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UNIFORM CONVERGENCE OF A COUPLED METHOD FOR CONVECTION-DIFFUSION PROBLEMS IN 2-D SHISHKIN MESH

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Abstract. In this paper, we introduce a coupled approach of local discontinuous Galerkin (LDG) and continuous finite element method (CFEM) for solving singularly perturbed convection-diffusion problems. When the coupled continuous-discontinuous linear FEM is used under the Shishkin mesh, a uniform convergence rate $\mathcal{O}(N^{-1}\ln N)$ in an associated norm is established, where N is the number of elements. Numerical experiments complement the theoretical results. Moreover, a uniform convergence rate $\mathcal{O}(N^{-2})$ in L^2 norm, is observed numerically on the Shishkin mesh.

Key words. convection diffusion equation, local discontinuous Galerkin method, finite element method, Shishkin mesh, uniform convergence

1. Introduction

In recent years, the numerical solutions of singularly perturbed boundary value problems have been received much attention and already studied in many papers and books, see for instance [6, 9, 11, 12]. One of the difficulties in numerically computing the solution of singularly perturbed problems lays in the so-called boundary layer behavior, i.e., the solution varies very rapidly in a very thin layer near the boundary. Traditional methods such as finite element and finite difference methods, do not work well for these problems as they often produce oscillatory solutions which are inaccurate if the perturbed parameter ϵ is small. When ϵ approaches zero, the problem changes from an elliptic equation to a hyperbolic one. Inspired by the great success of the discontinuous Galerkin (DG) method in solving hyperbolic equations, Cockburn and Shu [4], Celiker and Cockburn [3], Xie et al. [13, 14, 15] and Zhang et al. [19] adopted the local discontinuous Galerkin (LDG) method to solve convection-diffusion equations and analyzed the corresponding convergence properties. On the other hand, nonsymmetric discontinuous Galerkin method with interior penalty (the NIPG method), originally designed for elliptic equations, is analyzed by Zarin and Roos [16] for convection-diffusion problems with parabolic layers.

A disadvantage of DG method is that it produces more degrees of freedom than the continuous finite element method (CFEM). With this motivation, our work is to derive a coupled approach of LDG and CFEM and analyze the uniform convergence in a DG-norm under Shishkin mesh for singularly perturbed convection diffusion problems. The basic idea is to decompose the domain into coarse and fine part and the latter is used to simulate the boundary layer. Then the CFEM using linear

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elements is adopted in the fine part where the mesh size is comparable with ϵ, and LDG method is used in the coarse part for its stabilization.

A coupled LDG-CFEM approach has also been studied by Perugia and Schötzau [8] for the modeling of elliptic problems arising in electromagnetics. Roos and Zarin [10], Zarin [17] analyzed the NIPG-CFEM coupled method on Shishkin mesh for convection-diffusion problems with exponentially layers or characteristic layers. In this paper, the coupled LDG method is used for the singularly perturbed convection-diffusion equation for the first time to our knowledge. Moreover, distinguished from the general approaches for proving uniform convergence of numerical methods for singularly perturbed problem on layer-adapted meshes, in which solution decomposition is usually necessary, our analysis is based on the uniform error estimates for the interpolation under the Shishkin mesh, which can be reduced by the priori estimate of the solution, i.e.,

$$\left|\frac{\partial^{i+j}u(x,y)}{\partial x^i \partial y^j}(x,y)\right| \leq C\left(1+\epsilon^{-i}e^{-\beta_1(1-x)/\epsilon}\right)\times\left(1+\epsilon^{-j}e^{-\beta_2(1-y)/\epsilon}\right),$$

for i,j satisfying $0 \leq i+j \leq 2$. Our method can be generalized to all DG methods belong to the unify framework in [1], including the NIPG method.

The paper is organized as follows. In Section 2, we introduce the coupled LDG and CFEM for the singularly perturbed problems. Then stability and error analysis of the coupled method on Shishkin mesh is given in Section 3. The implementation of our coupled method on Shishkin mesh is presented in Section 4. It aims to validate our theoretical results. Furthermore, the uniform convergence rate $\mathcal{O}(N^{-2})$ in L^2 norm is observed numerically. This phenomenon is not found in [10] and [17].

In the sequel, with C we shall denote a generic positive constant independent of the perturbation parameter ϵ and mesh size.

2. Coupling the LDG and CFEM

Consider the following two-dimensional convection-diffusion problem

$$\begin{cases} -\epsilon\Delta\, u+\mathbf{b}\cdot\nabla\, u+cu=f & \text{in } \ \Omega=(0,1)^2, \\ u=0 & \text{on } \ \partial\Omega, \end{cases} \tag{2.1}$$

where $0<\epsilon\ll 1$ is a small positive parameter, $\mathbf{b},c$, and f are sufficiently smooth functions with the following properties

$$\begin{gathered} \mathbf{b}(x,y)=(b_1(x,y),b_2(x,y))\geq(\beta_1,\beta_2)>(0,0), c(x,y)\geq\ 0, \forall(x,y)\in\bar{\Omega}, \\ c_0^2(x,y)\equiv(c-\tfrac{1}{2}\nabla\cdot\mathbf{b})(x,y)\geq\gamma_0>0,\quad \forall(x,y)\in\bar{\Omega}, \\ f(0,0)=f(1,0)\ =\ f(0,1)\ =\ f(1,1)\ =\ 0, \end{gathered} \tag{2.2}$$

for some constants β_1,β_2 and γ_0. With the assumptions above, it is well-known that there exists a solution u of (2.1) that in general exhibits an exponentially boundary layer near $x=1$ and $y=1$.

The Shishkin Mesh. Define the transition parameter

$$\tau_x=\min\left(\frac{1}{2},\frac{\kappa}{\beta_1}\epsilon\ln N\right),\quad \tau_y=\min\left(\frac{1}{2},\frac{\kappa}{\beta_2}\epsilon\ln N\right),$$

with $\kappa\geq 2$ and divide Ω into four sub-domains

$$\Omega_0=(0,1-\tau_x)\times(0,1-\tau_y),\qquad \Omega_x=(1-\tau_x,1)\times(0,1-\tau_y),$$

$$\Omega_y=(0,1-\tau_x)\times(1-\tau_y,1),\qquad \Omega_{xy}=(1-\tau_x,1)\times(1-\tau_y,1).$$

Each sub-domain is then decomposed into $N/2\times N/2$ uniform rectangles (see Figure 1). Therefore, there are $(N+1)^2$ nodes (x_i, y_j), $i, j = 0, 1, \cdots, N$, and N^2 elements

$$K_{ij} = (x_{i-1}, x_i) \times (y_{j-1}, y_j), \quad i, j = 1, 2, \cdots, N.$$

Consequently,

$$x_j = \begin{cases} 2(1-\tau_x)j/N, & j = 0, 1, \cdots, N/2, \\ 1 - 2\tau_x(N-j)/N, & j = N/2+1, \cdots, N, \end{cases}$$

and

$$y_j = \begin{cases} 2(1-\tau_y)j/N, & j = 0, 1, \cdots, N/2, \\ 1 - 2\tau_y(N-j)/N, & j = N/2+1, \cdots, N. \end{cases}$$

Denote

$$H_x = 2(1-\tau_x)/N, \qquad h_x = 2\tau_x/N,$$
$$H_y = 2(1-\tau_y)/N, \qquad h_y = 2\tau_y/N.$$

Obviously

$$\max\{H_x, H_y\} \le CN^{-1}, \qquad \max\{h_x, h_y\} \le C\epsilon\, N^{-1} \ln N.$$

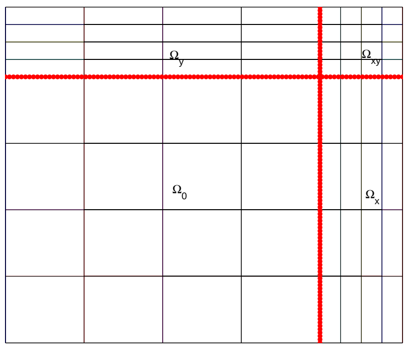

FIGURE 1. Shishkin Mesh with $N = 8$ and $\epsilon = 0.05$.

Set $\Omega_1 = \Omega_0$ and $\Omega_2 = \Omega_x \cup \Omega_y \cup \Omega_{xy}$, and the interface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. Let $\mathcal{T}_N^1 = \{K_{ij} : 1 \le i, j \le N/2\}$ and $\mathcal{T}_N^2 = \{K_{ij} : i > N/2 \,\text{or}\, j > N/2\}$. Obviously $\mathcal{T}_N^1$ and $\mathcal{T}_N^2$ are rectangular partitions of Ω_1 and Ω_2, respectively.

Weak formulation. Let $u_i = u|_{\Omega_i}, \Gamma_D^i = \partial\Omega \cap \partial\Omega_i$, $i = 1, 2$. Denote by $\mathbf{n}_i$ the unit outward normal vector to $\Gamma \cap \partial\Omega_i, i = 1, 2$, and by $\mathbf{n}$ the unit outward normal vector to $\partial\Omega$. As mentioned before, we discretize problem (2.1) by using the LDG method in Ω_1, and CFEM in Ω_2. For this purpose, we introduce an auxiliary variable $\mathbf{q} = \nabla u_1$ in Ω_1 and rewrite problem (2.1) as the following equivalent form, i.e.,

$$\begin{cases} \mathbf{q} - \nabla u_1 = 0 & \text{in } \Omega_1, \\ -\epsilon\nabla \cdot \mathbf{q} + \mathbf{b} \cdot \nabla u_1 + cu_1 = f & \text{in } \Omega_1, \\ -\epsilon\Delta u_2 + \mathbf{b} \cdot \nabla u_2 + cu_2 = f & \text{in } \Omega_2, \\ u_1 = u_2 & \text{on } \Gamma, \\ \mathbf{q} \cdot \mathbf{n}_1 = -\nabla u_2 \cdot \mathbf{n}_2 & \text{on } \Gamma, \end{cases} \tag{2.3}$$

with boundary conditions

$$u_i = 0 \quad \text{on} \quad \Gamma_D^i \quad i = 1, 2. \tag{2.4}$$

Multiplying the first three equations of (2.3) by test functions $\mathbf{w}, v_1, v_2$, respectively, and integrating by parts, it is noted that the solution $(\mathbf{q}, u_1, u_2)$, of problems (2.3) and (2.4) satisfies, for all $K \in \mathcal{T}_N^1$,

$$\int_K \mathbf{q} \cdot \mathbf{w} \mathrm{d}\mathbf{x} + \int_K u_1 \nabla \cdot \mathbf{w} \mathrm{d}\mathbf{x} - \int_{\partial K} u_1 \mathbf{w} \cdot \mathbf{n}_K \mathrm{d}s = 0, \tag{2.5}$$

$$\int_K (\epsilon\mathbf{q} - \mathbf{b}u_1) \cdot \nabla v_1 \mathrm{d}\mathbf{x} + \int_K (c - \nabla \cdot \mathbf{b}) u_1 v_1 \mathrm{d}\mathbf{x} - \int_{\partial K} v_1 (\epsilon\mathbf{q} - \mathbf{b}u_1) \cdot \mathbf{n}_K \mathrm{d}s = \int_K f v_1 \mathrm{d}\mathbf{x}, \tag{2.6}$$

for piecewise smooth functions $\mathbf{w}$ and v_1, with $\mathbf{n}_K$ the unit outward normal to ∂K, and

$$\int_{\Omega_2} (\epsilon \nabla u_2 - \mathbf{b}u_2) \cdot \nabla v_2 \mathrm{d}\mathbf{x} + \int_{\Omega_2} (c - \nabla \cdot \mathbf{b}) u_2 v_2 \mathrm{d}\mathbf{x} - \int_\Gamma v_2 (\epsilon \nabla u_2 - \mathbf{b}u_2) \cdot \mathbf{n}_2 \mathrm{d}s = \int_{\Omega_2} f v_2 \mathrm{d}x,$$

for smooth function v_2 with $v_2|_{\Gamma_D^2} = 0$. The above equations have to be coupled through the transmission conditions, i.e., the last two equations of (2.3). Replacing $-(\epsilon \nabla u_2 - \mathbf{b}u_2) \cdot \mathbf{n}_2$ by $(\epsilon\mathbf{q} - \mathbf{b}u_1) \cdot \mathbf{n}_1$ using the transmission conditions in the equation above, leads to

$$\int_{\Omega_2} (\epsilon \nabla u_2 - \mathbf{b}u_2) \cdot \nabla v_2 \mathrm{d}\mathbf{x} + \int_{\Omega_2} (c - \nabla \cdot \mathbf{b}) u_2 v_2 \mathrm{d}\mathbf{x} + \int_\Gamma v_2 (\epsilon\mathbf{q} - \mathbf{b}u_1) \cdot \mathbf{n}_1 \mathrm{d}s = \int_{\Omega_2} f v_2 \mathrm{d}\mathbf{x}. \tag{2.7}$$

It is worthwhile to point out that the first transmission condition is also imposed at the discrete level by choosing the numerical fluxes in the LDG method in a suitable way in the following. It will be seen that the combination of (2.5), (2.6) and (2.7) is the basis for the coupled continuous-discontinuous Galerkin approach.

Denote by $\mathbb{Q}^1(K)$ the space of bilinear functions defined on K, and define the finite element space Q_N, V_N^1 and V_N^2 as follows,

$$\begin{aligned} Q_N &= \left\{ \mathbf{q} \in L^2(\Omega_1)^2 : \mathbf{q}|_K \in \mathbb{Q}^1(K)^2, \forall K \in \mathcal{T}_N^1 \right\}, \\ V_N^1 &= \left\{ u_1 \in L^2(\Omega_1) : u_1|_K \in \mathbb{Q}^1(K), \forall K \in \mathcal{T}_N^1 \right\}, \\ V_N^2 &= \left\{ u_2 \in H^1(\Omega_2) : u_2|_{\Gamma_D^2} = 0, u_2|_K \in \mathbb{Q}^1(K), \forall K \in \mathcal{T}_N^2 \right\}. \end{aligned}$$

The space V_N^2 is a standard conforming finite element space, whereas the functions in V_N^1 and Q_N are completely discontinuous across interelement boundaries.

We will search for approximate solutions $(\mathbf{q}_N, u_{1,N}, u_{2,N})$ of (2.3) and (2.4) in the finite element space $Q_N \times V_N^1 \times V_N^2$ that satisfy (2.3) and (2.4) in a weak sense, i.e., we will find $(\mathbf{q}_N, u_{1,N}, u_{2,N}) \in Q_N \times V_N^1 \times V_N^2$ such that

$$\int_K \mathbf{q}_N \cdot \mathbf{w} \mathrm{d}\mathbf{x} + \int_K u_{1,N} \nabla \cdot \mathbf{w} \mathrm{d}\mathbf{x} - \int_{\partial K} \widehat{u}_{1,N} \mathbf{w} \cdot \mathbf{n}_K \mathrm{d}s = 0, \tag{2.8}$$

$$\int_K (\epsilon\mathbf{q}_N - \mathbf{b}u_{1,N}) \cdot \nabla v_1 \mathrm{d}\mathbf{x} + \int_K (c - \nabla \cdot \mathbf{b}) u_{1,N} v_1 \mathrm{d}\mathbf{x} - \int_{\partial K} v_1 (\epsilon\widehat{\mathbf{q}}_N - \mathbf{b}\widetilde{u}_{1,N}) \cdot \mathbf{n}_K \mathrm{d}s = \int_K f v_1 \mathrm{d}\mathbf{x}, \tag{2.9}$$

for any test function $(\mathbf{w}, v_1) \in Q_N \times V_N^1$ and $K \in \mathcal{T}_N^1$, and

$$\int_{\Omega_2} (\epsilon \nabla u_{2,N} - \mathbf{b}u_{2,N}) \cdot \nabla v_2 \mathrm{d}\mathbf{x} + \int_{\Omega_2} (c - \nabla \cdot \mathbf{b}) u_{2,N} v_2 \mathrm{d}\mathbf{x} + \int_\Gamma v_2 (\epsilon\widehat{\mathbf{q}}_N - \mathbf{b}\widetilde{u}_{1,N}) \cdot \mathbf{n}_1 \mathrm{d}s = \int_{\Omega_2} f v_2 \mathrm{d}\mathbf{x}, \tag{2.10}$$

for any test function $v_2 \in V_N^2$, where $\widehat{u}_{1,N}$, $\widetilde{u}_{1,N}$ and $\widehat{\mathbf{q}}_N$ are the numerical fluxes, which approximate the traces of $u_{1,N}$ and $\mathbf{q}_N$ on the boundary of the elements of $\mathcal{T}_N^1$. To complete the specification of the method, it only remains to define the numerical fluxes.

The numerical fluxes. In order to define the numerical fluxes, we need to introduce some notations. An interior face of the triangulation $\mathcal{T}_N^1$ is defined to be the (non-empty) interior of $\partial K_+ \cap \partial K_-$, where K_+ and K_- are two adjacent elements of $\mathcal{T}_N^1$. Analogously, a boundary face in $\mathcal{T}_N^1$ is the interior of $\partial K \cap \partial\Omega$ where K is a boundary element in $\mathcal{T}_N^1$. We assume the boundary faces are contained in either Γ_D^1 or Γ. Let $\mathcal{E}^o$ be the union of all interior faces, and $\mathcal{E}$ the union of all faces of $\mathcal{T}_N^1$. Let now $e \in \mathcal{E}^o$ be an interior face shared by K_+ and K_-, $\mathbf{n}_+$ and $\mathbf{n}_-$ the unit normal vectors on e pointing exterior to K_+ and K_-, respectively. The *average* and the *jump* of a scalar-valued function v on $e \in \mathcal{E}^o$ are given by

$$\{v\} := \frac{1}{2}(v_+ + v_-), \qquad [v] := v_+\mathbf{n}_+ + v_-\mathbf{n}_-,$$

respectively. For a vector-valued function $\mathbf{w}$, we define its corresponding average and jump analogously, i.e.,

$$\{\mathbf{w}\} := \frac{1}{2}(\mathbf{w}_+ + \mathbf{w}_-), \qquad [\mathbf{w}] := \mathbf{w}_+ \cdot \mathbf{n}_+ + \mathbf{w}_- \cdot \mathbf{n}_- \qquad \text{on} \quad e \in \mathcal{E}^o.$$

For $e \subset \partial\Omega_1 \setminus \Gamma$, each v and $\mathbf{w}$ has a uniquely defined restriction on e. We set

$$[v] := v\mathbf{n}_e, \qquad \{\mathbf{w}\} := \mathbf{w} \qquad \text{on} \quad e \subset \partial\Omega_1 \setminus \Gamma,$$

where $\mathbf{n}_e$ is the unit outward normal of e. We do not require either of the quantities $\{v\}$ or $[\mathbf{w}]$ on the boundary faces, and leave them undefined. Now the numerical fluxes $\widehat{u}_{1,N}$ and $\widehat{\mathbf{q}}_{1,N}$ are defined by

$$\widehat{u}_1 = \begin{cases} \{u_{1,N}\} + \beta \cdot [u_{1,N}] & \text{if } \ e \in \mathcal{E}^o, \\ u_{2,N} & \text{if } \ e \subset \Gamma, \\ 0 & \text{if } \ e \subset \Gamma_D^1, \end{cases} \tag{2.11}$$

and

$$\widehat{\mathbf{q}}_{1,N} = \begin{cases} \{\mathbf{q}_N\} - \alpha[u_{1,N}] - \beta[\mathbf{q}_N] & \text{if } \ e \in \mathcal{E}^o, \\ \mathbf{q}_N - \alpha(u_{1,N} - u_{2,N})\mathbf{n}_1 & \text{if } \ e \subset \Gamma, \\ \mathbf{q}_N - \alpha\, u_{1,N}\mathbf{n} & \text{if } \ e \subset \Gamma_D^1. \end{cases} \tag{2.12}$$

On the other hand, the numerical flux associated with the convection term is the classical upwinding one, which can be expressed as

$$\widetilde{u}_{1,N} = \begin{cases} \{u_{1,N}\} + \beta_0 \cdot [u_{1,N}] & \text{if } \ e \in \mathcal{E}^o, \\ u_{1,N} & \text{if } \ e \subset \Gamma, \\ 0 & \text{if } \ e \subset \Gamma_D^1. \end{cases} \tag{2.13}$$

Here the scalar function $\alpha = \alpha(\mathbf{x})$, the vector functions $\beta = \beta(\mathbf{x})$ and $\beta_0 = \beta_0(\mathbf{x})$ are auxiliary functions, which play a very important role in guaranteeing the stability and enhancing the accuracy of the LDG scheme (see [2] and [4]). In this paper we take

$$\alpha = \alpha(\mathbf{x}) = \mathcal{O}(1/H_x, 1/H_y),$$

for $\mathbf{x} \in \mathcal{E}$, and $\beta = \beta(\mathbf{x})$ and $\beta_0 = \beta_0(\mathbf{x})$, such that

$$\beta \cdot \mathbf{n}_K(\mathbf{x}) = \beta_0 \cdot \mathbf{n}_K(\mathbf{x}) = \frac{1}{2}\text{sign}(\mathbf{b}(\mathbf{x}) \cdot \mathbf{n}_K(\mathbf{x})),$$

for $\mathbf{x} \in \mathcal{E}^0$, where $\mathbf{n}_K(\mathbf{x})$ is the unit outward normal of K at $\mathbf{x}$.

3. Stability and error analysis of the coupled method.

This section is devoted to the existence and uniqueness of the solution of the coupled method (2.8)-(2.10) with numerical fluxes (2.11)-(2.13), and its corresponding error analysis. If the stabilization parameter α is taken of order $\mathcal{O}(1/H_x, 1/H_y)$ as we did in section 2, we can rewrite our method in a primal form by eliminating $\mathbf{q}$ following Arnold *et al.* [1].

Primal formulation. A straightforward computation shows that

$$\int_{\Omega_1} \nabla\cdot\mathbf{w} v \mathrm{d}\mathbf{x} = -\int_{\Omega_1} \mathbf{w}\cdot\nabla v \mathrm{d}\mathbf{x} + \int_{\mathcal{E}^o} ([\mathbf{w}]\{v\} + \{\mathbf{w}\}\cdot[v])\mathrm{d}s + \int_{\Gamma} v\mathbf{w}\cdot\mathbf{n}_1 \mathrm{d}s + \int_{\Gamma_D^1} v\mathbf{w}\cdot\mathbf{n}\mathrm{d}s, \tag{3.1}$$

for $\mathbf{w} \in H^1(\mathcal{T}_N^1)\times H^1(\mathcal{T}_N^1)$ and $v \in H^1(\mathcal{T}_N^1)$, where $H^1(\mathcal{T}_N^1)$ is the piecewise Sobolev space defined by

$$H^1(\mathcal{T}_N^1) := \left\{ \phi \in L^2(\Omega_1) : \phi|_K \in H^1(K) \text{ for all } K \in \mathcal{T}_N^1 \right\}.$$

Differential operators are understood to act on such a space piecewisely.

Summing up (2.8) for all $K \in \mathcal{T}_N^1$, combined with (2.11) and (3.1), we obtain

$$\begin{aligned} &\int_{\Omega_1} (\mathbf{q}_N - \nabla u_{1,N})\cdot\mathbf{w}\mathrm{d}\mathbf{x} + \int_{\mathcal{E}^o} (\{\mathbf{w}\} - \beta[\mathbf{w}])\cdot[u_{1,N}]\mathrm{d}s \\ &\quad + \int_{\Gamma} (u_{1,N} - u_{2,N})\mathbf{w}\cdot\mathbf{n}_1\mathrm{d}s + \int_{\Gamma_D^1} u_{1,N}\mathbf{w}\cdot\mathbf{n}\mathrm{d}s = 0. \end{aligned} \tag{3.2}$$

Denote $V(N) = \left[H^2(\Omega)\cap H^1_{\Gamma_D}(\Omega)\right] + V_N$, where $V_N :=\{ v = (v_1, v_2) : v_1 \in V_N^1, v_2 \in V_N^2 \}$, $H^1_{\Gamma_D}(\Omega) = \left\{ v = (v_1, v_2) \in H^1(\Omega) : v_1|_{\Gamma_D^1} = 0, v_2|_{\Gamma_D^2} = 0 \right\}$. For $v \in V(N)$, we define $\mathcal{L}_1(v)$ as the unique element in Q_N such that

$$\int_{\Omega_1} \mathcal{L}_1(v)\cdot\mathbf{r}\mathrm{d}\mathbf{x} = \int_{\mathcal{E}^o} (\{\mathbf{r}\} - \beta[\mathbf{r}])\cdot[v_1]\mathrm{d}s + \int_{\Gamma} (v_1 - v_2)\mathbf{r}\cdot\mathbf{n}_1\mathrm{d}s + \int_{\Gamma_D^1} v_1\mathbf{r}\cdot\mathbf{n}\mathrm{d}s, \tag{3.3}$$

for all $\mathbf{r} \in Q_N$. As a result, (3.2) can be rewritten as

$$\mathbf{q}_N = \nabla u_{1,N} - \mathcal{L}_1(u_N). \tag{3.4}$$

Summing up (2.9) for all $K \in \mathcal{T}_N^1$ and then adding up (2.10), we get

$$\begin{aligned} &\int_{\Omega_1} (\epsilon\mathbf{q}_N - \mathbf{b}u_{1,N})\cdot\nabla v_1\mathrm{d}\mathbf{x} + \int_{\Omega_2} (\epsilon\nabla u_{2,N} - \mathbf{b}u_{2,N})\cdot\nabla v_2\mathrm{d}\mathbf{x} + \int_{\Omega} (c - \nabla\cdot\mathbf{b})u_N v\mathrm{d}\mathbf{x} \\ &- \int_{\mathcal{E}\backslash\Gamma} (\epsilon\widehat{\mathbf{q}}_N - \mathbf{b}\widetilde{u}_{1,N})\cdot[v_1]\mathrm{d}s - \int_{\Gamma} (\epsilon\widehat{\mathbf{q}}_N - \mathbf{b}\widetilde{u}_{1,N})\cdot\mathbf{n}_1(v_1 - v_2)\mathrm{d}s = \int_{\Omega} f v\mathrm{d}\mathbf{x}. \end{aligned} \tag{3.5}$$

Inserting (2.12) and (2.13) into (3.5), and recalling the definition of $\mathcal{L}_1(\cdot)$, we obtain

$$\begin{aligned} &\int_{\Omega_1} \epsilon\mathbf{q}_N\cdot(\nabla v_1 - \mathcal{L}_1(v))\mathrm{d}\mathbf{x} + \int_{\Omega_2} \epsilon\nabla u_{2,N}\cdot\nabla v_2\mathrm{d}\mathbf{x} - \int_{\Omega} \mathbf{b}\cdot\nabla v u_N\mathrm{d}\mathbf{x} \\ &+ \int_{\Omega} (c - \nabla\cdot\mathbf{b})u_N v\mathrm{d}\mathbf{x} + \int_{\mathcal{E}\backslash\Gamma} \epsilon\alpha[u_{1,N}][v_1]\mathrm{d}s + \int_{\Gamma} \epsilon\alpha(u_{1,N} - u_{2,N})(v_1 - v_2)\mathrm{d}s \\ &+ \int_{\mathcal{E}^o} \mathbf{b}\cdot[v_1](\{u_{1,N}\} + \beta_0\cdot[u_{1,N}])\mathrm{d}s + \int_{\Gamma} \mathbf{b}\cdot\mathbf{n}_1 u_{1,N}(v_1 - v_2)\mathrm{d}s = \int_{\Omega} f\, v\mathrm{d}\mathbf{x}. \end{aligned} \tag{3.6}$$

Similar to the definition of $\mathcal{L}_1(\cdot)$, for $v \in V(N)$, we define $\mathcal{L}_2(v)$ as the unique element in Q_N such that

$$\int_{\Omega_1} \mathbf{b}\cdot\mathcal{L}_2(v)u\mathrm{d}\mathbf{x} = \int_{\mathcal{E}^o} \mathbf{b}\cdot[v_1](\{u_1\} + \beta_0\cdot[u_1])\mathrm{d}s + \int_{\Gamma} \mathbf{b}\cdot\mathbf{n}_1 u_1(v_1 - v_2)\mathrm{d}s, \tag{3.7}$$

for all $u \in V_N$. As a consequence, (3.6) can be rewritten as

$$\begin{aligned} &\int_{\Omega_1} \epsilon \mathbf{q}_N \cdot (\nabla v_1 - \mathcal{L}_1(v))\mathrm{d}\mathbf{x} + \int_{\Omega_2} \epsilon \nabla u_{2,N} \cdot \nabla v_2 \mathrm{d}\mathbf{x} \\ - &\int_{\Omega} \mathbf{b} \cdot \nabla v u_N \mathrm{d}\mathbf{x} + \int_{\Omega} (c - \nabla \cdot \mathbf{b}) u_N v \mathrm{d}\mathbf{x} + \int_{\mathcal{E} \setminus \Gamma} \epsilon\alpha [u_{1,N}][v_1] \mathrm{d}s \\ + &\int_{\Gamma} \epsilon\alpha (u_{1,N} - u_{2,N})(v_1 - v_2)\mathrm{d}s + \int_{\Omega_1} \mathbf{b} \cdot \mathcal{L}_2(v) u_N \mathrm{d}\mathbf{x} = \int_{\Omega} f\, v \mathrm{d}\mathbf{x}. \end{aligned} \tag{3.8}$$

Inserting (3.4) into (3.8), we obtain the so-called primal form of our coupled method which reads: find $u_N \in V_N$ such that

$$\mathcal{A}_N(u_N, v) := \mathcal{B}_N(u_N, v) + \mathcal{C}_N(u_N, v) + \mathcal{S}_N(u_N, v) = \mathcal{F}_N(v) \qquad \forall v \in V_N, \tag{3.9}$$

with

$$\begin{aligned} \mathcal{B}_N(u,v) &= \int_\Omega \epsilon(\nabla u - \mathcal{L}_1(u)) \cdot (\nabla v - \mathcal{L}_1(v))\mathrm{d}\mathbf{x}, \qquad \mathcal{F}_N(v) = \int_\Omega f v \mathrm{d}\mathbf{x}, \\ \mathcal{C}_N(u,v) &= -\int_\Omega \mathbf{b} u \cdot (\nabla v - \mathcal{L}_2(v))\mathrm{d}\mathbf{x} + \int_\Omega (c - \nabla \cdot \mathbf{b}) u v \mathrm{d}\mathbf{x}, \\ \mathcal{S}_N(u,v) &= \int_{\mathcal{E}\setminus\Gamma} \epsilon\alpha [u_1][v_1]\mathrm{d}s + \int_\Gamma \epsilon\alpha(u_1 - u_2)(v_1 - v_2)\mathrm{d}s. \end{aligned}$$

Here, $\mathcal{L}_1(u)$ and $\mathcal{L}_2(u)$ have been defined in $L^2(\Omega)$ by a trivial extension.

From the following lemma, the primal formulation is consistent.

Lemma 3.1. *Let u be the exact solution of (2.3) and (2.4), then we have the Galerkin orthogonality property, i.e.,*

$$\mathcal{A}_N(u - u_N, v) = 0, \qquad \textit{for all } v \in V_N. \tag{3.10}$$

Proof. Since u is the exact solution, we have $[u_1]_e = 0, [\nabla u_1] = 0$ for all $e \in \mathcal{E}^o$, $u_1 = 0$ on Γ_D^1, and $u_1 = u_2, \nabla u_1 \cdot \mathbf{n}_1 = -\nabla u_2 \cdot \mathbf{n}_2$ on Γ. Consequently,

$$\mathcal{L}_1(u) = 0, \qquad \mathcal{S}_N(u,v) = 0. \tag{3.11}$$

Then, for all $v \in V_N$, we have

$$\begin{aligned} \mathcal{B}_N(u,v) &= \int_\Omega \epsilon \nabla u \cdot (\nabla v - \mathcal{L}_1(v))\mathrm{d}\mathbf{x} \\ &= \int_{\Omega_1} \epsilon \nabla u_1 \cdot \nabla v_1 \mathrm{d}\mathbf{x} - \int_{\Omega_1} \epsilon \mathcal{L}_1(v) \cdot \nabla u_1 \mathrm{d}\mathbf{x} + \int_{\Omega_2} \epsilon \nabla u_2 \cdot \nabla v_2 \mathrm{d}\mathbf{x}. \end{aligned} \tag{3.12}$$

Taking $\mathbf{w} = \epsilon \nabla u_1$ and $v = v_1$ in (3.1), we have

$$\begin{aligned} \int_{\Omega_1} \epsilon \nabla u_1 \cdot \nabla v_1 \mathrm{d}\mathbf{x} &= -\int_{\Omega_1} \epsilon \Delta u_1 v_1 \mathrm{d}\mathbf{x} + \int_{\mathcal{E}^o} \epsilon \left([\nabla u_1]\{v_1\} + \{\nabla u_1\} \cdot [v_1] \right) \mathrm{d}s \\ &\quad + \int_\Gamma \epsilon\, v_1 \nabla u_1 \cdot \mathbf{n}_1 \mathrm{d}s + \int_{\Gamma_D^1} \epsilon\, v_1 \nabla u_1 \cdot \mathbf{n} \mathrm{d}s. \end{aligned} \tag{3.13}$$

Inserting (3.13) into (3.12), and integrating by parts in the third term in the right hand side of (3.12), in terms of the definition of $\mathcal{L}_1(\cdot)$, we have

$$\mathcal{B}_N(u,v) = \int_\Omega -\epsilon \Delta u v \mathrm{d}\mathbf{x} + \int_{\mathcal{E}^o} \epsilon [\nabla u_1](\beta \cdot [v_1] + \{v_1\})\mathrm{d}s + \int_{\Gamma_D^2} \epsilon\, v_2 \nabla u_2 \cdot \mathbf{n} \mathrm{d}s.$$

By the definition of V_N, we get $v_2 = 0$ on Γ_D^2. This, together with $[\nabla u_1] = 0$, yields

$$\mathcal{B}_N(u,v) = \int_\Omega -\epsilon\Delta\, uv\mathrm{d}\mathbf{x}, \quad \text{for all } v \in V_N. \tag{3.14}$$

Now we consider the term $\mathcal{C}_N(u,v)$. It can be rewritten as

$$\mathcal{C}_N(u,v) = -\int_{\Omega_1} \mathbf{b}u_1\cdot(\nabla v_1 - \mathcal{L}_2(v))\mathrm{d}\mathbf{x} - \int_{\Omega_2} \mathbf{b}\cdot\nabla v_2 u_2\mathrm{d}\mathbf{x} + \int_\Omega (c - \nabla\cdot\mathbf{b})uv\mathrm{d}\mathbf{x}. \tag{3.15}$$

Taking $\mathbf{w} = -\mathbf{b}u_1$ and $v = v_1$ in (3.1), we have

$$\begin{aligned} -\int_{\Omega_1} \mathbf{b}u_1\cdot\nabla v_1\mathrm{d}\mathbf{x} &= \int_{\Omega_1}(\mathbf{b}\cdot\nabla u_1 + \nabla\cdot\mathbf{b}u_1)v_1\mathrm{d}\mathbf{x} - \int_{\mathcal{E}^o}\mathbf{b}\cdot([u_1]\{v_1\} + \{u_1\}[v_1])\mathrm{d}s \\ &\quad -\int_\Gamma \mathbf{b}\cdot\mathbf{n}_1 u_1 v_1\mathrm{d}s - \int_{\Gamma_D^1}\mathbf{b}\cdot\mathbf{n}u_1v_1\mathrm{d}s. \end{aligned} \tag{3.16}$$

Inserting (3.16) into (3.15), and integrating by parts in the second term in the right hand side of (3.15), in terms of the definition of $\mathcal{L}_2(\cdot)$, we obtain

$$\begin{aligned} \mathcal{C}_N(u,v) &= \int_\Omega(\mathbf{b}\cdot\nabla u + cu)v\mathrm{d}\mathbf{x} + \int_{\mathcal{E}^o}(\mathbf{b}\cdot[v_1]\beta_0\cdot[u_1] - \mathbf{b}\cdot[u_1]\{v_1\})\mathrm{d}s \\ &= \int_\Omega(\mathbf{b}\cdot\nabla u + cu)v\mathrm{d}\mathbf{x} + \int_{\mathcal{E}^o}\mathbf{b}\cdot[u_1](\beta_0\cdot[v_1] - \{v_1\})\mathrm{d}s \\ &= \int_\Omega(\mathbf{b}\cdot\nabla u + cu)v\mathrm{d}\mathbf{x}, \end{aligned} \tag{3.17}$$

due to the fact $[u_1] = 0$ on $\mathcal{E}^o$. The combination of (3.9), (3.11), (3.14) and (3.17), leads to

$$\begin{aligned} \mathcal{A}_N(u,v) &= \int_\Omega(-\epsilon\Delta\, u + \mathbf{b}\cdot\nabla u + cu)\mathrm{d}\mathbf{x} \\ &= \int_\Omega f\, v\mathrm{d}\mathbf{x}, \quad \forall\, v\in V_N. \end{aligned}$$

In view of (3.9), (3.10) obviously holds. □

Stability analysis. To consider the stability of the primal form $\mathcal{A}_N$, define the following norms and seminorms for $v \in V(N)$:

$$|||v|||_\epsilon^2 = ||v||_{0,\Omega}^2 + \epsilon|v|_{1,N}^2 + \epsilon|v|_*^2 + |v|_c^2, \tag{3.18}$$

$$|v|_{1,N}^2 = |v_2|_{1,\Omega_2}^2 + \sum_{K\in\mathcal{T}_N^1}|v_1|_{1,K}^2, \quad |v|_*^2 = \int_{\mathcal{E}\setminus\Gamma}\alpha[v_1]^2\mathrm{d}s + \int_\Gamma \alpha(v_1 - v_2)^2\mathrm{d}s,$$

$$|v|_c^2 = \frac{1}{2}\sum_{e\in\mathcal{E}^o}\int_e|\mathbf{b}\cdot\mathbf{n}_e|\,[v_1]^2\mathrm{d}s + \frac{1}{2}\int_\Gamma|\mathbf{b}\cdot\mathbf{n}_1|\,(v_1-v_2)^2\mathrm{d}s + \frac{1}{2}\int_{\Gamma_D^1}|\mathbf{b}\cdot\mathbf{n}|\,v^2\mathrm{d}s.$$

Lemma 3.2. *There exists a constant $C_1 > 0$, such that*

$$\mathcal{A}_N(u_N,u_N) \geq C_1|||u_N|||_\epsilon^2, \quad \forall u_N \in V_N.$$

Proof. By direct computation, we obtain, for all $u_N \in V_N$,

$$\begin{aligned} \mathcal{B}_N(u_N,u_N) &= \int_\Omega \epsilon(\nabla u_N - \mathcal{L}_1(u_N))^2\mathrm{d}\mathbf{x}, \quad \mathcal{S}_N(u_N,u_N) = \epsilon|u_N|_*^2, \\ \mathcal{C}_N(u_N,u_N) &= \int_\Omega(c - \frac{1}{2}\nabla\cdot\mathbf{b})u_N^2\mathrm{d}\mathbf{x} + |u_N|_c^2. \end{aligned}$$

By (3.9) and (2.2), we have

$$
\begin{aligned}
&\mathcal{A}_N(u_N,u_N)\\
= &\int_\Omega \epsilon(\nabla u_N - \mathcal{L}_1(u_N))^2 \mathrm{d}\mathbf{x} + \int_\Omega c_0^2 u_N^2 \mathrm{d}\mathbf{x} + \epsilon|u_N|_*^2 + |u_N|_c^2\\
\geq &\ \epsilon\,|u_N|_{1,N}^2 - 2\epsilon \int_\Omega \mathcal{L}_1(u_N)\cdot \nabla u_N \mathrm{d}\mathbf{x} + \epsilon||\mathcal{L}_1(u_N)||_{0,\Omega}^2 + \gamma_0||u_N||_{0,\Omega}^2 + \epsilon|u_N|_*^2 + |u_N|_c^2.
\end{aligned}
$$

Applying the arithmetic-geometric mean inequality, we have, for every $\theta > 0$,

$$
\begin{aligned}
&\mathcal{A}_N(u_N,u_N)\\
\geq &\ \epsilon\left[(1-\theta)|u_N|_{1,N}^2 + (1-1/\theta)||\mathcal{L}_1(u_N)||_{0,\Omega}^2\right] + \gamma_0||u_N||_{0,\Omega}^2 + \epsilon|u_N|_*^2 + |u_N|_c^2.
\end{aligned} \tag{3.19}
$$

According to [8] (p422), when the coefficient $\alpha = \mathcal{O}(1/H_x, 1/H_y)$, there exists a constant $C > 0$, such that

$$
||\mathcal{L}_1(u)||_{0,\Omega} \leq C|u|_*, \quad \forall u \in V(N). \tag{3.20}
$$

Inserting (3.20) into (3.19), for any θ satisfies $\frac{C^2}{C^2+1} < \theta < 1$, we easily get

$$
\begin{aligned}
\mathcal{A}_N(u_N,u_N) &\geq \epsilon(1-\theta)|u_N|_{1,N}^2 + \epsilon[C^2(1-1/\theta)+1]|u_N|_*^2 + \gamma_0||u_N||_{0,\Omega}^2 + |u_N|_c^2\\
&\geq \gamma_1(\epsilon|u_N|_{1,N}^2 + \epsilon|u_N|_*^2) + \gamma_0||u_N||_{0,\Omega}^2 + |u_N|_c^2\\
&\geq \min\{\gamma_0,\gamma_1\}|||u_N|||_\epsilon^2,
\end{aligned}
$$

where $\gamma_1 = \min\{1-\theta, C^2(1-1/\theta)+1\}$. Taking $C_1 = \min\{\gamma_0,\gamma_1\}$, the proof is completed. □

From Lemma 3.2, we easily get

$$
|||u_N|||_\epsilon \leq C||f||_{0,\Omega},
$$

which implies the uniqueness of the solution to (3.9). Further, since (3.9) is a linear problem over the finite-dimensional space V_N, the existence of the solution follows from its uniqueness. Consequently, by (3.4), we get the existence and uniqueness of the solution to the problem (2.8)-(2.10) with numerical fluxes (2.11)-(2.13).

Remark 3.1. *In fact, following* [8] *or* [13]*, for any $\alpha \geq 0$, the existence and uniqueness of the solution to the problem (2.8)-(2.10) with numerical flux (2.11)-(2.13) can be proved. In this paper, we are only interested in the special case $\alpha = \mathcal{O}(1/H_x, 1/H_y)$.*

Error analysis. We are now going to provide an ϵ-uniform estimate for the error $u - u_N$ in the norm (3.18). First, we start with the error decomposition

$$
u - u_N = (u - u_I) + (u_I - u_N) \equiv \eta + \xi, \tag{3.21}
$$

where u_I be the standard bilinear interpolation of u.

The final estimate for $|||u-u_N|||_\epsilon$ will be derived by applying the triangle inequality to (3.21). For this purpose, we estimate $|||\eta|||_\epsilon$ and $|||\xi|||_\epsilon$, respectively. To bound η, we need some regularity results. Denote the operator $\mathcal{L}_i$, $i = 0, 1$, by

$$
\mathcal{L}_i v = \frac{\partial v}{\partial y}\frac{\partial^i}{\partial x^i}\Big(\frac{b_2}{b_1}\Big) + v\frac{\partial^i}{\partial x^i}\Big(\frac{c}{b_1}\Big). \tag{3.22}
$$

Lemma 3.3. [7, 18] *Let* $\mathbf{b}$ *and* c *be smooth, and let $f \in C^{4,\lambda}(\bar{\Omega})$ for some $\lambda \in (0,1)$. Further suppose that f satisfies the compatibility conditions*

$$
f(0,0) = f(0,1) = f(1,0) = f(1,1) = 0,
$$

and

$$\Big(\frac{f}{b_1}\Big)_y(0,0) = \Big(\frac{f}{b_2}\Big)_x(0,0),$$

$$\Big(\Big(\frac{f}{b_1}\Big)_x - \mathcal{L}_0\Big(\frac{f}{b_1}\Big)\Big)_y(0,0) = \Big(\frac{f}{b_2}\Big)_{xx}(0,0), \tag{3.23}$$

$$\Big(\Big(\frac{f}{b_1}\Big)_{xx} - \mathcal{L}_0\Big(\Big(\frac{f}{b_1}\Big)_x - \mathcal{L}_0\Big(\frac{f}{b_1}\Big)\Big) - 2\mathcal{L}_1\Big(\frac{f}{b_1}\Big)\Big)_y(0,0) = \Big(\frac{f}{b_2}\Big)_{xxx}(0,0), \tag{3.24}$$

$$\Big(b_2\Big(\frac{f}{b_2}\Big)_{xx}\Big)(0,0) = \Big(b_1\Big(\frac{f}{b_1}\Big)_{yy}\Big)(0,0). \tag{3.25}$$

Then the boundary value problem (2.1) has a classical solution $u \in C^{3,\lambda}(\bar{\Omega})$*, that can be bounded by*

$$\left|\frac{\partial^{i+j} u(x,y)}{\partial x^i \partial y^j}(x,y)\right| \le C\left(1+\epsilon^{-i}e^{-\beta_1(1-x)/\epsilon}\right) \times \left(1+\epsilon^{-j}e^{-\beta_2(1-y)/\epsilon}\right) \tag{3.26}$$

for all $(x,y) \in \bar{\Omega}$ *and* $0 \le i+j \le 2$.

Remark 3.2. *The following convection-diffusion problem*

$$-\epsilon\Delta u - \mathbf{b}\cdot\nabla u + cu = f \tag{3.27}$$

is considered in [5]*, which exhibits an exponential layer near* $x=0$ *and* $y=0$*. Let* $\hat{x} = 1-x, \hat{y} = 1-y$, $\hat{b}(\hat{x},\hat{y}) = b(1-\hat{x},1-\hat{y})$, $\hat{c}(\hat{x},\hat{y}) = c(1-\hat{x},1-\hat{y})$,$\hat{f}(\hat{x},\hat{y}) = f(1-\hat{x},1-\hat{y})$ *and* $\hat{u}(\hat{x},\hat{y}) = u(1-\hat{x},1-\hat{y})$*, then (2.1) can be rewritten as*

$$-\epsilon\Delta\,\hat{u} - \hat{\mathbf{b}}\cdot\nabla\,\hat{u} + \hat{c}\hat{u} = \hat{f},$$

which is the same as (3.27). So the formula (8) in [5] *is also valid for (2.1) in this paper, if we replace* x *and* y *by* $\hat{x} = 1-x$ *and* $\hat{y} = 1-y$*, respectively. That is exactly (3.26).*

Lemma 3.4. [5] *Let* u *be a solution of (2.1) and* u_I *the standard bilinear interpolation of* u *on the Shishkin mesh defined before. Under the conditions of Lemma 3.3, the interpolation error* $\eta = u - u_I$ *satisfies*

$$\begin{aligned}
||\eta||_{L^\infty(\Omega_1)} &\le CN^{-2}, \qquad ||\eta||_{L^\infty(\Omega_2)} \le CN^{-2}\ln^2 N,\\
||\eta||_{L^2(\Omega_1)} &\le CN^{-2}, \qquad ||\eta||_{L^2(\Omega_2)} \le CN^{-2}\ln^2 N,\\
||\nabla\eta||_{L^2(\Omega)} &\le C\epsilon^{-1/2}N^{-1}\ln N.
\end{aligned}$$

Remark 3.3. *According to* [7]*, under the assumption of Lemma 3.3, the solution of (2.1) has the usual solution decomposition* $u = S + E_1 + E_2 + E_{12}$*, which satisfies*

$$\begin{aligned}
\Big|\frac{\partial^{i+j} S}{\partial x^i \partial y^j(x,y)}\Big| &\le C,\\
\Big|\frac{\partial^{i+j} E_1}{\partial x^i \partial y^j(x,y)}\Big| &\le C\epsilon^{-i}e^{-\beta_1(1-x)/\epsilon},\\
\Big|\frac{\partial^{i+j} E_2}{\partial x^i \partial y^j(x,y)}\Big| &\le C\epsilon^{-j}e^{-\beta_2(1-y)/\epsilon},\\
\Big|\frac{\partial^{i+j} E_{12}}{\partial x^i \partial y^j(x,y)}\Big| &\le C\epsilon^{-(i+j)}e^{-(\beta_1(1-x)+\beta_2(1-y))/\epsilon},
\end{aligned}$$

for $0 \le i+j \le 2$*, which is stronger than (3.26) to some extent. Actually the estimates above imply (3.26). Based on the solution decomposition above, the conclusions of Lemma 3.4 can be found in* [11]*, i.e., (3.127) of Page 384, (3.128b)* *and (3.128c) of Page 385 with the additional assumption $\epsilon^{1/2}\ln^2 N \le C$. It is worthwhile to point out that there is no $\ln N$ factor in (3.128b) there.*

The following statement represents the direct consequence of Lemma 3.4.

Prosition 3.1. *Under the conditions of Lemma 3.4, we have*

$$|||\eta|||_\epsilon \le CN^{-1}\ln N. \tag{3.28}$$

Proof. Since $u - u_I$ is continuous in Ω, we have $|\eta|_* = 0, |\eta|_c = 0$. Then, $|||\eta|||_\epsilon^2 = ||\eta||_{0,\Omega}^2 + \epsilon|\eta|_{1,N}^2$. By Lemma 3.4, we easily conclude (3.28). □

Now we turn to estimate $|||\xi|||_\epsilon$.

Prosition 3.2. *Under the conditions of Lemma 3.4, assume $\alpha = \mathcal{O}(1/H_x, 1/H_y)$ and*

$$\ln^{3/2} N \le CN, \tag{3.29}$$

then $\xi = u_I - u_N$ satisfies

$$|||\xi|||_\epsilon \le C\,N^{-1}\ln N. \tag{3.30}$$

Proof. By Lemma 3.1 and Lemma 3.2, we first obtain

$$C_1|||\xi|||_\epsilon^2 \le \mathcal{A}_N(\xi,\xi) = -\mathcal{A}_N(\eta,\xi) = -\mathcal{B}_N(\eta,\xi) - \mathcal{C}_N(\eta,\xi) - \mathcal{S}_N(\eta,\xi). \tag{3.31}$$

By the definition of u_I, we have $[\eta]_e = 0$ for all $e \in \mathcal{E}$. Consequently,

$$\begin{aligned}
\mathcal{B}_N(\eta,\xi) &= \int_\Omega \epsilon\nabla\eta\cdot(\nabla\xi - \mathcal{L}_1(\xi))\mathrm{d}\mathbf{x},\\
\mathcal{C}_N(\eta,\xi) &= -\int_\Omega \mathbf{b}\cdot\nabla\xi\eta\mathrm{d}\mathbf{x} + \int_{\Omega_1}\mathbf{b}\cdot\mathcal{L}_2(\xi)\eta\mathrm{d}\mathbf{x} + \int_\Omega (c - \nabla\cdot\mathbf{b})\eta\xi\mathrm{d}\mathbf{x}\\
&\equiv I_1 + I_2 + I_3, \qquad (3.32)\\
\mathcal{S}_N(\eta,\xi) &= 0.
\end{aligned}$$

Then, by (3.20) and Lemma 3.4, we have

$$|\mathcal{B}_N(\eta,\xi)| \le C\epsilon||\nabla\eta||_{L^2(\Omega)}(|\xi|_{1,N} + |\xi|_*) \le C\epsilon^{1/2}||\nabla\eta||_{L^2(\Omega)}|||\xi|||_\epsilon \le CN^{-1}\ln N\,|||\xi|||_\epsilon. \tag{3.33}$$

The first term in the right hand side of (3.32) can be estimated by,

$$|I_1| \le C\left(||\eta||_{L^2(\Omega_1)}||\nabla\xi||_{L^2(\Omega_1)} + ||\eta||_{L^\infty(\Omega_2)}||\nabla\xi||_{L^1(\Omega_2)}\right).$$

On Ω_1, the implementation of an inverse inequality leads to

$$||\nabla\xi||_{L^2(\Omega_1)} \le CN||\xi||_{L^2(\Omega_1)} \le CN|||\xi|||_\epsilon.$$

On the other hand, on Ω_2, the Cauchy-Schwarz inequality yields

$$||\nabla\xi||_{L^1(\Omega_2)} \le |\Omega_2|^{1/2}||\nabla\xi||_{L^2(\Omega_2)} \le C\tau^{1/2}||\nabla\xi||_{L^2(\Omega_2)} \le C\ln^{1/2} N|||\xi|||_\epsilon.$$

Consequently,

$$|I_1| \le C\left(N^{-1} + N^{-2}\ln^{5/2} N\right)|||\xi|||_\epsilon \le C\left(N^{-1} + N^{-1}\ln N\right)|||\xi|||_\epsilon, \tag{3.34}$$

where we have used the assumption (3.29) and Lemma 3.4.

Due to $[\eta]_e = 0$ for all $e \in \mathcal{E}$, the second term in the right hand side of (3.32) can be rewritten as

$$I_2 = \int_{\mathcal{E}^o}\mathbf{b}\cdot[\xi_1]\{\eta_1\}\mathrm{d}s + \int_\Gamma \mathbf{b}\cdot\mathbf{n}_1\eta_1(\xi_1 - \xi_2)\mathrm{d}s.$$

By Cauchy-Schwarz inequality and Lemma 3.4, we have

$$
\begin{aligned}
|I_2| \quad &\leq \quad \left(\sum_{e\in\mathcal{E}^o}\int_e |\mathbf{b}\cdot\mathbf{n}_e|\,|\{\eta\}|^2 \mathrm{d}s\right)^{1/2}\left(\sum_{e\in\mathcal{E}^o}\int_e |\mathbf{b}\cdot\mathbf{n}_e||[\xi_1]|^2 \mathrm{d}s\right)^{1/2} \\
&\quad + \left(\int_\Gamma |\mathbf{b}\cdot\mathbf{n}_1||\eta_1|^2 \mathrm{d}s\right)^{1/2}\left(\int_\Gamma |\mathbf{b}\cdot\mathbf{n}_1||\xi_1-\xi_2|^2 \mathrm{d}s\right)^{1/2} \\
&\leq \quad C||\eta||_{L^2(\mathcal{E})}|\xi|_c \leq C|\mathcal{E}|^{1/2}||\eta||_{L^\infty(\Omega_1)}|\xi|_c \leq CN^{-3/2}|||\xi|||_\epsilon, \qquad (3.35)
\end{aligned}
$$

where $|\cdot|_{L^2(\mathcal{E})}$ means $\left(\sum_{e\in\mathcal{E}}|\cdot|^2_{L^2(e)}\right)^{1/2}$ and $|\mathcal{E}|$ means the sum of the lengths of all edges in $\mathcal{E}$, which can be estimated with

$$|\mathcal{E}| = [(1-\tau_x)+(1-\tau_y)](N/2+1) \leq 2N.$$

The third term in the right hand side of (3.32) can be easily estimated with

$$|I_3| \leq C||\eta||_{0,\Omega}\,||\xi||_{0,\Omega} \leq C\,N^{-2}\ln^2 N|||\xi|||_\epsilon.$$

This, combined with (3.34) and (3.35), yields

$$|\mathcal{C}_N(\eta,\xi)| \leq C\,N^{-1}\ln N|||\xi|||_\epsilon. \qquad (3.36)$$

Collecting (3.31), (3.32), (3.33) and (3.36), we have (3.30). □

The combination of Proposition 3.1 and Proposition 3.2 leads to our main result, i.e.,

Theorem 3.1. *Let u and u_N be the solutions of (2.1) and (3.9), respectively. Under the conditions of Lemma 3.4, assume $\ln^{3/2} N \leq CN$ and $\alpha = \mathcal{O}(1/H_x, 1/H_y)$, then*

$$|||u-u_N|||_\epsilon \leq C\,N^{-1}\ln N. \qquad (3.37)$$

Corollary 3.1. *Let (q_N,u_N) be the solution obtained by the coupled method (2.8)–(2.10) with numerical fluxes (2.11)–(2.13). Under the assumptions of Theorem 3.1, we have*

$$|(q-q_N,u-u_N)|_{\mathcal{A}_N} \leq C\,N^{-1}\ln N, \qquad (3.38)$$

where $|(\cdot,\cdot)|_{\mathcal{A}_N}$ is a problem-related norm , which is called $\mathcal{A}-$norm in this paper, defined by

$$|(q,u)|^2_{\mathcal{A}_N} = ||u||^2_{0,\Omega} + \epsilon||q||^2_{0,\Omega_1} + \epsilon|u_2|^2_{1,\Omega_2} + \epsilon|u|^2_* + |u|^2_c. \qquad (3.39)$$

Proof. From (3.4), we have $q-q_N = \nabla(u_1-u_{1,N}) + \mathcal{L}_1(u_N)$. Since $\mathcal{L}_1(u)=0$, we obtain $q-q_N = \nabla(u_1-u_{1,N}) - \mathcal{L}_1(u-u_N)$. In terms of (3.20), we conclude $|(q-q_N,u-u_N)|_{\mathcal{A}_N} \leq C|||u-u_N|||_\epsilon$, which implies the conclusion. □

4. Numerical experiments.

In this section, we numerically verify the sharpness of our theoretical findings and explore some situations not covered by them. Take $\alpha = 1/H_x$, $\beta = \beta_0 = \frac{1}{2}\mathrm{sign}(\mathbf{b}\cdot\mathbf{n}_K)\mathbf{n}_K$, in (2.11), (2.12) and (2.13), where $\mathbf{n}_K$ is the unit outward normal of the element K, and $\kappa = 2$ in the transition parameter for Shishkin mesh. By our numerical experiment, there is no difference between the results computed by taking $\alpha = 1/H_x$ and $\alpha = 1/H_y$.

Example. Take $\mathbf{b} = (2,3)$ and $c = 1$ in the model problem (2.1). f is chosen such that

$$u(x,y) = 2\sin x\,(1-e^{-2(1-x)/\epsilon})y^2(1-e^{-3(1-y)/\epsilon})$$

is the exact solution.

We display the history of convergence for our coupled method in Table 1. A Shishkin mesh with $N \times N$ elements is called mesh N. Let $err(N)$ denote the error of the approximation computed on the mesh N. Then the convergence order, i.e., $order(2N)$, is defined by

$$order(2N) := \begin{cases} \frac{\ln(err(N)/err(2N))}{\ln(2\ln(N)/\ln(2N))}, & \text{for the } \mathcal{A}\text{-norm}, \\ \frac{\ln(err(N)/err(2N))}{\ln(2)}, & \text{for the } L^2 \text{ norm}. \end{cases}$$

TABLE 1. History of convergence for the coupled method, under the $\mathcal{A}$-norm, Shishkin mesh with $\kappa = 2$.

	$\epsilon = 1.0e-04$		$\epsilon = 1.0e-05$		$\epsilon = 1.0e-06$	
N	error	order	error	order	error	order
8	4.363098e-01	–	4.363004e-01	–	4.362995e-01	–
16	2.937673e-01	0.98	2.937531e-01	0.98	2.937517e-01	0.98
32	1.847859e-01	0.99	1.847726e-01	0.99	1.847713e-01	0.99
64	1.111902e-01	0.99	1.111795e-01	0.99	1.111785e-01	0.99
128	6.493348e-02	1.00	6.492494e-02	1.00	6.492421e-02	1.00

TABLE 2. History of convergence for the coupled method, under the L^2 norm , Shishkin mesh with $\kappa = 2$.

	$\epsilon = 1.0e-04$		$\epsilon = 1.0e-05$		$\epsilon = 1.0e-06$	
N	error	order	error	order	error	order
8	7.828466e-03	–	7.823300e-03	–	7.822785e-03	–
16	2.004742e-03	1.97	1.998638e-03	1.97	1.998030e-03	1.97
32	5.090079e-04	1.98	5.047762e-04	1.99	5.043584e-04	1.99
64	1.293231e-04	1.98	1.269031e-04	1.99	1.266759e-04	1.99
128	3.343957e-05	1.95	3.185281e-05	1.99	3.174415e-05	2.00

From Table 1 and Figure 2, it is observed that the numerical results for $\mathcal{A}$-norm (3.39) of the error agree with those predicted in Corollary 3.1. Further, from Table 2 and Figure 2, it is numerically predicted that

$$||u - u_N||_{0,\Omega} \leq CN^{-2},$$

which is uniformly optimal. The reason behind it is worthwhile to investigate.

5. Conclusions.

In this paper, we introduce a coupled LDG-CFEM method for solving two-dimensional singularly perturbed convection-diffusion problems, whose stability and uniform convergence property on the Shishkin mesh is investiaged. For bilinear element, a rate $\mathcal{O}(N^{-1} \ln N)$ in an associated norm is established. Our numerical experiments indicate the sharpness of this error estimate. Moreover, a uniform convergence rate $\mathcal{O}(N^{-2})$ in L^2 norm are numerically observed on the Shishkin mesh.

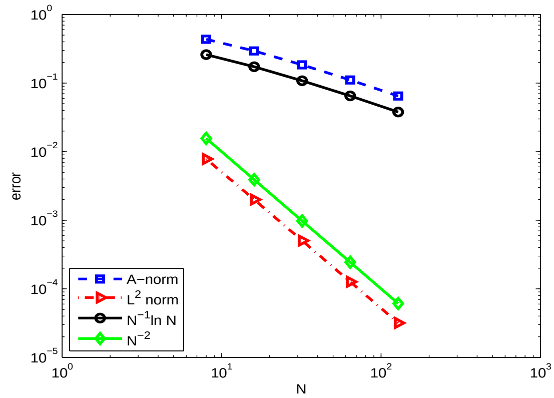

FIGURE 2. Convergence curve of $u - u_N$, the Shishkin mesh, $\epsilon = 10^{-6}$.

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