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NUMERICAL ANALYSIS OF MODULAR VMS METHODS WITH NONLINEAR EDDY VISCOSITY FOR THE NAVIER-STOKES EQUATIONS

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Abstract. This paper presents the stabilities for both two modular, projection-based variational multiscale (VMS) methods and the error analysis for only first one for the incompressible Naiver-Stokes equations, expanding the analysis in [39] to include nonlinear eddy viscosities. In VMS methods, the influence of the unresolved scales onto the resolved small scales is modeled by a Smagorinsky-type turbulent viscosity acting only on the marginally resolved scales. Different realization of VMS models arise through different models of fluctuations. We analyze a method of inducing a VMS treatment of turbulence in an existing NSE discretization through an additional, uncoupled projection step. We prove stability, identifying the VMS model and numerical dissipation and give an error estimate. Numerical tests are given that confirm and illustrate the theoretical estimates. One method uses a fully nonlinear step inducing the VMS discretization. The second induces a nonlinear eddy viscosity model with a linear solve of much less cost.

Key words. Navier-Stokes equations, eddy viscosity, projection-based VMS method, uncoupled approach.

1. Introduction

Variational multiscale (VMS) methods have proven to be an important approach to the numerical simulation of turbulent flows (see Section 1.1 for its genesis and some recent work). VMS methods are efficient, clever and simple realization of the idea of introducing eddy viscosity locally in scale space only on the marginally resolved scales. They add dissipation to mimic the loss of energy in the marginally resolved scales caused by breakdown of eddies to unresolved scales through a term of the form:

$$\Big(\nu_T(\mathbf{u}^h)\mathbb{D}(I-P_H)\mathbf{u}^h, \mathbb{D}(I-P_H)\mathbf{v}^h\Big), \tag{1}$$

where $\mathbb{D}(\mathbf{v}) = (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)/2$ is the velocity deformation tensor (symmetric part of the gradient), P_H is an elliptic projection onto the well-resolved velocities on a given mesh (so $(I-P_H)\mathbf{u}^h$ represents the marginally resolved velocity scales).

The success of VMS methods leads naturally to the question of how to introduce them into legacy codes and other multi-physics codes so large as to discourage abandoning a method or a model that is already implemented to reprogram another one. In [39], this question was addressed: a VMS method can be induced into a black box (even laminar) flow simulation by adding a modular projection step, uncoupled from the (possibly black box) flow code. Although the numerical tests were quite general, the mathematical/numerical analysis in [39] in support of modular VMS methods was for constant eddy viscosity parametrizations $\nu_T(\cdot)$. In this report we continue the mathematical support for modular VMS methods in two ways. First we expand the analysis of [39] to include the fully nonlinear, eddy

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viscosity case of the (ideal) "small-small" Smagorinsky model (1) above for which

$$\nu_T(\mathbf{u}^h) = (C_s\delta)^2 \big|\mathbb{D}(I-P_H)\mathbf{u}^h\big|_F, \tag{2}$$

The motivation of the Smagorinsky model is to replicate the decay of energy due to breakdown of eddies from resolved to unresolved scales in the energy cascade, [6, 7, 8, 15, 21, 22, 29, 31, 32, 34, 43, 45]. The ideal case of (2) has the most complete mathematical theory due to the strong monotonicity on the marginally resolved s-cales of (1) with (2), Section 3. Unfortunately, the choice (2) also increases the cost of implementing a VMS method in a modular Step 2. We therefore consider methods (i) whose realization is as close as possible to the ideal small-small S-magorinsky model, (ii) for which a complete and rigorous mathematical foundation can be given, and (iii) whose implementation is comparable in cost and complexity to the linear case of $\nu_T \equiv$constant. These issues lead to our second, related method with eddy viscosity term:

$$\Big(A_e\big(\nu_T(\mathbf{u}^h)\big)\mathbb{D}(I-P_H)\mathbf{u}^h, \mathbb{D}(I-P_H)\mathbf{v}^h\Big) = \Big(A_e\big(\nu_T(\mathbf{u}^h)\big)\mathbb{D}(I-P_H)\mathbf{u}^h, \mathbb{D}\mathbf{v}^h\Big) \tag{3}$$

where $A_e\big(\nu_T(\cdot)\big)$ is an element average over the elements (e.g. triangles in 2d) which define the well-resolved scales, see Definition 4.1. Because the eddy viscosity coefficient $A_e\big(\nu_T(\cdot)\big)$ is now elementwise constant, simplifications arise in the modular Step 2 below which enforces the VMS turbulence model. The restriction to elementwise constant eddy viscosities originates in the works of Lube and Roehe [44] on full (or monolithic) VMS methods.

To introduce the idea of [39] developed herein, suppose the Navier-Stokes equations are written as

$$\frac{\partial \mathbf{u}}{\partial t} + N(\mathbf{u}) + \nu A\mathbf{u} = \mathbf{f}(t). \tag{4}$$

Let Π denote a postprocessing operator. The method we extends and analyzes, adds one uncoupled postprocessing step to a given method (we select the commonly used Crank-Nicolson time discretization for Step 1 for specificity): given $\mathbf{u}^n \cong \mathbf{u}(t^n)$, compute $\mathbf{u}^{n+1}$ by

Step 1: Compute $\mathbf{w}^{n+1}$ via:

$$\frac{\mathbf{w}^{n+1}-\mathbf{u}^n}{\triangle t} + N\Big(\frac{\mathbf{w}^{n+1}+\mathbf{u}^n}{2}\Big) + \nu A\frac{\mathbf{w}^{n+1}+\mathbf{u}^n}{2} = \mathbf{f}^{n+\frac{1}{2}}. \tag{5}$$

Step 2: Postprocess $\mathbf{w}^{n+1}$ to obtain $\mathbf{u}^{n+1}$:

$$\mathbf{u}^{n+1} = \Pi\mathbf{w}^{n+1}. \tag{6}$$

Both steps can be done by uncoupled modules. Eliminating Step 2 gives:

$$\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\triangle t} + N\Big(\frac{\mathbf{w}^{n+1}+\mathbf{u}^n}{2}\Big) + \nu A\frac{\mathbf{w}^{n+1}+\mathbf{u}^n}{2} + \frac{1}{\triangle t}(\mathbf{w}^{n+1}-\Pi\mathbf{w}^{n+1}) = \mathbf{f}^{n+\frac{1}{2}}, \tag{7}$$

where $\mathbf{f}^{n+\frac{1}{2}} = (\mathbf{f}^{n+1}+\mathbf{f}^n)/2$. We define the operator Π in Step 2, following [39] so that the extra term is exactly a nonlinear Smagorinsky model acting on small resolved scales.

$$\frac{1}{\triangle t}(\mathbf{w}^{n+1}-\mathbf{u}^{n+1}, \mathbf{v}_h) = (Smagorinsky\ Model, \mathbf{v}_h). \tag{8}$$

We consider herein two algorithmic realizations of (8). The first method analyzed is a full Smagorinsky model. Let P_{L^H} denote an L^2 projection onto a space of "well resolved" deformations, see Definition 1.2 for a precise formulation in Section 1.2.

Method 1. Let $\nu_T(\cdot)$ be defined by (2). Define $\mathbf{u} = \Pi\mathbf{w}$ in Step 2, (6), by

$$\begin{aligned}&\frac{1}{\triangle t}\big(\mathbf{w}^{n+1} - \mathbf{u}^{n+1}, \mathbf{v}_h\big)\\ &= \Big(\nu_T\big(\frac{\mathbf{w}^{n+1} + \mathbf{u}^{n+1}}{2}\big)\big[I - P_{L^H}\big]\mathbb{D}\big(\frac{\mathbf{w}^{n+1} + \mathbf{u}^{n+1}}{2}\big), \big[I - P_{L^H}\big]\mathbb{D}(\mathbf{v}_h)\Big),\end{aligned} \tag{9}$$

where $C_s > 0$ is a Smagorinsky constant, $\delta > 0$ is the averaging radius, which is connected to the resolution of the finite element spaces involved in the VMS method and $|\cdot|_F$ denotes the usual Frobenius norm of a tensor defined by $|\mathbb{T}|_F^2 = \sum_{i,j=1,d}(\mathbb{T}_{ij})^2$. Computationally, Step 2 reduces to the following nonlinear problem at each time step: Given $\mathbf{w}^{n+1}$, solve the nonlinear system (9) for $\mathbf{u}^{n+1}$, subject to $\nabla \cdot \mathbf{u}^{n+1} = \mathbf{0}$.

One difficulty with the modular, ideal Smagorinsky VMS method is exactly the cost of this nonlinear solve each time step. To reduce this cost we also consider the following Method 2 which is closely related and much less expensive.

Method 2. (See Algorithm 4.1, Section 4) Define $\mathbf{u} = \Pi\mathbf{w}$ in Step 2 by solving

$$\begin{aligned}&\frac{1}{\triangle t}\Big(\mathbf{w}^{n+1} - \mathbf{u}^{n+1}, \mathbf{v}_h\Big)\\ &= \Big(A_e\big(\nu_T\big(\frac{\mathbf{w}^{n} + \mathbf{u}^{n}}{2}\big)\big)\big[I - P_{L^H}\big]\mathbb{D}\big(\frac{\mathbf{w}^{n+1} + \mathbf{u}^{n+1}}{2}\big), \big[I - P_{L^H}\big]\mathbb{D}(\mathbf{v}_h)\Big).\end{aligned} \tag{10}$$

There are two ideas behind (10). The first and obvious one is that lagging $\nu_T(\cdot)$ reduces the computational problem of (10) to solving one (multiscale) linear equation per time step for $\mathbf{u}^{n+1}$. The second one is that since the eddy viscosity coefficient is a piecewise constant average, (10) reduces to: given $\mathbf{w}^{n+1}$ solve for $\mathbf{u}^{n+1}$ subject to $\nabla \cdot \mathbf{u}^{n+1} = 0$:

$$\begin{aligned}&\Big(A_e\big(\nu_T(\cdot)\big)\big[I - P_{L^H}\big]\mathbb{D}(\mathbf{u}^{n+1}), \mathbb{D}(\mathbf{v}_h)\Big) + \frac{2}{\triangle t}(\mathbf{u}^{n+1}, \mathbf{v}_h)\\ &= \frac{2}{\triangle t}(\mathbf{w}^{n+1}, \mathbf{v}_h) - \Big(A_e\big(\nu_T(\cdot)\big)[I - P_{L^H}]\mathbb{D}(\mathbf{w}^{n+1}), \mathbb{D}(\mathbf{v}_h)\Big).\end{aligned} \tag{11}$$

Note in particular $\mathbb{D}(\mathbf{v}_h)$ replaces $[I - P_{L^H}]\mathbb{D}(\mathbf{v}_h)$ as the test function. This change simplifies the computational work of (10) substantially. With one common choice of P_{L^H}, (11) simplifies further to one uncoupled linear system per macroelement (defining the well-resolved scales), see Definition 4.1, Section 4.

For both methods we prove unconditional stability and delineate their energy balance (including induced model and numerical dissipation). This part of the analysis extends readily to eddy viscosities other than (2). We give a convergence analysis of Method 1 in Theorem 3.1. This analysis uses the discrete Gronwall inequality at the last step and thus inherits the limitation introduced by its use of small time step restriction and exponential growth in time. These consequences have recently been thoroughly analyzed in [27] for laminar flows. Confirming numerical experiments are given in Section 5. For more numerical tests of the modular/partitioned VMS approach, see [39].

1.1. Previous Work. The VMS method is an active, successful and rapidly developing approach to the simulation of turbulent flows pioneered by Hughes and collaborators, [16, 20, 21, 22]. Mathematical study of it has taken several approaches, see [14, 35] for early works and [15, 18, 29, 30, 31, 32, 34, 37, 43, 46] for some recent developments. The idea of imposing a VMS treatment of turbulence through an uncoupled Step 2 is from [39]. This work builds a work on time relaxation and filter based stabilization in [3, 6, 10, 12, 40, 41].

Turbulent flows have many challenging features for numerical simulations including convection dominance, vortex stretching (including exponential growth of noise), backscatter and possible equipartition of energy due to truncation of the energy cascade. Projection based VMS methods address nonlinear error growth and equipartition directly. They address convection dominance but not as well as stabilized methods developed studying convection diffusion equations. Thus one natural improvement would be to replace Step 1 by a stabilized method for the NSE. In VMS modeling, a fluctuation model is used to approximate the exact fluctuation equation. Projection based VMS model fluctuations an a projection into a space of marginally resolved scales, defined either through a coarse and fine mesh velocity space or through velocities of two different polynomial degrees on a sing mesh. The most common choice in VMS (not considered here) is to model fluctuations with bubble functions, see Bensoward Larson [5] and Hsu, Bazilev, Calo, Tezdvyaz and Hughes [19] for recent work on this original VMS method.

The idea of stabilization by a separate, modular step first appears in [12, 40], see also [3, 41]. This work is also connected to time relaxation stabilizations in numerical methods and continuum models via (7).

This paper is organized into four sections. In Section 1.2 we establish the notation and give a weak formulation of the Navier-Stokes equations. In Section 2, the uncoupled, projection-based VMS scheme is described and the stability of Method 1 is proven. In Section 3 we present its error estimate. We present Method 2 and analyze its stability in Section 4. Section 5 describes the implementation of two algorithms and presents some numerical results that confirm the theoretical analysis.

1.2. Notations. Let Ω be an open, bounded region in $\mathbb{R}^d, d = 2$ *or* 3 with a Lipschitz continuous boundary. Throughout this paper, standard notation is used for $L^p(\Omega)$ and the Sobolev spaces $W^{k,p}(\Omega), 1 < p < \infty, k = 0, 1, 2, \ldots$. The corresponding norms are denoted by $||\cdot||_{L^p}$ and $||\cdot||_{W^{k,p}}$, respectively. $H^k(\Omega)$ is used to represent the Sobolev space $W^{k,2}(\Omega)$, $|\cdot|_k$ and $||\cdot||_k$ denote the semi-norm and norm in $H^k(\Omega)$, respectively. The standard L^2 inner product is denoted by $(\cdot,\cdot)$ and L^2 norm by $||\cdot||$. The space $H^{-k}(\Omega)$ denotes the dual space of $H^k_0(\Omega)$. In addition, the vector spaces and vector functions will be indicated by boldface. For the velocity $\mathbf{v}(x,t)$ defined on $(0,T)$, we define

$$||\mathbf{v}||_{\infty,k} := EssSup_{[0,T]}||\mathbf{v}(t,\cdot)||_k \text{ and } ||\mathbf{v}||_{m,k} := \Big(\int_0^T ||\mathbf{v}(t,\cdot)||_k^m dt\Big)^{1/m}.$$

Define the velocity space $\mathbf{X}$, the pressure space Q and the deformation space $\mathbf{L}$ as follows:

$$\begin{aligned}
\mathbf{X} \quad &: \quad = \mathbf{H}^1_0(\Omega) = \{\mathbf{v} : \mathbf{v} \in \mathbf{H}^1(\Omega), \mathbf{v} = 0 \text{ on } \partial\Omega\},\\
Q \quad &: \quad = L^2_0(\Omega) := \{q \in L^2(\Omega), \int_\Omega q dx = 0\},\\
\mathbf{L} \quad &: \quad = \{\mathbb{L} \in (L^2(\Omega))^{d\times d}, \mathbb{L} = \mathbb{L}^T\}.
\end{aligned}$$

The closed sub-space of divergence free functions is given by

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{X} : (\nabla\cdot\mathbf{v}, q) = 0, \ \forall\, q \in Q\}.$$

Let $\mathcal{T}_H$ denotes a coarse finite element mesh which is refined (once, twice, ...) to produce the finer mesh $\mathcal{T}_h$, so $h < H$. Let $\mathbf{X}^h \times Q^h$ be a pair of conforming velocity-pressure finite element spaces satisfying the usual inf-sup condition (see Gunzburger [13]): there exists a constant β independent of h such that

$$\inf_{q^h \in Q^h} \sup_{\mathbf{v}^h \in \mathbf{X}^h} \frac{(q^h, \nabla \cdot \ \mathbf{v}^h)}{||q^h|| ||\nabla \mathbf{v}^h||} \geq \beta > 0, \tag{12}$$

and consisting of C^0 piecewise polynomials. Examples of such spaces include the mini-element [1] and the Taylor-Hood element [13]. We assume that the spaces $\mathbf{X}^h$ and Q^h contain piecewise continuous polynomials of degree k and $k-1$, respectively, and satisfy the following approximation properties:

$$\begin{aligned} \inf_{\mathbf{v}^h \in \mathbf{X}^h} \{||\mathbf{u} - \mathbf{v}^h|| + h||\nabla(\mathbf{u} - \mathbf{v}^h)||\} \qquad &\leq Ch^{k+1}|\mathbf{u}|_{k+1} \ \forall \ \mathbf{u} \in \mathbf{H}^{k+1}(\Omega) \cap \mathbf{X}, \\ \inf_{q^h \in Q^h} ||p - q^h|| \qquad &\leq Ch^k |p|_k \ \forall \ p \in H^k(\Omega) \cap Q. \end{aligned}$$

Through the paper, C and $\tilde{C}$ denote generic constants which independent of ν, h, H, δ and $\triangle t$ which have different values at its different occurrences.

Under (12), we introduce the discretely divergence free subspace of $\mathbf{X}^h$,

$$\mathbf{V}^h := \{\mathbf{v}^h \in \mathbf{X}^h : (\nabla \cdot \mathbf{v}^h, q^h) = 0, \ \forall \ q^h \in Q^h\}.$$

We shall use a space $\mathbf{L}^H$ of "well resolved" velocity deformations. There are two natural ways to define $\mathbf{L}^H$. If $\mathbf{X}^h$ is a higher order finite element space on a given mesh, one approach is to define the large scale space using lower order finite elements on the same mesh. The second option, and only one for low order elements, is to define the well-resolved space $\mathbf{L}^H$ on a coarse mesh leading to a two-level discretization. Both cases are included in the following definition by the assumption $\mathbf{X}^H \subset \mathbf{X}^h$. In our numerical tests, we choose Taylor-Hood element for $\mathbf{X}^h \times Q^h$ and piecewise constant element on the same mesh for $\mathbf{L}^H$. To present the method, we introduce the following projection operators.

Definition 1.1(L^2 projection) Let $\mathbf{X}^H \subset \mathbf{X}^h$. $P_{L^H} : \mathbf{L} \to \mathbf{L}^H$ is the L^2-orthogonal projection. We take a well-resolved velocity space, denoted by $\mathbf{X}^H$, and select

$$\mathbf{L}^H = \{\mathbb{D}(\mathbf{v}^h) : \forall \ \mathbf{v}^H \in \mathbf{X}^H\},$$

so that P_{L^H} satisfies

$$(P_{L^H}\mathbb{L}, \mathbb{L}^H) \ = \ (\mathbb{L}, \mathbb{L}^H) \ \forall \ \mathbb{L} \in \mathbf{L}, \mathbb{L}^H \in \mathbf{L}^H, \tag{13}$$

$$\big\|[I - P_{L^H}]\mathbb{L}\big\| \ \leq \ CH^l |\mathbb{L}|_l \ \forall \ \mathbb{L} \in \mathbf{L} \cap \mathbf{H}^l(\Omega), \tag{14}$$

where either $H > h$ and $k = l$ or $H = h$ and $l < k$.

Definition 1.2 (Elliptic projection). $P_H : \mathbf{X} \to \mathbf{X}^H$ is the projection operator satisfying

$$\big(\mathbb{D}[\mathbf{w} - P_H(\mathbf{w})], \mathbb{D}(\mathbf{v}^H)\big) = 0 \ \forall \ \mathbf{v}^H \in \mathbf{X}^H. \tag{15}$$

From [36], see also [37] and [29], we have the following.

Lemma 1.3 Let $\mathbf{v} \in \mathbf{X}$ and $\mathbf{L}^H = \mathbb{D}(\mathbf{X}^H)$, Then

$$P_{L^H}\big(\mathbb{D}(\mathbf{v})\big) = \mathbb{D}(P_H\mathbf{v}) \ \texttt{and} \ (I - P_{L^H})\mathbb{D}(\mathbf{v}) = \mathbb{D}\big((I - P_H)\mathbf{v}\big). \tag{16}$$

Remark 1.4(Computing P_{L^H} elementwise): *We have assumed that $\mathbf{X}^H$ consists of C^0 piecewise polynomials of degree $\leq l$. This means that $\mathbf{L}^H$ consists of discontinuous piecewise polynomials of degree $\leq l-1$. Let e_H denote a typical element associated with $\mathbf{L}^H$. Then computing the projection P_{L^H} in (13) uncoupled into one linear system per element e_H of the form*

$$(\mathbb{L} - P_{L^H}\mathbb{L}, \mathbb{L}^H)_{e_H} = 0 \ \forall \ \mathbb{L}^H \in \mathbf{L}^H \mid_{e_H} .$$

Weighted L^2 projections into $\mathbf{L}^H$ similarly uncouple provided the weights are constant on each e_H.

We are interested in approximating the solution of the evolutionary Naiver-Stokes equations

$$\begin{aligned}
&(17) & \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - 2\nu\nabla\cdot\mathbb{D}(\mathbf{u}) + \nabla p &= \mathbf{f} \texttt{ in } (0,T]\times\Omega,\\
&(18) & \nabla\cdot\mathbf{u} &= 0 \texttt{ in } [0,T]\times\Omega,\\
&(19) & \mathbf{u} &= 0 \texttt{ in } [0,T]\times\partial\Omega,\\
&(20) & \mathbf{u}(0,\mathbf{x}) &= \mathbf{u}_0 \texttt{ in } \Omega,\\
&(21) & \int_\Omega p dx &= 0 \texttt{ in } (0,T].
\end{aligned}$$

Here $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ is the body force, ν is the kinematic viscosity, $\mathbf{u}_0$ is the initial velocity, and $[0,T]$ is a finite time interval.

The variational formulation of the Navier-Stokes equations (17)-(21) is: find $\mathbf{u} : (0,T] \to \mathbf{X}, p : (0,T] \to Q$ satisfying

$$\begin{aligned}
&(22) & (\mathbf{u}_t,\mathbf{v}) + b_s(\mathbf{u},\mathbf{u},\mathbf{v}) + 2\nu(\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v})) - (p,\nabla\cdot\mathbf{v}) &= (\mathbf{f},\mathbf{v}) \ \forall\ \mathbf{v}\in\mathbf{X},\\
&(23) & (q,\nabla\cdot\mathbf{u}) &= 0 \ \forall\ q\in Q.
\end{aligned}$$

Here

$$b_s(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}) - \frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v})$$

is the skew-symmetric trilinear form of the convective term. It has the following properties:

$$(24) \qquad b_s(\mathbf{u},\mathbf{v},\mathbf{w}) = -b_s(\mathbf{u},\mathbf{w},\mathbf{v}) \ \forall\ \mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}$$

and consequently

$$\begin{aligned}
&(25) & b_s(\mathbf{u},\mathbf{v},\mathbf{v}) &= 0 \ \forall\ \mathbf{u},\mathbf{v}\in\mathbf{X},\\
&(26) & |b_s(\mathbf{u},\mathbf{v},\mathbf{w})| &\leq C||\mathbf{u}||^{1/2}||\nabla\mathbf{u}||^{1/2}||\nabla\mathbf{v}||||\nabla\mathbf{w}|| \ \forall\ \mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}.
\end{aligned}$$

In the divergence-free space (22)-(23) can be reformulated as follows: find $\mathbf{u} : [0,T] \to \mathbf{V}$ satisfying

$$(27) \qquad (\mathbf{u}_t,\mathbf{v}) + b_s(\mathbf{u},\mathbf{u},\mathbf{v}) + 2\nu(\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v})) = (\mathbf{f},\mathbf{v})$$

for all $\mathbf{v} \in \mathbf{V}$.

2. Uncoupled modular VMS method with nonlinear eddy viscosity

We give the precise formulation of the uncoupled algorithm with nonlinear eddy viscosity for the finite element discretization of Navier-Stokes equations (17)-(21). Let $t^n = n\triangle t, n = 0,1,2,\cdots,N_T$ and $T = N_T\triangle t$, and denote time averages by

$$\mathbf{v}^{n+\frac{1}{2}} = \frac{\mathbf{v}^{n+1} + \mathbf{v}^n}{2}.$$

The first FEM in space and Crank-Nicolson (CN) method in time with an additional postprocessing step is as follows.

Algorithm 2.1

Step 1: Given $(\mathbf{u}_h^n, p_h^n) \in \mathbf{X}^h \times Q^h$, compute $(\mathbf{w}_h^{n+1}, p_h^{n+1}) \in \mathbf{X}^h \times Q^h$ satisfying

$$(28)\quad \begin{cases} \big(\frac{\mathbf{w}_h^{n+1}-\mathbf{u}_h^n}{\triangle t}, \mathbf{v}_h\big) + b_s\big(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}, \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}, \mathbf{v}_h\big) \\ \qquad\qquad +2\nu\Big(\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}), \mathbb{D}(\mathbf{v}_h)\Big) - (p_h^{n+\frac{1}{2}}, \nabla\cdot\mathbf{v}_h) = (\mathbf{f}^{n+\frac{1}{2}}, \mathbf{v}_h), \\ (\nabla\cdot\mathbf{w}_h^{n+1}, q_h) = 0, \end{cases}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{X}^h \times Q^h$.

Step 2: Given $\mathbf{w}_h^{n+1} \in \mathbf{X}^h$, solve the following to obtain $(\mathbf{u}_h^{n+1}, \lambda_h^{n+1}) \in \mathbf{X}^h \times Q^h$:

(29)

$$\begin{cases} (\frac{\mathbf{u}_h^{n+1}-\mathbf{w}_h^{n+1}}{\triangle t}, \mathbf{v}_h) - (\lambda_h^{n+1}, \nabla\cdot\mathbf{v}_h) \\ +\Big((C_s\delta)^2\big|[I-P_{L^H}]\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})\big|_F[I-P_{L^H}]\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}), [I-P_{L^H}]\mathbb{D}(\mathbf{v}_h)\Big) = 0, \\ (\nabla\cdot\mathbf{u}_h^{n+1}, q_h) = 0, \end{cases}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{X}^h \times Q^h$, where P_{L^H} is a L^2-projection operator defined by (13).

Using (2) and (16), one can rewrite Step 2 in the following way.

Restated Step 2: Given $\mathbf{w}_h^{n+1} \in \mathbf{X}^h$, solve the following to obtain $(\mathbf{u}_h^{n+1}, \lambda_h^{n+1})$:

$$(30)\quad \begin{cases} \Big(\frac{\mathbf{u}_h^{n+1}-\mathbf{w}_h^{n+1}}{\triangle t}, \mathbf{v}_h\Big) - (\lambda_h^{n+1}, \nabla\cdot\mathbf{v}_h) \\ \qquad\qquad +\Big(\nu_T(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})\mathbb{D}[I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}, \mathbb{D}[I-P_H]\mathbf{v}_h\Big) = 0, \\ \big(\nabla\cdot\mathbf{u}_h^{n+1}, q_h\big) = 0, \end{cases}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{X}^h \times Q^h$, where P_H is an elliptic projector defined by (15).

Before discussing the stability of the method, we recall some important analytical tools in the analysis of the Smagorinsky model, see [35].

Lemma 2.1 (Strong monotonicity and local Lipschitz continuity) There is a constant $C > 0$ such that for all $\mathbf{u}, \mathbf{v} \in \mathbf{W}^{1,3}(\Omega)$,

$$(31)\qquad \Big(\big|\mathbb{D}(\mathbf{u})\big|_F\mathbb{D}(\mathbf{u}) - \big|\mathbb{D}(\mathbf{v})\big|_F\mathbb{D}(\mathbf{v}), \mathbb{D}(\mathbf{u}-\mathbf{v})\Big) \geq C\big\|\mathbb{D}(\mathbf{u}-\mathbf{v})\big\|_{L^3}^3.$$

There exists a constant $C > 0$ such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{W}^{1,3}(\Omega)$,

$$(32)\qquad \Big(\big|\mathbb{D}(\mathbf{u})\big|_F\mathbb{D}(\mathbf{u}) - \big|\mathbb{D}(\mathbf{v})\big|_F\mathbb{D}(\mathbf{v}), \mathbb{D}(\mathbf{w})\Big) \leq CC_L\big\|\mathbb{D}(\mathbf{u}-\mathbf{v})\big\|_{L^3}\big\|\mathbb{D}(\mathbf{w})\big\|_{L^3},$$

where $C_L = max\{||\mathbb{D}(\mathbf{u})||_{L^3}, ||\mathbb{D}(\mathbf{v})||_{L^3}\}$.

The global energy balance is derived in Proposition 2.3. Its proof utilizes the following lemma.

Lemma 2.2 Consider Step 2 in Algorithm 2.1. Let $C_s > 0, \delta > 0$. Given $\mathbf{w}_h^{n+1} \in \mathbf{V}^h$, any solution of (30) satisfies:

$$(33)\quad ||\mathbf{w}_h^{n+1}||^2 \;=\; ||\mathbf{u}_h^{n+1}||^2 + 2\triangle t(C_s\delta)^2\Big\|\Big|\mathbb{D}[I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}\Big|_F\Big\|_{L^3}^3.$$

Furthermore, the system (30) has a unique solution $(\mathbf{u}_h^{n+1}, \lambda_h^{n+1})$.

Proof. For the á priori bound, choosing $\mathbf{v}_h = \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}$ in (30) and $q_h = \lambda_h^{n+1}$ in the second equation of both (28) and (30), we get

$$\begin{aligned}
&\frac{1}{2\triangle t}\Big(||\mathbf{w}_h^{n+1}||^2 - ||\mathbf{u}_h^{n+1}||^2\Big)\\
&= \Big(\nu_T(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}), \mathbb{D}[I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}\Big)\\
&= (C_s\delta)^2\Big\|\big|\mathbb{D}[I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}\big|_F\Big\|_{L^3}^3.
\end{aligned}$$

Existence and uniqueness will (in essence) follow from Minty's Lemma [42] after Step 2 is split into two parts, one a solve with a nonlinear monotone operator and the second an orthogonal projection. This splitting introduces some (temporary) notation so we suppress superscripts $n+1$ and subscripts h where possible. Let $\Phi^h := (I-P_H)\mathbf{V}^h$ and $\Phi^{h\perp} := P_H\mathbf{V}^h$. Let

$$\begin{aligned}
\phi &:= (I-P_H)\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2} \in \Phi^h,\\
\phi^\perp &:= P_H\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2} \in \Phi^{h\perp}.
\end{aligned}$$

Existence and uniqueness of $\mathbf{u}_h^{n+1}$ is equivalent to the same question for ϕ and $\phi^\perp$. Using orthogonality, algebraic rearrangement and setting alternately $\mathbf{v}_h \in \Phi^h$ and $\mathbf{v}_h \in \Phi^{h\perp}$, equations (30) in Step 2 of Algorithm 2.1 becomes the pair of equations: given $\mathbf{w} = \mathbf{w}_h^{n+1}$ find $\phi \in \Phi^h, \phi^\perp \in \Phi^{h\perp}$ satisfying

$$(C_s\delta)^2\Big(\big|\mathbb{D}(\phi)\big|\mathbb{D}(\phi), \mathbb{D}(\psi)\Big) + \frac{1}{\triangle t}(\phi,\psi) = \frac{1}{\triangle t}\Big([I-P_H]\mathbf{w},\psi\Big), \forall\ \psi\in\Phi^h, \tag{34}$$

$$\frac{1}{\triangle t}(\phi^\perp,\psi^\perp) = \frac{1}{\triangle t}(P_H\mathbf{w},\psi^\perp),\ \forall\ \psi^\perp \in \Phi^{h\perp}. \tag{35}$$

Existence and uniqueness of the solution to equation (34) follows from Minty's lemma by noting the LHS defines (via the Riesz representation theorem in a standard way) a monotone operator on Φ^h. Equation (35) simply states that $\phi^\perp = P_H\mathbf{w}$.

We thus have existence and uniqueness of ϕ and thus $\mathbf{u}_h^{n+1}$. Existence and uniqueness of the associated pressure-like Lagrange multiplier λ_h^{n+1} follows from the discrete inf-sup condition as in the discrete Stokes problem. □

Now, we prove the strong energy equality and the strong, unconditional stability of the method.

Proposition 2.3 Let $C_s > 0, \delta > 0$. The approximate velocity $\mathbf{u}_h^{n+1}$ given by the Algorithm 2.1 satisfies the energy equality

$$\begin{aligned}
&\frac{1}{2}||\mathbf{u}_h^N||^2 + \triangle t\sum_{n=0}^{N-1}\Big\{(C_s\delta)^2\Big\|\big|\mathbb{D}[I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}\big|_F\Big\|_{L^3}^3 + 2\nu\Big\|\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2})\Big\|^2\Big\}\\
&\qquad = \frac{1}{2}||\mathbf{u}_h^0||^2 + \triangle t\sum_{n=0}^{N-1}\big(\mathbf{f}^{n+\frac{1}{2}}, \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}\big),
\end{aligned} \tag{36}$$

and the stability bound

$$\frac{1}{2}||\mathbf{u}_h^N||^2 + \triangle t \sum_{n=0}^{N-1} \Big\{ (C_s\delta)^2 \Big\| |\mathbb{D}[I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}|_F \Big\|_{L^3}^3 + \nu \Big\| \mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}) \Big\|^2 \Big\}$$

$$\le \frac{1}{2}||\mathbf{u}_h^0||^2 + \frac{\triangle t}{4\nu}\sum_{n=0}^{N-1} ||\mathbf{f}^{n+\frac{1}{2}}||_{H^{-1}}^2, \tag{37}$$

where $1 \le N \le N_T$.

Proof. Setting $\mathbf{v}_h = \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}$ in (28), and $q_h = p_h^{n+\frac{1}{2}}$ in the second equation of both (28) and (30), this gives

$$\frac{1}{2\triangle t}\big(||\mathbf{w}_h^{n+1}||^2 - ||\mathbf{u}_h^n||^2\big) + 2\nu \Big\| \mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}) \Big\|^2 = \Big(\mathbf{f}^{n+\frac{1}{2}}, \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2} \Big).$$

Lemma 2.2 then gives

$$\begin{aligned} \frac{1}{2\triangle t}\big(||\mathbf{u}_h^{n+1}||^2 - ||\mathbf{u}_h^n||^2\big) &+ (C_s\delta)^2 \Big\| |\mathbb{D}[I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}|_F \Big\|_{L^3}^3 \\ &+ 2\nu \Big\| \mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}) \Big\|^2 = \Big(\mathbf{f}^{n+\frac{1}{2}}, \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2} \Big). \end{aligned}$$

Summing over n establishes the energy equality. Using the Cauchy-Schwarz and Young's inequalities on the right-hand side, subsuming one term into the left-hand side gives

$$\begin{aligned} \frac{1}{2\triangle t}(||\mathbf{u}_h^{n+1}||^2 - ||\mathbf{u}_h^n||^2) + (C_s\delta)^2 \Big\| |\mathbb{D}[I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}|_F \Big\|_{L^3}^3 \\ +\nu \Big\| \mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}) \Big\|^2 \le \frac{1}{4\nu}||\mathbf{f}^{n+\frac{1}{2}}||_{H^{-1}}^2. \end{aligned}$$

Summing over n, the global stability estimate follows. □

The viscous and VMS model dissipation in the method are respectively

$$\begin{aligned} Viscous\ dissipation := &\quad \nu \Big\| \mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}) \Big\|^2, \\ VMS\ model\ dissipation := &\quad (C_s\delta)^2 \Big\| |\mathbb{D}[I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}|_F \Big\|_{L^3}^3. \end{aligned}$$

Furthermore, we also prove the stability for $\mathbf{w}_h^N$.

Proposition 2.4 The approximate velocity $\mathbf{w}_h^{n+1}$ of Algorithm 2.1 satisfies

$$\frac{1}{2}||\mathbf{w}_h^N||^2 + \triangle t \sum_{n=0}^{N-2} (C_s\delta)^2 \Big\| |\mathbb{D}[I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}|_F \Big\|_{L^3}^3$$

$$\le \frac{1}{2}||\mathbf{u}_h^0||^2 + \frac{\triangle t}{4\nu}\sum_{n=0}^{N-1} ||\mathbf{f}^{n+\frac{1}{2}}||_{H^{-1}}^2. \tag{38}$$

Proof. For $n = N-1$, a directly application of Lemma 2.2 gives

$$\frac{1}{2}||\mathbf{w}_h^N||^2 = \frac{1}{2}||\mathbf{u}_h^N||^2 + 2\triangle t(C_s\delta)^2 \Big\| |\mathbb{D}[I-P_H]\frac{\mathbf{w}_h^N+\mathbf{u}_h^N}{2}|_F \Big\|_{L^3}^3.$$

Using this in Proposition 2.3 proves the claim. □

3. Error Estimate

We give an error analysis for the uncoupled method. It is difficult within present tools to develop an error analysis that is directly relevant to the case of turbulent flows. Turbulence is a flow phenomena that develops from smooth data and initial conditions over longer time intervals due to the nonlinear energy cascade. Simulations of turbulent flows are commonly initialized by smooth, compatible (in the sense used by Heywood and Rannacher [23, 24, 25]), statistically stationary initial conditions. The generation of these initial conditions requires a separate spin-up procedure, see, e.g., [4, 17] for some examples. We shall thus make the assumption that such initial conditions are given:

Initialization Assumption: The initial conditions are generated so the solution is regular down to $t = 0$, in particular satisfying:

$$\begin{aligned}
\mathbf{u} &\in L^\infty(0,T;W_4^{k+1}(\Omega)) \cap H^1(0,T;H^{k+1}(\Omega)) \\
&\quad \cap H^3(0,T;L^2(\Omega)) \cap W_4^2(0,T;H^1(\Omega)), \qquad (39)\\
p &\in L^\infty(0,T;H^k(\Omega)),\ \mathbf{f} \in H^2(0,T;L^2(\Omega)). \qquad (40)
\end{aligned}$$

We also introduce the following discrete norms:

$$\begin{aligned}
|||\mathbf{v}|||_{\infty,k} &= \max_{0\le n\le N^T} ||\mathbf{v}^n||_k, \\
|||\mathbf{v}|||_{m,k} &= \Big(\triangle t \sum_{n=0}^{N_T} ||\mathbf{v}^n||_k^m\Big)^{1/m}, \\
|||\mathbf{v}_{1/2}|||_{m,k} &= \Big(\triangle t \sum_{n=0}^{N_T} ||\mathbf{v}^{n+1/2}||_k^m\Big)^{1/m}.
\end{aligned}$$

For compactness, we denote

$$\tilde{\mathbf{w}}_h^{n+\frac{1}{2}} = \frac{\mathbf{w}_h^{n+1} + \mathbf{u}_h^n}{2}.$$

To begin the analysis we rewrite Algorithm 2.1 as: find $\mathbf{w}_h^{n+1}, \mathbf{u}_h^{n+1} \in \mathbf{V}^h$ such that for all $\mathbf{v}_h \in \mathbf{V}^h$:

$$\begin{aligned}
&(\mathbf{w}_h^{n+1} - \mathbf{u}_h^n, \mathbf{v}_h) + \triangle t b_s(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \mathbf{v}_h) \\
&\qquad + 2\triangle t\nu\big(\mathbb{D}(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}), \mathbb{D}(\mathbf{v}_h)\big) = \triangle t(\mathbf{f}^{n+\frac{1}{2}}, \mathbf{v}_h), \qquad (41)\\
&\Big(\frac{\mathbf{u}_h^{n+1} - \mathbf{w}_h^{n+1}}{\triangle t}, \mathbf{v}_h\Big) \\
&\quad + \Big(\nu_T\big(\frac{\mathbf{w}_h^{n+1} + \mathbf{u}_h^{n+1}}{2}\big)\mathbb{D}[I - P_H]\frac{\mathbf{w}_h^{n+1} + \mathbf{u}_h^{n+1}}{2}, \mathbb{D}[I - P_H]\mathbf{v}_h\Big) = 0. \qquad (42)
\end{aligned}$$

Let $\mathbf{u}$ satisfy the weak formulation in the form (27). Then, at $t^{n+\frac{1}{2}}$ we have

$$\begin{aligned}
(43)\quad &(\mathbf{u}^{n+1} - \mathbf{u}^n, \mathbf{v}_h) + 2\triangle t\nu(\mathbb{D}(\mathbf{u}^{n+\frac{1}{2}}), \mathbb{D}(\mathbf{v}_h)) + \triangle t b_s(\mathbf{u}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}}, \mathbf{v}_h) \\
&\qquad + (p(t^{n+\frac{1}{2}}), \nabla \cdot \mathbf{v}_h) = \triangle t(\mathbf{f}^{n+\frac{1}{2}}, \mathbf{v}_h) + \triangle t R(\mathbf{u}^{n+1}, \mathbf{v}_h),
\end{aligned}$$

for all $\mathbf{v}_h \in \mathbf{V}^h$, where $R(\mathbf{u}^{n+1}, \mathbf{v}_h)$ represents the consistency and interpolation error, i.e.

$$\begin{aligned} R(\mathbf{u}^{n+1}, \mathbf{v}_h) = &\Big(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\triangle t} - \mathbf{u}_t(t^{n+\frac{1}{2}}), \mathbf{v}_h\Big) + \big(\mathbf{f}(t^{n+\frac{1}{2}}) - \mathbf{f}^{n+\frac{1}{2}}, \mathbf{v}_h\big) \\ &+ b_s\big(\mathbf{u}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}}, \mathbf{v}_h\big) - b_s\big(\mathbf{u}(t^{n+\frac{1}{2}}), \mathbf{u}(t^{n+\frac{1}{2}}), \mathbf{v}_h\big) \\ &+ 2\nu\Big(\mathbb{D}(\mathbf{u}^{n+\frac{1}{2}}) - \mathbb{D}(\mathbf{u}(t^{n+\frac{1}{2}})), \mathbb{D}(\mathbf{v}_h)\Big). \end{aligned} \tag{44}$$

We split the errors into the following parts:

$$\mathbf{u}^{n+1} - \mathbf{w}_h^{n+1} = (\mathbf{u}^{n+1} - I_h\mathbf{u}^{n+1}) + (I_h\mathbf{u}^{n+1} - \mathbf{w}_h^{n+1}) \triangleq \Lambda^{n+1} + \varepsilon_h^{n+1}, \tag{45}$$

$$\mathbf{u}^{n+1} - \mathbf{u}_h^{n+1} = (\mathbf{u}^{n+1} - I_h\mathbf{u}^{n+1}) + (I_h\mathbf{u}^{n+1} - \mathbf{u}_h^{n+1}) \triangleq \Lambda^{n+1} + e_h^{n+1}, \tag{46}$$

where $I_h\mathbf{u}^{n+1} \in \mathbf{X}^h$ will be an interpolation of $\mathbf{u}^{n+1}$ in $\mathbf{X}^h$.

Theorem 3.1 Let $\mathbf{u}, p$ and $\mathbf{f}$ satisfy the regularity assumptions (39)-(40). Let $\mathbf{u}_h^n, \mathbf{w}_h^n$ be given by Algorithm 2.1. Then, for $\triangle t$ sufficiently small, i.e.,

$$C_4\triangle t\Big(1 + \frac{C(\Omega)}{\nu^3}||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4\Big) \le 1,$$

we have

$$\begin{aligned} &\frac{1}{2}||\mathbf{u}^N - \mathbf{u}_h^N||^2 + \frac{1}{2}||\mathbf{u}^N - \mathbf{w}_h^N||^2 + \nu\triangle t\sum_{n=0}^{N-1}\Big\|\mathbb{D}(\mathbf{u}(t^{n+\frac{1}{2}}) - \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2})\Big\|^2 \\ &\le C(T)E(\triangle t, h, H, \nu, \delta) + Ch^{2k+2}|||\mathbf{u}|||^2_{\infty,k+1} + C\nu h^{2k}|||\mathbf{u}_{1/2}|||^2_{2,k+1}, \end{aligned}$$

for all $1 \le N \le N_T$, where $C(T)$ is a constant depending on T and

$$\begin{aligned} E(\triangle t, h, H, \nu, \delta) = &\tilde{C}\nu^{-1}\big(h^{2k+1}|||\mathbf{u}|||^4_{4,k+1} + h^{2k+1}|||\nabla\mathbf{u}_{1/2}|||^4_{4,0}\big) \\ &+\tilde{C}h^{2k+2}|||\mathbf{u}_t|||^2_{2,k+1} + \tilde{C}\nu^{-2}h^{2k}|||\mathbf{u}|||^2_{\infty,k+1} + \tilde{C}\nu h^{2k}|||\mathbf{u}|||^2_{2,k+1} \\ &+\tilde{C}(C_s\delta)^2H^{3l}h^{-\frac{d}{2}}|||\mathbf{u}|||^3_{3,l+1} + \tilde{C}\triangle t^4\nu^{-1}\big(|||\nabla\mathbf{u}|||^4_{4,0} + |||\nabla\mathbf{u}_{1/2}|||^4_{4,0}\big) \\ &+\tilde{C}\nu^{-1}\triangle t^5|||\nabla\mathbf{u}_{tt}(s)|||^4_{\infty,0} + \tilde{C}\triangle t^5|||\mathbf{u}_{tttt}|||^2_{2,0} \\ &+\tilde{C}\nu\triangle t^5|||\nabla\mathbf{u}_{tt}(s)|||^2_{\infty,0} + \tilde{C}\triangle t^5|||\mathbf{f}_{tt}(s)|||^2_{\infty,0}. \end{aligned}$$

Proof. First by subtracting (41) from (43) gives

$$\begin{aligned} &\Big((\mathbf{u}^{n+1} - \mathbf{w}_h^{n+1}) + (\mathbf{u}_h^n - \mathbf{u}^n), \mathbf{v}_h\Big) + \triangle t b_s\big(\mathbf{u}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}}, \mathbf{v}_h\big) \\ &+2\triangle t\nu\Big(\mathbb{D}\big(\frac{(\mathbf{u}^{n+1} - \mathbf{w}_h^{n+1}) + (\mathbf{u}^n - \mathbf{u}_h^n)}{2}\big), \mathbb{D}(\mathbf{v}_h)\Big) - \triangle t b_s\big(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \mathbf{v}_h\big) \\ &= \triangle t(p(t^{n+\frac{1}{2}}) - q_h, \nabla\cdot\mathbf{v}_h) + \triangle t R(\mathbf{u}^{n+1}, \mathbf{v}_h). \end{aligned} \tag{47}$$

Choosing $\mathbf{v}_h = \frac{\varepsilon_h^{n+1}+e_h^n}{2}$ derives

$$\begin{aligned} &\frac{1}{2}\big(||\varepsilon_h^{n+1}||^2 - ||e_h^n||^2\big) + 2\triangle t\nu\Big\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big\|^2 \\ &= -\Big(\Lambda^{n+1} - \Lambda^n, \frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big) - 2\triangle t\nu\Big(\mathbb{D}(\Lambda^{n+\frac{1}{2}}), \mathbb{D}\big(\frac{\varepsilon_h^{n+1}+e_h^n}{2}\big)\Big) \\ &\quad -\triangle t b_s\Big(\mathbf{u}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big) + \triangle t b_s\Big(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big) \\ &\quad +\triangle t\Big(p(t^{n+\frac{1}{2}}) - q_h, \nabla\cdot\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big) + \triangle t R(\mathbf{u}^{n+1}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}). \end{aligned} \tag{48}$$

We want to bound the terms on the right-hand side of (48). Consider first the convection term in (48). Adding and subtracting the term $b_s(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2})$, taking (25) into account, then the trilinear terms can be rewritten as follows:

$$
\begin{aligned}
& b_s\Big(\mathbf{u}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big) - b_s\Big(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big) \\
&= b_s\Big(\frac{\mathbf{u}^{n+1}-\mathbf{w}_h^{n+1}+\mathbf{u}^n-\mathbf{u}_h^n}{2}, \mathbf{u}^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big) \\
&\quad + b_s\Big(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \frac{\mathbf{u}^{n+1}-\mathbf{w}_h^{n+1}+\mathbf{u}^n-\mathbf{u}_h^n}{2}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big) \\
&= b_s\Big(\Lambda^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big) + b_s\Big(\frac{\varepsilon_h^{n+1}+e_h^n}{2}, \mathbf{u}^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big) \\
&\quad + b_s\Big(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \Lambda^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big).
\end{aligned}
$$

By using (26), Young's and Korn's inequalities, we have

$$
\begin{aligned}
& b_s(\Lambda^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}) \\
&\le C\Big\|\Lambda^{n+\frac{1}{2}}\Big\|^{\frac{1}{2}}\Big\|\nabla\Lambda^{n+\frac{1}{2}}\Big\|^{\frac{1}{2}}\Big\|\nabla\mathbf{u}^{n+\frac{1}{2}}\Big\|\Big\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big\| \\
&\le \frac{\nu}{10}\Big\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big\|^2 + \frac{C}{\nu}\|\Lambda^{n+\frac{1}{2}}\|\|\nabla\Lambda^{n+\frac{1}{2}}\|\|\nabla\mathbf{u}^{n+\frac{1}{2}}\|^2,
\end{aligned}
\tag{49}
$$

and

$$
\begin{aligned}
& b_s\Big(\frac{\varepsilon_h^{n+1}+e_h^n}{2}, \mathbf{u}^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big) \\
&\le C\Big\|\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big\|^{\frac{1}{2}}\Big\|\nabla\mathbf{u}^{n+\frac{1}{2}}\Big\|\Big\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big\|^{\frac{3}{2}} \\
&\le \frac{\nu}{10}\Big\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big\|^2 + \frac{C}{\nu^3}||\nabla\mathbf{u}^{n+\frac{1}{2}}||^4\Big(||\varepsilon_h^{n+1}||^2 + ||e_h^n||^2\Big).
\end{aligned}
\tag{50}
$$

as well as

$$
\begin{aligned}
b_s\Big(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \Lambda^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big) &\le C\|\nabla\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}\|\|\nabla\Lambda^{n+\frac{1}{2}}\|\Big\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big\| \\
&\le \frac{\nu}{10}\Big\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big\|^2 + \frac{C}{\nu}\|\nabla\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}\|^2|\nabla\Lambda^{n+\frac{1}{2}}\|^2.
\end{aligned}
\tag{51}
$$

The remaining terms in (48) are estimated by Cauchy-Schwarz, Young's and Minkowski's inequalities as follows:

$$
\begin{aligned}
\Big(\Lambda^{n+1}-\Lambda^n, \frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big) &= \triangle t\Big(\frac{\Lambda^{n+1}-\Lambda^n}{\triangle t}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big) \\
&\le \frac{\triangle t}{4}\Big\|\frac{\Lambda^{n+1}-\Lambda^n}{\triangle t}\Big\|^2 + \triangle t\Big\|\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big\|^2 \\
&= \frac{\triangle t}{4}\int_\Omega\Big(\frac{1}{\triangle t}\int_{t^n}^{t^{n+1}}\Lambda_t dt\Big)^2 d\mathbf{x} + \triangle t\Big\|\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big\|^2 \\
&\le \frac{\triangle t}{4}\int_\Omega\Big(\frac{1}{\triangle t}\int_{t^n}^{t^{n+1}}|\Lambda_t|^2 dt\Big) d\mathbf{x} + \triangle t\Big\|\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big\|^2 \\
&\le \frac{1}{4}\int_{t^n}^{t^{n+1}}||\Lambda_t||^2 dt + \frac{\triangle t}{2}\big(||\varepsilon_h^{n+1}||^2 + ||e_h^n||^2\big),
\end{aligned}
\tag{52}
$$

and

$$\begin{aligned}
&2\triangle t\nu\Big(\mathbb{D}(\Lambda^{n+\frac{1}{2}}),\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big)\\
&\leq \frac{\nu\triangle t}{10}\Big\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big\|^2+C\nu\triangle t\big\|\nabla\Lambda^{n+\frac{1}{2}}\big\|^2, \qquad (53)
\end{aligned}$$

and

$$\begin{aligned}
&\triangle t\Big(p(t^{n+\frac{1}{2}})-q_h,\nabla\cdot\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big)\\
&\leq \triangle t\big\|p(t^{n+\frac{1}{2}})-q_h\big\|\Big\|\nabla\cdot\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big\|\\
&\leq \frac{\nu\triangle t}{10}\Big\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big\|^2+\frac{C}{\nu}\big\|p(t^{n+\frac{1}{2}})-q_h\big\|^2. \qquad (54)
\end{aligned}$$

For the last term in (48), we present its estimate in the following lemma.

Lemma 3.2 There holds

$$\begin{aligned}
&\triangle tR\Big(\mathbf{u}^{n+1},\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big)\\
&\leq \frac{\triangle t}{2}\Big(||\varepsilon_h^{n+1}||^2+||e_h^n||^2\Big)+\frac{\nu\triangle t}{4}\Big\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big\|^2\\
&\quad+\frac{C\triangle t^5}{\nu}\Big(||\nabla\mathbf{u}^{n+\frac{1}{2}}||^4+||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4+\max_{t^n\leq t\leq t^{n+1}}||\nabla\mathbf{u}_{tt}(t)||^4\Big)\\
&\quad+C\triangle t^5\max_{t^n\leq t\leq t^{n+1}}||\mathbf{u}_{ttt}(t)||^2+\frac{\nu\triangle t^5}{2}\max_{t^n\leq t\leq t^{n+1}}||\nabla\mathbf{u}_{tt}(t)||^2\\
&\quad+C\triangle t^5\max_{t^n\leq t\leq t^{n+1}}||\mathbf{f}_{tt}(t)||^2. \qquad (55)
\end{aligned}$$

Proof. We estimate every term in the definition (44) of $R(\cdot,\cdot)$ as follows. Since

$$\Big\|\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\triangle t}-\mathbf{u}_t(t^{n+\frac{1}{2}})\Big\|\leq\frac{\triangle t^2}{48}\max_{t^n\leq t\leq t^{n+1}}||\mathbf{u}_{ttt}(t)||, \qquad (56)$$

then by using the Young's and Cauchy-Schwarz inequalities, we have

$$\begin{aligned}
&\Big(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\triangle t}-\mathbf{u}_t(t^{n+\frac{1}{2}}),\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big)\\
&\qquad\leq\frac{1}{4}||\varepsilon_h^{n+1}||^2+\frac{1}{4}||e_h^n||^2+C\triangle t^4\max_{t^n\leq t\leq t^{n+1}}||\mathbf{u}_{ttt}(t)||^2.
\end{aligned}$$

Similarly, since

$$||\nabla\mathbf{u}^{n+\frac{1}{2}}-\nabla\mathbf{u}(t^{n+\frac{1}{2}})||\leq\frac{\triangle t^2}{8}\max_{t^n\leq t\leq t^{n+1}}||\nabla\mathbf{u}_{tt}(t)||, \qquad (57)$$

then

$$\begin{aligned}
&2\nu\Big(\mathbb{D}(\mathbf{u}^{n+\frac{1}{2}})-\mathbb{D}(\mathbf{u}(t^{n+\frac{1}{2}})),\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big)\\
&\qquad\leq\frac{\nu}{8}\Big\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big\|^2+\frac{\nu\triangle t^4}{2}\max_{t^n\leq t\leq t^{n+1}}\big\|\nabla\mathbf{u}_{tt}(t)\big\|^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}&\Big(\mathbf{f}(t^{n+\frac12})-\mathbf{f}^{n+\frac12},\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big)\\
&\le\frac14||\varepsilon_h^{n+1}||^2+\frac14||e_h^n||^2+C\triangle t^4\max_{t^n\le t\le t^{n+1}}||\mathbf{f}_{tt}(t)||^2.\end{aligned}\tag{58}$$

For the trilinear form, we add and subtract the term $b_s\Big(\mathbf{u}(t^{n+\frac12}),\mathbf{u}^{n+\frac12},\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big)$ first, then by using (26), Young's and Minkowski's inequalities gives

$$\begin{aligned}&b_s\Big(\mathbf{u}^{n+\frac12},\mathbf{u}^{n+\frac12},\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big)-b_s\Big(\mathbf{u}(t^{n+\frac12}),\mathbf{u}(t^{n+\frac12}),\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big)\\
&=b_s\Big(\mathbf{u}^{n+\frac12}-\mathbf{u}(t^{n+\frac12}),\mathbf{u}^{n+\frac12},\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big)\\
&\quad+b_s\Big(\mathbf{u}(t^{n+\frac12}),\mathbf{u}^{n+\frac12}-\mathbf{u}(t^{n+\frac12}),\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big)\\
&\le C\Big\|\nabla\big(\mathbf{u}^{n+\frac12}-\mathbf{u}(t^{n+\frac12})\big)\Big\|\Big\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big\|\Big(\big\|\nabla\mathbf{u}^{n+\frac12}\big\|+\big\|\nabla\mathbf{u}(t^{n+\frac12})\big\|\Big)\\
&\le\frac{\nu}{8}\Big\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big\|^2\\
&\quad+\frac{C\triangle t^4}{\nu}\max_{t^n\le t\le t^{n+1}}\big\|\nabla\mathbf{u}_{tt}(t)\big\|^2\Big(\big\|\nabla\mathbf{u}^{n+\frac12}\big\|+\big\|\nabla\mathbf{u}(t^{n+\frac12})\big\|\Big)^2\\
&\le\frac{\nu}{8}\Big\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big\|^2\\
&\quad+\frac{C\triangle t^4}{\nu}\Big(\max_{t^n\le t\le t^{n+1}}||\nabla\mathbf{u}_{tt}(t)||^4+||\nabla\mathbf{u}^{n+\frac12}||^4+||\nabla\mathbf{u}(t^{n+\frac12})||^4\Big).\end{aligned}\tag{59}$$

Combining the estimates (57)-(59) together gives the Lemma. □

Next, by combining Lemma 3.2 with (48)-(53) gives

$$\begin{aligned}&\frac12\Big(||\varepsilon_h^{n+1}||^2-||e_h^n||^2\Big)+\triangle t\nu\Big\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big\|^2\\
&\le\triangle t\Big(1+\frac{C}{\nu^3}||\nabla\mathbf{u}(t^{n+\frac12})||^4\Big)\Big(||\varepsilon_h^{n+1}||^2+||e_h^n||^2\Big)\\
&\quad+C\triangle t\nu\big\|\nabla\Lambda^{n+\frac12}\big|^2+\frac{C}{\nu}\big\|\nabla\tilde{\mathbf{w}}_h^{n+\frac12}\big\|^2\big\|\nabla\Lambda^{n+\frac12}\big\|^2+C\triangle t^5\max_{t^n\le t\le t^{n+1}}||\mathbf{u}_{ttt}(t)||^2\\
&\quad+\frac{C}{\nu}\big\|\Lambda^{n+\frac12}\big\|\big\|\nabla\Lambda^{n+\frac12}\big\|\big\|\nabla\mathbf{u}^{n+\frac12}\big\|^2+\frac14\int_{t^n}^{t^{n+1}}|\Lambda_t||^2dt+\frac{C}{\nu}\big\|p(t^{n+\frac12})-q_h\big\|^2\\
&\quad+\frac{C\triangle t^5}{\nu}\Big(\big\|\nabla\mathbf{u}^{n+\frac12}\big\|^4+\big\|\nabla\mathbf{u}(t^{n+\frac12})\big\|^4+\max_{t^n\le t\le t^{n+1}}\big\|\nabla\mathbf{u}_{tt}(t)\big\|^4\Big)\\
&\quad+\frac{\nu\triangle t^5}{2}\max_{t^n\le t\le t^{n+1}}\big\|\nabla\mathbf{u}_{tt}(t)\big\|^2+C\triangle t^5\max_{t^n\le t\le t^{n+1}}\big\|\mathbf{f}_{tt}(t)\big\|^2.\end{aligned}\tag{60}$$

To estimate e_h^n, we formulate the relationship between ε_h^n and e_h^n in the next step. Since $\mathbf{u}_h^{n+1}$ and $\mathbf{w}_h^{n+1}$ are connected through the variational multiscale equation, we take $\mathbf{v}_h=\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2}$ in (42). Note that $\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}=I_h\mathbf{u}^{n+1}-\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2}$, with $I_h\mathbf{u}^{n+1}=\mathbf{u}^{n+1}-\Lambda^{n+1}$. For notational simplicity, we denote

$$\alpha=[I-P_H]I_h\mathbf{u}^{n+1},\ \beta=[I-P_H]\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2}.$$

By using monotonicity (31) and Lipschitz continuity (32) as well as Young's inequality with exponents 3 and 3/2 , we get

$$\begin{aligned}
&\frac{1}{2\triangle t}\Big(||e_h^{n+1}||^2-||\varepsilon_h^{n+1}||^2\Big)=(C_s\delta)^2\Big(\big|\mathbb{D}(\alpha-\beta)\big|_F\mathbb{D}(\alpha-\beta),\mathbb{D}(\beta)\Big)\\
= \;& -(C_s\delta)^2\Big(\big|\mathbb{D}(\beta-\alpha)\big|_F\mathbb{D}(\beta-\alpha)-\big|\mathbb{D}(-\alpha)\big|_F\mathbb{D}(-\alpha),\mathbb{D}(\beta)\Big)\\
&+(C_s\delta)^2\Big(\big|\mathbb{D}(\alpha)\big|_F\mathbb{D}(\alpha),\mathbb{D}(\beta)\Big)\\
\le \;& -C(C_s\delta)^2\big\|\mathbb{D}(\beta)\big\|_{L^3}^3+C(C_s\delta)^2\big\|\mathbb{D}(\alpha)\big\|_{L^3}^2\big\|\mathbb{D}(\beta)\big\|_{L^3}\\
\le \;& -C(C_s\delta)^2\big\|\mathbb{D}(\beta)\big\|_{L^3}^3+\frac{C(C_s\delta)^2}{2}\big\|\mathbb{D}(\beta)\big\|_{L^3}^3+C(C_s\delta)^2\big\|\mathbb{D}(\alpha)\big\|_{L^3}^3\\
\le \;& -\frac{C(C_s\delta)^2}{2}\big\|\mathbb{D}(\beta)\big\|_{L^3}^3+C(C_s\delta)^2\big\|\mathbb{D}(\alpha)\big\|_{L^3}^3.
\end{aligned}$$

This means

$$\begin{aligned}
(61)\quad \frac{1}{2}\big\|\varepsilon_h^{n+1}\big\|^2\ge\frac{1}{2}\big\|e_h^{n+1}\big\|^2 \;&+\; \frac{C(C_s\delta)^2\triangle t}{2}\Big\|\mathbb{D}[I-P_H](\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2})\Big\|_{L^3}^3\\
&-\; C(C_s\delta)^2\triangle t\Big\|\mathbb{D}([I-P_H]I_h\mathbf{u}^{n+1})\Big\|_{L^3}^3.
\end{aligned}$$

On the other hand, note $\beta=\alpha-\beta-(\alpha-2\beta)$. By repeated application of monotonicity (31) and Lipschitz continuity (32), as well as Young's inequality with exponents 3 and 3/2 and Minkowski's inequality gives

$$\begin{aligned}
&\frac{1}{2\triangle t}\Big(||e_h^{n+1}||^2-||\varepsilon_h^{n+1}||^2\Big)=(C_s\delta)^2\Big(\big|\mathbb{D}(\alpha-\beta)\big|_F\mathbb{D}(\alpha-\beta),\mathbb{D}(\beta)\Big)\\
&=(C_s\delta)^2\Big(\big|\mathbb{D}(\alpha-\beta)\big|_F\mathbb{D}(\alpha-\beta),\mathbb{D}(\alpha-\beta)\Big)\\
&\quad-(C_s\delta)^2\Big(\big|\mathbb{D}(\alpha-\beta)\big|_F\mathbb{D}(\alpha-\beta),\mathbb{D}(\alpha-2\beta)\Big)\\
&\ge C(C_s\delta)^2\big\|\mathbb{D}(\alpha-\beta)\big\|_{L^3}^3-C(C_s\delta)^2\big\|\mathbb{D}(\alpha-\beta)\big\|_{L^3}^2\big\|\mathbb{D}(\alpha-2\beta)\big\|_{L^3}\\
&\ge C(C_s\delta)^2\big\|\mathbb{D}(\alpha-\beta)\big\|_{L^3}^3-C(C_s\delta)^2\big\|\mathbb{D}(\alpha-\beta)\big\|_{L^3}^3-C(C_s\delta)^2\big\|\mathbb{D}(\alpha-2\beta)\big\|_{L^3}^3\\
&\ge -C(C_s\delta)^2\Big(\big\|\mathbb{D}(\alpha)\big\|_{L^3}+2\big\|\mathbb{D}(\beta)\big\|_{L^3}\Big)^3\\
&\ge -C(C_s\delta)^2\big\|\mathbb{D}(\alpha)\big\|_{L^3}^3-C(C_s\delta)^2\big\|\mathbb{D}(\beta)\big\|_{L^3}^3.
\end{aligned}$$

Here we use $(a+b)^3\le 4(a^3+b^3), a\ge 0, b\ge 0$. The above inequality implies

$$\begin{aligned}
(62)\qquad ||\varepsilon_h^{n+1}||^2\le||e_h^{n+1}||^2 \;&+\; \triangle tC(C_s\delta)^2\big\|\mathbb{D}([I-P_H]I_h\mathbf{u}^{n+1})\big\|_{L^3}^3\\
&+\; \triangle tC(C_s\delta)^2\Big\|\mathbb{D}([I-P_H]\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2})\Big\|_{L^3}^3.
\end{aligned}$$

Substitute (61)-(62) into (60) and assume $||e_h^0|| = 0, i.e., u^0 \in \mathbf{X}^h$. We obtain

$$
\begin{aligned}
&\frac{1}{2}\Big(||e_h^{n+1}||^2 - ||e_h^n||^2\Big) + \triangle t\nu\Big\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\Big\|^2 \\
&\quad +2\triangle tC(C_s\delta)^2\Big\|\mathbb{D}([I-P_H]\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2})\Big\|_{L^3}^3 \\
&\le \triangle t\Big(1+\frac{C}{\nu^3}||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4\Big)\Big(||e_h^{n+1}||^2+||e_h^n||^2\Big) + C\nu\triangle t||\nabla\Lambda^{n+\frac{1}{2}}||^2 \\
&\quad +\triangle t^2C(C_s\delta)^2\Big(1+\frac{C(\Omega)}{\nu^3}||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4\Big)\big\|\mathbb{D}([I-P_H](\mathbf{u}^{n+1}-\Lambda^{n+1}))\big\|_{L^3}^3 \\
&\quad +\triangle t^2C(C_s\delta)^2\Big(1+\frac{C(\Omega)}{\nu^3}||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4\Big)\Big\|\mathbb{D}([I-P_H]\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2})\Big\|_{L^3}^3 \\
&\quad +\triangle tC(C_s\delta)^2\big\|\mathbb{D}([I-P_H](\mathbf{u}^{n+1}-\Lambda^{n+1}))\big\|_{L^3}^3 + \frac{C\triangle t}{\nu}\big\|\nabla\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}\big\|^2\big\|\nabla\Lambda^{n+\frac{1}{2}}\big\|^2 \\
&\quad +\frac{C\triangle t}{\nu}\big\|\Lambda^{n+\frac{1}{2}}\big\|\big\|\nabla\Lambda^{n+\frac{1}{2}}\big\|\big\|\nabla\mathbf{u}^{n+\frac{1}{2}}\big\|^2 + \frac{1}{4}\int_{t^n}^{t^{n+1}}||\Lambda_t||^2dt + \frac{C}{\nu}\big\|p(t^{n+\frac{1}{2}})-q_h\big\|^2 \\
&\quad +\frac{C\triangle t^5}{\nu}\Big(\big\|\nabla\mathbf{u}^{n+\frac{1}{2}}\big\|^4+\big\|\nabla\mathbf{u}(t^{n+\frac{1}{2}})\big\|^4+\max_{t^n\le t\le t^{n+1}}\big\|\nabla\mathbf{u}_{tt}(t)\big\|^4\Big) \\
&\quad +\frac{\nu\triangle t^5}{2}\max_{t^n\le t\le t^{n+1}}\big\|\nabla\mathbf{u}_{tt}(t)\big\|^2 + C\triangle t^5\max_{t^n\le t\le t^{n+1}}\big\|\mathbf{f}_{tt}(t)\big\|^2 \\
&\quad +C\triangle t^5\max_{t^n\le t\le t^{n+1}}||\mathbf{u}_{ttt}(t)||^2. \qquad (63)
\end{aligned}
$$

We assume that $\triangle t$ is small enough such that $\triangle t\Big(1+\frac{C}{\nu^3}||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4\Big) \le 1$. We can "absorb" the terms stemming from the VMS method on the right-hand side into the last term on the left-hand side. Summing (63) from $n=0$ to $n=N-1$ and using Minkowski's inequality results in

$$
\begin{aligned}
&\frac{1}{2}||e_h^N||^2 + \triangle t\sum_{n=0}^{N-1}\Big(\nu\Big\|\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big\|^2 + C(C_s\delta)^2\Big\|\mathbb{D}([I-P_H]\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2})\Big\|_{L^3}^3\Big) \\
&\le \triangle t\Big(1+\frac{C}{\nu^3}||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4\Big)\sum_{n=0}^{N-1}\big\|e_h^{n+1}\big\|^2 + C\nu\triangle t\sum_{n=0}^{N-1}\big\|\nabla\Lambda^{n+\frac{1}{2}}\big\|^2 \\
&\quad +\triangle tC(C_s\delta)^2\sum_{n=0}^{N-1}\Big\{\big\|\mathbb{D}([I-P_H]\mathbf{u}^{n+1}\big\|_{L^3}^3 + \big\|\mathbb{D}([I-P_H]\Lambda^{n+1}\big\|_{L^3}^3\Big\} \\
&\quad +\frac{1}{4}\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}||\Lambda_t||^2dt + \frac{C}{\nu}\big\|p(t^{n+\frac{1}{2}})-q_h\big\|^2 + C\triangle t^5\sum_{n=0}^{N-1}\max_{t^n\le t\le t^{n+1}}||\mathbf{f}_{tt}(t)||^2 \\
&\quad +\frac{C\triangle t}{\nu}\sum_{n=0}^{N-1}\big\|\nabla\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}\big\|^2\big\|\nabla\Lambda^{n+\frac{1}{2}}\big\|^2 + \frac{C\triangle t}{\nu}\sum_{n=0}^{N-1}\big\|\Lambda^{n+\frac{1}{2}}\big\|\big\|\nabla\Lambda^{n+\frac{1}{2}}||||\nabla\mathbf{u}^{n+\frac{1}{2}}\big\|^2 \\
&\quad +\frac{C\triangle t^5}{\nu}\sum_{n=0}^{N-1}\Big(||\nabla\mathbf{u}^{n+\frac{1}{2}}||^4+||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4+\max_{t^n\le t\le t^{n+1}}||\nabla\mathbf{u}_{tt}(t)||^4\Big) \\
&\quad +C\triangle t^5\sum_{n=0}^{N-1}\max_{t^n\le t\le t^{n+1}}||\mathbf{u}_{ttt}(t)||^2 + \frac{\nu\triangle t^5}{2}\sum_{n=0}^{N-1}\max_{t^n\le t\le t^{n+1}}||\nabla\mathbf{u}_{tt}(t)||^2.
\end{aligned}
$$

The terms on the right-hand side of the above inequality can be further simplified as follows,

$$C\nu\triangle t\sum_{n=0}^{N-1}||\nabla\Lambda^{n+\frac{1}{2}}||^2\leq C\nu\triangle t\sum_{n=0}^{N-1}h^{2k}|\mathbf{u}|_{k+1}^2\leq\tilde{C}\nu h^{2k}|||\mathbf{u}|||_{2,k+1}^2.$$

By using the boundedness of $\nu\triangle t\sum_{n=0}^{N-1}||\mathbb{D}\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}||^2$(Proposition 2.3) and Korn's inequality, we have

$$\begin{aligned}\frac{C\triangle t}{\nu}\sum_{n=0}^{N-1}\big\|\nabla\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}\big\|^2\big\|\nabla\Lambda^{n+\frac{1}{2}}\big\|^2 &\leq \frac{C\triangle t}{\nu}\sum_{n=0}^{N-1}||\nabla\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}||^2\big(||\nabla\Lambda^{n+1}||^2+||\nabla\Lambda^n||^2\big)\\ &\leq \frac{\tilde{C}\triangle t}{\nu}\sum_{n=0}^{N-1}\big\|\nabla\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}\big\|^2h^{2k}\big(|\mathbf{u}^{n+1}|_{k+1}^2+|\mathbf{u}^n|_{k+1}^2\big)\\ &\leq \tilde{C}\nu^{-2}h^{2k}|||\mathbf{u}|||_{\infty,k+1}^2.\end{aligned}$$

Next, by using Young's inequality, we have

$$\begin{aligned}&\frac{C\triangle t}{\nu}\sum_{n=0}^{N-1}\big\|\Lambda^{n+\frac{1}{2}}\big\|\big\|\nabla\Lambda^{n+\frac{1}{2}}\big\|\big\|\nabla\mathbf{u}^{n+\frac{1}{2}}\big\|^2\\ &\leq\frac{C\triangle t}{2\nu}\sum_{n=0}^{N-1}\Big(||\Lambda^{n+1}||||\nabla\Lambda^{n+1}||+||\Lambda^n||||\nabla\Lambda^n||\\ &\qquad+||\Lambda^n||||\nabla\Lambda^{n+1}||+||\Lambda^{n+1}||||\nabla\Lambda^n||\Big)||\nabla\mathbf{u}^{n+\frac{1}{2}}||^2\\ &\leq\tilde{C}\nu^{-1}h^{2k+1}\big(\triangle t\sum_{n=0}^{N-1}(|\mathbf{u}^{n+1}|_{k+1}^2+|\mathbf{u}^n|_{k+1}^2+|\mathbf{u}^{n+1}|_{k+1}|\mathbf{u}^n|_{k+1})||\nabla\mathbf{u}^{n+\frac{1}{2}}||^2\big)\\ &\leq\tilde{C}\nu^{-1}h^{2k+1}\big(\triangle t\sum_{n=0}^{N}|\mathbf{u}^n|_{k+1}^4+\triangle t\sum_{n=0}^{N-1}||\nabla\mathbf{u}^{n+\frac{1}{2}}||^4\big)\\ &\leq\tilde{C}\nu^{-1}h^{2k+1}(|||\mathbf{u}|||_{4,k+1}^4+|||\nabla\mathbf{u}_{1/2}|||_{4,0}^4),\end{aligned}$$

as well as

$$\frac{1}{4}\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}||\Lambda_t||^2dt\leq\frac{\tilde{C}}{4}\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}h^{2k+2}|\mathbf{u}_t|_{k+1}^2dt\leq\tilde{C}h^{2k+2}|||\mathbf{u}_t|||_{2,k+1}^2,$$

and

$$\frac{C\triangle t}{\nu}\sum_{n=0}^{N-1}||p(t^{n+\frac{1}{2}})-q_h||^2\leq\tilde{C}\triangle t\nu^{-1}\sum_{n=0}^{N-1}h^{2k}|p(t^{n+\frac{1}{2}})|_k^2\leq\tilde{C}\nu^{-1}h^{2k}|||p|||_{2,k}^2.$$

We use an inverse type inequality which relates $L^p(\Omega)^d-$norms of the gradients of finite element functions to be $L^2(\Omega)^d-$norms of the gradients: there exists a constant $C=C(p)$ such that for $2\leq p<\infty, d\in\{2,3\}$,

$$||\nabla\mathbf{v}^h||_{L^p}\leq Ch^{\frac{d}{2}(\frac{2-p}{p})}||\nabla\mathbf{v}^h||. \tag{64}$$

See, e.g., [35] for a proof. Combining with the property of projection P_{L^H} (13), we get

$$\begin{aligned}
&\triangle t C(C_s\delta)^2 \sum_{n=0}^{N-1} \left\|\mathbb{D}[I-P_H]\Lambda^{n+1}\right\|_{L^3}^3 \\
\leq\ & C(C_s\delta)^2 \triangle t h^{-\frac{d}{2}} \sum_{n=0}^{N-1} \left\|[I-P_{L^H}]\nabla\Lambda^{n+1}\right\|^3 \\
\leq\ & \tilde{C}(C_s\delta)^2 H^{3l} h^{-\frac{d}{2}} \triangle t \sum_{n=0}^{N-1} |\mathbf{u}^{n+1}|_{l+1}^3 \\
\leq\ & \tilde{C}(C_s\delta)^2 H^{3l} h^{-\frac{d}{2}} |||\mathbf{u}|||_{3,l+1}^3,
\end{aligned}$$

and

$$\triangle t C(C_s\delta)^2 \sum_{n=0}^{N-1} \left\|\mathbb{D}[I-P_H]\mathbf{u}^{n+1}\right\|_{L^3}^3 \leq \tilde{C}(C_s\delta)^2 H^{3l} h^{-\frac{d}{2}} |||\mathbf{u}|||_{3,l+1}^3.$$

Combining all above estimates, we derive

$$\begin{aligned}
\frac{1}{2}||e_h^N||^2 \quad + \quad & \triangle t \sum_{n=0}^{N-1} \Big(\nu\Big\|\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big\|^2 + C(C_s\delta)^2 \Big\|\mathbb{D}([I-P_H]\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2})\Big\|_{L^3}^3\Big) \\
& \leq \triangle t\Big(1+\frac{C}{\nu^3}||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4\Big) \sum_{n=0}^{N-1} \left\|e_h^{n+1}\right\|^2 \\
& +\tilde{C}\nu^{-1}\big(h^{2k+1}|||\mathbf{u}|||_{4,k+1}^4 + h^{2k+1}|||\nabla\mathbf{u}_{1/2}|||_{4,0}^4\big) + \tilde{C}\nu^{-1}h^{2k}|||p|||_{2,k}^2 \\
& +\tilde{C}h^{2k+2}|||\mathbf{u}_t|||_{2,k+1}^2 + \tilde{C}\nu^{-2}h^{2k}|||\mathbf{u}|||_{\infty,k+1}^2 + \tilde{C}\nu h^{2k}|||\mathbf{u}|||_{2,k+1}^2 \\
& +\tilde{C}(C_s\delta)^2 H^{3l} h^{-\frac{d}{2}} |||\mathbf{u}|||_{3,l+1}^3 + \tilde{C}\triangle t^4\nu^{-1}\big(|||\nabla\mathbf{u}|||_{4,0}^4 + |||\nabla\mathbf{u}_{1/2}|||_{4,0}^4\big) \\
& +\tilde{C}\nu^{-1}\triangle t^5|||\nabla\mathbf{u}_{tt}|||_{\infty,0}^4 + \tilde{C}\triangle t^5|||\mathbf{u}_{ttt}|||_{\infty,0}^2 \\
& +\tilde{C}\nu\triangle t^5|||\nabla\mathbf{u}_{tt}|||_{\infty,0}^2 + \tilde{C}\triangle t^5|||\mathbf{f}_{tt}|||_{\infty,0}^2.
\end{aligned} \tag{65}$$

Hence, with $\triangle t$ sufficiently small, i.e., $\triangle t < \Big(1+\frac{C}{\nu^3}||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4\Big)^{-1}$, from the Gronwall's inequality, we have

$$\begin{aligned}
& \frac{1}{2}||e_h^N||^2 + \triangle t \sum_{n=0}^{N-1} \Big(\nu\Big\|\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}\Big\|^2 + C(C_s\delta)^2 \Big\|\mathbb{D}([I-P_H]\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2})\Big\|_{L^3}^3\Big) \\
& \leq C(T)\Big\{\tilde{C}\nu^{-1}(h^{2k+1}|||\mathbf{u}|||_{4,k+1}^4 + h^{2k+1}|||\nabla\mathbf{u}_{1/2}|||_{4,0}^4) + \tilde{C}\nu^{-1}h^{2k}|||p|||_{2,k}^2 \\
& +\tilde{C}h^{2k+2}|||\mathbf{u}_t|||_{2,k+1}^2 + \tilde{C}\nu^{-2}h^{2k}|||\mathbf{u}|||_{\infty,k+1}^2 + \tilde{C}\nu h^{2k}|||\mathbf{u}|||_{2,k+1}^2 \\
& +\tilde{C}(C_s\delta)^2 H^{3l} h^{-\frac{d}{2}} |||\mathbf{u}|||_{3,l+1}^3 + \tilde{C}\triangle t^4\nu^{-1}\big(|||\nabla\mathbf{u}|||_{4,0}^4 + |||\nabla\mathbf{u}_{1/2}|||_{4,0}^4\big) \\
& +\tilde{C}\nu^{-1}\triangle t^5|||\nabla\mathbf{u}_{tt}|||_{\infty,0}^4 + \tilde{C}\triangle t^5|||\mathbf{u}_{ttt}|||_{\infty,0}^2 \\
& +\tilde{C}\nu\triangle t^5|||\nabla\mathbf{u}_{tt}|||_{\infty,0}^2 + \tilde{C}\triangle t^5|||\mathbf{f}_{tt}|||_{\infty,0}^2\Big\},
\end{aligned} \tag{66}$$

where $C(T)$ is a constant which depends on T. The estimate given in Theorem 3.1 for $||\mathbf{u}^N - \mathbf{u}_h^N||^2$ then follows from the Minkowski's inequality and (66). The estimate for $||\mathbf{u}-\mathbf{w}_h||^2$ follows from (62), and Minkowski's inequality,

$$\begin{aligned}||\mathbf{u}^N-\mathbf{w}_h^N||^2 &= ||\Lambda^N+\varepsilon_h^N||^2 \le 2||\Lambda^N||^2+2||\varepsilon_h^N||^2\\ &\le 2||\Lambda^N||^2+2||e_h^N||^2+2\triangle tC(C_s\delta)^2\left\|\mathbb{D}([I-P_H]I_h\mathbf{u}^N)\right\|_{L^3}^3\\ &\quad +2\triangle tC(C_s\delta)^2\left\|\mathbb{D}([I-P_H]\frac{\varepsilon_h^N+e_h^N}{2})\right\|_{L^3}^3\\ &\le Ch^{2k+2}|\mathbf{u}^N|_{k+1}^2+C\triangle t(C_s\delta)^2h^{-\frac{d}{2}}H^{3l}|\mathbf{u}^N|_{l+1}^3\\ &\quad +2||e_h^N||^2+\triangle tC(C_s\delta)^2\left\|\mathbb{D}([I-P_H]\frac{\varepsilon_h^N+e_h^N}{2})\right\|_{L^3}^3, \end{aligned} \tag{67}$$

and

$$\begin{aligned}&\left\|\mathbb{D}(\mathbf{u}(t^{n+\frac{1}{2}})-\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2})\right\|^2\\ &\le C\Big\{\left\|\mathbb{D}(\mathbf{u}^{n+\frac{1}{2}}-\mathbf{u}(t^{n+\frac{1}{2}}))\right\|^2+\left\|\nabla\Lambda^{n+\frac{1}{2}}\right|^2+\left\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\right\|^2\Big\}\\ &\le \frac{C\triangle t^4}{4}\max_{t^n\le s\le t^{n+1}}\left\|\nabla\mathbf{u}_{tt}(s)\right\|^2+Ch^{2k}\left|\mathbf{u}^{n+\frac{1}{2}}\right|_{k+1}^2+C\left\|\mathbb{D}(\frac{\varepsilon_h^{n+1}+e_h^n}{2})\right\|^2. \end{aligned} \tag{68}$$

Finally, combining (67)-(68) with (66) yields the result of the theorem. □

For the case of Taylor-Hood approximating elements, i.e., $k=l=2$, we have the following estimate.

Corollary 3.3 Under the assumptions of Theorem 3.1, with $\triangle t=Ch, \delta=Ch, h=H^2, d=2$ and $\mathbf{X}^h\times Q^h$ given by the Taylor-Hood approximation elements, the there exist a constant $C(T)$ which depends on T such that

$$\begin{aligned}\frac{1}{2}\left\|\mathbf{u}^N-\mathbf{u}_h^N\right\|^2+\left\|\mathbf{u}^N-\mathbf{w}_h^N\right\|^2+\triangle t\nu\sum_{n=0}^{N-1}\left\|\mathbb{D}(\mathbf{u}(t^{n+\frac{1}{2}})-\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2})\right\|^2&\\ \le C(T)(\triangle t^4+h^4).&\end{aligned} \tag{69}$$

4. A variant with reduced complexity

Since the Step 2 in Algorithm 2.1 is nonlinear which costs more in simulation process. To reduce this cost we present a variant on Algorithm 2.1 which is closely related and requires the solution of a linear system instead.

Step 2 of Algorithm 2.1 presents two computational difficulties. One difficulty is the nonlinearity of the eddy viscosity. The second is the coupling of fine mesh elements and coarse mesh elements which is caused by the projection, especially in the factor $(I-P_{L^H})\mathbb{D}(\mathbf{v})$. The first and obvious modification is lagging $\nu_T(\cdot)$ to reduce the complexity to solving a linear equation per time step. We replace the eddy viscosity coefficient $\nu_T(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})$ by $\nu_T(\frac{\mathbf{w}_h^{n}+\mathbf{u}_h^{n}}{2})$. The second modification is to replace $\nu_T(\cdot)$ element by element by:

$$A_e\Big(\nu_T(\frac{\mathbf{w}_h^n+\mathbf{u}_h^n}{2})\Big)=\frac{1}{|e|}\int_e\nu_T(\frac{\mathbf{w}_h^n+\mathbf{u}_h^n}{2})dx,$$

where $|e|$ represents the area of e. The definition of e depends on the choice of $\mathbf{L}^H$.

Definition 4.1 If $\mathbf{L}^H$ is a lower order finite element space on the same mesh, then $e=e_h$ represents the element of the single mesh. If $\mathbf{L}^H$ is defined by the same finite element space on a coarse mesh with mesh width $H>h$, then $e=e_H$ represents the element of the coarse mesh.

Obviously, now the new eddy viscosity coefficient is piecewise constant, so it can be commuted with the operator $[I-P_{L^H}]\mathbb{D}$, see Remark 1.4 in Section 1.2. Thanks to the orthogonality of projection P_{L^H}, we can simplify Step 2 in Algorithm 2.1 as follow:

$$\Big(\frac{\mathbf{w}_h^{n+1}-\mathbf{u}_h^{n+1}}{\triangle t},\mathbf{v}_h\Big) = (\lambda_h^{n+1},\nabla\cdot\mathbf{v}_h) + \Big(A_e\big(\nu_T(\frac{\mathbf{w}_h^{n}+\mathbf{u}_h^{n}}{2})\big)[I-P_{L^H}]\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2},\mathbb{D}(\mathbf{v}_h)\Big).$$

Note in particular that $\mathbb{D}(\mathbf{v}_h)$ replaces $[I-P_{L^H}]\mathbb{D}(\mathbf{v}_h)$. This change simplifies the computational work of Step 2 substantially. Rearranging terms we can rewrite last equation as follows: given $\mathbf{w}_h^{n+1}$ solve for $(\mathbf{u}_h^{n+1},\lambda_h^{n+1})\in\mathbf{X}^h\times Q^h$

$$\Big(A_e\big(\nu_T(\frac{\mathbf{w}_h^{n}+\mathbf{u}_h^{n}}{2})\big)[I-P_{L^H}]\mathbb{D}(\mathbf{u}_h^{n+1}),\mathbb{D}(\mathbf{v}_h)\Big) + \frac{2}{\triangle t}(\mathbf{u}_h^{n+1},\mathbf{v}_h)+2(\lambda_h^{n+1},\nabla\cdot\mathbf{v}_h)$$

$$= \frac{2}{\triangle t}(\mathbf{w}_h^{n+1},\mathbf{v}_h)-\Big(A_e\big(\nu_T(\frac{\mathbf{w}_h^{n}+\mathbf{u}_h^{n}}{2})\big)[I-P_{L^H}]\mathbb{D}(\mathbf{w}_h^{n+1}),\mathbb{D}(\mathbf{v}_h)\Big), \tag{70}$$

$$(\nabla\cdot\mathbf{u}_h^{n+1},q_h)=0. \tag{71}$$

Next, we present this variant on the method.

Algorithm 4.1

Step 1: Given $(\mathbf{u}_h^n,p_h^n)\in\mathbf{X}^h\times Q^h$, compute $\mathbf{w}_h^{n+1}\in\mathbf{X}^h, p_h^{n+1}\in Q^h$ satisfying

$$\begin{cases} \Big(\frac{\mathbf{w}_h^{n+1}-\mathbf{u}_h^{n}}{\triangle t},\mathbf{v}_h\Big)+b_s\Big(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n}}{2},\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n}}{2},\mathbf{v}_h\Big) \\ \qquad +2\nu\Big(\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n}}{2}),\mathbb{D}(\mathbf{v}_h)\Big)-\Big(p_h^{n+\frac12},\nabla\cdot\mathbf{v}_h\Big)=\Big(\mathbf{f}^{n+\frac12},\mathbf{v}_h\Big), \\ \big(\nabla\cdot\mathbf{w}_h^{n+1},q_h\big)=0, \end{cases} \tag{72}$$

for all $(\mathbf{v}_h,q_h)\in\mathbf{X}^h\times Q^h$,

Step 2: Given $\mathbf{w}_h^{n+1}\in\mathbf{X}^h$, solve the following system to obtain $(\mathbf{u}_h^{n+1},\lambda_h^{n+1})$:

$$\begin{cases} \Big(\frac{\mathbf{w}_h^{n+1}-\mathbf{u}_h^{n+1}}{\triangle t},\mathbf{v}_h\Big)=(\lambda_h^{n+1},\nabla\cdot\mathbf{v}_h) \\ \qquad +\Big((C_s\delta)^2A_e\big(\nu_T(\frac{\mathbf{w}_h^{n}+\mathbf{u}_h^{n}}{2})\big)[I-P_{L^H}]\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2},\mathbb{D}(\mathbf{v}_h)\Big), \\ (\nabla\cdot\mathbf{u}_h^{n+1},q_h)=0, \end{cases}$$

for all $(\mathbf{v}_h,q_h)\in\mathbf{X}^h\times Q^h$.

Now, we prove a strong energy equality and associated strong, unconditional stability of this variant.

Theorem 4.2 Let $C_s>0,\delta>0$. The approximate velocity $\mathbf{u}_h^{n+1}$ given by the Algorithm 4.1 satisfies the energy equality

$$\frac12||\mathbf{u}_h^N||^2+\triangle t\sum_{n=0}^{N-1}(C_s\delta)^2A_e\big(\nu_T(\frac{\mathbf{w}_h^{n}+\mathbf{u}_h^{n}}{2})\big)\Big\|[I-P_{L^H}]\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})\Big\|^2 +2\triangle t\sum_{n=0}^{N-1}\nu\Big\|\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n}}{2})\Big\|^2=\frac12\big\|\mathbf{u}_h^0\big\|^2+\triangle t\sum_{n=0}^{N-1}\Big(\mathbf{f}^{n+\frac12},\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n}}{2}\Big), \tag{73}$$

and the stability bound

$$\begin{aligned}(74)\quad &\frac{1}{2}||\mathbf{u}_h^N||^2 + \triangle t\sum_{n=0}^{N-1}(C_s\delta)^2 A_e\big(\nu_T(\frac{\mathbf{w}_h^n+\mathbf{u}_h^n}{2})\big)\Big\|[I-P_{L^H}]\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})\Big\|^2\\ &+\triangle t\sum_{n=0}^{N-1}\nu\Big\|\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2})\Big\|^2 \le \frac{1}{2}\big\|\mathbf{u}_h^0\big\|^2 + \frac{\triangle t}{4\nu}\sum_{n=0}^{N-1}\big\|\mathbf{f}^{n+\frac{1}{2}}\big\|^2_{H^{-1}}.\end{aligned}$$

Proof. From the orthogonality of the projection P_{L^H}, we can rewrite the first equation in (73) as:

$$\begin{aligned}\Big(\frac{\mathbf{w}_h^{n+1}-\mathbf{u}_h^{n+1}}{\triangle t},\mathbf{v}_h\Big) &= (\lambda_h^{n+1},\nabla\cdot\mathbf{v}_h)\\ &+\Big((C_s\delta)^2A_e\big(\nu_T(\frac{\mathbf{w}_h^n+\mathbf{u}_h^n}{2})\big)[I-P_{L^H}]\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}),[I-P_{L^H}]\mathbb{D}(\mathbf{v}_h)\Big).\end{aligned}$$

Setting $\mathbf{v}_h = \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}$ and using the divergence-free property for both $\mathbf{w}_h^{n+1}$ and $\mathbf{u}_h^{n+1}$, then we have

$$\begin{aligned}\frac{1}{2\triangle t}||\mathbf{w}_h^{n+1}||^2 &= \frac{1}{2\triangle t}||\mathbf{u}_h^{n+1}||^2\\ &+(C_s\delta)^2A_e\big(\nu_T(\frac{\mathbf{w}_h^n+\mathbf{u}_h^n}{2})\big)\Big\|[I-P_{L^H}]\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})\Big\|^2.\end{aligned}$$

Setting $\mathbf{v}_h = \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}$ in (72) and using the divergence-free property for both $\mathbf{w}_h^{n+1}$ and $\mathbf{u}_h^n$, then from the Cauchy-Schwarz and Young's inequalities, we get

$$\frac{1}{2\triangle t}\big(||\mathbf{w}_h^{n+1}||^2-||\mathbf{u}_h^n||^2\big)+2\nu\Big\|\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2})\Big\|^2 = \Big(\mathbf{f}^{n+\frac{1}{2}},\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}\Big).$$

Combining the above two equations gives

$$\begin{aligned}\frac{1}{2\triangle t}\big(||\mathbf{u}_h^{n+1}||^2-||\mathbf{u}_h^n||^2\big)+(C_s\delta)^2A_e\big(\nu_T(\frac{\mathbf{w}_h^n+\mathbf{u}_h^n}{2})\big)\Big\|[I-P_{L^H}]\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})\Big\|^2\\ +2\nu\Big\|\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2})\Big\|^2 = \Big(\mathbf{f}^{n+\frac{1}{2}},\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}\Big).\end{aligned}$$

Summing establishes the energy equality. Using the Cauchy-Schwarz inequality and Young's inequality on the right-hand side, subsuming one term into the left-hand side gives

$$\begin{aligned}\frac{1}{2}\big(||\mathbf{u}_h^{n+1}||^2-||\mathbf{u}_h^n||^2\big)+\triangle t(C_s\delta)^2A_e\big(\nu_T(\frac{\mathbf{w}_h^n+\mathbf{u}_h^n}{2})\big)\Big\|[I-P_{L^H}]\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})\Big\|^2\\ +\triangle t\nu\Big\|\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2})\Big\|^2 \le \frac{\triangle t\nu}{4}\sum_{n=0}^{N-1}||\mathbf{f}^{n+\frac{1}{2}}||^2_{H^{-1}}.\end{aligned}$$

Summing over the index n, the global stability estimate follows. □

5. Numerical results

In all experiments, the algorithms are implemented by using public domain finite element software Freefem++ [26].

5.1. Convergence study. Let Ω be the unit square in $\mathbb{R}^2$. The uniform mesh is obtained by dividing Ω into squares and then drawing a diagonal in each square in the same direction. The Taylor-Hood element are chosen for the velocity-pressure finite element space $\mathbf{X}^h \times Q^h$, the large scale space $\mathbb{L}^H$ is using the piecewise constant space on the same grid.

Then, choose the true solution ($\mathbf{u} = (u_1, u_2)$, p) as follows:

$$\begin{aligned} u_1 &= -cos(\pi x)sin(\pi y)exp(-2\pi^2 t/Re), \\ u_2 &= sin(\pi x)cos(\pi y)exp(-2\pi^2 t/Re), \\ p &= -0.25(cos(2\pi x) + cos(2\pi y))exp(-4\pi^2 t/Re), \end{aligned}$$

which is the Green-Taylor vortex. It was used as a numerical test in Chorin [9], Tafti [48] and John and Layton [33] among many others. Nonhomogeneous boundary conditions are imposed based on the given exact solution.

First, we compare the uncoupled VMS method in Algorithm 2.1 and Algorithm 4.1 with the classical, monolithic VMS method. We choose $C_S = 0.1$, $\delta = h$. In Table 1, we display the errors of the classical VMS method for $\mathbf{u}_h$ and p_h, while Table 2 and 3 give the results of both Algorithm 2.1 and Algorithm 4.1 for $\mathbf{w}_h$, $\mathbf{u}_h$ and p_h. Here we denote the errors tabulated by

$$e_w = \mathbf{u} - \mathbf{w}_h, \; e_u = \mathbf{u} - \mathbf{u}_h, \; e_p = p - p_h.$$

Table 1. Errors of convergence using classical VMS, Re=1000

$\frac{h}{\Delta t}$	$\|\|e_u\|\|_{L^2(0,T;L^2)}$	$\|\|e_u\|\|_{L^2(0,T;H^1)}$	$\|\|e_p\|\|_{L^2(0,T;L^2)}$
$\frac{0.1}{0.05}$	0.0113960	0.6584710	0.00542664
$\frac{0.05}{0.025}$	0.0009871	0.1329400	0.00095956
$\frac{0.025}{0.0125}$	6.42669e-5	0.019094	0.00022907

Table 2. Errors of convergence using Algorithm 2.1 of uncoupled VMS, Re=1000

$\frac{h}{\Delta t}$	$\|\|e_w\|\|_{L^2(0,T;L^2)}$	$\|\|e_u\|\|_{L^2(0,T;L^2)}$	$\|\|e_w\|\|_{L^2(0,T;H^1)}$	$\|\|e_u\|\|_{L^2(0,T;H^1)}$	$\|\|e_p\|\|_{L^2(0,T;L^2)}$
$\frac{0.1}{0.05}$	0.0114657	0.0114564	0.663174	0.662166	0.0054379
$\frac{0.05}{0.025}$	0.0009905	0.0009904	0.133443	0.133411	0.0009597
$\frac{0.025}{0.0125}$	6.43394e-5	6.43362e-5	0.019117	0.019116	0.0002291

Table 3. Errors of convergence using Algorithm 4.1 of uncoupled VMS, Re=1000

$\frac{h}{\Delta t}$	$\|\|e_w\|\|_{L^2(0,T;L^2)}$	$\|\|e_u\|\|_{L^2(0,T;L^2)}$	$\|\|e_w\|\|_{L^2(0,T;H^1)}$	$\|\|e_u\|\|_{L^2(0,T;H^1)}$	$\|\|e_p\|\|_{L^2(0,T;L^2)}$
$\frac{0.1}{0.05}$	0.01147343	0.0114676	0.6637040	0.663071	0.0054390
$\frac{0.05}{0.025}$	0.00099064	0.0009905	0.1333457	0.133438	0.0009598
$\frac{0.025}{0.0125}$	6.43402e-5	6.43386e-5	0.0191171	0.019117	0.0002291

From these tables, we notice that all three algorithms obtain similar accuracy. This indicates that the uncoupled VMS method is almost comparably accurate to the one-step, classical VMS method.

5.2. Flow around a cylinder. The second example is the 'flow around a cylinder' benchmark problem from Shafer and Turek [47] and John [28]. The domain with meshes is presented in Figure 1.

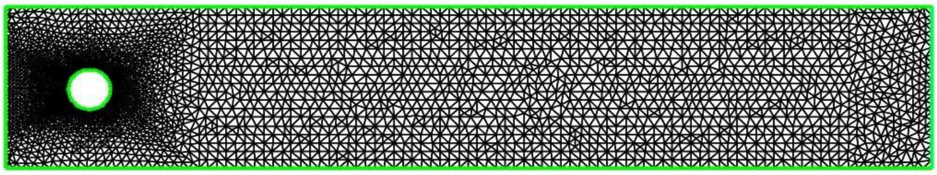

FIGURE 1. The triangulation of the computational domain for uncoupled VMS method.

The time-dependent inflow and outflow profiles are

$$u_1(0,y,t) = u_1(2.2,y,t) = \frac{6}{0.41^2} sin(\frac{\pi t}{8}) y(0.41-y),$$
$$u_2(0,y,t) = u_2(2.2,y,t) = 0.$$

No-slip conditions are prescribed at the other boundaries. Computations are performed for the Reynolds number corresponding to $\nu = 10^{-3}$, and the external force $\mathbf{f} = 0$. A mesh with 7510 triangles is used, and $C_S = 0.1$, $\quad h = \min_{T \in \tau_h} \{\mathrm{diam}(T)\}$.

The development of the flows by both uncoupled VMS algorithms are depicted in Figure 2, 3, respectively. From these figures, we notice that from $t = 2$ to $t = 4$, along with the flow increasing, two vortices start to develop behind the cylinder. Then, the vortices separate from the cylinder between $t = 4$ and $t = 5$, and a vortex street develops, and they continue to be visible through the final time $t = 8$, which agrees with the results of [10, 47, 28].

TABLE 4. Results maximal drag $c_{d,max}$, maximal lift $c_{l,max}$ and $\Delta p(8s)$ for different time step size by Algorithm 2.1

Δt	$t(c_{d,max})$	$c_{d,max}$	$t(c_{l,max})$	$c_{l,max}$	$\Delta p(8s)$
0.025	3.95	2.92769	5.775	0.436106	-0.0941906
0.01	3.94	2.93828	5.72	0.460058	-0.106839
0.005	3.93	2.94126	5.715	0.463752	-0.10902
0.0025	3.94	2.94222	5.7125	0.464803	-0.110134

TABLE 5. Results maximal drag $c_{d,max}$, maximal lift $c_{l,max}$ and $\Delta p(8s)$ for different time step size by Algorithm 4.1

Δt	$t(c_{d,max})$	$c_{d,max}$	$t(c_{l,max})$	$c_{l,max}$	$\Delta p(8s)$
0.025	3.95	2.93908	5.775	0.436462	-0.0941091
0.01	3.94	2.94295	5.72	0.460243	-0.106833
0.005	3.93	2.94366	5.715	0.463825	-0.109021
0.0025	3.94	2.94352	5.7125	0.464835	-0.110135

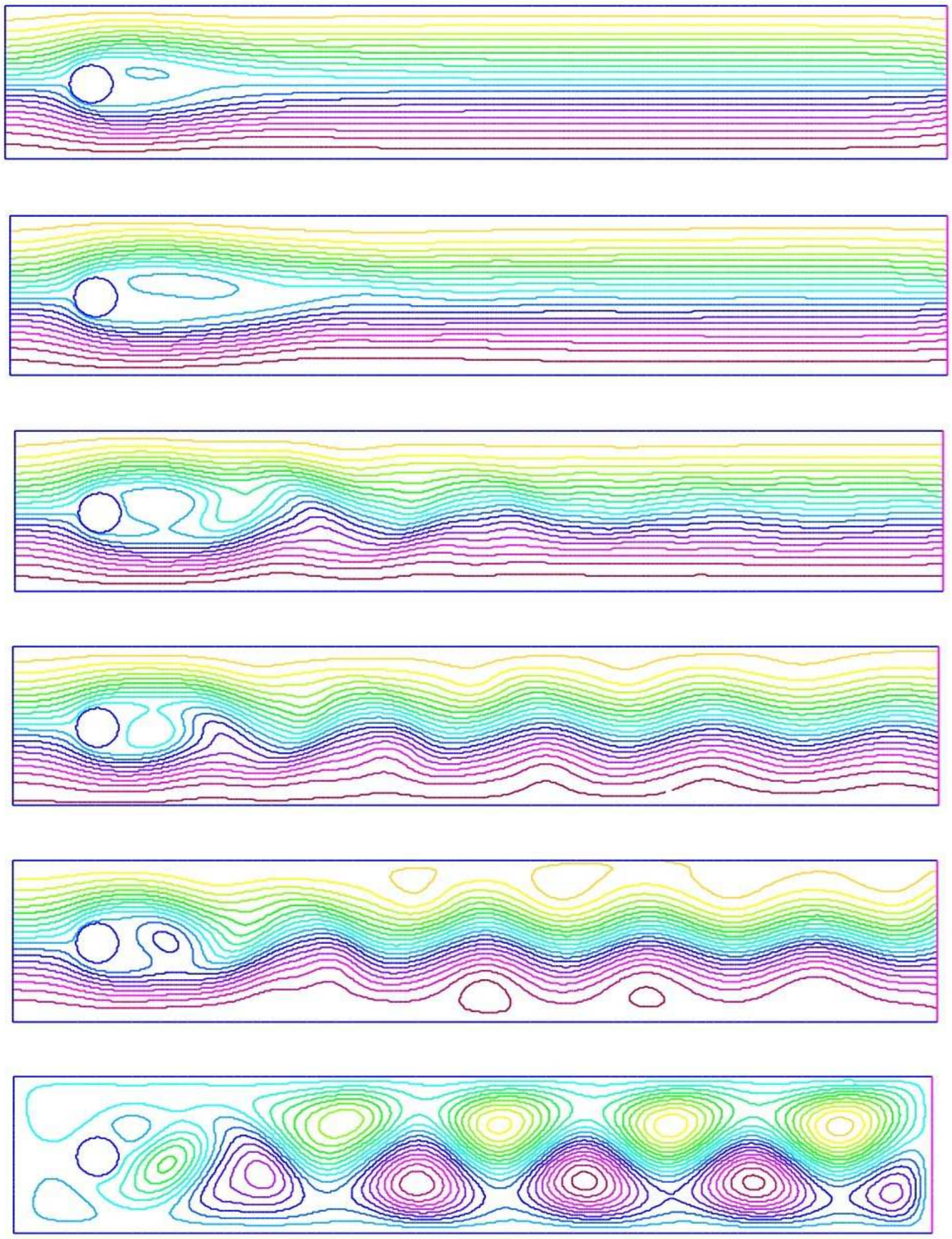

FIGURE 2. The streamline at $t = 2,\ 4,\ 5,\ 6,\ 7,\ 8$ by Algorithm 2.1 of uncoupled VMS method with $\triangle t = 0.0025$.

The evolutions of $c_{d,max},\ c_{l,max}$ and Δp with $\Delta t = 0.0025$ for Algorithm 2.1 and 4.1 are presented in Figure 4 and 5, respectively. The values for the maximal drag $c_{d,max}$, maximal lift $c_{l,max}$ and $\Delta p(8s)$ (here $\Delta p(t) = p(t; 0.15, 0.2) - p(t; 0.25, 0.2)$) with different time step size Δt for Algorithm 2.1 and Algorithm 4.1 are presented

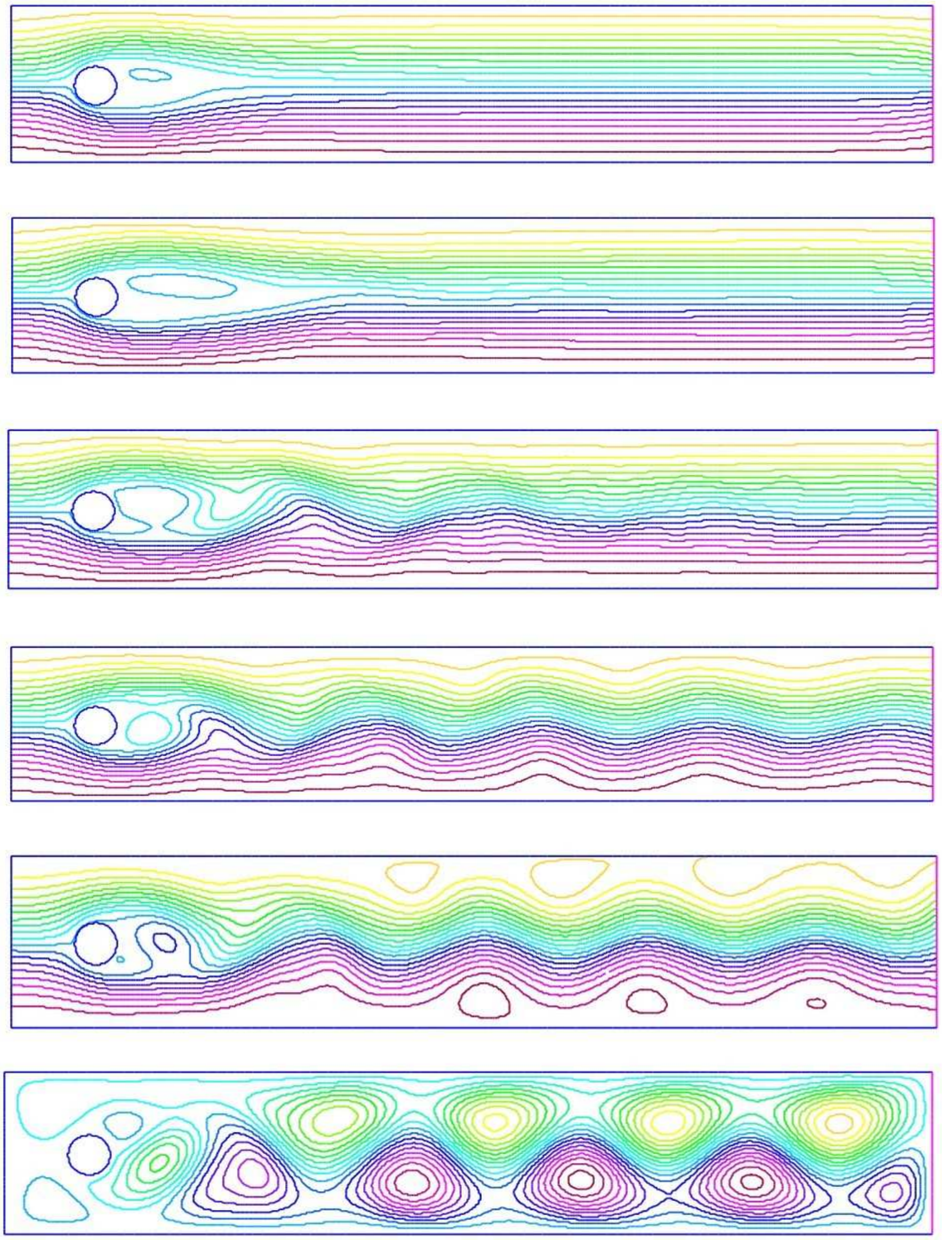

FIGURE 3. The streamline at $t = 2,\ 4,\ 5,\ 6,\ 7,\ 8$ by Algorithm 4.1 of uncoupled VMS method with $\triangle t = 0.0025$.

in Table 4 and 5, respectively. The following reference intervals are given in [47],

$$c_{d,max}^{ref} \in [2.93, 2.97], \quad c_{l,max}^{ref} \in [0.47, 0.49], \quad \Delta p(8s)^{ref} \in [-0.115, -0.105].$$

The computation results in both tables show that when the time step size decreases, all coefficients computed by Algorithm 2.1 and 4.1 approach the reference

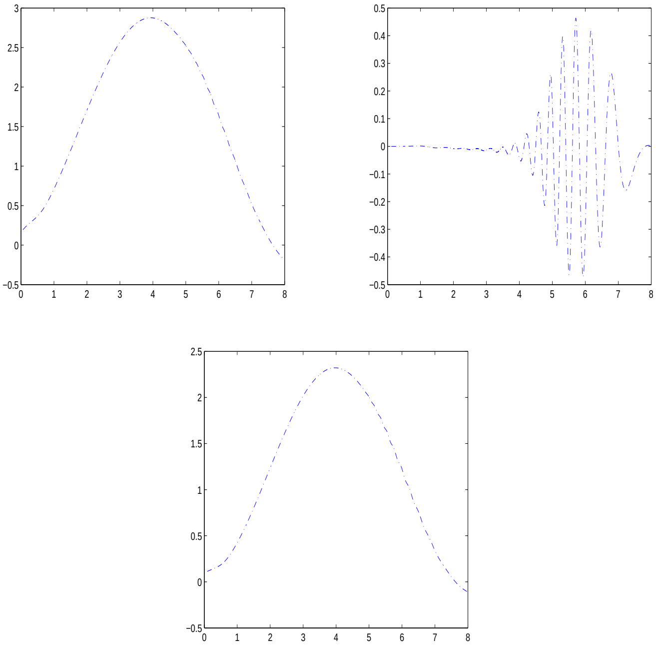

FIGURE 4. The evolutions of $c_{d,max}$, $c_{l,max}$ and Δp by Algorithm 2.1 of uncoupled VMS method with $\triangle t = 0.0025$.

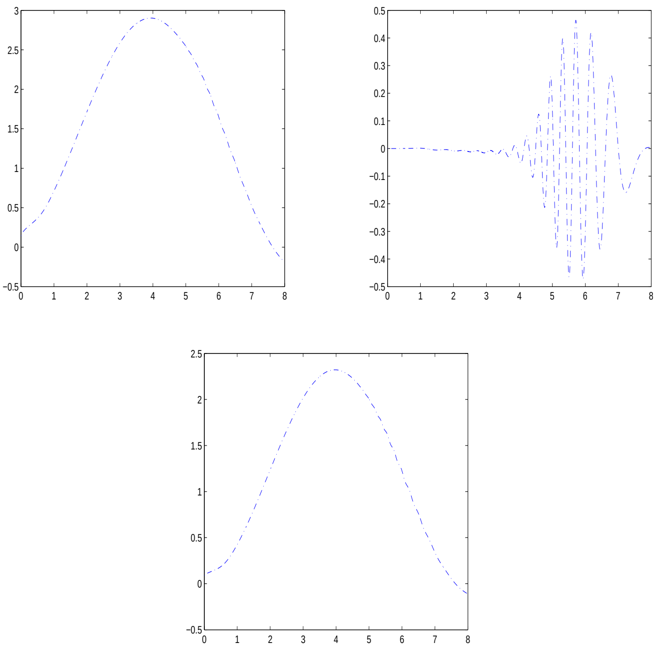

FIGURE 5. The evolutions of $c_{d,max}$, $c_{l,max}$ and Δp by Algorithm 4.1 of uncoupled VMS method with $\triangle t = 0.0025$.

results. Surprisingly, for this test, Algorithm 4.1 is a little more accurate than Algorithm 2.1.

6. Conclusions

In this paper, we have analyzed two modular, uncoupled variational multiscale methods focusing on analysis specifically on the case of nonlinear eddy viscosity for the Navier-Stokes equations. We separated the VMS treatment as a separate step, which means one can utilize legacy codes to deal with the NSE in Step 1 or adapt a laminar code to a VMS model by adding Step 2. We proved stability and performed an error analysis of the method. Numerical tests were given that confirm and illustrate the theoretical results as well.

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