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NONCONFORMING MIXED FINITE ELEMENT METHODS FOR STATIONARY INCOMPRESSIBLE MAGNETOHYDRODYNAMICS

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Abstract. The main aim of this paper is to study the approximation of nonconforming mixed finite element methods for stationary, incompressible magnetohydrodynamics (MHD) equations in 3D. A family of nonconforming finite elements are taken as the approximation spaces for the velocity field, the piecewise constant element for the pressure and the Nédélec's element with the lowest order for the magnetic field on hexahedra or tetrahedra. A new simple method is adopted to prove the discrete Poincaré-Friedrichs inequality instead of the discrete Helmholtz decomposition method. The existence and uniqueness of the approximate solutions are shown. The convergence analysis is presented and the optimal order error estimates for the pressure in L^2-norm, as well as those in a broken H^1-norm for the velocity field and $H(curl)-$norm for the magnetic field are derived.

Key words. Incompressible MHD equations, Nonconforming mixed finite element methods, Optimal error estimates

1. Introduction

Magnetohydrodynamics (MHD) equations is the complicated coupling problem which is composed of electrically conducting fluid and electromagnetic fields. The MHD equations arises in several applications, for example, astronomy and geophysics as well as the associated numerous engineering problems, such as liquid-metal cooling of nuclear reactors, electromagnetic casting of metals, MHD power generation and MHD ion propulsion [1]. Many studies have been already devoted to the incompressible MHD equations. For theoretical results, G. Duvaut and J.-L. Lions [2] first established the existence and uniqueness results for weak and strong solutions of the MHD equations. M. Sermange and R. Temam [3] then analyzed the large time behavior, the regularity properties and bound on the solutions to the MHD equations which are valid for all time. On one hand, a considerable amount of research activity has been devoted to the analysis of numerical methods for the simulation of MHD flows by using finite difference methods (FDMs) [4]-[7]. On the other hand, most of the numerical solutions of the MHD equations are performed with the finite element methods (FEMs) [8]-[12], [14]-[22].

More precisely, in [8]-[12], the studies required that the magnetic field belongs to $H^1(\Omega)^3$. However, in the presence of reentrant corners or edges, setting the magnetic unknowns of the incompressible MHD equations in $H^1(\Omega)^3$ leads to a well-posed problem where the magnetic field cannot be correctly approximated because the magnetic field may have regularity below $H^1(\Omega)^3$ [13]. In order to overcome this difficulty, M. Costabel and M. Dauge [13] first presented a method of regularizing

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time harmonic Maxwell equations where the magnetic field belongs to $H(curl;\Omega)$. This observation recently motivated the work [14], where a mixed formulation of the stationary incompressible MHD problem based on $H(curl)-$conforming (edge) elements to approximate the magnetic field was proposed, and the convergence to weak solutions of the discrete problem was proved. A different strategy to achieve convergence in general polyhedral domains was realized in [15], where an interior penalty discontinuous Galerkin method for a linearized incompressible MHD model problem was applied. Based on the mixed formulation introduced in [14], there have been numerical analysis for schemes available, whose convergence was shown either in the context of existing weak or strong solutions [16]-[26]. A recent summary of known results for the MHD equations, including modeling, analysis, and numerics is [1]. But, almost all the analysis in (FEMs) [8]-[12], [14]-[1] are about the conforming FEMs except [16]. To the best of our knowledge, until now there are few papers focusing on the nonconforming finite element methods (NFEMs).

It is well known that NFEMs play an important role in the numerical approximation of partial differential equations. Firstly, NFEMs have been used effectively especially in fluid and solid mechanics when conforming FEMs and others seem too costly or unstable. Secondly, the mixed finite element approximation to MHD equations needs the stability and the compatibility between the velocity and the pressure finite element spaces satisfying the discrete inf-sup condition [27], NFEMs are much easier to be constructed to satisfy the above condition than conforming FEMs. Thirdly, for some Crouzeix-Raviart type finite elements with the degrees of freedom defined on the edges (or faces) of element or element itself, since the unknowns are associated with the element faces, each degree of freedom belongs to at most two elements, the use of the nonconforming finite elements facilitates the exchange of information across each subdomain and provides spectral radius estimates for the iterative domain decomposition operator [28]. Furthermore, NFEMs for the resolution of a wide range of linear and nonlinear boundary value problems have a great development in the last years [29]-[40]. The authors [41] also proposed a family of low-order nonconforming mixed FEMs to approximate stationary MHD equations and obtained the optimal error estimates in convex polyhedral domains, or domains with a boundary $C^{1,1}$.

As an attempt, we are concerned with NFEMs for nonlinear, fully coupled stationary incompressible MHD equations by the mixed formulation in general Lipschitz polyhedra. We will adopt a family of nonconforming finite elements as approximation space for the velocity field, the piecewise constant element for the pressure and the first kind Nédélec's elements on tetrahedra or hexahedra with the lowest order for the magnetic field. A new method is introduced to prove the discrete Poincar*é*-Friedrichs inequality, which is much easier than the methods used in [27, 42, 43]. Finally, we will show the existence and uniqueness of the approximate solutions and obtain the optimal order error estimates.

This paper is organized as follows: In Section 2, we introduce the variational formulation for the MHD equations. Section 3 will give the nonconforming finite element spaces. In Section 4, we state some important lemmas and prove the existence and uniqueness of discrete solutions. Section 5 will present the convergence analysis and derive the optimal order error estimates.

In this paper, we will use the notations $\|\cdot\|_l, \|\cdot\|_{l,K}$ for $H^l(\Omega)^3, H^l(K)^3$-norm, $|\cdot|_m, |\cdot|_{m,K}$ for $H^m(\Omega)^3, H^m(K)^3$-seminorm, where $H^0(\Omega)^3 = L^2(\Omega)^3$ and $H^0(K)^n = L^2(K)^3, l \geq 0, m \geq 0$ are integer numbers. Throughout the paper, C indicates a positive constant, possibly different at different occurrences, which is independent of the mesh parameter h, but may depend on Ω and other parameters introduced in this paper. Notations without especially explained are used with their usual meanings.

2. Equations and the mixed variational formulation

We consider the the following stationary, incompressible MHD equations in a bounded domain in three dimensions [14]:

Problem (**I**): Find the velocity field $u=(u_1,u_2,u_3)$, the pressure p, the magnetic field $B=(B_1,B_2,B_3)$ and the scalar function r satisfying

$$\begin{cases} -R_s^{-1}\triangle u+(u\cdot\nabla)u+\nabla p-S_c(curlB)\times B=f \quad in \quad \Omega,\\ R_m^{-1}S_c curl(curlB)-S_c curl(u\times B)-\nabla r=g \quad in \quad \Omega,\\ divu=0, divB=0 \quad in \quad \Omega,\\ u=0 \quad on \quad \partial\Omega,\\ B\cdot n=0, n\times curlB=0 \quad on \quad \partial\Omega. \end{cases} \tag{2.1}$$

Here, we take Ω to be a bounded Lipschitz polyhedron in R^3 with outward normal unit vector $n=(n_1,n_2,n_3)$ on $\Gamma=\partial\Omega$, R_s is the hydrodynamic Reynolds number, R_m the magnetic Reynolds number, S_c the coupling number, respectively, and $f\in H^{-1}(\Omega)^3$ and $g\in L^2(\Omega)^3$ are given source terms.

The mixed variational formulation for Problem (**I**) is written as:

Problem ($\mathbf{I}_1$): Find $(u,B)\in W,(p,r)\in V$ such that

$$\begin{cases} a((u,B),(u,B),(v,\Psi))+b((v,\Psi),(p,r))=F((v,\Psi)),\ \forall(v,\Psi)\in W,\\ b((u,B),(\chi,q))=0,\ \forall(\chi,q)\in V, \end{cases}$$

where $W=H_0^1(\Omega)^3\times H(curl;\Omega), V=L_0^2(\Omega)\times H_0^1(\Omega)$.

Set

$$H(curl;\Omega)=\{v\in L^2(\Omega)^3; curlv\in L^2(\Omega)^3\},$$
$$H(div;\Omega)=\{w\in L^2(\Omega)^3; divw\in L^2(\Omega)\},$$
$$H_0(div;\Omega)=\{w\in H(div;\Omega); w\cdot n=0 \ on\ \partial\Omega\},$$
$$H(div^0;\Omega)=\{c\in H(div;\Omega); divc=0\}, H(\Omega)=H(curl;\Omega)\cap H_0(div;\Omega),$$
$$J=H_0^1(\Omega)^3\cap H(div^0;\Omega), X=H(\Omega)\cap H(div^0;\Omega),$$
$$L_0^2(\Omega)=\{q\in L^2(\Omega); \int_\Omega q d\mathbf{x}=0\},$$

hereafter, $\mathbf{x}=(x,y,z)$.

We first equip the product spaces W and V with the norms

$$\|(w,\Phi)\|_W:=(\|w\|_1^2+\|\Phi\|_{curl}^2)^{\frac{1}{2}},$$
$$\|(\chi,q)\|_V:=(\|\chi\|_0^2+\|q\|_1^2)^{\frac{1}{2}}$$

and the spaces $H(\Omega)$ and $H(curl;\Omega)$ with the norms

$$\|\Phi\|_H^2=\|\Phi\|_0^2+\|curl\Phi\|_0^2+\|div\Phi\|_0^2,$$
$$\|\Phi\|_{H(curl;\Omega)}^2=\|\Phi\|_0^2+\|curl\Phi\|_0^2,$$

respectively.

Next, we introduce the bilinear and trilinear forms as:

$$a((u,B),(v,\Psi),(w,\Phi)):=a_0((v,\Psi),(w,\Phi))+a_1((u,B),(v,\Psi),(w,\Phi)), \tag{2.2}$$

$$a_0((v,\Psi),(w,\Phi)):=R_s^{-1}\int_\Omega \nabla v:\nabla w d\mathbf{x}+R_m^{-1}S_c\int_\Omega curl\Psi\cdot cur\Phi d\mathbf{x}, \tag{2.3}$$

$$a_1((u,B),(v,\Psi),(w,\Phi)):=c_0(u;v,w)-c_1(B;w,\Psi)+c_2(B;v,\Phi), \tag{2.4}$$

$$c_0(u;v,w):=\frac{1}{2}\int_\Omega(u\cdot\nabla)v\cdot w d\mathbf{x}-\frac{1}{2}\int_\Omega(u\cdot\nabla)w\cdot v d\mathbf{x}, \tag{2.5}$$

$$c_1(B; w, \Psi) := S_c \int_\Omega (curl\Psi \times B) \cdot w d\mathbf{x}, \quad (2.6)$$

$$c_2(B; v, \Phi) := S_c \int_\Omega (curl\Phi \times B) \cdot v d\mathbf{x}, \quad (2.7)$$

$$b((v, \Psi), (\chi, q)) := b_1(v, \chi) + b_2(\Psi, q), \quad (2.8)$$

$$b_1(v, \chi) := -\int_\Omega \chi divv d\mathbf{x}, b_2(\Psi, q) := -\int_\Omega \nabla q \cdot \Psi d\mathbf{x}, \quad (2.9)$$

$$F((v, \Psi)) := \int_\Omega f v d\mathbf{x} + \int_\Omega g \Psi d\mathbf{x}. \quad (2.10)$$

Set

$$\|F\|_* := \sup_{(0,0)\neq(v,\Psi)\in J\times X} \frac{F((v, \Psi))}{\|(v, \Psi)\|_W},$$

$$\|F\|_{-1} := [\|f\|_{-1}^2 + \|g\|_0^2]^{\frac{1}{2}}, \quad (2.11)$$

$$Z = \{(v, \Psi) \in W \mid b((v, \Psi), (\chi, q)) = 0, \forall(\chi, q) \in V\}.$$

Then the following conclusions can be found in [14]:

(I) $\|F\|_* \leq \|F\|_{-1}$ and $Z = J \times X$.

(II) for $u, v, w \in H^1(\Omega)^3, B \in X, \Psi, \Phi \in H(curl; \Omega)$, there hold

$$c_0(w; v, v) = 0, \quad (2.12)$$

$$a_1((u, B), (v, \Psi), (w, \Phi)) = -a_1((u, B), (w, \Phi), (v, \Psi)), \quad (2.13)$$

$$a_1((u, B), (v, \Psi), (v, \Psi)) = 0. \quad (2.14)$$

(III) If $f \in H^{-1}(\Omega)^3, g \in L^2(\Omega)^3$, then Problem $(\mathbf{I}_1)$ has at least a solution

$$((u, B), (p, r)) \in Z \times V,$$

in addition, is unique provided that

$$C_2\gamma_3(C_1\gamma_1)^{-2}\|F\|_{-1} < 1,$$

and has the stability bound

$$\|(u, B)\|_W \leq (C_1\gamma_1)^{-1}\|F\|_{-1},$$

where

$$\gamma_1 = \min\{R_s^{-1}, R_m^{-1}S_c\}, \gamma_2 = \max\{R_s^{-1}, R_m^{-1}S_c\}, \gamma_3 = \max\{1, S_c\}$$

and C_1, C_2 are two positive constants only depending on Ω.

3. Nonconforming finite element spaces

We consider the regular and quasi-uniform meshes Γ^h of mesh-size h that partiting Ω into tetrahedra or hexahedra K. Let $X_{1h} \not\subseteq H_0^1(\Omega)^3$ and $M_h \subset L_0^2(\Omega)$ be finite element spaces for approximating the unknowns u and p, the interpolation operator associated to X_{1h} is denoted by Π^1. Let $\Pi_K = \Pi^1|_K$ for $K \in \Gamma^h$, $P_1(K)$ be the polynomial space of degree less than or equal to 1 on K, and satisfy the following assumptions [41]:

(A) $\forall K \in \Gamma^h, v \in P_1(K)^3 \subset X_{1h}, \Pi_K v = v$;

(B) $M_h = \{\chi^h \in L_0^2(\Omega); \chi^h|_K$ is a constant, $\forall K \in \Gamma^h\}$;

(C) $\|\cdot\|_{1h} = (\sum_{K\in\Gamma^h} |\cdot|_{1,K}^2)^{\frac{1}{2}}$ is a norm over X_{1h};

(D) $\forall v^h \in X_{1h}, \int_F [v^h]ds = 0, \forall F \subset \partial K, \int_F v^h ds = 0, \forall F \subset \partial\Omega$;

(E) $b_{1h}(v - \Pi^1 v, q^h) = 0, \|\Pi^1 v\|_{1h} \leq C|v|_1, \forall v \in H_0^1(\Omega)^3, q^h \in M_h$.

It can be checked the nonconforming finite elements studied in [29]-[41], [44]-[48] satisfy the above assumptions.

To approximate the unknowns B and r in Problem ($\mathbf{I_1}$), the associated finite element spaces on hexahedra K are defined by

$$X_{2h} = \{\Psi^h \in H(curl;\Omega) \mid \Psi^h|_K \in Q_{0,1,1}(K) \times Q_{1,0,1}(K) \times Q_{1,1,0}(K), \forall K \in \Gamma^h\},$$
$$N_h = \{q^h \in H_0^1(\Omega) \mid q^h|_K \in Q_1(K), \forall K \in \Gamma^h\},$$

and on tetrahedra K

$$X_{2h} = \{\Psi^h \in H(curl;\Omega) \mid \Psi^h|_K = a + b \times \mathbf{x}, a, b \in R^3, \forall K \in \Gamma^h\},$$
$$N_h = \{q^h \in H_0^1(\Omega) \mid q^h|_K \in P_1(K), \forall K \in \Gamma^h\},$$

where $Q_{i,i,i}(K)$ is a space of polynomials whose degrees for x, y, z are equal to i, respectively. Here we point out that X_{2h} is the first kind Nédélec's element with the lowest order [49, 50].

4. The existence and uniqueness of the approximate solutions and some lemmas

In this section, we will first introduce the mixed finite element approximation of MHD equations in (2.1).

Now we let $W_h = X_{1h} \times X_{2h}, V_h = M_h \times N_h$ and introduce the trilinear forms a_h, a_{1h} and the bilinear forms a_{0h}, b_h as:

for $(u^h, B^h), (v^h, \Psi^h), (w^h, \Phi^h) \in W_h$ and $(\chi^h, q^h) \in V_h$,

$$\begin{aligned} a_h((u^h, B^h), (v^h, \Psi^h), (w^h, \Phi^h)) &:= a_{0h}((v^h, \Psi^h), (w^h, \Phi^h)) \\ &+ a_{1h}((u^h, B^h), (v^h, \Psi^h), (w^h, \Phi^h)), \end{aligned} \tag{4.1}$$

$$\begin{aligned} a_{0h}((v^h, \Psi^h), (w^h, \Phi^h)) &:= \sum_{K \in \Gamma^h} \{R_s^{-1} \int_K \nabla v^h : \nabla w^h \\ &+ R_m^{-1} S_c \int_K curl\Psi^h \cdot curl\Phi^h\} d\mathbf{x}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} a_{1h}((u^h, B^h), (v^h, \Psi^h), (w^h, \Phi^h)) &:= c_{0h}(u^h; v^h, w^h) \\ &- c_{1h}(B^h; w^h, \Psi^h) + c_{2h}(B^h; v^h, \Phi^h), \end{aligned} \tag{4.3}$$

$$c_{0h}(u^h; v^h, w^h) := \frac{1}{2} \sum_{K \in \Gamma^h} \int_K [(u^h \cdot \nabla) v^h \cdot w^h - (u^h \cdot \nabla) w^h \cdot v^h] d\mathbf{x}, \tag{4.4}$$

$$c_{1h}(B^h; w^h, \Psi^h) := S_c \sum_{K \in \Gamma^h} \int_K (curl\Psi^h \times B^h) \cdot w^h d\mathbf{x}, \tag{4.5}$$

$$c_{2h}(B^h; v^h, \Phi^h) := S_c \sum_{K \in \Gamma^h} \int_K (curl\Phi^h \times B^h) \cdot v^h d\mathbf{x}, \tag{4.6}$$

$$b_h((v^h, \Psi^h), (\chi^h, q^h)) := b_{1h}(v^h, \chi^h) + b_{2h}(\Psi^h, q^h), \tag{4.7}$$

$$b_{1h}(v^h, \chi^h) := -\sum_{K \in \Gamma^h} \int_K \chi^h div v^h d\mathbf{x}, b_{2h}(\Psi^h, q^h) := -\sum_{K \in \Gamma^h} \int_K \nabla q^h \cdot \Psi^h d\mathbf{x}, \tag{4.8}$$

respectively.

Then the approximation formulation of Problem ($\mathbf{I_1}$) reads as:

Problem ($\mathbf{I_2}$): Find $(u^h, B^h) \in W_h, (p^h, r^h) \in V_h$ such that

$$\begin{cases} a_h((u^h, B^h), (u^h, B^h), (v^h, \Psi^h)) + b_h((v^h, \Psi^h), (p^h, r^h)) \\ = F((v^h, \Psi^h)), \forall (v^h, \Psi^h) \in W_h, \\ b_h((u^h, B^h), (\chi^h, q^h)) = 0, \ \forall (\chi^h, q^h) \in V_h. \end{cases} \tag{4.9}$$

From the definition of (4.3), a_{1h} satisfies the following antisymmetric properties [14]:

$$a_{1h}((u^h, B^h), (v^h, \Psi^h), (v^h, \Psi^h)) = 0, \tag{4.10}$$

Let

$$J_h = \{v^h \in X_{1h}, b_{1h}(v^h, \chi^h) = 0, \forall \chi^h \in M_h\},$$
$$X_h = \{\Psi_h \in X_{2h} \mid b_{2h}(\Psi^h, q^h) = 0, \forall q^h \in N_h\}$$

and

$$Z_h = \{(v^h, \Psi^h) \in W_h \mid b_h((v^h, \Psi^h), (\chi^h, q^h)) = 0, \forall (\chi^h, q^h) \in V_h\},$$

we have

$$Z_h = J_h \times X_h \subset W_h.$$

For $v^h = (v_1^h, v_2^h, v_3^h) \in X_{1h}, \Psi^h = (\Psi_1^h, \Psi_2^h, \Psi_3^h) \in X_{2h}$, we define

$$\| v^h \|_{0h} = (\sum_{K \in \Gamma^h} \| v^h \|_{0,K}^2)^{\frac{1}{2}},$$

$$\| v^h \|_{1h} = (\sum_{K \in \Gamma^h} |v^h|_{1,K}^2)^{\frac{1}{2}} = (\sum_{K \in \Gamma^h} \int_K \nabla v^h : \nabla v^h d\mathbf{x})^{\frac{1}{2}},$$

$$\| (v^h, \Psi^h) \|_h = (\|v^h\|_{1h}^2 + \|\Psi^h\|_{H(curl;\Omega)}^2)^{\frac{1}{2}},$$

$$\|F\|_{*h} := \sup_{(0,0) \neq (v^h, \Psi^h) \in J_h \times X_h} \frac{F((v^h, \Psi^h))}{\|(v^h, \Psi^h)\|_h}$$

and

$$\|F\|_h := \sup_{(0,0) \neq (v^h, \Psi^h) \in X_{1h} \times X_{2h}} \frac{F((v^h, \Psi^h))}{\|(v^h, \Psi^h)\|_h},$$

respectively. Then it is easy to see that $\| \cdot \|_{0h}$ and $\| \cdot \|_{1h}$ are the norms over X_{1h}, and $\| (\cdot, \cdot) \|_h$ is the norm over W_h.

In order to prove the existence and uniqueness of the solutions of (4.9), first of all, we need to prove the following important lemmas.

Lemma 4.1. The following discrete Poincaré-Friedrichs inequality holds

$$\|\Psi^h\|_0 \leq C \|curl \Psi^h\|_0, \qquad \forall \Psi^h \in X_{2h}. \tag{4.11}$$

Proof. We consider the following problem

$$\begin{cases} curl(curl \widetilde{B}) = f \ in \ \Omega, \\ div \widetilde{B} = 0 \ in \ \Omega, \\ \widetilde{B} \cdot n = 0 \ on \ \partial\Omega, \\ curl \widetilde{B} \times n = 0 \ on \ \partial\Omega. \end{cases} \tag{4.12}$$

Due to the regularity of the solution [3], we have

$$\|curl \widetilde{B}\|_0 \leq C \|f\|_0. \tag{4.13}$$

By Green's formula and Hölder's inequality, we deduce that

$$|\int_\Omega f \Psi^h d\mathbf{x}| = |\sum_{K \in \Gamma^h} \int_K curl \widetilde{B} \cdot curl \Psi^h d\mathbf{x}|$$
$$\leq \|curl \widetilde{B}\|_0 \|curl \Psi^h\|_0. \tag{4.14}$$

Then, using (4.13) and (4.14) and choosing $f = \Psi^h$ implies

$$\|\Psi^h\|_0 \leq C \|curl \Psi^h\|_0.$$

The proof is completed. □

Remark 1: The technique used in this lemma is different from [27, 42] and simplifies the proof of the above inequality (4.11) compared with the discrete Helmholtz decomposition method.

Lemma 4.2. For $(u^h, B^h) \in X_{1h} \times X(h)$, and $(v^h, \Psi^h), (w^h, \Phi^h) \in X_{1h} \times X_{2h}$, we have

(1) $|c_{0h}(u^h; v^h, w^h)| \leq C\|u^h\|_{1h}\|v^h\|_{1h}\|w^h\|_{1h}$,

(2) $|c_{1h}(B^h; w^h, \Psi^h)| \leq CS_c\|curl\Psi^h\|_0\|B^h\|_{H(curl;\Omega)}\|w^h\|_{1h}$,

(3) $|c_{2h}(B^h; v^h, \Phi^h)| \leq CS_c\|curl\Phi^h\|_0\|B^h\|_{H(curl;\Omega)}\|v^h\|_{1h}$.

Proof. The first result is well-known [14, 27, 42]. To prove the second result, we start by noting that $\|B^h\|_{H(\Omega)} = \|B^h\|_{curl}$ (for $B^h \in X$). Furthermore, due to the imbedding property $H(\Omega) \hookrightarrow L^{3+\delta_1}(\Omega)^3$ [14] and the discrete imbedding inequality over X_{1h} in, e.g., [32, 34, 35]

$$\|v^h\|_{0,2k,\Omega} \leq C(k)\|v^h\|_{1h}, \forall v^h \in X_{1h}, k = 1, 2. \tag{4.15}$$

Firstly, by choosing $\delta_1 = 1, k = 2$ and using Hölder's inequality, we get

$$\begin{aligned}|c_{1h}(B^h; w^h, \Psi^h)| \quad &\leq \sum_{K\in\Gamma^h}\int_K S_c|(curl\Psi^h \times B^h)\cdot w^h|d\mathbf{x}\\ &\leq S_c\|curl\Psi^h\|_0\|B^h\|_{0,4}\|w^h\|_{0,4}\\ &\leq CS_c\|curl\Psi^h\|_0\|B^h\|_{H(curl;\Omega)}\|w^h\|_{1h}.\end{aligned}$$

Secondly, for $B^h \in X_h$, set $X(h) = X + X_h$. Then there exists a linear mapping $F : X_h \to X$ satisfying $curlB^h = curl(FB^h)$ and the Poincaré-Friedrichs inequality in X (see [14, 50, 51]):

$$\|curlFB^h\|_0 \geq C\|FB^h\|_0.$$

We have

$$|c_{1h}(B^h; w^h, \Psi^h)| \leq |c_{1h}(B^h - FB^h; w^h, \Psi^h)| + |c_{1h}(FB^h; w^h, \Psi^h)|,$$

$$\begin{aligned}|c_{1h}(FB^h; w^h, \Psi^h)| \quad &\leq CS_c\|curl\Psi^h\|_0\|FB^h\|_{H(curl;\Omega)}\|w^h\|_{1h}\\ &\leq CS_c\|curl\Psi^h\|_0\|curl(FB^h)\|_0\|w^h\|_{1h}\\ &= CS_c\|curl\Psi^h\|_0\|curlB^h\|_0\|w^h\|_{1h}\\ &\leq C\|curl\Psi^h\|_0\|B^h\|_{H(curl;\Omega)}\|w^h\|_{1h}.\end{aligned}$$

On the other hand, for all piecewise polynomial functions φ, $B^h \in X_h$ and $l > \frac{1}{2}$, [53] and [14, 50] have shown that the following inverse estimate

$$\|\phi\|_{0,q} \leq Ch^{3(\frac{1}{q}-\frac{1}{p})}\|\phi\|_{0,q}, 1 \leq p \leq q \leq \infty \tag{4.16}$$

and the inequality

$$\|B^h - FB^h\|_0 \leq Ch^l\|curlB^h\|_0 \tag{4.17}$$

hold, respectively.

So, by Hölder's inequality and let $q = \infty, p = 2$ in (4.16), $l = \frac{3}{2}$ in (4.17) and $k = 1$ in (4.15), respectively, we deduce that

$$\begin{aligned}&|c_{1h}(B^h - FB^h; w^h, \Psi^h)|\\ &\leq S_c\|w^h\|_{0,\infty}\|B^h - FB^h\|_0\|curl\Psi^h\|_0\\ &\leq CS_ch^{-\frac{3}{2}}h^{\frac{3}{2}}\|w^h\|_{0,2}\|curlB^h\|_0\|curl\Psi^h\|_0\\ &\leq CS_c\|w^h\|_{1h}\|curlB^h\|_0\|curl\Psi^h\|_0.\end{aligned}$$

Similarly, choosing $p = 2, q = 4$ in (4.16), $k = 2$ in (4.15) and $l = \frac{3}{4}$ in (4.17) leads to

$$
\begin{aligned}
&|c_{1h}(B^h - \digamma B^h; w^h, \Psi^h)| \\
&\le S_c\|w^h\|_{0,4}\|B^h - \digamma B^h\|_0\|curl\Psi^h\|_{0,4} \\
&\le CS_c h^{-\frac{3}{4}} h^{\frac{3}{4}}\|w^h\|_{0,4}\|curlB^h\|_0\|curl\Psi^h\|_0 \\
&\le CS_c\|w^h\|_{1h}\|curlB^h\|_0\|curl\Psi^h\|_0.
\end{aligned}
$$

Thus, the assertion for c_{1h} is proved, the proof for c_{2h} is analogous. □

Lemma 4.3. Let $(u^h, B^h) \in X_{1h} \times X(h)$, and $(v^h, \Psi^h), (w^h, \Phi^h) \in X_{1h} \times X_{2h}$, then the following results hold:

(1) $|a_{1h}((u^h, B^h), (v^h, \Psi^h), (w^h, \Phi^h))| \le C_c\gamma_3\|(u^h, B^h)\|_h\|(v^h, \Psi^h)\|_h\|(w^h, \Phi^h)\|_h$,

(2) $|a_{0h}((u^h, B^h), (u^h, B^h))| \ge C_a\gamma_1\|(u^h, B^h)\|_h^2$,

(3) $|a_{0h}((u^h, B^h), (v^h, \Psi^h))| \le C\gamma_2\|(u^h, B^h)\|_h\|(v^h, \Psi^h)\|_h$,

where C_c, C_a are two positive constants, independent of h.

Proof. Using the triangle inequality and Lemma 4.2, we get

$$
\begin{aligned}
&|a_{1h}((u^h, B^h), (v^h, \Psi^h), (w^h, \Phi^h))| \\
&\le |c_{0h}(u^h; v^h, w^h)| + |c_{1h}(B^h; w^h, \Psi^h)| + |c_{2h}(B^h; v^h, \Phi^h)| \\
&\le C_c max\{1, S_c\}\|(u^h, B^h)\|_h\|(v^h, \Psi^h)\|_h\|(w^h, \Phi^h)\|_h \\
&= C_c\gamma_3\|(u^h, B^h)\|_h\|(v^h, \Psi^h)\|_h\|(w^h, \Phi^h)\|_h.
\end{aligned}
$$

Applying Lemma 4.1 gives

$$
\begin{aligned}
&a_{0h}((u^h, B^h), (u^h, B^h)) \\
&= \sum_{K\in\Gamma^h}\{R_s^{-1}\int_K \nabla u^h : \nabla u^h + R_m^{-1}S_c\int_K curlB^h \cdot curlB^h\}d\mathbf{x} \\
&= R_s^{-1}\|\nabla u^h\|_0^2 + R_m^{-1}S_c\|curlB^h\|_0^2 \\
&\ge C_a min\{R_s^{-1}, R_m^{-1}S_c\}[\|u^h\|_{1h}^2 + \|B^h\|_{H(curl;\Omega)}^2] \\
&= C_a\gamma_1\|(u^h, B^h)\|_h^2.
\end{aligned}
$$

By Hölder's inequality yields and the definition of $\|\cdot\|$, we have

$$
\begin{aligned}
&|a_{0h}((u^h, B^h), (v^h, \Psi^h))| \\
&\le \sum_{K\in\Gamma^h}\{R_s^{-1}\int_K |\nabla u^h : \nabla v^h|d\mathbf{x} + R_m^{-1}S_c\int_K |curlB^h \cdot curl\Psi^h|\}d\mathbf{x} \\
&\le \{R_s^{-1}\|\nabla u^h\|_0\|\nabla v^h\|_0 + R_m^{-1}S_c\|curlB^h\|_0\|curl\Psi^h\|_0\} \\
&\le Cmax\{R_s^{-1}, R_m^{-1}S_c\}\|(u^h, B^h)\|_h\|(v^h, \Psi^h)\|_h \\
&= C\gamma_2\|(u^h, B^h)\|_h\|(v^h, \Psi^h)\|_h.
\end{aligned}
$$

The proof is completed. □

Lemma 4.4. The spaces W_h and V_h satisfy the discrete inf-sup condition, i.e.,

$$
\inf_{(\chi^h, q^h)\in V_h}\sup_{(v^h, \Psi^h)\in W_h}\frac{b_h((v^h, \Psi^h), (\chi^h, q^h))}{\|(v^h, \Psi^h)\|_h\|(\chi^h, q^h)\|_V} \ge \beta^\star, \tag{4.18}
$$

where $\beta^\star$ is a positive constant independent of h .

Proof. On one hand, by [27, 42], there exists a constant $\beta_1 > 0$ such that

$$\inf_{0\neq\chi\in L_0^2(\Omega)} \sup_{0\neq v\in H_0^1(\Omega)^3} \frac{b_1(v,\chi)}{\|v\|_1\|\chi\|_0} \geq \beta_1. \tag{4.19}$$

Therefore, by the assumption (E) and (4.20), we get

$$\sup_{v^h\in X_{1h}} \frac{b_{1h}(v^h,\chi^h)}{\|v^h\|_{1h}} \geq \sup_{v\in H_0^1(\Omega)^3} \frac{b_{1h}(\Pi^1 v,\chi^h)}{\|\Pi^1 v\|_{1h}} = \sup_{v\in H_0^1(\Omega)^3} \frac{b_{1h}(v,\chi^h)}{\|\Pi^1 v\|_{1h}}$$

$$\geq \frac{1}{C} \sup_{v\in H_0^1(\Omega)^3} \frac{b_{1h}(v,\chi^h)}{\|v\|_{1h}} = \frac{1}{C} \sup_{v\in H_0^1(\Omega)^3} \frac{b_1(v,\chi^h)}{\|v\|_1} \geq \frac{\beta_1}{C}\|\chi^h\|_0. \tag{4.20}$$

On the other hand, there exists a constant $\beta_2 > 0$ such that [14, 50, 51]

$$\inf_{0\neq q^h\in N_h} \sup_{0\neq \Psi^h\in X_{2h}} \frac{b_{2h}(\Psi^h,q^h)}{\|\Psi^h\|_{H(curl;\Omega)}\|q^h\|_1} \geq \beta_2. \tag{4.21}$$

Combining (4.20) and (4.21) yields the desired result. □

From Lemmas 4.3-4.4, we get the following conclusion.

Theorem 4.1. For $f \in H^{-1}(\Omega)^3$, Problem ($\mathbf{I}_2$) has at least one solution

$$((u^h,B^h),(p^h,r^h)) \in Z_h \times V_h$$

satisfying the stability bound

$$\|(u^h,B^h)\|_h \leq (C_a\gamma_1)^{-1}\|F\|_h. \tag{4.22}$$

Moreover, Problem ($\mathbf{I}_2$) has a unique solution provided that

$$C_c\gamma_3(C_a\gamma_1)^{-2}\|F\|_{*h} < 1.$$

5. Convergence analysis

Based on the lemmas in Section 4, we state the main results of this paper in this section.

Theorem 5.1. Assume that

$$\frac{C_c\gamma_3\|F\|_{-1}}{C_aC_1\gamma_1^2} < \frac{1}{2}.$$

Let $((u,B),(p,r)) \in Z\times V$ and $((u^h,B^h),(p^h,r^h)) \in Z_h\times V_h$ denote the solutions of Problem ($\mathbf{I_1}$) and Problem ($\mathbf{I_2}$), respectively, then there exist two positive constants C_3, C_4 independent of h such that

$$\begin{aligned}(1)\|(u,B)-(u^h,B^h)\|_h \leq C_3\{&\inf_{(v^h,\Psi^h)\in W_h}\|(u,B)-(v^h,\Psi^h)\|_h\\ &+\inf_{(\chi^h,q^h)\in V_h}\|(p,r)-(\chi^h,q^h)\|_V\\ &+\sup_{(v^h,\Psi^h)\in Z_h}\frac{|E((v^h,\Psi^h))|}{\|(v^h,\Psi^h)\|_h}\}\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}(2)\|(p,r)-(p^h,r^h)\|_V \leq C_4\{&\inf_{(v^h,\Psi^h)\in W_h}\|(u,B)-(v^h,\Psi^h)\|_h\\ &+\inf_{(\chi^h,q^h)\in V_h}\|(p,r)-(\chi^h,q^h)\|_V\\ &+\sup_{(v^h,\Psi^h)\in W_h}\frac{|E((v^h,\Psi^h))|}{\|(v^h,\Psi^h)\|_h}\},\end{aligned} \tag{5.2}$$

where

$$E((v^h,\Psi^h)) = \sum_{K\in\Gamma^h}\int_{\partial K}[R_s^{-1}\frac{\partial u}{\partial n}v^h - pv^h\cdot n - \frac{1}{2}(u\cdot n)(u\cdot v^h)]ds.$$

Proof. We proceed in two steps.

Step 1 For $(v^h,\Psi^h)\in W_h$, using Green's formula, we get

$$\begin{aligned}
&a_{0h}((u,B),(v^h,\Psi^h)) + a_{1h}((u,B),(u,B),(v^h,\Psi^h))\\
&\quad + b_h((v^h,\Psi^h),(p,r)) - F((v^h,\Psi^h))\\
&= \sum_{K\in\Gamma^h}\{\int_K R_s^{-1}\nabla u\cdot\nabla v^h d\mathbf{x} + R_m^{-1}S_c\int_K curlB\cdot curl\Psi^h d\mathbf{x}\\
&+\frac{1}{2}\int_K[(u\cdot\nabla)u\cdot v^h - (u\cdot\nabla)v^h\cdot u]d\mathbf{x}\\
&-S_c\int_K[(curlB\times B)\cdot v^h - (curl\Psi^h\times B)\cdot u]d\mathbf{x}\\
&-\int_K pdivv^h d\mathbf{x} - \int_K\nabla r\cdot\Psi^h d\mathbf{x} - \int_K fv^h d\mathbf{x} - \int_K g\Psi^h d\mathbf{x}\}\\
&= \sum_{K\in\Gamma^h}\{\int_K -R_s^{-1}\triangle u\cdot v^h d\mathbf{x} + \int_{\partial K}R_s^{-1}\frac{\partial u}{\partial n}v^h ds\\
&+\int_K R_m^{-1}S_c curl(curlB)\cdot\Psi^h d\mathbf{x} + \int_{\partial K}R_m^{-1}S_c(curlB\times n)\cdot\Psi^h ds\\
&+\int_K(u\cdot\nabla)u\cdot v^h d\mathbf{x} - \int_{\partial K}\frac{1}{2}(u\cdot n)(u\cdot v^h)ds\\
&-S_c\int_K(curlB\times B)\cdot v^h d\mathbf{x} - S_c\int_K(curlu\times B)\cdot\Psi^h d\mathbf{x}\\
&+\int_{\partial K}(u\times B\times n)\cdot\Psi^h ds\\
&+\int_K\nabla p\cdot v^h d\mathbf{x} - \int_{\partial K}pv^h\cdot nds - \int_K\nabla r\cdot\Psi^h d\mathbf{x}\\
&-\int_K fv^h d\mathbf{x} - \int_K g\Psi^h d\mathbf{x}\}\\
&= \sum_{K\in\Gamma^h}\{\int_K(-R_s^{-1}\triangle u + (u\cdot\nabla)u + \nabla p - S_c(curlB)\times B - f)\cdot v^h d\mathbf{x}\\
&+\int_K(R_m^{-1}S_c curl(curlB) - S_c curl(u\times B) - \nabla r - g)\cdot\Psi^h d\mathbf{x}\\
&+\int_{\partial K}[R_s^{-1}\frac{\partial u}{\partial n}v^h - \frac{1}{2}(u\cdot n)(u\cdot v^h) - pv^h\cdot n]ds\}\\
&= E((v^h,\Psi^h)).
\end{aligned}$$

So there holds

$$\begin{aligned}
a_{0h}((u,B),(v^h,\Psi^h)) + a_{1h}((u,B),(u,B),(v^h,\Psi^h)) + b_h((v^h,\Psi^h),(p,r))\\
= F((v^h,\Psi^h)) + E((v^h,\Psi^h)).
\end{aligned} \tag{5.3}$$

From (4.9), we see that

$$a_{0h}((u^h,B^h),(v^h,\Psi^h)) + a_{1h}((u^h,B^h),(u^h,B^h),(v^h,\Psi^h))$$

$$+b_h((v^h,\Psi^h),(p^h,r^h))=F((v^h,\Psi^h)). \tag{5.4}$$

Subtraction of (5.4) from (5.3) yields

$$a_{0h}((u,B)-(u^h,B^h),(v^h,\Psi^h))+a_{1h}((u,B)-(u^h,B^h),(u,B),(v^h,\Psi^h))$$
$$+a_{1h}((u^h,B^h),(u,B)-(u^h,B^h),(v^h,\Psi^h))$$
$$+b_h((v^h,\Psi^h),(p,r)-(p^h,r^h))=E((v^h,\Psi^h)). \tag{5.5}$$

Setting (w^h,Φ^h) be an arbitrary element of Z_h, i.e.,

$$b_h((w^h,\Phi^h),(\chi^h,q^h))=0,\forall(\chi^h,q^h)\in V_h. \tag{5.6}$$

Then, for $(\chi^h,q^h)\in V_h$,

$$b_h((u^h-w^h,B^h-\Phi^h),(\chi^h,q^h))$$
$$=b_h((u^h,B^h),(\chi^h,q^h))-b_h((w^h,\Phi^h),(\chi^h,q^h))=0, \tag{5.7}$$

so $(u^h-w^h,B^h-\Phi^h)\in Z_h$ and by (5.5), we get

$$a_{0h}((w^h,\Phi^h)-(u^h,B^h),(v^h,\Psi^h))$$
$$+a_{1h}((w^h,\Phi^h)-(u^h,B^h),(u,B),(v^h,\Psi^h))$$
$$+a_{1h}((u^h,B^h),(w^h,\Phi^h)-(u^h,B^h),(v^h,\Psi^h))$$
$$+b_h((v^h,\Psi^h),(\chi^h,q^h)-(p^h,r^h))$$
$$=a_{0h}((w^h,\Phi^h)-(u,B),(v^h,\Psi^h))$$
$$+a_{1h}((w^h,\Phi^h)-(u,B),(u,B),(v^h,\Psi^h))$$
$$+a_{1h}((u^h,B^h),(w^h,\Phi^h)-(u,B),(v^h,\Psi^h))$$
$$+b_h((v^h,\Psi^h),(\chi^h,q^h)-(p,r))+E((v^h,\Psi^h)),$$
$$\forall(v^h,\Psi^h)\in W_h,(\chi^h,q^h)\in V_h. \tag{5.8}$$

Note that

$$a_{1h}((u^h,B^h),(w^h,\Phi^h)-(u^h,B^h),(w^h,\Phi^h)-(u^h,B^h))=0,$$
$$b_h((u^h-w^h,B^h-\Phi^h),(\chi^h,q^h)-(p^h,r^h))=0,$$

set $(v^h,\Psi^h)=(w^h,\Phi^h)-(u^h,B^h)$ and by (5.8), we obtain

$$a_{0h}((w^h,\Phi^h)-(u^h,B^h),(w^h,\Phi^h)-(u^h,B^h))$$
$$+a_{1h}((w^h,\Phi^h)-(u^h,B^h),(u,B),(w^h,\Phi^h)-(u^h,B^h))$$
$$=a_{0h}((w^h,\Phi^h)-(u,B),(w^h,\Phi^h)-(u^h,B^h))$$
$$+a_{1h}((w^h,\Phi^h)-(u,B),(u,B),(w^h,\Phi^h)-(u^h,B^h))$$
$$+a_{1h}((u^h,B^h),(w^h,\Phi^h)-(u,B),(w^h,\Phi^h)-(u^h,B^h))$$
$$+b_h((w^h,\Phi^h)-(u^h,B^h),(\chi^h,q^h)-(p,r))+E((w^h,\Phi^h)-(u^h,B^h)). \tag{5.9}$$

Thus, using the continuity of the forms a_{0h}, a_{1h} and the stability bounds for $\|(u,B)\|_W$ and $\|(u^h,B^h)\|_h$, the right-hand side of (5.9) can be bounded by

$$\begin{aligned} r.h.s. \quad & \le \|(w^h,\Phi^h)-(u^h,B^h)\|_h[C\gamma_2\|(w^h,\Phi^h)-(u,B)\|_h \\ & +C_c\|(w^h,\Phi^h)-(u,B)\|_h\|(u,B)\|_W \\ & +C_c\|(w^h,\Phi^h)-(u,B)\|_h\|(u^h,B^h)\|_h \\ & +C\|(\chi^h,q^h)-(p,r)\|_V+\frac{E((w^h,\Phi^h)-(u^h,B^h)}{\|(w^h,\Phi^h)-(u^h,B^h)\|_h})] \\ & \le C\|(w^h,\Phi^h)-(u^h,B^h)\|_h[\|(w^h,\Phi^h)-(u,B)\|_h \\ & +\|(\chi^h,q^h)-(p,r)\|_V+\frac{E((w^h,\Phi^h)-(u^h,B^h)}{\|(w^h,\Phi^h)-(u^h,B^h)\|_h})]. \end{aligned}$$

Similarly, employing the coercivity property of the form a_{0h}, continuity of a_{1h} in Lemma 4.3, the stability bound for $\|(u,B)\|_W$ and the assumption $\frac{C_c\gamma_3\|F\|_{-1}}{C_aC_1\gamma_1^2}<\frac{1}{2}$ allows us to bound the left-hand side of (5.9) as

$$\begin{aligned} l.h.s. \quad & \ge C_a\gamma_1\|(w^h,\Phi^h)-(u^h,B^h)\|_h^2-C_c\gamma_3\|(w^h,\Phi^h)-(u^h,B^h)\|_h^2\|(u,B)\|_W \\ & \ge \frac{1}{2}C_a\gamma_1\|(w^h,\Phi^h)-(u^h,B^h)\|_h^2 \\ & \ge C\|(w^h,\Phi^h)-(u^h,B^h)\|_h^2. \end{aligned}$$

Combining these bounds yields

$$\begin{aligned} \|(w^h,\Phi^h)-(u^h,B^h)\|_h \le C[&\|(w^h,\Phi^h)-(u,B)\|_h+\|(\chi^h,q^h)-(p,r)\|_V \\ &+\frac{E((w^h,\Phi^h)-(u^h,B^h))}{\|(w^h,\Phi^h)-(u^h,B^h)\|_h})]. \end{aligned}$$

Therefore, applying the triangle inequality gives

$$\begin{aligned} \|(u,B)-(u^h,B^h)\|_h \le C[&\|(w^h,\Phi^h)-(u,B)\|_h+\|(\chi^h,q^h)-(p,r)\|_V \\ &+\frac{E((w^h,\Phi^h)-(u^h,B^h))}{\|(w^h,\Phi^h)-(u^h,B^h)\|_h}]. \end{aligned} \tag{5.10}$$

Now, for $(w^h,\Phi^h)\in Z_h, (\chi^h,q^h)\in V_h$, taking the infimum of (5.10) leads to

$$\begin{aligned} \|(u,B)-(u^h,B^h)\|_h \le C[&\inf_{(w^h,\Phi^h)\in Z_h}\|(w^h,\Phi^h)-(u,B)\|_h \\ &+\inf_{(\chi^h,q^h)\in V_h}\|(\chi^h,q^h)-(p,r)\|_V+\sup_{(v^h,\Psi^h)\in Z_h}\frac{E((v^h,\Psi^h))}{\|(v^h,\Psi^h)\|_h}]. \end{aligned} \tag{5.11}$$

With the same argument as [27, 42], we get

$$\inf_{(w^h,\Phi^h)\in Z_h}\|(w^h,\Phi^h)-(u,B)\|_h \le C\inf_{(v^h,\Psi^h)\in W_h}\|(w^h,\Phi^h)-(u,B)\|_h. \tag{5.12}$$

Substituting (5.12) into (5.11) implies (5.1).

Step 2 For $(\chi^h,q^h)\in V_h$ and by (5.8), we have

$$\begin{aligned} & b_h((v^h,\Psi^h),(\chi^h,q^h)-(p^h,r^h)) \\ & = b_h((v^h,\Psi^h),(\chi^h,q^h)-(p,r))+b_h((v^h,\Psi^h),(p,r)-(p^h,r^h)) \\ & = b_h((v^h,\Psi^h),(\chi^h,q^h)-(p,r))-a_{0h}((u,B)-(u^h,B^h),(v^h,\Psi^h)) \\ & -a_{1h}((u,B)-(u^h,B^h),(u,B),(v^h,\Psi^h)) \\ & -a_{1h}((u^h,B^h),(u,B)-(u^h,B^h),(v^h,\Psi^h)) \\ & +E((v^h,\Psi^h)), \quad \forall (v^h,\Psi^h)\in W_h. \end{aligned}$$

Thus, taking into account the stability bounds for a_{0h} and a_{1h} and the discrete inf-sup condition (4.19) of Lemma 4.4 yields

$$\begin{aligned}\|(\chi^h, q^h) - (p^h, r^h)\|_V \quad & \le \frac{1}{\beta^\star}\{C\|(\chi^h, q^h) - (p, r)\|_V + [C\gamma_2 + C_c(\|(u, B)\|_W \\ & +\|(u^h, B^h)\|_h)]\|(u, B) - (u^h, B^h)\|_h \\ & +\frac{E((v^h, \Psi^h))}{\|(v^h, \Psi^h)\|_h}\}.\end{aligned}$$

Then, with the help of the triangle inequality and (5.1), we complete the proof of (5.2). □

Theorem 5.2. Let

$$u \in (H_0^1(\Omega)^3 \cap H^2(\Omega)^3), B, curlB \in H^1(\Omega)^3, r \in H^2(\Omega), p \in (L_0^2(\Omega) \cap H^1(\Omega))$$

and $((u^h, B^h), (p^h, r^h)) \in Z_h \times V_h$ denote the solutions of Problem ($\mathbf{I_1}$) and Problem ($\mathbf{I_2}$), respectively, then we get

$$\begin{aligned}&\|(u, B) - (u^h, B^h)\|_h + \|(p, r) - (p^h, r^h)\|_V \\ &\le Ch(|u|_2 + \|p\|_1 + \|B\|_1 + \|curlB\|_1 + \|r\|_2). \qquad (5.13)\end{aligned}$$

Proof. On one hand, the interpolation theory gives

$$\inf_{v^h \in X_{1h}} \|u - v^h\|_{1h}^2 \le \|u - \Pi^1 u\|_{1h}^2 = \sum_{K \in \Gamma^h} |u - \Pi_K^1 u|_{1,K}^2 \le Ch^2 |u|_2^2 \qquad (5.14)$$

and [1, 14, 50, 51, 52]

$$\inf_{\Psi^h \in X_{2h}} \|B - \Psi^h\|_{H(curl;\Omega)}^2 \le Ch^2[\|B\|_1 + \|curlB\|_1]^2. \qquad (5.15)$$

Therefore, by (5.14)-(5.15), we obtain

$$\inf_{(v^h, \Psi^h) \in W_h} \|(u, B) - (v^h, \Psi^h)\|_h \le Ch(|u|_2 + \|B\|_1 + \|curlB\|_1). \qquad (5.16)$$

At the same time, for any $p \in L_0^2(\Omega)$, we define the interpolation $R_0^h p \in M_h$ on each element K as

$$\int_K (p - R_0^h p) d\mathbf{x} = 0.$$

Then we have

$$\inf_{\chi^h \in M_h} \|p - \chi^h\|_0 \le \|p - R_0^h p\|_0 \le Ch\|p\|_1. \qquad (5.17)$$

Similarly,

$$\inf_{q^h \in N_h} \|r - q^h\|_1 \le Ch\|r\|_2. \qquad (5.18)$$

On the other hand, by the similar techniques to [29]-[41], we can estimate the consistency error as

$$|E((v^h, \Psi^h))| \le Ch(|u|_2 + \|p\|_1)\|(v^h, \Psi^h)\|_h. \qquad (5.19)$$

Substituting (5.14)-(5.18) into (5.1) and (5.2) yields the desired result. □

Remark 2: The results obtained in this work are also valid to the MHD problems with the boundary condition $u = 0, n \times B = 0, r = 0$ on $\partial\Omega$ when $B \in H_0(curl;\Omega), r \in H_0^1(\Omega)$.

References

[1] J.-F. Gerbeau, C. Le Bris and T. Leli'evre, Mathematical method for the magnetohydrodynamics of liquid metals, Oxford University Press, Oxford, 2006.

[2] G. Duvaut, J. -L. Lions, Inéquations en thermoélasticité et magnétohydrodynamique, Arch. Rat. Mech. Anal. 46 (1972) 241-279.

[3] M. Sermange, R. Temam, Some mathematical questions related to the MHD equations, Comm. Pure Appl. Math. XXXVI 36 (1983) 635-664.

[4] T. V. S. Sekhar, T. V. R Ravikumar, H. Kumar, MHD flow past a sphere at low and moderate Reynolds numbers, Comput. Mech. 31(2003) 437-444.

[5] T. W. H. Sheu, R. K. Lin, Development of a convection-diffusion-reaction magnetohydrodynamic solver on a non staggered grids, Int. J. Numer. Methods Fluids 45 (2004) 1209-1233.

[6] H. A. Navarro, L. Cabezas-Gómez, R. Césarda Silva, A. N. Montagnoli, A generalized alternating-direction implicit scheme for incompressible magnetohydrodynamic viscous flows at low magnetic Reynolds number, Appl. Math. Comput. 189 (2007) 1601-1613.

[7] H. Kumamaru, S. Kodama, H. Hirano, K. Itoh, Three-dimensional numerical calculations on liquid-metal magnetohydrodynamic flow in magnetic-field inlet-region, J. Nuclear Sci. Tech. 41(5)(2004) 624-631.

[8] M. D. Gunzburger. A. J. Meir and J. S. Peterson, On the existence and uniqueness and finite element approximation of solutions of the equations of stationary incompressible magnetohydrodynamics, Math. Comp. 56 (1991) 523-563.

[9] M. Wiedmer, Finite element approximation for equations of magnetohydrodynamics, Math. Comput. 69(229) (1999) 83-101.

[10] N. B. Salah, A. Soulaimani, W. G. Habashi, A finite element method for magnetohydrodynamics, Comput. Methods Appl. Mech. Engrg. 190 (2001) 5867-5892.

[11] J. -F. Gerbeau, A stabilized finite element method for the incompressible magnetohydrodynamic equations, Numer. Math. 87(2000) 83-111.

[12] R. Codina, N. H. Silva, Stabilized finite element approximation of the stationary magnetohydrodynamics equations, Comput. Mech. 38 (2006) 344-355.

[13] M. Costabel, M. Dauge, Weighted regularization of Maxwell equations in polyhedral domains, Numer. Math. 93 (2002) 239-277.

[14] D. Schötzau, Mixed finite element methods for stationary incompressible magnetohydrodynamics. Numer. Math. 96 (2004) 771-800.

[15] P. Houston, D. Schötzau, X. X. Wei, A Mixed DG Method for Linearized Incompressible Magnetohydrodynamics, J. Sci. Comput. 40 (2009) 281-314.

[16] C. Greif, D. Li, D. Schötzau, X. X. Wei, A mixed finite element method with exactly divergence-free velocities for incompressible magnetohydrodynamics, Comput. Methods Appl. Mech. Engrg. 199 (2010) 2840-2855.

[17] S. H. Aydin, A. I. Nesliturk, M. Tezer-Sezgin, Two-level finite element method with a stabilizing subgrid for the incompressible MHD equations, Int. J. Numer. Meth. Fluids. 62(2010) 188-210.

[18] S. Q. Gao, H. Y. Duan, Negative norm least-squares methods for the incompressible Magnetohydrodynamic equations, Acta Math. Sci. 28(3) (2008) 675-684.

[19] F. Armero, J. C. Simo, Long-term dissipativity of time-stepping algorithms for an abstract evolution equation with applications to the incompressible MHD and Navier-Stokes equations, Comput. Methods Appl. Mech. Engrg. 131(1996) 41-90.

[20] A. Prohl, Convergent finite element discretizations of the nonstationary incompressible magnetohydrodynamic system, ESAIM: Math. Model. Numer. Anal., Math. Model. Numer. Anal. 42 (2008) 1065-1087.

[21] L. Baňas, A. Prohl, Convergent finite element discretization of the multi-fluid nonstationary incompressible magnetohydrodynamics equations, Math. Comp. 79 (2010) 1957-1999.

[22] N. Besse, D. Kröner, Convergence of locally divergence-free discontinuous-Galerkin methods for the induction equations of the 2D sysytem, ESIAM: Math. Model. Numer. Anal. 39(6) (2005) 1177-1202.

[23] S. Yakovlev, L. Xu, F. Li, Locally divergence-free central discontinuous Galerkin methods for ideal MHD equations, J. Comput. Sci., Doi:10.1016/j.jocs.2012.05.002.

[24] F. Li, L. Xu, Arbitrary order exactly divergence-free central discontinuous Galerkin methods for ideal MHD equations, J. Comput. Phys. 231 (2012)2655-2675

[25] F. Li, L. Xu, S. Yakovlev, Central discontinuous Galerkin methods for ideal MHD with the exactly divergence-free magnetic flield, J. Comput. Phys. 230 (2011) 4828-4847

[26] F. Li, C.-W. Shu, Locally divergence-free discontinuous Galerkin methods for MHD equations, J. Sci. Comput. 22-23 (2005) 413-442
[27] V. Girault, P. A. Raviart, Finite Element Method for Navier-Stokes Equations: Theory and Algorithms,Springer-Verlag, New York, 1986.
[28] J. Douglas Jr., J. E. Santos, D. Sheen, X. Ye, Nonconforming Galerkin methods based on quadrilateral elements for second order elliptic problems, RAIRO Math, Model. Anal. Numer. 33(4) (1999) 747-770.
[29] M. Crouzeix, P. A. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations, RAIRO Numer. Anal. 7(R-3) (1973) 33-76.
[30] D. Y. Shi, L. F. Pei. Low order C-R type nonconforming finite element methods for approximating Maxwell equations, Int. J. Numer. Anal. Model. 5(3) (2008) 373-385.
[31] D. Y. Shi, S. P. Mao, Analysis of an anisotropic nonconforming mixed finite element for 3D Stokes problem, Acta Math. Appl. 23(3) (2005) 261-274.
[32] D. Y. Shi, J. C. Ren, Nonconforming mixed finite element approximation to the stationary Navier-Stokes equations on anisotropic meshes, Nonlinear Anal: TMA. 71(9) (2009) 3842-3852.
[33] D. Y. Shi, L. F. Pei, Low order Crouzeix-Raviar type nonconforming finite element for the 3-D time-department Maxwell's equation, Appl. Math. Comp. 211(1) (2009) 1-9.
[34] D. Y. Shi, J. C. Ren, Nonconforming mixed finite element method for the stationary conduction-convection problem, Int. J. Numer. Anal. Model. 6(2) (2009) 293-310.
[35] D. Y. Shi, Y. R. Zhang, A nonconforming anisotropic finite element approximation with moving grids for Stokes problem, J. Comput. Math. 24(5) (2006) 561-578.
[36] D. Y. Shi, C. Wang, A new low-order nonconforming mixed finite element scheme for second order elliptic problems. Inter. J. Comput. Math. 88(10) (2011) 2167-2177.
[37] D. Y. Shi, J. Ren, W. Gong, A new nonconforming mixed finite element scheme for the stationary Navier-Stokes equations. Acta Math. Sci. 31 (2) (2011) 367-382.
[38] F. S. Luo, Q. Lin, H. H. Xie, Computing the lower and upper bounds of Laplace eigenvalue problem: by combining conforming and nonconforming finite element methods. Sci. China Math. 55 (5) (2012) 1069-1082.
[39] Q. Lin, H. H. Xie, F. S. Luo, Y. Li, Y. D. Yang, Stokes eigenvalue approximations from below with nonconforming mixed finite element methods. Math. Pract. Theory 40 (19) (2010) 157-168.
[40] Z. Qiao, C. Yao, S. Jia, Superconvergence and extrapolation analysis of a nonconforming mixed finite element approximation for time-Harmonic Maxwell's equations. J. Sci. Comput. 46 (2011) 1-19.
[41] D. Y. Shi, Z. Y. Yu, Low-order nonconforming fixed finite element methods for stationary incompressible magnetohydrodynamics equations, J. Appl. Math. 2012, doi:10.1155/2012/825609.
[42] R. Temam, Navier-Stokes Equation Theory and Numerical Analysis, North-Holland, Amsterdam, New York, 1984.
[43] W. Y. Zheng, H. Qi, On Friedrichs-Poincar-type inequalities, J. Math. Anal. Appl., 304 (2005) 542-551.
[44] Th. Apel, S. Nicaise, J. Schöberl, Crouzeix-Raviart type finite elements on anisotropic meshes, Numer. Math. 89 (2001) 193-223.
[45] C. Park, D. Sheen, P^1-nonconforming quadrilateral finite element methods for second-order elliptic problems, SIAM J. Numer. Anal. 41(2)(2003) 624-640.
[46] S. Q. Zhang, X. P. Xie, Y. M. Chen, Low order nonconforming rectangular finite element methods for Darcy-Stokes problems, J. Comput. Math. 27(2-3) (2009) 400-424.
[47] Z. X. Chen, Finite element methods and their applications, Springer-Verlag, Berlin Heidelberg, 2005.
[48] Q. Lin, L. Tobiska, A. Zhou, Superconvergence and extrapolation of nonconforming low order elements applied to the Poisson equation, IMA J. Numer. Anal. 25(1) (2005) 160-181.
[49] J. C. Nédé lec, Mixed finite elements in R^3, Numer. Math. 35 (1980) 315-341.
[50] P. Monk, Finite element methods for Maxwell's equations, Oxford University Press, New York, 2003.
[51] C. Amrouche, C. Bernardi, M. Dauge, V. Girault, Vector potentials in three-dimensional non-smooth domains, Math. Models Appl. Sci. 21(1998) 823-864.
[52] A. Alonso, A. Valli, An optimal domain decomposition preconditioner for low-frequency time-harmonic Maxwell equations, Math. Comput. 68 (1999) 607-631.

[53] P. G. Ciarlet, The finite element method for elliptic problems, North-Holland, Amsterdam, New York, 1978.

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