \tag{2.7}$$

Indeed since J_ϵ is convex,

$$\text{for } \boldsymbol{u}, \boldsymbol{v} \text{ elements of } \boldsymbol{V} \text{ and } 0 < t < 1,\ J_\epsilon(t\boldsymbol{u} + (1-t)\boldsymbol{v}) \leq tJ_\epsilon(\boldsymbol{u}) + (1-t)J_\epsilon(\boldsymbol{v})\ ,$$

which can be re-written as

$$\frac{J_\epsilon(\boldsymbol{v} + t(\boldsymbol{u} - \boldsymbol{v}))}{t} \leq J_\epsilon(\boldsymbol{u}) - J_\epsilon(\boldsymbol{v}).$$

Then taking the limit $t \to 0$ on both sides yields

$$\langle K_\epsilon(\boldsymbol{v}), \boldsymbol{u} - \boldsymbol{v}\rangle \leq J_\epsilon(\boldsymbol{u}) - J_\epsilon(\boldsymbol{v}).$$

Interchanging the role of $\boldsymbol{u}$ and $\boldsymbol{v}$, one gets instead

$$\langle K_\epsilon(\boldsymbol{u}), \boldsymbol{v} - \boldsymbol{u}\rangle \leq J_\epsilon(\boldsymbol{v}) - J_\epsilon(\boldsymbol{u}).$$

Finally, putting together the latter and former inequalities,one obtains the monotonicity property (2.7).

(c) $J_\epsilon(\boldsymbol{v}) \to J(\boldsymbol{v})$ as $\epsilon \to 0$, uniformly with respect to $\boldsymbol{v} \in \boldsymbol{V}$. Indeed

$$J_\epsilon(\boldsymbol{v}) - J_\epsilon(\boldsymbol{u}) = \int_S g \frac{\epsilon^2}{\sqrt{|\boldsymbol{v}_{\boldsymbol{\tau}}|^2 + \epsilon^2} + |\boldsymbol{v}_{\boldsymbol{\tau}}|}\ ds.$$

The regularized problem then takes the following form. Find $\boldsymbol{u}_\epsilon$ satisfying

$$\boldsymbol{u}_\epsilon(0) = \boldsymbol{u}_0, \tag{2.8}$$

such that

(2.9)

$$\begin{cases} \boldsymbol{u}_\epsilon \in L^\infty(0,T; Z(\Omega))\ ,\ \boldsymbol{u}'_\epsilon \in L^2(0,T; Z(\Omega)'), \\ \text{for almost all } t, \text{ and for all } \boldsymbol{v} \in Z(\Omega), \\ \langle \boldsymbol{u}'_\epsilon(t), \boldsymbol{v} - \boldsymbol{u}_\epsilon(t)\rangle + a(\boldsymbol{u}_\epsilon(t), \boldsymbol{v} - \boldsymbol{u}_\epsilon(t)) + J_\epsilon(\boldsymbol{v}) - J_\epsilon(\boldsymbol{u}_\epsilon(t)) \geq \ell(\boldsymbol{v} - \boldsymbol{u}_\epsilon(t)). \end{cases}$$

Since J_ϵ is differentiable, the variational inequality problem (2.9) reduces to the variational equation: Find $\boldsymbol{u}_\epsilon$ satisfying (2.8) such that

$$\begin{cases} \boldsymbol{u}_\epsilon \in L^\infty(0,T; Z(\Omega))\ ,\ \boldsymbol{u}'_\epsilon \in L^2(0,T; Z(\Omega)'), \\ \text{for almost all } t, \text{ and for all } \boldsymbol{v} \in Z(\Omega), \\ \langle \boldsymbol{u}'_\epsilon(t), \boldsymbol{v}\rangle + a(\boldsymbol{u}_\epsilon(t), \boldsymbol{v}) + \langle K_\epsilon(\boldsymbol{u}_\epsilon(t)), \boldsymbol{v}\rangle = \ell(\boldsymbol{v}). \end{cases} \tag{2.10}$$

We now show how the pressure is constructed from the velocity, solution of the regularized problems (2.8), and (2.10). To that end we let

$$G_t(\boldsymbol{v}) = \int_0^t [\ell(\boldsymbol{v}) - a(\boldsymbol{u}_\epsilon(s), \boldsymbol{v}) - \langle K_\epsilon(\boldsymbol{u}_\epsilon(s)), \boldsymbol{v}\rangle]\ ds - (\boldsymbol{u}_\epsilon(t), \boldsymbol{v}) + (\boldsymbol{u}_0, \boldsymbol{v})$$

For $t \in [0,T]$, G_t is linear and continuous on V, and from (2.8), and (2.10), it vanishes on $Z(\Omega)$. Hence from [1] (see Chap 1, theorem 2.3), for each $t \in [0,T]$, there exists a function $P_\epsilon(t)$ in $L^2(\Omega)$, and a positive constant C depending only on Ω such that:

$$\begin{cases} G_t(\boldsymbol{v}) = b(P_\epsilon(t), \boldsymbol{v})\ ,\ \forall \boldsymbol{v} \in V, \\ C\|P_\epsilon(t)\| \leq \sup\limits_{\boldsymbol{v} \in V} \dfrac{G_t(\boldsymbol{v})}{\|\boldsymbol{v}\|_1}. \end{cases} \tag{2.11}$$

Now, taking the time derivative of $(2.11)_1$, we find

$$\langle \boldsymbol{u}'_\epsilon(t), \boldsymbol{v}\rangle + a(\boldsymbol{u}_\epsilon(t), \boldsymbol{v}) + \langle K_\epsilon(\boldsymbol{u}_\epsilon(t)), \boldsymbol{v}\rangle + b(P'_\epsilon(t), \boldsymbol{v}) = \ell(\boldsymbol{v})\ ,\ \forall\ \boldsymbol{v} \in Z(\Omega).$$

Finally, for $p_\epsilon(t) = -P'_\epsilon(t)$, one realizes that $(\boldsymbol{u}_\epsilon(t), p_\epsilon(t))$ satisfies the following variational problem for a.e t, with $0 \le t \le T$, such that for all $(\boldsymbol{v}, q) \in \boldsymbol{V} \times L^2(\Omega)$,

$$\begin{cases} \langle \boldsymbol{u}'_\epsilon(t), \boldsymbol{v}\rangle + a(\boldsymbol{u}_\epsilon(t), \boldsymbol{v}) + \langle K_\epsilon(\boldsymbol{u}_\epsilon(t)), \boldsymbol{v}\rangle - b(p_\epsilon(t), \boldsymbol{v}) = \ell(\boldsymbol{v}), \\ b(q, \boldsymbol{u}_\epsilon(t)) = 0. \end{cases} \tag{2.12}$$

Next, we study the existence and uniqueness of a solution of (2.10), and (2.8), and then pass to the limit when ϵ tends to zero to recover the existence of solution of (2.2) and (2.3).

step 2: Existence and uniqueness of (2.8) and (2.10). We let

$$\boldsymbol{H} = \{\boldsymbol{v} \in L^2(\Omega)^2 \ , \quad \operatorname{div} \boldsymbol{v} = 0 \quad , \quad \boldsymbol{v} \cdot \boldsymbol{n}|_S = 0\}.$$

One readily observes that $\boldsymbol{V}$ is compact in $\boldsymbol{H}$. Next, we introduce the Stokes operator defined on a subspace of $\boldsymbol{V}$ constructed in [27] as follows; for every $\boldsymbol{f}$ in $\boldsymbol{H}$, there exists a unique $\boldsymbol{v}$ in $\boldsymbol{V}$ such that

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{w}) = (\boldsymbol{f}, \boldsymbol{w}) \text{ for all } \boldsymbol{w} \in \boldsymbol{V}. \tag{2.13}$$

Moreover, for every $\boldsymbol{v}$ in $\boldsymbol{V}$, there is a unique $\boldsymbol{f}$ in $\boldsymbol{H}$ such that (2.13) holds. Then (2.13) defines a one to one mapping between $\boldsymbol{f}$ in $\boldsymbol{H}$ and $\boldsymbol{v}$ in $D(A)$, where $D(A)$ is a subspace of $\boldsymbol{V}$. Hence, $A\boldsymbol{v} = \boldsymbol{f}$ defines the Stokes operator $A : D(A) \to \boldsymbol{H}$. Its inverse is compact and self adjoint as a mapping from $\boldsymbol{H}$ to $\boldsymbol{H}$ and possesses an orthogonal sequence of eigenfunctions ϕ_k which are complete in $\boldsymbol{H}$ and $\boldsymbol{V}$;

$$A\phi_k = \lambda_k \phi_k \ . \tag{2.14}$$

Let $\boldsymbol{V}_m$ be the subspace of $\boldsymbol{V}$ spanned by $\phi_1, \cdots, \phi_m$, that is

$$\boldsymbol{V}_m = \{\phi_1, \phi_2, \cdots, \phi_m\}.$$

We consider the following ordinary differential equation:

$$\begin{cases} \text{Find } \boldsymbol{u}_{\epsilon,m}(t) \in \boldsymbol{V}_m \text{ such that for all } \boldsymbol{v} \in \boldsymbol{V}_m, \\ \langle \boldsymbol{u}'_{\epsilon,m}(t), \boldsymbol{v}\rangle + a(\boldsymbol{u}_{\epsilon,m}(t), \boldsymbol{v}) + \langle K_\epsilon(\boldsymbol{u}_{\epsilon,m}(t)), \boldsymbol{v}\rangle = \ell(\boldsymbol{v}), \\ \boldsymbol{u}_{\epsilon,m}(0) \longrightarrow \boldsymbol{u}_\epsilon(0) = \boldsymbol{u}_0 \in \boldsymbol{V}_m. \end{cases} \tag{2.15}$$

One can show that the mapping

$$T : \boldsymbol{w} \longmapsto \ell(\boldsymbol{w}) - a(\boldsymbol{w}, \boldsymbol{v}) + \langle K_\epsilon(\boldsymbol{w}), \boldsymbol{v}\rangle,$$

is locally Lipschitz thanks to the nature of the operators involved. It then follows from Cauchy-Lipschitz's theorem that (2.15) has a unique solution $\boldsymbol{u}_{\epsilon,m}(t)$. Next, we establish a priori estimates independent of ϵ, and then pass to the limit in ϵ.

step 3: a priori estimate.

Lemma 2.1. *Assume that $(\boldsymbol{f}, g)$ belong to $L^\infty(0,T; L^2(\Omega)^2) \times L^\infty(S)$, with the initial velocity $\boldsymbol{u}_0$ in $H^1(\Omega)^2$, and satisfying (2.1). Then the solution $\boldsymbol{u}_\epsilon(t)$ of (2.8), and (2.10) satisfies the bounds*

$$\begin{aligned} \|\boldsymbol{u}_\epsilon(t)\| &\le \|\boldsymbol{u}_0\| \exp(-Ct) + C\|\boldsymbol{f}\|_{L^\infty(0,T;L^2(\Omega)^2)}, \\ \|\boldsymbol{u}'_\epsilon\|_{L^2(0,T;L^2(\Omega)^2)} &\le \|\boldsymbol{f}\|_{L^2(0,T;L^2(\Omega)^2)} + C(\Omega,\nu)\|\boldsymbol{u}_0\|_1 + C\|g\|_{L^\infty(S)} \\ &+ C\|g\|^{1/2}_{L^\infty(S)}, \\ \nu\|\nabla \boldsymbol{u}_\epsilon\|^2_{L^\infty(0,T;L^2(\Omega)^2)} &\le \|\boldsymbol{f}\|^2_{L^2(0,T;L^2(\Omega)^2)} + C(\Omega,\nu)\|\boldsymbol{u}_0\|^2_1 + C\|g\|^2_{L^\infty(S)} \\ &+ C\|g\|_{L^\infty(S)}. \end{aligned} \tag{2.16}$$

Proof. It should be mentioned that these a priori estimates can be established rigorously by working on a Galerkin approximation of (2.8) and (2.10) in which the approximation of $\partial_t \boldsymbol{u}_\epsilon$ indeed belongs to $L^2(0,T;\boldsymbol{V}_m)$ and $\boldsymbol{u}_\epsilon$ in $L^\infty(0,T;\boldsymbol{V}_m)$. First,taking $\boldsymbol{v} = \boldsymbol{u}_\epsilon(t)$ in (2.10), we find

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_\epsilon(t)\|^2 + \nu\|\varepsilon(\boldsymbol{u}_\epsilon(t))\|^2 + \langle K_\epsilon(\boldsymbol{u}_\epsilon(t)), \boldsymbol{u}_\epsilon(t)\rangle = \ell(\boldsymbol{u}_\epsilon(t)). \tag{2.17}$$

But (2.7) implies that $\langle K_\epsilon(\boldsymbol{u}_\epsilon(t)), \boldsymbol{u}_\epsilon(t)\rangle \geq 0$, so that (2.16) gives

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_\epsilon(t)\|^2 + \nu\|\varepsilon(\boldsymbol{u}_\epsilon(t))\|^2 \leq \|\boldsymbol{f}(t)\|\|\boldsymbol{u}_\epsilon(t)\|. \tag{2.18}$$

Now, since $\boldsymbol{u}_\epsilon$ is in $\boldsymbol{V}$, Korn's inequality [1] applies. That is there exists $C(\Omega)$, such that

$$\int_\Omega \left[|\boldsymbol{u}_\epsilon(t)| + |\nabla \boldsymbol{u}_\epsilon(t)|\right]^2 dx \leq C(\Omega)\int_\Omega |\varepsilon(\boldsymbol{u}_\epsilon(t))|^2 dx, \tag{2.19}$$

moreover, in order to control from below the of the gradient, Poincare-Freidrichs' inequality should be used. That is there exists $C(\Omega)$, such that

$$\int_\Omega |\boldsymbol{u}_\epsilon(t)|^2 dx \leq C(\Omega)\int_\Omega |\nabla(\boldsymbol{u}_\epsilon(t))|^2 dx. \tag{2.20}$$

We then infer from (2.18), and (2.19) that

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_\epsilon(t)\|^2 + \nu\|\varepsilon(\boldsymbol{u}_\epsilon(t)\|^2 \leq C(\Omega)\|\boldsymbol{f}(t)\|\|\varepsilon(\boldsymbol{u}_\epsilon(t))\|.$$

With Young's inequality, (2.19) and (2.20) yield

$$\frac{d}{dt}\|\boldsymbol{u}_\epsilon(t)\|^2 + C(\Omega,\nu)\|\boldsymbol{u}_\epsilon(t)\|^2 \leq C(\nu,\Omega)\|\boldsymbol{f}(t)\|^2.$$

Thus the first inequality in (2.16) is obtained by integration.
Next, setting $\boldsymbol{v} = \boldsymbol{u}'_\epsilon(t)$ in (2.10), one gets

$$\|\boldsymbol{u}'_\epsilon(t)\|^2 + \frac{d}{dt}\left[\frac{1}{2}a(\boldsymbol{u}_\epsilon(t),\boldsymbol{u}_\epsilon(t)) + J_\epsilon(\boldsymbol{u}_\epsilon(t))\right] = \ell(\boldsymbol{u}'_\epsilon(t)),$$

which gives (after utilization of Cauchy-Schwarz and Young's inequalities),

$$\frac{1}{2}\|\boldsymbol{u}'_\epsilon(t)\|^2 + \frac{d}{dt}\left[\frac{1}{2}a(\boldsymbol{u}_\epsilon(t),\boldsymbol{u}_\epsilon(t)) + J_\epsilon(\boldsymbol{u}_\epsilon(t))\right] \leq \frac{1}{2}\|\boldsymbol{f}(t)\|^2.$$

Integration of the latter inequality over the time interval $(0,t)$ yields

$$\begin{aligned}\frac{1}{2}\|\boldsymbol{u}'_\epsilon\|^2_{L^2(0,t;L^2(\Omega)^2)} &+ \frac{1}{2}a(\boldsymbol{u}_\epsilon(t),\boldsymbol{u}_\epsilon(t)) + J_\epsilon(\boldsymbol{u}_\epsilon(t)) \\ &\leq \frac{1}{2}\|\boldsymbol{f}\|^2_{L^2(0,t;L^2(\Omega)^2)} + \frac{1}{2}a(\boldsymbol{u}_0,\boldsymbol{u}_0) + J_\epsilon(\boldsymbol{u}_0).\end{aligned} \tag{2.21}$$

Now, one realizes that for $\boldsymbol{v}$ fixed, the mapping $\epsilon \in (0,1) \longmapsto J_\epsilon(\boldsymbol{v}) \in \mathbb{R}_+$ is non decreasing so that (2.21) gives

$$\begin{aligned}&\|\boldsymbol{u}'_\epsilon\|^2_{L^2(0,t;L^2(\Omega)^2)} \leq \|\boldsymbol{f}\|^2_{L^2(0,t;L^2(\Omega)^2)} + \nu\|\nabla\boldsymbol{u}_0\|^2 + 2\int_S g\left(|\boldsymbol{u}_{0,\boldsymbol{\tau}}| + 1\right), \\ &\nu\|\varepsilon(\boldsymbol{u}_\epsilon)\|^2_{L^\infty(0,t;L^2(\Omega)^2)} + 2J_\epsilon(\boldsymbol{u}_\epsilon(t)) \leq \\ &\qquad\qquad \|\boldsymbol{f}\|^2_{L^2(0,t;L^2(\Omega)^2)} + \nu\|\nabla\boldsymbol{u}_0\|^2 + 2\int_S g\left(|\boldsymbol{u}_{0,\boldsymbol{\tau}}| + 1\right),\end{aligned}$$

from which we deduce the second and third inequalities in (2.16). From the second relation of (2.11) and (2.16) we readily derive the a priori estimate for P_ϵ defined

by (2.11).

step 4: passage to the limit in ϵ. According to lemma 2.1, we can select from the sequence $(\boldsymbol{u}_\epsilon)_\epsilon$, a subsequence, again denoted by $(\boldsymbol{u}_\epsilon)_\epsilon$, such that

$$\boldsymbol{u}_\epsilon \to \boldsymbol{u} \text{ weak star in } L^\infty(0,T;L^2(\Omega)^2) \tag{2.22}$$

$$\boldsymbol{u}'_\epsilon \to \boldsymbol{u}' \text{ weakly in } L^2(0,T;L^2(\Omega)^2), \tag{2.23}$$

which imply in particular that

$$\boldsymbol{u}_\epsilon \quad \text{remains in a bounded set of } H^1(Q). \tag{2.24}$$

But from Rellich-Kondrachoff's result, the embedding $H^1(Q) \longmapsto L^2(Q)$ is compact. So one can extract from $(\boldsymbol{u}_\epsilon)$ a subsequence, denoted again by $(\boldsymbol{u}_\epsilon)$ such that

$$\boldsymbol{u}_\epsilon \longrightarrow \boldsymbol{u} \text{ strongly in } L^2(0,T;L^2(\Omega)^2) \text{ and a.e. in } Q. \tag{2.25}$$

From (2.10), it follows that

$$\begin{aligned} &\langle \boldsymbol{u}'_\epsilon(t), \boldsymbol{v}-\boldsymbol{u}_\epsilon(t)\rangle + a(\boldsymbol{u}_\epsilon(t), \boldsymbol{v}-\boldsymbol{u}_\epsilon(t)) + J_\epsilon(\boldsymbol{v}) - J_\epsilon(\boldsymbol{u}_\epsilon(t)) - \ell(\boldsymbol{v}-\boldsymbol{u}_\epsilon(t)) \\ &= J_\epsilon(\boldsymbol{v}) - J_\epsilon(\boldsymbol{u}_\epsilon(t)) - \langle K_\epsilon(\boldsymbol{u}_\epsilon(t)), \boldsymbol{v}-\boldsymbol{u}_\epsilon(t)\rangle \geq 0 \end{aligned} \tag{2.26}$$

since J_ϵ is convex and Gateaux differentiable. Next, taking in (2.26), $\boldsymbol{v} = \boldsymbol{v}(t)$ where $t \mapsto \boldsymbol{v}(t)$ is arbitrary in $L^2(0,T;Z(\Omega))$, it follows that

$$\begin{aligned} &\int_0^T [\langle \boldsymbol{u}'_\epsilon(t), \boldsymbol{v}\rangle + a(\boldsymbol{u}_\epsilon(t), \boldsymbol{v}) + J_\epsilon(\boldsymbol{v}) - \ell(\boldsymbol{v}-\boldsymbol{u}_\epsilon(t))]dt \\ &\geq \int_0^T \left[\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_\epsilon(t)\|^2 + a(\boldsymbol{u}_\epsilon(t), \boldsymbol{u}_\epsilon(t)) + J_\epsilon(\boldsymbol{u}_\epsilon(t))\right] dt \\ &= \frac{1}{2}\left[\|\boldsymbol{u}_\epsilon(T)\|^2 - \|\boldsymbol{u}_0\|^2\right] + \int_0^T [a(\boldsymbol{u}_\epsilon(t), \boldsymbol{u}_\epsilon(t)) + J_\epsilon(\boldsymbol{u}_\epsilon(t))]\, dt. \end{aligned} \tag{2.27}$$

But from (2.22),

$$\begin{aligned} &\liminf_\epsilon \left\{\frac{1}{2}\left[\|\boldsymbol{u}_\epsilon(T)\|^2 - \|\boldsymbol{u}_0\|^2\right] + \int_0^T [a(\boldsymbol{u}_\epsilon(t), \boldsymbol{u}_\epsilon(t)) + J_\epsilon(\boldsymbol{u}_\epsilon(t))]\, dt\right\} \\ &\geq \frac{1}{2}\left[\|\boldsymbol{u}(T)\|^2 - \|\boldsymbol{u}_0\|^2\right] + \int_0^T [a(\boldsymbol{u}(t), \boldsymbol{u}(t)) + J(\boldsymbol{u}(t))]\, dt. \end{aligned}$$

We then deduce from (2.22) and (2.23), that

$$\begin{cases} \text{for all } \boldsymbol{v} \in L^2(0,T;Z(\Omega)), \\ \displaystyle\int_0^T [\langle \boldsymbol{u}'(t), \boldsymbol{v}-\boldsymbol{u}(t)\rangle + a(\boldsymbol{u}(t), \boldsymbol{v}-\boldsymbol{u}(t)) + J(\boldsymbol{v}) - J(\boldsymbol{u}(t)) - \ell(\boldsymbol{v}-\boldsymbol{u}(t))]\, dt \geq 0, \end{cases}$$

which by arguing as in [5](see p. 56-58) yields

$$\begin{cases} \text{for all } \boldsymbol{v} \in Z(\Omega), \\ \langle \boldsymbol{u}'(t), \boldsymbol{v}-\boldsymbol{u}(t)\rangle + a(\boldsymbol{u}(t), \boldsymbol{v}-\boldsymbol{u}(t)) + J(\boldsymbol{v}) - J(\boldsymbol{u}(t)) \geq \ell(\boldsymbol{v}-\boldsymbol{u}(t)), \end{cases} \tag{2.28}$$

which ends the construction of the velocity for the problem (2.2), and (2.3). Having obtained the velocity, we shall indicate how the pressure can be constructed. We first observed from (2.10) that

$$b(p_\varepsilon(t), \boldsymbol{v}) = -\ell(\boldsymbol{v}) + \langle \boldsymbol{u}'_\epsilon(t), \boldsymbol{v}\rangle + a(\boldsymbol{u}_\epsilon(t), \boldsymbol{v}) + \langle K_\epsilon(\boldsymbol{u}_\epsilon(t)), \boldsymbol{v}\rangle.$$

But since $p_\epsilon(t) \in L^2_0(\Omega)$, following [1], one can find a positive constant C such that

$$C\|p_\epsilon(t)\| \leq \sup_{\boldsymbol{v}\in \boldsymbol{V}} \frac{b(p_\epsilon(t), \boldsymbol{v})}{\|\boldsymbol{v}\|_1}.$$

Hence

$$\begin{aligned} C\|p_\epsilon(t)\| &\leq \|\boldsymbol{u}'_\epsilon(t)\| + \nu\|\nabla \boldsymbol{u}_\epsilon(t)\| + \|K_\epsilon(\boldsymbol{u}_\epsilon(t))\|_{\boldsymbol{V}'} + \|\boldsymbol{f}(t)\| \\ &\leq \|\boldsymbol{u}'_\epsilon(t)\| + \nu\|\nabla \boldsymbol{u}_\epsilon(t)\| + C(\Omega)\|g\|_{L^\infty(S)}\|\nabla \boldsymbol{u}_\epsilon(t)\| + \|\boldsymbol{f}(t)\| \end{aligned}$$

which by Young's inequality, and integrating the resulting inequality over $[0, T]$, yields (after utilization of (2.16))

$$\begin{aligned} \int_0^T \|p_\epsilon(t)\|^2 dt &\leq C\int_0^T \|\boldsymbol{u}'_\epsilon(t)\|^2 + C\int_0^T \|\nabla \boldsymbol{u}_\epsilon(t)\|^2 dt \\ &+ C\|g\|^2_{L^\infty(S)}\int_0^T \|\nabla \boldsymbol{u}_\epsilon(t)\|^2 + C\int_0^T \|\boldsymbol{f}(t)\|^2 dt < \infty, \end{aligned} \tag{2.29}$$

C being a positive constant depending on the parameters and the domain of the problem. Then we can select from $p_\epsilon(t)$ a sequence, again denoted by $p_\epsilon(t)$,such that

$$p_\epsilon \longrightarrow p \text{ weakly in } L^2(0, T; L^2(\Omega)). \tag{2.30}$$

Next, one observes that (2.10) can be re-written as

$$\begin{aligned} \langle \boldsymbol{u}'_\epsilon(t), \boldsymbol{v} - \boldsymbol{u}_\epsilon(t)\rangle + a(\boldsymbol{u}_\epsilon(t), \boldsymbol{v} - \boldsymbol{u}_\epsilon(t)) - b(p_\epsilon(t), \boldsymbol{v} - \boldsymbol{u}_\epsilon(t)) \\ + J_\epsilon(\boldsymbol{v}) - J_\epsilon(\boldsymbol{u}_\epsilon(t)) - \ell(\boldsymbol{v} - \boldsymbol{u}_\epsilon(t)) \geq 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{V}, \\ b(q, \boldsymbol{u}_\epsilon(t)) = 0 \quad \text{for all } q \in L^2(\Omega), \end{aligned}$$

which by integration over the time interval $[0, T]$ and passage to the limit (as $\epsilon \to 0$) yields, (after utilization of the identity $b(q, \boldsymbol{u}_\epsilon(t)) = 0$ for all $q \in L^2(\Omega)$)

$$\begin{aligned} \int_0^T \big[\langle \boldsymbol{u}'(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle + a(\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))\big] dt \\ + \int_0^T \big[- b(p(t), \boldsymbol{v} - \boldsymbol{u}(t)) + J(\boldsymbol{v}) - J(\boldsymbol{u}(t)) - \ell(\boldsymbol{v} - \boldsymbol{u}(t))\big] dt \geq 0, \end{aligned}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$. Also, $b(q, \boldsymbol{u}(t)) = 0$ for all $q \in L^2(\Omega)$. Finally, arguing as in [5] (see p 56-57), one obtains

$$\begin{cases} \text{for all } (\boldsymbol{v}, q) \in \boldsymbol{V} \times M, \\ \langle \boldsymbol{u}'(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle + a(\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) - b(p(t), \boldsymbol{v} - \boldsymbol{u}(t)) \\ + J(\boldsymbol{v}) - J(\boldsymbol{u}(t)) \geq \ell(\boldsymbol{v} - \boldsymbol{u}(t)), \\ b(q, \boldsymbol{u}(t)) = 0, \end{cases} \tag{2.31}$$

which ensures the existence of solutions claimed in Theorem 2.1. Of course the estimates (2.4) are readily obtained from (2.16) when ϵ goes to zero. As far as the uniqueness of solutions is concerned,we assume that $(\boldsymbol{u}_1, p_1)$ and $(\boldsymbol{u}_2, p_2)$ are two set of solutions of (2.2),(2.3) with $\boldsymbol{u}_{12} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $p_{12} = p_1 - p_2$. Then

$$\begin{cases} \langle \boldsymbol{u}'_1(t), \boldsymbol{u}_{12}(t)\rangle + a(\boldsymbol{u}_1(t), \boldsymbol{u}_{12}(t)) - b(p_1(t), \boldsymbol{u}_{12}(t)) \\ -J(\boldsymbol{u}_2(t)) + J(\boldsymbol{u}_1(t)) \leq \ell(\boldsymbol{u}_{12}(t)), \\ b(q, \boldsymbol{u}_1(t)) = 0 \\ \boldsymbol{u}_{12}(0) = \boldsymbol{0}. \end{cases}$$

and

$$\begin{cases} \langle \boldsymbol{u}_2'(t), \boldsymbol{u}_{12}(t)\rangle + a(\boldsymbol{u}_2(t), \boldsymbol{u}_{12}(t)) - b(p_2(t), \boldsymbol{u}_{12}(t)) \\ +J(\boldsymbol{u}_1(t)) - J(\boldsymbol{u}_2(t)) \geq \ell(\boldsymbol{u}_{12}(t)), \\ b(q, \boldsymbol{u}_2(t)) = 0, \end{cases}$$

which after subtracting the former from the latter gives

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_{12}(t)\|^2 + a(\boldsymbol{u}_{12}(t), \boldsymbol{u}_{12}(t)) - b(p_{12}(t), \boldsymbol{u}_{12}(t)) = 0 \\ b(q, \boldsymbol{u}_{12}(t)) = 0 \text{ and } \boldsymbol{u}_{12}(0) = \mathbf{0}. \end{cases}$$

So for $q = p_{12}(t)$, one gets

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_{12}(t)\|^2 + a(\boldsymbol{u}_{12}(t), \boldsymbol{u}_{12}(t)) = 0 \ , \\ \boldsymbol{u}_{12}(0) = \mathbf{0}, \end{cases}$$

which, after integration over time, gives $\boldsymbol{u}_{12}(t) = \mathbf{0}$. It is immediate to see that the pressure is defined up to a constant in $L^2(\Omega)$, but uniquely defined if one works in $L^2_0(\Omega)$. □

Remark 2.1. *The uniqueness of solution implies that the whole sequence $(\boldsymbol{u}_\epsilon)_\epsilon$ converges.*

Next, we estimate the difference between $\boldsymbol{u}(t)$ and $\boldsymbol{u}_\epsilon(t)$ in the $L^2(\Omega)$ norm.

Lemma 2.2. *Let $\boldsymbol{u}(t)$ be the solution of (2.6), and $\boldsymbol{u}_\epsilon(t)$ the solution of (2.10). Assume that $g \in L^\infty(\Omega)$. Then, there exists a positive constant C depending on Ω and ν, such that*

$$\|\boldsymbol{u}(t) - \boldsymbol{u}_\epsilon(t)\| \leq \epsilon^{1/2} C(\nu, \Omega, S)[1 - \exp(-Ct)]\|g\|^{1/2}_{L^\infty(S)}. \tag{2.32}$$

Proof. First, take $\boldsymbol{v} = \boldsymbol{u}_\epsilon(t)$ in (2.6), this gives

$$\langle \boldsymbol{u}'(t), \boldsymbol{u}_\epsilon(t) - \boldsymbol{u}(t)\rangle + a(\boldsymbol{u}(t), \boldsymbol{u}_\epsilon(t) - \boldsymbol{u}(t)) + J(\boldsymbol{u}_\epsilon(t)) - J(\boldsymbol{u}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{u}_\epsilon(t) - \boldsymbol{u}(t)) \, .$$

Secondly, for $\boldsymbol{v} = \boldsymbol{u}(t)$ in (2.10), one has

$$\langle \boldsymbol{u}_\epsilon'(t), \boldsymbol{u}(t) - \boldsymbol{u}_\epsilon(t)\rangle + a(\boldsymbol{u}_\epsilon(t), \boldsymbol{u}(t) - \boldsymbol{u}_\epsilon(t)) + J_\epsilon(\boldsymbol{u}(t)) - J_\epsilon(\boldsymbol{u}_\epsilon(t)) \geq (\boldsymbol{f}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\epsilon(t)) \, .$$

Adding the previous two inequalities,we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t) - \boldsymbol{u}_\epsilon(t)\|^2 + 2\nu\|\varepsilon(\boldsymbol{u}(t) - \boldsymbol{u}_\epsilon(t))\|^2 \\ &\leq [J(\boldsymbol{u}_\epsilon(t)) - J(\boldsymbol{u}(t))] + [J_\epsilon(\boldsymbol{u}(t)) - J_\epsilon(\boldsymbol{u}_\epsilon(t))] \\ &\leq C\epsilon\|g\|_{L^\infty(S)}|S|, \end{aligned}$$

which together with (2.19) and (2.20), and integration over time, leads to the desired result. □

3. A time discrete approximation: Implicit Euler scheme

3.1. Variational formulation. Our objectives here are, first to show the existence of solutions for the time discrete scheme associated with (2.2) and (2.3), and then to establish the discrete counterpart to (2.4).

We introduce a partition of the interval $[0, T]$ into sub-intervals $[t_n, t_{n+1}]$ of equal size $k = t_{n+1} - t_n$, and $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$. We then introduce the following implicit scheme for calculating the approximation of $(\boldsymbol{u}(t), p(t))$ defined

by (2.2), and (2.3).
Find $(\boldsymbol{u}^n, p^n) \in \boldsymbol{V} \times M$ such that

$$\boldsymbol{u}^0 = \boldsymbol{u}_0 \ , \text{ in } \Omega, \tag{3.1}$$

and such that, for all n, $1 \leq n \leq N$,

$$\begin{cases} \text{for all } (\boldsymbol{v}, q) \in \boldsymbol{V} \times M, \\ \left(\dfrac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{k}, \boldsymbol{v} - \boldsymbol{u}^n\right) + a(\boldsymbol{u}^n, \boldsymbol{v} - \boldsymbol{u}^n) - b(p^n, \boldsymbol{v} - \boldsymbol{u}^n) \\ + J(\boldsymbol{v}) - J(\boldsymbol{u}^n) \geq (\boldsymbol{f}^n, \boldsymbol{v} - \boldsymbol{u}^n), \\ b(q, \boldsymbol{u}^n) = 0, \end{cases} \tag{3.2}$$

where,

$$\boldsymbol{f}^n = \frac{1}{k} \int_{t_{n-1}}^{t_n} \boldsymbol{f}(t) dt.$$

Following our presentation of the continuous problem, we claim that (see [8], theorem 3-1), the variational problem (3.1) and (3.2) is equivalent to: Find the sequence $(\boldsymbol{u}^n)_{0 \leq n \leq N}$ in $Z(\Omega)^{N+1}$ that satisfies (3.1), (2.1), and such that, for $1 \leq n \leq N$,

$$\begin{cases} \left(\dfrac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{k}, \boldsymbol{v} - \boldsymbol{u}^n\right) + a(\boldsymbol{u}^n, \boldsymbol{v} - \boldsymbol{u}^n) + J(\boldsymbol{v}) - J(\boldsymbol{u}^n) \\ \geq (\boldsymbol{f}^n, \boldsymbol{v} - \boldsymbol{u}^n) \quad \text{for all } \boldsymbol{v} \in Z(\Omega). \end{cases} \tag{3.3}$$

3.2. Existence of solutions. In this paragraph, we construct the solutions of (3.1) and (3.2) by adopting the algorithm presented in section 2. We also establish the discrete a priori estimates, which mimic the energy law proved in lemma 2.1. Note that not every numerical method possesses such a property.
To claim the existence of solutions of (3.1) and (3.3), we state the following result

Theorem 3.1. *Let $g \in L^\infty(S)$, $\boldsymbol{f} \in L^2(0, T; L^2(\Omega)^2)$ and $\boldsymbol{u}_0 \in H^1(\Omega)^2$. Then the problem (3.1) and (3.3) has a unique solution $\boldsymbol{u}^n$ in $Z(\Omega)$.*

Proof. The existence result is obtained in several steps.
Step 1: Regularized problem. We first introduce the following regularized variational problem: Find the sequence $(\boldsymbol{u}_\epsilon^n)_{0 \leq n \leq N} \in Z(\Omega)^{N+1}$ such that

$$\boldsymbol{u}_\epsilon^0 = \boldsymbol{u}_0, \tag{3.4}$$

and for all n, with $1 \leq n \leq N$,

$$\begin{cases} \left(\dfrac{\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^{n-1}}{k}, \boldsymbol{v}\right) + a(\boldsymbol{u}_\epsilon^n, \boldsymbol{v}) + \langle K_\epsilon(\boldsymbol{u}_\epsilon^n), \boldsymbol{v} \rangle = (\boldsymbol{f}^n, \boldsymbol{v}), \\ \text{for all } \boldsymbol{v} \in Z(\Omega), \end{cases} \tag{3.5}$$

and indicate how the regularized pressure is obtained from the velocity $\boldsymbol{u}_\epsilon^n$. For that purpose, for any $\boldsymbol{v}$ in $\boldsymbol{V}$, we set

$$G_n(\boldsymbol{v}) = (\boldsymbol{f}^n, \boldsymbol{v}) - a(\boldsymbol{u}_\epsilon^n, \boldsymbol{v}) - \langle K_\epsilon(\boldsymbol{u}_\epsilon^n), \boldsymbol{v} \rangle - \frac{1}{k}(\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^{n-1}, \boldsymbol{v}).$$

It can be checked that the mapping $\boldsymbol{v} \longmapsto G_n(\boldsymbol{v})$ is a continuous linear functional on $\boldsymbol{V}$. Moreover if $\boldsymbol{u}_\epsilon^n$ is a solution of (3.4) and (3.5), then $G_n(\boldsymbol{v})$ vanishes on $Z(\Omega)$. Hence, there exists a function $p_\epsilon^n \in L^2(\Omega)$ such that

$$\begin{cases} b(p_\epsilon^n, \boldsymbol{v}) = G_n(\boldsymbol{v}), \text{ for all } \boldsymbol{v} \in \boldsymbol{V}, \text{ and} \\ C\|p_\epsilon^n\| \leq \dfrac{G_n(\boldsymbol{v})}{\|\boldsymbol{v}\|_1}. \end{cases} \tag{3.6}$$

Thus the couple $(\boldsymbol{u}_\epsilon^n, p_\epsilon^n)$ satisfies: for all n, with $1 \le n \le N$,

$$\begin{cases} \text{for all } (\boldsymbol{v}, q) \in \boldsymbol{V} \times L^2(\Omega) \\ \left(\dfrac{\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^{n-1}}{k}, \boldsymbol{v}\right) + a(\boldsymbol{u}_\epsilon^n, \boldsymbol{v}) - b(p_\epsilon^n, \boldsymbol{v}) + \langle K_\epsilon(\boldsymbol{u}_\epsilon^n), \boldsymbol{v}\rangle = (\boldsymbol{f}^n, \boldsymbol{v}), \\ b(q, \boldsymbol{u}_\epsilon^n) = 0, \end{cases} \tag{3.7}$$

Step 2: Existence, uniqueness of (3.4), (3.5). Now, we shall show how existence and uniqueness of the weak solution $\boldsymbol{u}_\epsilon^n$ of (3.5) is obtained. For that purpose, we observe that the variational problem (3.5) is a particular case of nonlinear monotone type. Hence following [17, 28], we introduce the mapping $\boldsymbol{v} \longmapsto \mathcal{H}_\epsilon \boldsymbol{v}$ defined by

$$\begin{aligned} &\mathcal{H}_\epsilon \boldsymbol{v} = \boldsymbol{v} + kA\boldsymbol{v} + kK_\epsilon(\boldsymbol{v}) \\ &\text{with } (A\boldsymbol{v}, \boldsymbol{w}) = a(\boldsymbol{v}, \boldsymbol{w}) \text{ for } \boldsymbol{w}, \boldsymbol{v} \in Z(\Omega). \end{aligned} \tag{3.8}$$

It is then immediate to observe that the existence and uniqueness of the solution of (3.5) will be achieved if one establishes as in [17, 28], that the mapping $\mathcal{H}_\epsilon$ is monotone, coercive and hemi-continuous in $Z(\Omega)$. We then claim that

Lemma 3.1. *The problem (3.4) and (3.5) has a unique solution sequence* $(\boldsymbol{u}_\epsilon^n)_{0\le n\le N} \subset Z(\Omega)^{N+1}$.

Proof. As mentioned, it will be enough to show that $\mathcal{H}_\epsilon$ is monotone, coercive and hemi-continuous in $Z(\Omega)$.
Step 1: $\mathcal{H}_\epsilon$ is monotone. Indeed, for $\boldsymbol{v}, \boldsymbol{w}$ in $Z(\Omega)$ one has

$$\begin{aligned} &\int_\Omega (\mathcal{H}_\epsilon(\boldsymbol{v}) - \mathcal{H}_\epsilon(\boldsymbol{w})) \cdot (\boldsymbol{v} - \boldsymbol{w}) dx \\ &= \|\boldsymbol{v} - \boldsymbol{w}\|^2 + k\|\nabla(\boldsymbol{v} - \boldsymbol{w})\|^2 + k\int_\Omega (K_\epsilon(\boldsymbol{v}) - K_\epsilon(\boldsymbol{w})) \cdot (\boldsymbol{v} - \boldsymbol{w}) dx \\ &= \|\boldsymbol{v} - \boldsymbol{w}\|^2 + k\|\nabla(\boldsymbol{v} - \boldsymbol{w})\|^2 + k\int_0^1 \underbrace{J_\epsilon''(\boldsymbol{v} + \theta(\boldsymbol{v} - \boldsymbol{w})) \cdot (\boldsymbol{v} - \boldsymbol{w}, \boldsymbol{v} - \boldsymbol{w})}_{\ge 0} d\theta \\ &\ge \|\boldsymbol{v} - \boldsymbol{w}\|^2 + \rho_n\|\nabla(\boldsymbol{v} - \boldsymbol{w})\|^2 \ge \min(1, k)\|\boldsymbol{v} - \boldsymbol{w}\|_1^2. \end{aligned}$$

Step 2: $\mathcal{H}_\epsilon$ is coercive. Indeed, for $\boldsymbol{v} \in Z(\Omega)$, one has

$$\begin{aligned} \lim_{\|\boldsymbol{v}\|_1 \to \infty} \left(\frac{1}{\|\boldsymbol{v}\|_1}(\mathcal{H}_\epsilon \boldsymbol{v}, \boldsymbol{v})\right) &= \lim_{\|\boldsymbol{v}\|_1 \to \infty} \frac{1}{\|\boldsymbol{v}\|_1}\left(\|\boldsymbol{v}\|^2 + k\, a(\boldsymbol{v}, \boldsymbol{v}) + k(K_\epsilon(\boldsymbol{v}), \boldsymbol{v})\right) \\ &\ge \lim_{\|\boldsymbol{v}\|_1 \to \infty} \frac{1}{\|\boldsymbol{v}\|_1}\left(\|\boldsymbol{v}\|^2 + \rho_n\|\nabla \boldsymbol{v}\|^2\right) \\ &\ge \min(1, k) \lim_{\|\boldsymbol{v}\|_1 \to \infty} \|\boldsymbol{v}\|_1 = \infty\,. \end{aligned}$$

Step 3: $\mathcal{H}_\epsilon$ is hemi-continuous. We need to show that for $\boldsymbol{v}, \boldsymbol{u} \in Z(\Omega)$, the mapping

$$t \mapsto (\mathcal{H}_\epsilon(\boldsymbol{u} + t\boldsymbol{v}), \boldsymbol{v}) \text{ is continuous in } \mathbb{R}.$$

For any t_1, t_2 in $\mathbb{R}$

$$\begin{aligned}
&\int_\Omega (\mathcal{H}_\epsilon(\boldsymbol{u}+t_1\boldsymbol{v}) - \mathcal{H}_\epsilon(\boldsymbol{u}+t_2\boldsymbol{v}))\cdot \boldsymbol{v}\ dx \\
&= (t_1-t_2)\|\boldsymbol{v}\|^2 + k(t_1-t_2)\|\nabla \boldsymbol{v}\|^2 + k\int_\Omega (K_\epsilon(\boldsymbol{u}+t_1\boldsymbol{v}) - K_\epsilon(\boldsymbol{u}+t_2\boldsymbol{v}))\cdot \boldsymbol{v}\ dx \\
&= (t_1-t_2)\|\boldsymbol{v}\|^2 + k(t_1-t_2)\|\nabla \boldsymbol{v}\|^2 \\
&+k(t_1-t_2)\int_0^1 J_\epsilon^{''}(\boldsymbol{u}-t_2\boldsymbol{v}-\theta(t_2-t_1)\boldsymbol{v})\cdot(\boldsymbol{v},\boldsymbol{v})d\theta. \qquad (3.9)
\end{aligned}$$

Now, as $\boldsymbol{v}$ and $\boldsymbol{u}$ are fixed, and given that J_ϵ is convex, it then follows that the right hand side terms in (3.9) tends to zero with $t_1 - t_2$. Thus (3.4) and (3.5) has a unique solution. □

Step 3: Some a priori estimates. In this paragraph, we would like to obtain some a priori estimates associated to the variational problem (3.4) and (3.5). Indeed we would like to have a priori estimates of the quantities $\|\boldsymbol{u}_\epsilon^n\|$, $\|\nabla \boldsymbol{u}_\epsilon^n\|$ and $\left\|\dfrac{\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^{n-1}}{k}\right\|$. By doing so, we also establish discrete a priori estimates, which mimic the energy law proved in Lemma 2.1. Note that not every numerical method possesses such a property. Hence, we claim that

Lemma 3.2. *Assume that $(\boldsymbol{f}, g)$ belong to $L^\infty\left(0,t_n;L^2(\Omega)^2\right)\times L^\infty(S)$, with the initial velocity $\boldsymbol{u}_0 \in H^1(\Omega)^2$, and satisfies (2.1). Then, the solution $\boldsymbol{u}_\epsilon^n$ enjoys the following a priori estimates*

$$\begin{aligned}
&\|\boldsymbol{u}_\epsilon^n\|^2 \le \frac{1}{(1+kC\nu)^n}\|\boldsymbol{u}_0\|^2 + C\|\boldsymbol{f}\|^2_{L^\infty(0,T;L^2(\Omega)^2)}[1-(1+kC\nu)^{-n}], \\
&\sum_{m=1}^n k\left\|\frac{\boldsymbol{u}_\epsilon^m - \boldsymbol{u}_\epsilon^{m-1}}{k}\right\|^2 + 2J_\epsilon(\boldsymbol{u}_\epsilon^n) + \nu\|\varepsilon(\boldsymbol{u}_\epsilon^n)\|^2 \qquad (3.10)\\
&\le C(\Omega,\nu)\|\boldsymbol{u}_0\|_1^2 + C\|g\|^2_{L^\infty(S)} + C\|g\|_{L^\infty(S)} + \|\boldsymbol{f}\|^2_{L^2(0,t_n;L^2(\Omega)^2)}.
\end{aligned}$$

Proof. We first recall that

$$2(\boldsymbol{u}-\boldsymbol{v},\boldsymbol{u}) = \|\boldsymbol{u}\|^2 - \|\boldsymbol{v}\|^2 + \|\boldsymbol{u}-\boldsymbol{v}\|^2 \text{ for all } \boldsymbol{v},\boldsymbol{u}\in L^2(\Omega)^2\ . \qquad (3.11)$$

Setting $\boldsymbol{v} = 2k\boldsymbol{u}_\epsilon^n$ in (3.5), and using (3.11), (2.7), we obtain

$$\|\boldsymbol{u}_\epsilon^n\|^2 + \|\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^{n-1}\|^2 + 2k\nu\|\varepsilon(\boldsymbol{u}_\epsilon^n)\|^2 \le \|\boldsymbol{u}_\epsilon^{n-1}\|^2 + 2k(\boldsymbol{f}^n,\boldsymbol{u}_\epsilon^n). \qquad (3.12)$$

Using the Cauchy-Schwarz inequality and Korn's inequality (2.19), we can dominate the right hand side of (3.12) as follows

$$2k(\boldsymbol{f}^n,\boldsymbol{u}_\epsilon^n) \le 2Ck\|\boldsymbol{f}^n\|\|\varepsilon(\boldsymbol{u}_\epsilon^n)\| \le \frac{Ck}{\nu}\|\boldsymbol{f}^n\|^2 + k\nu\|\varepsilon(\boldsymbol{u}_\epsilon^n)\|^2. \qquad (3.13)$$

Relations (3.12) and (3.13) give

$$\|\boldsymbol{u}_\epsilon^n\|^2 + \|\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^{n-1}\|^2 + k\nu\|\varepsilon(\boldsymbol{u}_\epsilon^n)\|^2 \le \|\boldsymbol{u}_\epsilon^{n-1}\|^2 + \frac{k\ C}{\nu}\|\boldsymbol{f}^n\|^2\,. \qquad (3.14)$$

Using again Korn's inequality (2.19), we obtain from (3.14)

$$\|\boldsymbol{u}_\epsilon^n\|^2 \le \frac{1}{1+kC\nu}\|\boldsymbol{u}_\epsilon^{n-1}\|^2 + \frac{k\ C}{\nu(1+kC\nu)}\|\boldsymbol{f}^n\|^2\,, \qquad (3.15)$$

which by induction over n, yields

$$\begin{aligned}\|\boldsymbol{u}_\epsilon^n\|^2 &\leq \frac{1}{(1+kC\nu)^n}\|\boldsymbol{u}_0\|^2 + \frac{k\,C}{\nu}\sum_{m=1}^{n}\frac{1}{(1+kC\nu)^m}\|\boldsymbol{f}^{n+1-m}\|^2\\ &\leq \frac{1}{(1+kC\nu)^n}\|\boldsymbol{u}_0\|^2 + C\|\boldsymbol{f}\|^2_{L^\infty(0,T;L^2(\Omega)^2)}[1-(1+kC\nu)^{-n}],\end{aligned}$$

which is the first inequality announced.

Next we set $\boldsymbol{v} = \dfrac{\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^{n-1}}{k}$. Then (3.5) becomes (with the aid of (3.11))

$$\begin{aligned}&\left\|\frac{\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^{n-1}}{k}\right\|^2 + \frac{\nu\,k}{2}\left\|\frac{\varepsilon(\boldsymbol{u}_\epsilon^n)-\varepsilon(\boldsymbol{u}_\epsilon^{n-1})}{k}\right\|^2 + \frac{\nu}{2k}\|\varepsilon(\boldsymbol{u}_\epsilon^n)\|^2\\ &= \frac{\nu}{2k}\|\varepsilon(\boldsymbol{u}_\epsilon^{n-1})\|^2 + \left\langle K_\epsilon(\boldsymbol{u}_\epsilon^n), \frac{\boldsymbol{u}_\epsilon^{n-1}-\boldsymbol{u}_\epsilon^n}{k}\right\rangle + \left(\boldsymbol{f}^n, \frac{\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^{n-1}}{k}\right).\end{aligned} \tag{3.16}$$

Since J_ϵ is convex and differentiable,

$$K_\epsilon(\boldsymbol{u}_\epsilon^n)\cdot(\boldsymbol{u}_\epsilon^{n-1}-\boldsymbol{u}_\epsilon^n) \leq J_\epsilon(\boldsymbol{u}_\epsilon^{n-1}) - J_\epsilon(\boldsymbol{u}_\epsilon^n).$$

Then, (3.16) becomes

$$\begin{aligned}&\left\|\frac{\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^{n-1}}{k}\right\|^2 + \frac{\nu\,k}{2}\left\|\frac{\varepsilon(\boldsymbol{u}_\epsilon^n)-\varepsilon(\boldsymbol{u}_\epsilon^{n-1})}{k}\right\|^2 + \frac{1}{k}J_\epsilon(\boldsymbol{u}_\epsilon^n) + \frac{\nu}{2k}\|\varepsilon(\boldsymbol{u}_\epsilon^n)\|^2\\ &= \frac{\nu}{2k}\|\varepsilon(\boldsymbol{u}_\epsilon^{n-1})\|^2 + \frac{1}{k}J_\epsilon(\boldsymbol{u}_\epsilon^{n-1}) + \left(\boldsymbol{f}^n, \frac{\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^{n-1}}{k}\right)\\ &\leq \frac{\nu}{2k}\|\varepsilon(\boldsymbol{u}_\epsilon^{n-1})\|^2 + \frac{1}{k}J_\epsilon(\boldsymbol{u}_\epsilon^{n-1}) + \|\boldsymbol{f}^n\|\left\|\frac{\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^{n-1}}{k}\right\|\\ &\leq \frac{\nu}{2k}\|\varepsilon(\boldsymbol{u}_\epsilon^{n-1})\|^2 + \frac{1}{k}J_\epsilon(\boldsymbol{u}_\epsilon^{n-1}) + \frac{1}{2}\|\boldsymbol{f}^n\|^2 + \frac{1}{2}\left\|\frac{\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^{n-1}}{k}\right\|\ .\end{aligned}$$

Thus

$$\begin{aligned}&\left\|\frac{\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^{n-1}}{k}\right\|^2 + \nu\,k\left\|\frac{\varepsilon(\boldsymbol{u}_\epsilon^n)-\varepsilon(\boldsymbol{u}_\epsilon^{n-1})}{k}\right\|^2 + \frac{2}{k}J_\epsilon(\boldsymbol{u}_\epsilon^n) + \frac{\nu}{k}\|\varepsilon(\boldsymbol{u}_\epsilon^n)\|^2\\ &\leq \frac{\nu}{k}\|\varepsilon(\boldsymbol{u}_\epsilon^{n-1})\|^2 + \frac{2}{k}J_\epsilon(\boldsymbol{u}_\epsilon^{n-1}) + \|\boldsymbol{f}^n\|^2\ .\end{aligned}$$

Summing this inequality over n, we obtain

$$\begin{aligned}&\sum_{m=1}^{n} k\left\|\frac{\boldsymbol{u}_\epsilon^m - \boldsymbol{u}_\epsilon^{m-1}}{k}\right\|^2 + \nu\,k^2\sum_{m=1}^{n}\left\|\frac{\varepsilon(\boldsymbol{u}_\epsilon^m)-\varepsilon(\boldsymbol{u}_\epsilon^{m-1})}{k}\right\|^2 + 2J_\epsilon(\boldsymbol{u}_\epsilon^n) + \nu\|\varepsilon(\boldsymbol{u}_\epsilon^n)\|^2\\ &\leq \nu\|\varepsilon(\boldsymbol{u}_0)\|^2 + 2J_\epsilon(\boldsymbol{u}_0) + k\sum_{m=1}^{n}\|\boldsymbol{f}^m\|^2\\ &\leq \nu\|\varepsilon(\boldsymbol{u}_0)\|^2 + 2\int_S g\sqrt{|\boldsymbol{u}_{0,\tau}|^2+1} + \int_0^{t_n}\|\boldsymbol{f}(s)\|^2 ds\\ &\leq C(\Omega,\nu)\|\boldsymbol{u}_0\|_1^2 + C\|g\|^2_{L^\infty(S)} + C\|g\|_{L^\infty(S)} + \|\boldsymbol{f}\|^2_{L^2(0,t_n;L^2(\Omega)^2)},\end{aligned}$$

which is the desired inequality after dropping some positive terms. □

Step 4: passage to the limit in ϵ. According to lemma 3.2, and the result of Rellich-Kondrachoff, we can select from the $(\boldsymbol{u}_\epsilon^n)_\epsilon$ a subsequence, denoted again by $(\boldsymbol{u}_\epsilon^n)_\epsilon$, such that

$$\boldsymbol{u}_\epsilon^n \to \boldsymbol{u}^n \quad \text{strongly in} \quad L^2(\Omega)^2 \quad \text{and} \quad \boldsymbol{u}_\epsilon^n \to \boldsymbol{u}^n \quad \text{weakly in} \quad H^1(\Omega)^2. \tag{3.17}$$

Since $Z(\Omega)$ is closed, and $(\boldsymbol{u}_\epsilon^n)_n \subset Z(\Omega)^{N+1}$, we $\boldsymbol{u}^n \in Z(\Omega)$. Next, from (3.5), and taking $\boldsymbol{v} \in Z(\Omega)$, there holds

$$\begin{aligned} &\frac{1}{k}(\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^{n-1}, \boldsymbol{v} - \boldsymbol{u}_\epsilon^n) + a(\boldsymbol{u}_\epsilon^n, \boldsymbol{v} - \boldsymbol{u}_\epsilon^n) + J_\epsilon(\boldsymbol{v}) - J_\epsilon(\boldsymbol{u}_\epsilon^n) - (\boldsymbol{f}^n, \boldsymbol{v} - \boldsymbol{u}_\epsilon^n) \\ &= \underbrace{J_\epsilon(\boldsymbol{v}) - J_\epsilon(\boldsymbol{u}_\epsilon^n) + \langle K_\epsilon(\boldsymbol{u}_\epsilon^n), \boldsymbol{v} - \boldsymbol{u}_\epsilon^n\rangle}_{\geq 0} . \end{aligned} \tag{3.18}$$

Next, using (3.17), and from the fact that J_ϵ is convex and lower semi-continuous, we have

$$\liminf_\epsilon J_\epsilon(\boldsymbol{u}_\epsilon^n) \geq J(\boldsymbol{u}^n).$$

But on the other hand

$$\lim_\epsilon J_\epsilon(\boldsymbol{v}) = J(\boldsymbol{v}).$$

Hence, returning to (3.18), we obtain

$$\begin{cases} \left(\dfrac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{k}, \boldsymbol{v} - \boldsymbol{u}^n\right) + a(\boldsymbol{u}^n, \boldsymbol{v} - \boldsymbol{u}^n) + J(\boldsymbol{v}) - J(\boldsymbol{u}^n) \geq (\boldsymbol{f}^n, \boldsymbol{v} - \boldsymbol{u}^n), \\ \text{for all} \quad \boldsymbol{v} \in Z(\Omega), \end{cases}$$

which ensures that (3.1) and (3.3) admits at least one solution $\boldsymbol{u}^n$ in $Z(\Omega)$. But from the property of the bilinear form $a(\cdot,\cdot)$ and the nonlinear functional J, it can be shown that there is only one solution. Thus we have constructed a solution $\boldsymbol{u}^n$ of (3.1) and (3.3) announced in Theorem 3.1. □

Let us indicate how the pressure is constructed from the velocity. First we use the regularized pressure obtained in (3.6), get some bounds uniform with respect to ϵ, and thirdly passed to the limit in ϵ using appropriate compactness results. Denoting by p^n the limit (in some sense) of p_ϵ^n when ϵ goes to zero, we finally check that the couple $(\boldsymbol{u}^n, p^n)$ is a solution of (3.2).

Finally, passing to the limit in lemma 3.2, one has the following result.

Corollary 3.1. *Under the hypotheses of lemma 3.2, the variational problem (3.1), (3.2) has a unique solution $(\boldsymbol{u}^n, p^n) \in \boldsymbol{V} \times L_0^2(\Omega)$, which moreover satisfies the bounds*

$$\|\boldsymbol{u}^n\|^2 \leq \frac{1}{(1+kC\nu)^n}\|\boldsymbol{u}_0\|^2 + C\|\boldsymbol{f}\|^2_{L^\infty(0,T;L^2(\Omega)^2)}[1-(1+kC\nu)^{-n}], \tag{3.19}$$

$$\begin{aligned} &\sum_{m=1}^n k\left\|\frac{\boldsymbol{u}^m - \boldsymbol{u}^{m-1}}{k}\right\|^2 + 2J(\boldsymbol{u}^n) + \nu\|\varepsilon(\boldsymbol{u}^n)\|^2 \\ &\leq C(\Omega,\nu)\|\boldsymbol{u}_0\|_1^2 + C\|g\|^2_{L^\infty(S)} + C\|g\|_{L^\infty(S)} + \|\boldsymbol{f}\|^2_{L^2(0,t_n;L^2(\Omega)^2)}. \end{aligned} \tag{3.20}$$

Remark 3.1. (a) *Of course using the energy method, it can be shown that (3.2) admits only one solution.*
(b) *The uniqueness of solution implies that the whole sequence $(\boldsymbol{u}_\epsilon^n)_\epsilon$ converges.*
(c) *The a priori estimates in corollary 3.1 should be viewed as discrete version of estimates obtained in theorem 2.1.*

4. Error Analysis

The goal in this section is to estimate the quantity $\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|$ using two different approaches. First, we use the solution of the regularized problems; that is we estimate first $\|\boldsymbol{u}_\epsilon(t_n) - \boldsymbol{u}_\epsilon^n\|$ and then pass to the limit. Next, we directly estimate $\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|$ based on the variational problems (2.2), (2.6) and (3.1),

(3.3). It is observed that the rate of convergence for both methods is of order one. Hence for convergence analysis of the time discretization of (2.2) and (2.3), there is no apparent advantage as long as the regularized parameter is very small. We first introduce some notations. Starting with the regularized problems, the error equation can be obtained by subtracting (3.5) from (2.5) at time $t = t_n$. Thus the sequence $(\boldsymbol{e}_\epsilon^n)_{0\le n\le N}$ defined by $\boldsymbol{e}_\epsilon^n = \boldsymbol{u}_\epsilon(t_n) - \boldsymbol{u}_\epsilon^n$, where $\boldsymbol{u}_\epsilon(t_n)$ is the regularized solution obtained from (2.5) at $t = t_n$, and $\boldsymbol{u}_\epsilon^n$ is the numerical solution defined by (3.5), satisfies:

$$
\begin{cases}
\boldsymbol{e}_\epsilon^0 = 0, \text{ for } 1 \le n \le N, \\
\text{and for all } \boldsymbol{v} \in Z(\Omega), \\
\left(\dfrac{\boldsymbol{e}_\epsilon^n - \boldsymbol{e}_\epsilon^{n-1}}{k}, \boldsymbol{v}\right) + a(\boldsymbol{e}_\epsilon^n, \boldsymbol{v}) + \langle K_\epsilon(\boldsymbol{u}_\epsilon(t_n)) - K_\epsilon(\boldsymbol{u}_\epsilon^n), \boldsymbol{v}\rangle \\
= \left(\dfrac{\boldsymbol{u}_\epsilon(t_n) - \boldsymbol{u}_\epsilon(t_{n-1})}{k} - (\partial_t \boldsymbol{u}_\epsilon)(t_n), \boldsymbol{v}\right) + (\boldsymbol{f}(t_n) - \boldsymbol{f}^n, \boldsymbol{v}) \ .
\end{cases}
\tag{4.1}
$$

Remark 4.1. *The quantity* $\varepsilon = \left(\dfrac{\boldsymbol{u}_\epsilon(t_n) - \boldsymbol{u}_\epsilon(t_{n-1})}{k} - (\partial_t \boldsymbol{u}_\epsilon)(t_n), \boldsymbol{v}\right) + (\boldsymbol{f}(t_n) - \boldsymbol{f}^n, \boldsymbol{v})$ *may be regarded as the consistency error.*

We then claim that

Proposition 4.1. *Assume that the solution* $\boldsymbol{u}_\epsilon$ *of problem (2.5) belongs to* $H^2(0,T;L^2(\Omega)^2)$*, with* $\boldsymbol{f} \in H^1(0,T;L^2(\Omega)^2)$*. Then the following a priori error estimate holds for* $1 \le n \le N$*,*

$$
\|\boldsymbol{u}_\epsilon(t_n) - \boldsymbol{u}_\epsilon^n\| \le Ck\left[\|\boldsymbol{u}_\epsilon\|_{H^2(t_{n-1},t_n;L^2(\Omega)^2)} + \|\boldsymbol{f}\|_{H^1(t_{n-1},t_n;L^2(\Omega)^2)}\right] \ .
$$

Proof. We take $\boldsymbol{v} = 2k\boldsymbol{e}_\epsilon^n = 2k(\boldsymbol{u}_\epsilon(t_n) - \boldsymbol{u}_\epsilon^n)$ in (4.1). This gives

$$
\begin{aligned}
&2(\boldsymbol{e}_\epsilon^n - \boldsymbol{e}_\epsilon^{n-1}, \boldsymbol{e}_\epsilon^n) + 2ka(\boldsymbol{e}_\epsilon^n, \boldsymbol{e}_\epsilon^n) + 2k\underbrace{\langle K_\epsilon(\boldsymbol{u}_\epsilon(t_n)) - K_\epsilon(\boldsymbol{u}_\epsilon^n), \boldsymbol{u}_\epsilon(t_n) - \boldsymbol{u}_\epsilon^n\rangle}_{\ge 0} \\
&\quad = 2k\left(\frac{\boldsymbol{u}_\epsilon(t_n) - \boldsymbol{u}_\epsilon(t_{n-1})}{k} - (\partial_t \boldsymbol{u}_\epsilon)(t_n), \boldsymbol{e}_\epsilon^n\right) + 2k(\boldsymbol{f}(t_n) - \boldsymbol{f}^n, \boldsymbol{e}_\epsilon^n).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\|\boldsymbol{e}_\epsilon^n\|^2 - \|\boldsymbol{e}_\epsilon^{n-1}\|^2 + \|\boldsymbol{e}_\epsilon^n - \boldsymbol{e}_\epsilon^{n-1}\|^2 + 2k\nu\|\varepsilon(\boldsymbol{e}_\epsilon^n)\|^2 \\
&\le 2k\left\|\frac{\boldsymbol{u}_\epsilon(t_n) - \boldsymbol{u}_\epsilon(t_{n-1})}{k} - (\partial_t \boldsymbol{u}_\epsilon)(t_n)\right\| \|\boldsymbol{e}_\epsilon^n\| + 2k\|\boldsymbol{f}(t_n) - \boldsymbol{f}^n\|\|\boldsymbol{e}_\epsilon^n\|.
\end{aligned}
\tag{4.2}
$$

Using Taylor's expansion:

$$
\boldsymbol{u}_\epsilon(t_n) - \boldsymbol{u}_\epsilon(t_{n-1}) = k(\partial_t \boldsymbol{u}_\epsilon)(t_n) - \int_{t_{n-1}}^{t_n} (t - t_{n-1})(\partial_{tt}^2 \boldsymbol{u}_\epsilon)(t)dt,
$$

so

$$
\frac{\boldsymbol{u}_\epsilon(t_n) - \boldsymbol{u}_\epsilon(t_{n-1})}{k} - (\partial_t \boldsymbol{u}_\epsilon)(t_n) = -\frac{1}{k}\int_{t_{n-1}}^{t_n} (t - t_{n-1})(\partial_{tt}^2 \boldsymbol{u}_\epsilon)(t)dt\,.
$$

From Cauchy-Schwarz inequality in the former equation, one has

$$
\left\|\frac{\boldsymbol{u}_\epsilon(t_n) - \boldsymbol{u}_\epsilon(t_{n-1})}{k} - (\partial_t \boldsymbol{u}_\epsilon)(t_n)\right\| \le Ck^{1/2}\|\boldsymbol{u}_\epsilon\|_{H^2(t_{n-1},t_n;L^2(\Omega)^2)}.
\tag{4.3}
$$

Next, by the mean value theorem,

$$\boldsymbol{f}(t_n) - \boldsymbol{f}^n = -\frac{1}{k}\int_{t_{n-1}}^{t_n} (t - t_n)\boldsymbol{f}_t(d_n)dt.$$

Again, the Cauchy-Schwarz inequality in this formula gives

$$\|\boldsymbol{f}(t_n) - \boldsymbol{f}^n\| \leq Ck^{1/2}\|\boldsymbol{f}\|_{H^1(t_{n-1},t_n;L^2(\Omega)^2)}. \tag{4.4}$$

Returning to (4.2) with (4.3) and (4.4), one gets

$$\begin{aligned} &\|\boldsymbol{e}_\epsilon^n\|^2 + \|\boldsymbol{e}_\epsilon^n - \boldsymbol{e}_\epsilon^{n-1}\|^2 + 2k\nu\|\varepsilon(\boldsymbol{e}_\epsilon^n)\|^2 \\ &\leq \|\boldsymbol{e}_\epsilon^{n-1}\|^2 + Ck^{3/2}\left[\|\boldsymbol{u}_\epsilon\|_{H^2(t_{n-1},t_n;L^2(\Omega)^2)} + \|\boldsymbol{f}\|_{H^1(t_{n-1},t_n;L^2(\Omega)^2)}\right]\|\boldsymbol{e}_\epsilon^n\|. \end{aligned} \tag{4.5}$$

Using (2.19), (2.20), and (4.5) gives (after utilization of Young's inequality and dropping some positive terms)

$$\begin{aligned} \|\boldsymbol{e}_\epsilon^n\|^2 \quad \leq \quad & \frac{1}{1+Ck\nu}\|\boldsymbol{e}_\epsilon^{n-1}\|^2 \frac{Ck^3}{1+Ck\nu}[\|\boldsymbol{u}_\epsilon\|^2_{H^2(t_{n-1},t_n;L^2(\Omega)^2)} \\ & +\|\boldsymbol{f}\|^2_{H^1(t_{n-1},t_n;L^2(\Omega)^2)}]. \end{aligned} \tag{4.6}$$

Using (4.6) recursively, we find (recall that $\boldsymbol{e}_\epsilon^0 = 0$)

$$\begin{aligned} \|\boldsymbol{e}_\epsilon^n\|^2 \quad \leq \quad & \frac{1}{(1+C\,k)^n}\|\boldsymbol{e}_\epsilon^0\|^2 \\ & +Ck^3\left[\|\boldsymbol{u}_\epsilon\|^2_{H^2(t_{n-1},t_n;L^2(\Omega)^2)} + \|\boldsymbol{f}\|^2_{H^1(t_{n-1},t_n;L^2(\Omega)^2)}\right]\sum_{m=1}^{n}\frac{1}{(1+C\,k)^m} \\ \leq \quad & Ck^2\left[\|\boldsymbol{u}_\epsilon\|^2_{H^2(t_{n-1},t_n;L^2(\Omega)^2)} + \|\boldsymbol{f}\|^2_{H^1(t_{n-1},t_n;L^2(\Omega)^2)}\right]\left[1-(1+C\,k)^{-n}\right] \ , \end{aligned}$$

which is the result announced in proposition 4.1 . □

Corollary 4.1. *Under the assumptions of proposition 4.1, assume that*

(a) $\boldsymbol{u}(t_n)$ *is the weak solution at* t_n *defined by the variational problem (2.2) and (2.6).*
(b) $\boldsymbol{u}^n$ *is the weak solution defined by (3.1), and (3.3).*

Then the following a priori error estimates hold for $1 \leq n \leq N$

$$\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\| \leq Ck\left[\liminf_\epsilon \|\boldsymbol{u}_\epsilon\|_{H^2(t_{n-1},t_n;L^2(\Omega)^2)} + \|\boldsymbol{f}\|_{H^1(t_{n-1},t_n;L^2(\Omega)^2)}\right].$$

Proof. It suffices to pass to the limit as $\epsilon \to 0$ in Proposition 4.1.We recall that

$$\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\| \leq \liminf_\epsilon \|\boldsymbol{u}_\epsilon(t_n) - \boldsymbol{u}_\epsilon^n\|.$$

□

Next, we estimate the error $\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|$ based on the variational problems (2.2), (2.6), (3.1), and (3.3)

Proposition 4.2. *Let* $\boldsymbol{u}^n$ *be the solution of (3.3). If the solution* $\boldsymbol{u}$ *of problem (2.2) belongs to* $H^2(0,T;L^2(\Omega)^2)$*, with* $\boldsymbol{f} \in H^1(0,T;L^2(\Omega)^2)$*, then, the following a priori error estimate holds for* $1 \leq n \leq N$*,*

$$\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\| \leq Ck\left[\|\boldsymbol{u}\|_{H^2(t_{n-1},t_n;L^2(\Omega)^2)} + \|\boldsymbol{f}\|_{H^1(t_{n-1},t_n;L^2(\Omega)^2)}\right].$$

Proof. We consider the equation (2.2) when $t = t_n$, and we take $\boldsymbol{v} = \boldsymbol{u}^n$, thus
$(\partial_t(\boldsymbol{u}(t_n)), \boldsymbol{u}^n - \boldsymbol{u}(t_n)) + a(\boldsymbol{u}(t_n), \boldsymbol{u}^n - \boldsymbol{u}(t_n)) + J(\boldsymbol{u}^n) - J(\boldsymbol{u}(t_n)) \geq (\boldsymbol{f}(t_n), \boldsymbol{u}^n - \boldsymbol{u}(t_n))$.
Secondly, consider (3.3) with $\boldsymbol{v} = \boldsymbol{u}(t_n)$, then

$$\begin{aligned}&\left(-\frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{k}, \boldsymbol{u}^n - \boldsymbol{u}(t_n)\right)\\ &+a(-\boldsymbol{u}^n, \boldsymbol{u}^n - \boldsymbol{u}(t_n)) + J(\boldsymbol{u}(t_n)) - J(\boldsymbol{u}^n) \geq (-\boldsymbol{f}^n, \boldsymbol{u}^n - \boldsymbol{u}(t_n))\,.\end{aligned}$$

Putting together, the former and later inequalities, one gets

$$\begin{aligned}\left(\frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{k} - \partial_t(\boldsymbol{u}(t_n)), \boldsymbol{u}^n - \boldsymbol{u}(t_n)\right) \quad &+ \quad a(\boldsymbol{u}^n - \boldsymbol{u}(t_n), \boldsymbol{u}^n - \boldsymbol{u}(t_n))\\ &\leq \quad (\boldsymbol{f}^n - \boldsymbol{f}(t_n), \boldsymbol{u}^n - \boldsymbol{u}(t_n)),\end{aligned}$$

which can be re-written as (for $\boldsymbol{e}^n = \boldsymbol{u}^n - \boldsymbol{u}(t_n)$);

$$\begin{aligned}&\left(\frac{\boldsymbol{e}^n - \boldsymbol{e}^{n-1}}{k}, \boldsymbol{e}^n\right) + a(\boldsymbol{e}^n, \boldsymbol{e}^n)\\ &\leq -\left(\frac{\boldsymbol{u}(t_n) - \boldsymbol{u}(t_{n-1})}{k} - \partial_t(\boldsymbol{u}(t_n)), \boldsymbol{e}^n\right) + (\boldsymbol{f}^n - \boldsymbol{f}(t_n), \boldsymbol{e}^n)\\ &\leq \left\|\frac{\boldsymbol{u}(t_n) - \boldsymbol{u}(t_{n-1})}{k} - \partial_t(\boldsymbol{u}(t_n))\right\| \|\boldsymbol{e}^n\| + \|\boldsymbol{f}^n - \boldsymbol{f}(t_n)\|\|\boldsymbol{e}^n\|\\ &\leq \underbrace{Ck^{1/2}\|\boldsymbol{u}\|_{H^2(t_{n-1},t_n;L^2(\Omega)^2)}}_{\text{from (4.3)}} \|\boldsymbol{e}^n\| + \underbrace{Ck^{1/2}\|\boldsymbol{f}\|_{H^1(t_{n-1},t_n;L^2(\Omega)^2)}}_{\text{from (4.4)}} \|\boldsymbol{e}^n\|\,. \qquad (4.7)\end{aligned}$$

For the left hand side of (4.7), we find

$$\begin{aligned}&\left(\frac{\boldsymbol{e}^n - \boldsymbol{e}^{n-1}}{k}, \boldsymbol{e}^n\right) + a(\boldsymbol{e}^n, \boldsymbol{e}^n)\\ = \quad &\frac{1}{2k}\left[\|\boldsymbol{e}^n\|^2 - \|\boldsymbol{e}^{n-1}\|^2 + \|\boldsymbol{e}^n - \boldsymbol{e}^{n-1}\|^2\right] + 2\nu\|\nabla \boldsymbol{e}^n\|^2\\ \geq \quad &\frac{1}{2k}\left[\|\boldsymbol{e}^n\|^2 - \|\boldsymbol{e}^{n-1}\|^2 + \|\boldsymbol{e}^n - \boldsymbol{e}^{n-1}\|^2\right] + 2C\nu\|\boldsymbol{e}^n\|^2,\end{aligned}$$

which together with (4.7) gives in particular

$$\begin{aligned}&\|\boldsymbol{e}^n\|^2 + Ck\nu\|\boldsymbol{e}^n\|^2\\ \leq \quad &\|\boldsymbol{e}^{n-1}\|^2 + Ck^{3/2}[\|\boldsymbol{u}\|_{H^2(t_{n-1},t_n;L^2(\Omega)^2)} + \|\boldsymbol{f}\|_{H^1(t_{n-1},t_n;L^2(\Omega)^2)}]\|\boldsymbol{e}^n\|.\end{aligned}$$

We then proceed as in (4.5), and the proof is completed. □

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