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AN OPTIMAL UNIFORM A PRIORI ERROR ESTIMATE FOR AN UNSTEADY SINGULARLY PERTURBED PROBLEM

MILOSLAV VLASAK AND HANS–GÖRG ROOS

Abstract. A time–dependent convection–diffusion problem is discretized by the Galerkin finite element method in space with bilinear elements on a general layer adapted mesh and in time by discontinuous Galerkin method. We present optimal error estimates. The estimates hold true for consistent stabilization too.

Key words. discontinuous Galerkin, convection–diffusion, layer adapted mesh, error estimate

1. Introduction

We focus ourselves on the analysis of the solution of unsteady linear 2D singularly perturbed convection–diffusion equation. This type of equation can be considered as simplified model problem to many important problems, especially to Navier–Stokes equations.

The space discretization of such a problem is a difficult task and it stimulated development of many stabilization methods (e.g. streamline upwind Petrov–Galerkin (SUPG) method, local projection stabilization methods) and layer–adapting techniques (e.g. Shishkin meshes, Bakhvalov meshes). For the overview see [9] or [8].

In order to achieve optimal diffusion–uniform error estimates we employ layer adapted meshes. On these general layer adapted meshes we assume a general space discretization covering standard conforming finite element method (FEM) or consistent stabilization methods. The resulting system of ordinary differential equations is solved by discontinuous Galerkin (DG) method.

Considering the space discretization on Shishkin meshes, we will follow the theory for stationary singularly perturbed problems based on the solution decomposition, which enables us to derive a priori error estimates independent of the diffusion parameter even with respect to the norms (seminorms) of the exact solution, which can be also highly dependent on the diffusion parameter. For the details see [9].

The discontinuous Galerkin (DG) method is a very popular approach for solving ordinary differential equations arising from space discretization of parabolic problems, which is based on piecewise polynomial approximation in time. Among important advantages we should mention unconditional stability for arbitrary order, which allows us to solve stiff problems efficiently, and good smoothing property, which enables us to work with inexact or rough data. For introduction to DG time discretization see e.g. [11].

In [6] and [1] the authors study DG in time and DG and local projection stabilization method, respectively, in space on standard meshes for singularly perturbed problems. The error estimates in these papers contain norms of the exact solutions which go to infinity if diffusion parameter goes to zero.

There are only few papers dealing with finite elements in space on the special meshes combined with any discretization in time. While in [7] the θ–scheme as discretization in time is used, in [5] the authors study BDF time discretization.

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In [7] the authors also study DG time discretization and derive suboptimal error estimates.

Our aim is improving some results from [7] and proving optimal a priori diffusion–uniform error estimates for DG time discretization in $L^\infty(L^2)$ norm.

The main difficulty in proving optimal diffusion–uniform error estimates for DG time discretization is the fact that we cannot employ standard technique of the proof, which is based on the construction of a suitable projection, which enables us to eliminate discrete time derivative in the error equation, see e.g. [10]. This technique enforces us to do some upper bound of the projection error contained in stationary terms, which depends on a higher time derivative of the exact solution in H^1 seminorm, which depends on the diffusion parameter.

2. Continuous problem

Let $\Omega = (0,1)^2$ be a computational domain and $T > 0$. Then let us consider parabolic singularly perturbed problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \varepsilon \Delta u + b \cdot \nabla u + cu &= f, \quad \forall x \in \Omega, t \in (0,T), \\ u &= 0, \quad \forall x \in \partial\Omega, t \in (0,T), \\ u(x,0) &= u^0(x), \quad \forall x \in \Omega, \end{aligned} \tag{1}$$

where function $u^0 \in L^2(\Omega)$, $0 < \varepsilon << 1$ and functions $f(x,t)$, $b(x)$ and $c(x)$ are sufficiently smooth with $b_1(x) > \beta_1 > 0$ and $b_2(x) > \beta_2 > 0$. By substitution in time variable we can achieve

$$c - \frac{1}{2}\nabla \cdot b \geq c_0 > 0. \tag{2}$$

To simplify the text we will use the following notation. $(.,.)$ and $\|.\|$ are $L^2(\Omega)$ scalar product and norm, $|.|_1$ and $\|.\|_1$ are $H^1(\Omega)$ seminorm and norm. Let us define bilinear form

$$a(u,v) = \varepsilon(\nabla u, \nabla v) + (b \cdot \nabla u + cu, v). \tag{3}$$

Definition 1. *We say that the function $u \in L^2(0,T,H_0^1(\Omega))$ with the time derivative $\frac{\partial u}{\partial t} \in L^2(0,T,H^{-1}(\Omega))$ is the weak solution of (1), if the following conditions are satisfied*

$$\begin{aligned} \left(\frac{\partial u(t)}{\partial t}, v\right) + a(u(t),v) &= (f(t),v) \quad \forall t \in (0,T), \forall v \in H_0^1(\Omega), \\ u(0) &= u^0. \end{aligned} \tag{4}$$

It is possible to show that the solution has in general boundary layer around the border of Ω at $x = 1$ and $y = 1$. Assuming sufficiently compatible data we can avoid the existence of interior layers, which enables us to concentrate on the boundary layers only, see [9] or [4]. Moreover, it is possible to guarantee the S–decomposition of the solution: $u = S + V_1 + V_2 + V_{12}$, where

$$\left|\frac{\partial^{i+j+k} S(x_1,x_2,t)}{\partial x_1^i \partial x_2^j \partial t^k}\right| \leq C, \tag{5}$$

$$\left|\frac{\partial^{i+j+k} V_1(x_1,x_2,t)}{\partial x_1^i \partial x_2^j \partial t^k}\right| \leq C\varepsilon^{-i} e^{-\beta_1(1-x_1)/\varepsilon}, \tag{6}$$

$$\left|\frac{\partial^{i+j+k} V_2(x_1,x_2,t)}{\partial x_1^i \partial x_2^j \partial t^k}\right| \leq C\varepsilon^{-j} e^{-\beta_2(1-x_2)/\varepsilon}, \tag{7}$$

$$\left|\frac{\partial^{i+j+k} V_{12}(x_1,x_2,t)}{\partial x_1^i \partial x_2^j \partial t^k}\right| \leq C\varepsilon^{-i-j} \min\{e^{-\beta_1(1-x_1)/\varepsilon}, e^{-\beta_2(1-x_2)/\varepsilon}\}, \tag{8}$$

where i, j, k are nonnegative integers such that $i + j \leq 3$ and $k \leq q + 2$, where q denotes the degree of the intended polynomial approximation in time. S represents the smooth part of the solution, V_1 and V_2 represent boundary layers and V_{12} represents the corner layer. This result shows dependence of space derivatives on ε, which complicates deriving standard a priori error estimates.

2.1. Discretization. We want to discretize the problem (1) by either standard finite element method or some consistent stabilization method on general layer adapted meshes in space. This technique allows us to derive a priori error estimates that are independent of ε.

We will start with the construction of the general layer adapted mesh. To do this we will follow the approach described in [8] or [9]. Let us denote N, space mesh parameter, as an even number. Then let us set

$$0 = x_0 < x_1 < \ldots < x_N = 1, \quad 0 = y_0 < y_1 < \ldots < y_N = 1. \tag{9}$$

The final mesh arises as tensor product mesh with mesh points (x_i, y_j). Since the idea of distribution of mesh points is the same in both direction (using either parameter β_1 or β_2), we describe the idea only in x_1 direction. Let us introduce the mesh generating function ϕ satisfying $\phi(0) = 0$ and $\phi(1/2) = \ln(N)$, moreover we assume ϕ be continuous, increasing and differentiable. Let the mesh points are equally distributed in $[0, x_{N/2}]$ and graded according to the function ϕ in $[x_{N/2}, 1]$:

$$x_i = \frac{2i}{N}\left(1 - \frac{\sigma\varepsilon}{\beta_1}\phi\left(\frac{1}{2}\right)\right), \quad \forall i = 0, \ldots, N/2 \tag{10}$$

$$x_i = 1 - \frac{\sigma\varepsilon}{\beta_1}\phi\left(\frac{N-i}{N}\right), \quad \forall i = N/2, \ldots, N. \tag{11}$$

The parameter σ is chosen to satisfy $\sigma \geq 5/2$. These meshes can be called S–type meshes. For instance, the special choice of the function $\phi(s) = 2\ln(N)s$ leads to classical Shishikin mesh and the choice $\phi(s) = -\ln(1 - 2s(1 - N^{-1}))$ leads to Bakhvalov–type meshes.

Let us define the conforming bilinear finite element space V_N on our mesh. We denote $a_{st}(.,.)$ the space discretization bilinear form and f_{st} the corresponding right–hand side. In the case of classical finite element method the form $a_{st}(.,.)$ and the right–hand side f_{st} are identical to former bilinear form $a(.,.)$ and former right–hand side f, but they can differ in the case of stabilization methods. Moreover, we assume that the new bilinear form is consistent, i.e., the exact solution u satisfies

$$\left(\frac{\partial u}{\partial t}, v\right) + a_{st}(u, v) = (f_{st}, v), \quad \forall v \in V_N. \tag{12}$$

The semi–discrete problem reads: find $u_N \in C^1(0,T,V_N)$ satisfying

$$\begin{aligned} \left(\frac{\partial u_N(t)}{\partial t}, v\right) + a_{st}(u_N(t), v) &= (f_{st}(t), v), \quad \forall v \in V_N, \forall t \in (0,T), \\ (u_N(0), v) &= (u^0, v). \quad \forall v \in V_N \end{aligned} \tag{13}$$

To discretize this problem in time we assume time partition $0 = t_0 < t_1 < \ldots < t_r = T$ with time intervals $I_m = (t_{m-1}, t_m)$, time steps $\tau_m = |I_m| = t_m - t_{m-1}$ and $\tau = \max_{m=1,\ldots,r} \tau_m$. We denote the function values at the nodes as $v^m = v(t_m)$. To be able to use the Galerkin type of discretization we denote the space of piecewise polynomial functions

$$V_N^\tau = \{v \in L^2(0,T,V_N) : v|_{I_m} = \sum_{j=0}^{q} v_{j,m} t^j, \ v_{j,m} \in V_N\}. \tag{14}$$

For the functions from such a space we need to define the values at the nodes of time partition

$$v_\pm^m = v(t_m \pm) = \lim_{t \to t_m \pm} v(t) \tag{15}$$

and the jumps

$$\{v\}_m = v_+^m - v_-^m. \tag{16}$$

Definition 2. *We say that the function $U \in V_N^\tau$ is the approximate solution to the problem (1) if*

$$\begin{aligned} \int_{I_m} (U', v) + a_{st}(U, v) dt + (\{U\}_{m-1}, v_+^{m-1}) &= \int_{I_m} (f_{st}, v) dt, \\ &\quad \forall v \in V_N^\tau, \ \forall m = 1, \ldots, r \\ (U_-^0, v) &= (u^0, v) \quad \forall v \in V_N. \end{aligned} \tag{17}$$

3. Error analysis

We define energy norm

$$|||v|||^2 = a_{st}(v,v), \quad \forall v \in H^1(\Omega). \tag{18}$$

3.1. Stationary problem. In this part we want to go through some well known results for the singularly perturbed problems (for the details see [9]). Let us assume related stationary problem

$$a_{st}(u,v) = (f_{st}^*, v), \quad \forall v \in H_0^1(\Omega), \tag{19}$$

with some $f_{st}^* \in L^2(\Omega)$, and corresponding discrete finite element problem on layer–adapted mesh. Let us define the Ritz projection $R : H_0^1(\Omega) \to V_N$ satisfying

$$a_{st}(u - Ru, v) = 0, \quad \forall v \in V_N. \tag{20}$$

We assume that on layer–adapted mesh following error estimates hold true:

$$|||u - Ru||| \le C g_1(N), \tag{21}$$

$$\|u - Ru\| \le C g_2(N), \tag{22}$$

$$\|u' - Ru'\| \le C g_2(N), \tag{23}$$

with C independent of ε. In the case of classical finite element method on Shishkin mesh we obtain these results with $g_1(N) = N^{-1}\ln(N)$ and $g_2(N) = (N^{-1}\ln(N))^2$. The same situation with Bakhvalov mesh leads to the estimates $g_1(N) = N^{-1}$ and $g_2(N) = N^{-2}$. Remark that the estimates in L^2–norm are based on supercloseness results, because it is not possible to use the Nitsche–duality trick. See [9] for more detailed informations. From this follows easily

Lemma 1. *Let u be the exact solution of (1). Then*

$$\|Ru(s_1) - u(s_1) - Ru(s_2) + u(s_2)\| \leq C|s_1 - s_2|g_2(N). \tag{24}$$

Proof.

$$\begin{aligned} \|Ru(s_1) - u(s_1) - Ru(s_2) + u(s_2)\| &= \|\int_{s_2}^{s_1} Ru'(t) - u'(t)dt\| \\ &\leq |s_1 - s_2| \sup_{I_m} \|Ru' - u'\| \\ &\leq C|s_1 - s_2|g_2(N) \end{aligned} \tag{25}$$

□

3.2. Radau quadrature. Let us define Radau quadrature on each interval I_m

$$\int_{I_m} f\,dt \approx Q[f] = \sum_{i=0}^{q} w_i f(t_{m,i}), \tag{26}$$

where $t_{m,i}$ are Radau quadrature nodes in I_m with $t_{m,0} = t_m$. Such a quadrature has algebraic order $2q$ and the coefficients of the quadrature satisfy $0 \leq w_i \leq \tau_m$ and

$$\sum_{i=0}^{q} w_i = \tau_m. \tag{27}$$

Let us assume for simplicity that right–hand side f (and therefore f_{st}) of our continuous problem (1) is polynomial up to the degree q. Otherwise, we will need to use additionally error estimate of following type

$$\int_{I_m} (f, v)dt - Q[(f, v)] \leq \tau_m C \tau^{q+1} \sup_{I_m} \|v\|, \quad \forall v \in V_N^\tau, \tag{28}$$

which holds true for f sufficiently smooth in time. Then it is possible to express our method (17) by

$$Q[(U', v)] + Q[a_{st}(U, v)] + (\{U\}_{m-1}, v_+^{m-1}) = Q[(f_{st}, v)], \quad \forall v \in V_N^\tau. \tag{29}$$

Since the equation for continuous solution (1) is defined at every point $t \in I_m$, we can see that the exact solution satisfy (29) too.

3.3. Projections. Let us set the space

$$V^\tau = \{v \in L^2(0, T, H_0^1(\Omega)) : v|_{I_m} = \sum_{j=0}^{q} v_{j,m} t^j,\ v_{j,m} \in H_0^1(\Omega)\}. \tag{30}$$

We define time projection $P : C([0, T], H_0^1(\Omega)) \to V^\tau$, such that

$$Pu(t) = \sum_{i=0}^{q} \ell_i(t) u(t_{m,i}), \tag{31}$$

where ℓ_i is Lagrange interpolation basis function for the quadrature node $t_{m,i}$. Since

$$RPu(t) = R \sum_{i=0}^{q} \ell_i(t) u(t_{m,i}) = \sum_{i=0}^{q} \ell_i(t) Ru(t_{m,i}) = PRu(t), \tag{32}$$

we can see that projections P and R commute. We define the space–time projection $\pi = PR : C(0, T, H_0^1(\Omega)) \to V_N^\tau$.

Now, we present some basic approximation properties of our projections P and π.

Lemma 2. *Let u be the exact solution of (1). Then*

$$\sup_{I_m} \|Pu - u\| \leq C\tau^{q+1}, \tag{33}$$

$$\sup_{I_m} \|Pu' - u'\| \leq C\tau^{q+1}, \tag{34}$$

where the constant C does not depend on τ.

Proof. The proof can be made by standard arguments. It is an analogy to e.g. [3, Theorem 3.1.4] in Bochner spaces. □

Lemma 3. *Let u be the exact solution of (1). Then*

$$\sup_{I_m} \|\pi u - u\| \leq C(\tau^{q+1} + g_2(N)), \tag{35}$$

where the constant C does not depend on τ or N.

Proof. Since $|\ell_i(t)| \leq C$, where the constant C depends only on q, we obtain

$$\begin{aligned} \sup_{I_m} \|\pi u - u\| &\leq \sup_{I_m} \|Pu - u\| + \sup_{I_m} \|PRu - Pu\| \\ &\leq C\tau^{q+1} + C\|\sum_{i=0}^{q} Ru(t_{m,i}) - u(t_{m,i})\| \\ &\leq C\tau^{q+1} + C(q+1) \max_{i=0,\dots,q} \|Ru(t_{m,i}) - u(t_{m,i})\| \\ &\leq C(\tau^{q+1} + g_2(N)). \end{aligned} \tag{36}$$

□

3.4. Auxiliary result. We subtract the equation for exact solution from (29) and divide the error into projection part $\eta = \pi u - u$ and $\xi = U - \pi u \in V_N^\tau$. We obtain

$$\begin{aligned} &\int_{I_m} (\xi', v) + a_{st}(\xi, v)dt + (\{\xi\}_{m-1}, v_+^{m-1}) \\ &= -Q[(\eta', v)] - (\{\eta\}_{m-1}, v_+^{m-1}) - Q[a_{st}(\eta, v)]. \end{aligned} \tag{37}$$

Since

$$Q[a_{st}(\eta, v)] = \sum_{i=0}^{q} w_i a_{st}(Ru(t_{m,i}) - u(t_{m,i}), v) = 0 \tag{38}$$

we need to estimate the rest of the right–hand side only.

Lemma 4. *Let u be an exact solution of (1). Then*

$$Q[(\eta', v)] + (\{\eta\}_{m-1}, v_+^{m-1}) \leq \tau_m C\left(\tau^{q+1} + g_2(N)\right) \sup_{I_m} \|v\|, \quad \forall v \in V_N^\tau. \tag{39}$$

Proof.

$$\begin{aligned} &Q[(\eta', v)] + (\{\eta\}_{m-1}, v_+^{m-1}) \\ &= \int_{I_m} ((\pi u)', v)dt - Q[(u', v)] + (\{\eta\}_{m-1}, v_+^{m-1}) \\ &= \int_{I_m} (\eta', v)dt + (\{\eta\}_{m-1}, v_+^{m-1}) + \int_{I_m} (u', v)dt - Q[(u', v)] \end{aligned} \tag{40}$$

We estimate first two terms and last two terms (quadrature error) individually.

$$\int_{I_m} (\eta', v)dt + (\{\eta\}_{m-1}, v_+^{m-1}) = -\int_{I_m} (\eta, v') + (\eta_-^m, v_-^m) - (\eta_-^{m-1}, v_+^{m-1}) \tag{41}$$

We can see that $v_-^m = v_+^{m-1} + \int_{I_m} v' dt$. Using this fact we obtain

$$-\int_{I_m} (\eta, v') + (\eta_-^m, v_-^m) - (\eta_-^{m-1}, v_+^{m-1}) = \int_{I_m} (\eta_-^m - \eta, v')dt + (\eta_-^m - \eta_-^{m-1}, v_+^{m-1}) \tag{42}$$

We estimate these terms individually. The first term we can rewrite in the following way

$$\int_{I_m} (\eta_-^m - \eta, v')dt = \int_{I_m} (Ru^m - u^m - RPu + u, v')dt = \int_{I_m} (Ru^m - u^m - RPu + Pu, v')dt + \int_{I_m} (u - Pu, v')dt \tag{43}$$

Since all the terms in the first integral on the right–hand side are polynomials we can apply Radau quadrature exactly and using (27), Lemma 1 and inverse inequality we get

$$\begin{aligned} &\int_{I_m} (Ru^m - u^m - RPu + Pu, v')dt \\ &= Q[(Ru^m - u^m - RPu + Pu, v')] \\ &\le \tau_m \sup_i \|Ru^m - u^m - Ru(t_{m,i}) + u(t_{m,i})\| \sup_{I_m} \|v'\| \\ &\le \tau_m C g_2(N) \sup_{I_m} \|v\| \end{aligned} \tag{44}$$

We need to estimate $\int_{I_m} (u - Pu, v')dt$. To do this we define interpolation operator $\hat{P}$ such that $\hat{P}u$ is a polynomial of degree $q+1$ in time which interpolates u in Radau quadrature nodes $t_{m,i}$ and (in addition) t_{m-1}. Then we get

$$\int_{I_m} (\hat{P}u, v')dt = \int_{I_m} (Pu, v')dt. \tag{45}$$

It is possible to show that $\sup_{I_m} \|u - \hat{P}u\| \le C\tau_m^{q+2}$ by the same arguments as for interpolation operator P. Then we get with the inverse inequality on the test function v

$$\begin{aligned} \int_{I_m} (u - Pu, v')dt &= \int_{I_m} (u - \hat{P}u, v')dt \\ &\le \tau_m C \tau_m^{q+2} \sup_{I_m} \|v'\| \le \tau_m C \tau^{q+1} \sup_{I_m} \|v\|. \end{aligned} \tag{46}$$

The estimate for the second term follows directly from Lemma 1

$$\begin{aligned} &(\eta_-^m - \eta_-^{m-1}, v_+^{m-1}) = (Ru^m - u^m - Ru^{m-1} + u^{m-1}, v_+^{m-1}) \\ &\le \tau_m \sup_{I_m} \|Ru' - u'\| \sup_{I_m} \|v\| \le \tau_m C g_2(N) \sup_{I_m} \|v\|. \end{aligned} \tag{47}$$

Finally, we need to estimate quadrature error.

(48)
$$\begin{aligned}\int_{I_m}(u',v)dt - Q[(u',v)] &= \int_{I_m}(u'-Pu',v)dt \\ &\leq \tau_m C\tau^{q+1}\sup_{I_m}\|v\|\end{aligned}$$

□

Remark 1. *Lemma 4 can be easily generalized to any time projection that interpolates the end points of the intervals and to any space projection that commutes with the time projection. Then the result will take the following form*

(49)
$$\begin{aligned}&Q[(\eta',v)] + (\{\eta\}_{m-1}, v_+^{m-1}) \\ &\leq \tau_m C\,(\text{'time error'} + \text{'space error'})\sup_{I_m}\|v\|, \quad \forall v\in V_N^\tau.\end{aligned}$$

For the estimates of supremum term we will need the following lemma.

Lemma 5. *Let* $\xi \in V_N^\tau$ *and*

(50)
$$\tilde{\xi} = P\left(\frac{\tau_m\xi(t)}{t-t_{m-1}}\right)\in V_N^\tau.$$

Then

(51)
$$\int_{I_m}(\xi', 2\tilde{\xi})dt + (\xi_+^{m-1}, 2\tilde{\xi}_+^{m-1}) = \|\xi_-^m\|^2 + \frac{1}{\tau_m}\int_{I_m}\|\tilde{\xi}\|^2dt.$$

Proof. The proof can be made as a simple extension of [2, Lemma 2.1], which describes the same result for scalar polynomials and on unit time interval. □

3.5. Main result. We are ready to present the main result.

Theorem 1. *Let* u *be an exact solution of (1) and* $U\in V_N^\tau$ *be its discrete approximation given by (17). Then*

(52)
$$\max_{m=1,\dots,r}\sup_{I_m}\|U-u\| \leq C\left(g_2(N)+\tau^{q+1}\right).$$

Proof. We can estimate right–hand side of (37) by Lemma 4. Then we obtain

(53)
$$\begin{aligned}&\int_{I_m}(\xi',v) + a_{st}(\xi,v)dt + (\{\xi\}_{m-1}, v_+^{m-1}) \\ &\leq \tau_m C\left(\tau^{q+1}+g_2(N)\right)\sup_{I_m}\|v\|.\end{aligned}$$

Setting $v = 2\xi$ we get

(54)
$$\begin{aligned}&\|\xi_-^m\|^2 - \|\xi_-^{m-1}\|^2 + \|\{\xi\}_{m-1}\|^2 + 2\int_{I_m}|\!|\!|\xi|\!|\!|^2dt \\ &\leq \tau_m C\left(\tau^{q+1}+g_2(N)\right)\sup_{I_m}\|\xi\| \\ &\leq \tau_m C\left(\tau^{2q+2}+g_2(N)^2\right) + \frac{\tau_m}{2}\sup_{I_m}\|\xi\|^2\end{aligned}$$

We need to deal with the last term at the right–hand side.

It is simple to see that for $\tilde{\xi}$ defined by (50) we get

$$
\begin{aligned}
(55) \qquad \int_{I_m} |||\xi|||^2 dt &= \int_{I_m} a_{st}(\xi,\xi) dt \\
&= Q[a_{st}(\xi,\xi)] = \sum_{i=0}^{q} w_i a_{st}(\xi(t_{m,i}),\xi(t_{m,i})) \\
&\le \sum_{i=0}^{q} w_i \frac{\tau_m}{t_{m,i}-t_{m-1}} a_{st}(\xi(t_{m,i}),\xi(t_{m,i})) \\
&= Q[a_{st}(\xi,\tilde{\xi})] = \int_{I_m} a_{st}(\xi,\tilde{\xi}) dt,
\end{aligned}
$$

since $\tau_m/(t_{m,i}-t_{m-1}) \ge 1$.

Since the terms $\sup_{I_m}\|\xi\|^2$, $\frac{1}{\tau_m}\int_{I_m}\|\tilde{\xi}\|^2 dt$ and $\sup_{I_m}\|\tilde{\xi}\|^2$ are equivalent, we get by setting $v = 2\tilde{\xi}$ in (53) with the aid of Lemma 5

$$
\begin{aligned}
(56) \qquad \sup_{I_m}\|\xi\|^2 &\le C\frac{1}{\tau_m}\int_{I_m}\|\tilde{\xi}\|^2 dt \\
&\le C\left(\|\xi_-^m\|^2 + \frac{1}{\tau_m}\int_{I_m}\|\tilde{\xi}\|^2 dt + 2\int_{I_m}|||\xi|||^2 dt\right) \\
&\le C\left((\xi_-^{m-1},\tilde{\xi}_+^{m-1}) + C\left(\tau^{q+1}+g_2(N)\right)\sup_{I_m}\|\xi\|\right) \\
&\le C\left(\|\xi_-^{m-1}\|^2 + \tau^{2q+2} + g_2(N)^2\right) + \frac{1}{2}\sup_{I_m}\|\xi\|^2
\end{aligned}
$$

We can substitute this result into our error inequality (54) and we obtain

$$\|\xi_-^m\|^2 - \|\xi_-^{m-1}\|^2 \le \tau_m C\left(\tau^{2q+2}+g_2(N)^2\right) + \tau_m C\|\xi_-^{m-1}\|^2.$$

Now, it is sufficient to employ the forward difference form of the discrete Gronwall lemma to obtain nodal error estimates. Estimates inside of intervals I_m follows from nodal estimates and from (56). □

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Charles University in Prague, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Prague, Czech Republic

E-mail: `vlasak@karlin.mff.cuni.cz`

URL: `http://www.karlin.mff.cuni.cz/∼vlasak/`

Technical University of Dresden, Institute of Numerical Mathematics, Helmholzstrasse 10, 01069 Dresden, Germany

E-mail: `hans-goerg.roos@tu-dresden.de`

URL: `http://www.math.tu-dresden.de/∼roos/`