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A COMMENT ON LEAST-SQUARES FINITE ELEMENT METHODS WITH MINIMUM REGULARITY ASSUMPTIONS

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Abstract. Least-squares(LS) finite element methods are applied successfully to a wide range of problems arising from science and engineering. However, there are reservations to use LS methods for problems with low regularity solutions. In this paper, we consider LS methods for second-order elliptic problems using the minimum regularity assumption, i.e. the solution only belongs to H^1 space. We provide a theoretical analysis showing that LS methods are competitive alternatives to mixed and standard Galerkin methods by establishing that LS solutions are bounded by the mixed and standard Galerkin solutions.

Key words. least-squares, finite element methods, Galerkin methods.

1. Introduction

The purpose of this paper is to show that LS methods could compete favorably with mixed Galerkin methods under minimum regularity assumptions, i.e. the solution only belongs to H^1 space. We consider first-order LS methods for second-order elliptic problems proposed by Cai et. al [6] and Pehlivanov et. al [10]. They transformed the second-order equations into a system of first-order by introducing a new variable(flux) $\boldsymbol{\sigma} = -\mathcal{A}\nabla u$. Least-squares type methods applied to the system lead to a minimization problem and resulting algebraic equation involves a symmetric and positive definite matrix. The approximate spaces do not require the inf-sup condition and any conforming finite element spaces can be used as approximate spaces.

While the LS methods are successfully applied to a wide range of problems in science and engineering, there are reservations to use LS methods for problems with low regularity solutions such as problems with discontinuous coefficients, problems on nonconvex domains etc. This is due to the fact that most of the error estimates concerning LS methods require high regularity solutions.

In this paper, we consider errors of LS solutions, and mixed and standard Galerkin solutions under the assumption, $\|u\|_1 \le C\|f\|_{-1}$, and establish

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|u - u_h\|_0 \le C(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m\|_0 + \|u - u_h^G\|_0),$$

where $(u_h, \boldsymbol{\sigma}_h)$ is the LS solution for $(u, \boldsymbol{\sigma} = -\mathcal{A}\nabla u)$, and $\boldsymbol{\sigma}_h^m$ and u_h^G is the mixed and Galerkin solution for $\boldsymbol{\sigma}$ and u, respectively. From this estimate, we observe that LS solutions compete favorably with mixed and Galerkin solutions. For other results concerning error estimates with minimum regularity assumptions, we refer the reader to [12], where the error estimates for the Ritz-Galerkin methods are presented with the minimum regularity assumption considered here. For results concerning smooth problems, we refer to [1, 3] and references therein.

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2. Problem formulation

Let Ω be bounded domain in $\mathbb{R}^n, n = 2, 3$ with $\partial\Omega = \Gamma_D \bigcup \Gamma_N$, where $\Gamma_D \neq \emptyset$. Let $\mathbf{n}$ denote the unit outward normal vector to the boundary. We consider the following boundary value problem:

$$-\nabla \cdot (\mathcal{A}\nabla u) = f \quad \text{in } \Omega, \tag{1}$$

with boundary conditions

$$u = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot \mathcal{A}\nabla u = 0 \quad \text{on } \Gamma_N, \tag{2}$$

where the symbols $\nabla\cdot$ and ∇ stand for the divergence and gradient operators, respectively; $f \in L^2(\Omega)$ and $\mathcal{A} = (a_{ij}(x))_{i,j=1}^n, x \in \Omega$. We shall assume that $a_{ij} \in L^\infty(\Omega)$ and the matrix $\mathcal{A}$ is symmetric and uniformly positive definite, i.e., there exist positive constants $\alpha_0 > 0$ and $\alpha_1 > 0$ such that

$$\alpha_0 \zeta^T \zeta \leq \zeta^T \mathcal{A} \zeta \leq \alpha_1 \zeta^T \zeta, \tag{3}$$

for all $\zeta \in \mathbb{R}^n$ and all $x \in \Omega$. We assume that there exists a unique solution to (1).

Let $H^s(\Omega)$ denote the Sobolev space of order s defined on Ω. The norm in $H^s(\Omega)$ will be denoted by $\|\cdot\|_s$. For $s = 0, H^s(\Omega)$ coincides with $L_2(\Omega)$. We shall use the spaces

$$\begin{aligned} V &= \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\}, \\ \mathbf{W} &= \{\boldsymbol{\sigma} \in (L^2(\Omega))^n : \nabla \cdot \boldsymbol{\sigma} \in L^2(\Omega) \text{ and } \mathbf{n} \cdot \boldsymbol{\sigma} = 0 \text{ on } \Gamma_N\}, \end{aligned}$$

with norms

$$\|u\|_1^2 = (u, u) + (\nabla u\,,\, \nabla u) \;\text{ and }\; \|\boldsymbol{\sigma}\|_{H(\mathrm{div})}^2 = (\nabla \cdot \boldsymbol{\sigma}\,,\, \nabla \cdot \boldsymbol{\sigma}) + (\boldsymbol{\sigma}\,,\, \boldsymbol{\sigma}).$$

We assume the following *a priori* estimate for u satisfying (1): there exists a positive constant C independent of f satisfying

$$\|u\|_1 \leq C\, \|f\|_{-1}. \tag{4}$$

Here, $\|f\|_{-1}$ is defined in the standard way by $\|v\|_{-1} = \sup_{\phi \in V} \frac{(v,\phi)}{\|\phi\|_1}$. As noted in [12], it is not known in general whether $u \in H^{1+s}(\Omega)$ for some $s > 0$ even if $f \in C^\infty(\Omega)$, under the assumption on the coefficients a_{ij}.

Here and thereafter, we use C with or without subscripts to denote a generic positive constant, possibly different at different occurrences, that is independent of the mesh size h and f.

By introducing a new variable $\boldsymbol{\sigma} = -\mathcal{A}\nabla u \in \mathbf{W}$, we transform the original problem into a system of first-order

$$\begin{aligned} \boldsymbol{\sigma} + \mathcal{A}\nabla u &= 0 \qquad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{\sigma} &= f \qquad \text{in } \Omega, \end{aligned} \tag{5}$$

with boundary conditions

$$u = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot \boldsymbol{\sigma} = 0 \quad \text{on } \Gamma_N, \tag{6}$$

Then, the least-squares method for the first-order system (5) is: Find $u \in V, \boldsymbol{\sigma} \in \mathbf{W}$ such that

$$\begin{aligned} b(u, \boldsymbol{\sigma}; v, \mathbf{q}) &\equiv (\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \mathbf{q}) + \big(\mathcal{A}^{-1}(\boldsymbol{\sigma} + \mathcal{A}\nabla u), \mathbf{q} + \mathcal{A}\nabla v\big) \\ &= (f, \nabla \cdot \mathbf{q}), \end{aligned} \tag{7}$$

for all $v \in V, \mathbf{q} \in \mathbf{W}$.

3. Finite element approximation

Let $\mathcal{T}_h$ be a regular triangulation of Ω (see [7]) with triangular/tetrahedra elements of size $h = \max\{diam(K); K \in \mathcal{T}_h\}$. Let $P_k(K)$ be the space of polynomials of degree k on triangle K and denote the local Raviart-Thomas space of order k on K:

$$RT_k(K) = P_k(K)^n + \mathbf{x}\, P_k(K)$$

with $\mathbf{x} = (x_1, ..., x_n)$. Then the standard (conforming) continuous piecewise polynomials of degree r and the standard $H(\text{div})$ conforming Raviart-Thomas space of index k [11] are defined, respectively, by

$$V_h^r = \{q \in V \,:\, q|_K \in P_r(K) \;\; \forall\, K \in \mathcal{T}_h\}, \tag{8}$$

$$\mathbf{W}_h^k = \{\boldsymbol{\tau} \in \mathbf{W} \,:\, \boldsymbol{\tau}|_K \in RT_k(K) \;\; \forall\, K \in \mathcal{T}_h\}. \tag{9}$$

The finite element approximation to (7) is: Find $u_h \in V_h^r$ and $\boldsymbol{\sigma}_h \in \mathbf{W}_h^k$ such that

$$b(u_h, \boldsymbol{\sigma}_h; v_h, \mathbf{q}_h) = (f, \nabla \cdot\, \mathbf{q}_h), \tag{10}$$

for all $v_h \in V_h^r, \mathbf{q}_h \in \mathbf{W}_h^k$. Note that the bilinear form $b(\cdot,\cdot;\cdot,\cdot)$ satisfies the following coercivity property: For any $(v, \boldsymbol{\tau}) \in V \times \mathbf{W}$,

$$\|v\|_1^2 + \|\boldsymbol{\tau}\|_{H(\text{div})}^2 \le C\, b(v, \boldsymbol{\tau}; v, \boldsymbol{\tau}). \tag{11}$$

From the above coercivity property and Lax-Milgram Theorem, it can be easily shown that (10) has a unique solution. Moreover, the error has the orthogonality property

$$b(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h; v_h, \mathbf{q}_h) = 0, \text{ for all } v_h \in V_h^r, \mathbf{q}_h \in \mathbf{W}_h^k. \tag{12}$$

3.1. Mixed and Standard Galerkin methods. We briefly introduce the mixed and standard Galerkin methods. First, define

$$Q_h^r = \{q \in L^2(\Omega) \,:\, q_{|K} \in P_r(K), \text{ for each } K \in \mathcal{T}_h\}. \tag{13}$$

The mixed finite element method corresponding to (5) is defined as follows: Find a pair $(u_h^m, \boldsymbol{\sigma}_h^m) \in Q_h^k \times \mathbf{W}_h^k$ such that

$$\begin{aligned} (\mathcal{A}^{-1}\boldsymbol{\sigma}_h^m, \mathbf{q}_h) - (\nabla \cdot \mathbf{q}_h, u_h^m) &= 0, \\ (\nabla \cdot \boldsymbol{\sigma}_h^m, v_h) &= (f, v_h) \end{aligned} \tag{14}$$

for all $(v_h, \mathbf{q}_h) \in Q_h^k \times \mathbf{W}_h^k$. The pair $(u - u_h^m, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m)$ satisfy the following error equations

$$(\mathcal{A}^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m), \mathbf{q}_h) - (\nabla \cdot \mathbf{q}_h, u - u_h^m) = 0, \tag{15}$$

$$(\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m), v_h) = 0. \tag{16}$$

The standard Galerkin solution $u_h^G \in V_h^r$ is defined as

$$(\mathcal{A}\nabla(u - u_h^G), \nabla\psi) = 0, \text{ for all } \psi \in V_h^r. \tag{17}$$

4. Upper bound of LS solution with mixed and Galerkin solutions

We provide our main result showing that LS methods provide competitive alternatives to mixed and Galerkin methods. In other words, we establish that the LS solution is bounded by the mixed and Galerkin solutions under minimum regularity assumption, i.e. $u \in H^1(\Omega)$. The proof given here is based on the technique developed in [5].

Theorem 1. *Let $(u_h, \boldsymbol{\sigma}_h)$ be the LS solutions satisfying (12), $(u_h^m, \boldsymbol{\sigma}_h^m)$ be the mixed solution satisfying (14), and u_h^G be the Galerkin solution satisfying (17). Then,*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|u - u_h\|_0 \le C(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m\|_0 + \|u - u_h^G\|_0). \tag{18}$$

Proof. By the triangle inequality and the definition of norms, we have

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|u - u_h\|_0 &\le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m\|_0 + \|u - u_h^G\|_0 \\ &\qquad + \|\boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}_h\|_0 + \|u_h^G - u_h\|_0 \\ &\le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m\|_0 + \|u - u_h^G\|_0 + \|\boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}_h\|_{H(\mathrm{div})} + \|u_h^G - u_h\|_1. \qquad (19) \end{aligned}$$

Using coercivity (11) and orthogonal property (12), we have

$$\begin{aligned} \|\boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}_h\|_{H(\mathrm{div})}^2 + \|u_h^G - u_h\|_1^2 &\le C\, b(u_h^G - u_h, \boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}_h\,; u_h^G - u_h, \boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}_h) \\ &= C b(u_h^G - u, \boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}\,; u_h^G - u_h, \boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}_h). \qquad (20) \end{aligned}$$

Now, by the definition of $b(\cdot;\cdot)$ in (7), and (16), (17), integration by parts and Cauchy-Schwarz inequality, we have

$$\begin{aligned} b(u_h^G - u, \boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}\,; u_h^G - u_h, \boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}_h) &= (\nabla \cdot (\boldsymbol{\sigma}_h^m - \boldsymbol{\sigma})\,, \nabla \cdot (\boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}_h)) \\ &\quad + (\mathcal{A}^{-1}(\boldsymbol{\sigma}_h^m - \boldsymbol{\sigma} + \mathcal{A}\nabla(u_h^G - u))\,, \boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}_h + \mathcal{A}\nabla(u_h^G - u_h)) \\ &= (\mathcal{A}^{-1}(\boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}), \boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}_h) + (\boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}, \nabla(u_h^G - u_h)) - (u_h^G - u, \nabla \cdot (\boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}_h)) \\ &\le C_1(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m\|_0^2 + \|u - u_h^G\|_0^2) + \frac{1}{C_1}(\|\boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}_h\|_{H(\mathrm{div})}^2 + \|u_h^G - u_h\|_1^2). \end{aligned}$$

Plugging the above inequality into (20) and using a kickback argument for sufficiently large C_1, we obtain

$$\|\boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}_h\|_{H(\mathrm{div})} + \|u_h^G - u_h\|_1 \le C(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m\|_0 + \|u - u_h^G\|_0).$$

Finally, plugging the above estimate into (19), we obtain the desired inequality. This completes the proof. □

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