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SPATIAL ERROR ESTIMATES FOR A FINITE ELEMENT VISCOSITY-SPLITTING SCHEME FOR THE NAVIER-STOKES EQUATIONS

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Abstract. In this paper, we obtain optimal first order error estimates for a fully discrete fractional-step scheme applied to the Navier-Stokes equations. This scheme uses decomposition of the viscosity in time and finite elements (FE) in space.

In [15], optimal first order error estimates (for velocity and pressure) for the corresponding time-discrete scheme were obtained, using in particular $\mathbf{H}^2 \times H^1$ estimates for the approximations of the velocity and pressure. Now, we use this time-discrete scheme as an auxiliary problem to study a fully discrete finite element scheme, obtaining optimal first order approximation for velocity and pressure with respect to the max-norm in time and the $\mathbf{H}^1 \times L^2$-norm in space.

The proof of these error estimates are based on three main points: a) provide some new estimates for the time-discrete scheme (not proved in [15]) which must be now used, b) give a discrete version of the $\mathbf{H}^2 \times H^1$ estimates in FE spaces, using stability in the $\mathbf{W}^{1,6} \times L^6$-norm of the FE Stokes projector, and c) the use of a weight function vanishing at initial time will let to hold the error estimates without imposing global compatibility for the exact solution.

Key words. Navier-Stokes Equations, splitting in time schemes, fully discrete schemes, error estimates, mixed formulation, stable finite elements.

1. Introduction

We consider the Navier-Stokes system, modelling viscous and incompressible fluids filling a bounded domain $\Omega \subset I\!R^3$ in a time interval $(0,T)$:

$$(P) \qquad \left\{ \begin{array}{rcll} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\,\Delta\mathbf{u} + \nabla p &=& \mathbf{f} & \text{in } \Omega\times(0,T), \\ \nabla\cdot\mathbf{u} &=& 0 & \text{in } \Omega\times(0,T), \\ \mathbf{u} &=& 0 & \text{on } \partial\Omega\times(0,T), \\ \mathbf{u}_{|t=0} &=& \mathbf{u}_0 & \text{in } \Omega. \end{array} \right.$$

where $\mathbf{u} : (\mathbf{x},t) \in \Omega\times(0,T) \to I\!R^3$ the velocity field and $p : (\mathbf{x},t) \in \Omega\times(0,T) \to I\!R^3$ the pressure are the unknowns, and data are $\nu > 0$ the viscosity coefficient (which is assumed constant for simplicity) and $\mathbf{f} : (\mathbf{x},t) \in \Omega\times(0,T) \to I\!R^3$ the external forces. We denote by ∇ the gradient operator and Δ the Laplace operator.

Considering a (regular) partition of $[0,T]$ of diameter $k = T/M$: $(t_m = mk)_{m=0}^M$, for a given vector $u = (u^m)_{m=0}^M$ with $u^m \in X$ (a Banach space), let us to introduce the following notation for discrete in time norms:

$$\|u\|_{l^2(X)} = \left(k \sum_{m=0}^M \|u^m\|_X^2 \right)^{1/2} \quad \text{and} \quad \|u\|_{l^\infty(X)} = \max_{m=0,\dots,M} \|u^m\|_X$$

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For simplicity, we will denote $H^1 = H^1(\Omega)$ etc., $L^2(H^1) = L^2(0, T; H^1)$ etc., and $\mathbf{H^1} = H^1(\Omega)^3$ etc.

The numerical analysis for the Navier-Stokes problem (P) has received much attention in the last decades and many numerical schemes are now available. The main (numerical) difficulties in this problem are the coupling between the pressure and the incompressibility condition and the nonlinearity of the convective terms.

Fractional step methods in time are becoming widely used in this context, allowing us to separate the effects of different operators appearing in the problem. For instance, the projection schemes decompose the convection-diffusion operators to the incompresibility ([20], [21], [19], [13]). These projection schemes are two-step schemes where the second step is a free divergence projection step. The main drawbacks of projection methods are that the end-of-step velocity does not satisfy the exact boundary conditions and the discrete pressure satisfies "artificial" boundary conditions.

Another class of fractional step methods, so-called θ-schemes (where viscosity is not fully decoupled from incompressibility), were introduced by Glowinski and his co-authors in the 1980's (see for instance a review in [12]). Afterwards, some analytical results were given, see for instance [8] where stability and convergence of two fully discrete θ-schemes were proved.

In this paper, we study a fractional step method (so-called viscosity-splitting) which can be seen as an special case of the θ-scheme. This scheme was inspired in the previous projection schemes and θ-schemes, jointly to the predictor-corrector argument applied to incompressible fluids ([6]). This viscosity-splitting method was studied in [1], [2], [3] and [4]. It is a two-step scheme splitting the nonlinearity and the incompressibility of the problem into two different steps (but keeping viscosity term and boundary conditions in both steps). Essentially, in this viscosity-splitting scheme, given $\mathbf{u}_h^m$ an approximation of $\mathbf{u}(t_m)$, first one computes an intermediate velocity $\mathbf{u}_h^{m+1/2}$ (as a first approximation of $\mathbf{u}(t_{m+1})$) by means of a convection-diffusion problem, and afterwards $(\mathbf{u}_h^{m+1}, p_h^{m+1})$ (as approximation of $(\mathbf{u}(t_{m+1}), p(t_{m+1})))$ is obtained solving a generalized Stokes problem. On the other hand, the θ-scheme is a three-step method; the first and third step (or generalized Stokes problem) accounts for viscous effect together with incompressibility, but it also includes an explicit convective term; the second step (or regularized Burger's problem) also includes an implicit viscous term and a non-linear implicit approximation of convection together with an explicit pressure gradient but not the incompressibility condition.

In [1], [2], Blasco, Codina and Huerta prove the convergence of the time-discrete viscosity splitting scheme. Afterwards, also for the time-discrete case, error estimates of order $O(k)$ in $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$ for the end-of-step velocity $\mathbf{u}^{m+1}$ and order $O(k^{1/2})$ in $l^2(L^2)$ for the pressure p^{m+1} are obtained in [3]. Moreover, in [4] these error estimates are used to obtain the following error estimates for a fully discrete scheme based on $O(h)$ finite element approximations in $\mathbf{H}^1 \times L^2$ for the velocity and pressure:

$$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)} \le C\,(k + h),$$

under the constraint $h^2 \le C\,k$.

On the other hand, in [2] numerical computations with this viscosity-splitting scheme drive to order $O(k)$ in $L^2(\Omega)$ for velocity and pressure. In [11], this time scheme is studied jointly to Galerkin discontinuous FE methods in space with $P_1 \times P_0$ approximation. From the analytical point of view, order $O(k + h)$ in $l^\infty(\mathbf{L}^2)$ for the velocity and order $O(\sqrt{k} + h)$ in $l^2(L^2)$ for the pressure were obtained.

Moreover, numerical computations give order $O(h)$ in a discrete $\mathbf{H}^1$-norm for the end-of-step velocity and order $O(h)$ in L^2 for the pressure.

Previous error estimates for the time-discrete viscosity-splitting scheme are improved in [15], obtaining the following sharp error estimates:

$$\|\sqrt{\sigma(t_{m-1})}\,(p(t_m)-p^m)\|_{l^2(L^2)} \le C\,k,$$
$$\|\sigma(t_m)\,(\mathbf{u}(t_m)-\mathbf{u}^m, p(t_m)-p^m)\|_{l^\infty(\mathbf{H}^1\times L^2)} \le C\,k$$

where $\sigma(t)=\min\{t,1\}$, that is a weight function vanishing at $t=0$.

Now, in this paper, we use this time-discrete scheme as an auxiliary problem, in order to obtain error estimates for a fully discrete FE scheme.

Basically, the task of this work is to extend the approximation of order $O(k)$ in velocity and pressure of the time-discrete scheme obtained in [15], to order $O(k+h)$ for a fully discrete scheme. It seems natural to think that this extension could be possible at least for small enough h in function of k.

More concretely, assuming the constraint:

$$(\mathbf{H}) \qquad h \le C\,k$$

we will obtain the following optimal error estimates:

$$\|\sqrt{\sigma(t_{m-1})}\,(p(t_m)-p_h^m)\|_{l^2(L^2)} \le C\,(k+h),$$
$$\|\sigma(t_m)\,(\mathbf{u}(t_m)-\mathbf{u}_h^m, p(t_m)-p_h^m)\|_{l^\infty(\mathbf{H}^1\times L^2)} \le C\,(k+h).$$

Due to these improvements, projection scheme with incremental pressure and viscosity-splitting scheme are comparable, where the viscosity-splitting scheme presents some analytical advantages (although since viscosity-splitting scheme solves a mixed method which request higher computational cost that projection schemes):

(1) In pressure incremental schemes, an initial pressure p^0 must be introduced as approximation of $p(0)$, which is not possible to compute from the problem, being necessary to begin with an auxiliary initial step by means of another scheme.
(2) The viscosity-splitting scheme has not numerical boundary layer for the pressure due to this scheme includes a residual diffusion term in the incompressibility step, hence the imposition of the fully boundary conditions for the velocity is possible in the two sub-steps, while needing no boundary condition at all for the pressure.

On the other hand, the viscosity-splitting scheme has the same analytical results than Euler's type schemes [22] (in the sense to provide optimal error estimates without imposing non-local compatibility conditions), improving their numerical treatment because the main difficulties are split. Also, the second step of the viscosity-splitting scheme is a modified symmetric problem that can be formulated as a minimization problem, which can be approximated by using many solvers related to the numerical optimization, as the Uzawa's method, the Augmented Lagrangian method, etc ([12]).

Some results of this paper have already been announced (without proofs) in [14], but imposing regularity hypothesis on the exact solution of (P) related to non-local compatibility conditions on the data, and in order to deduce optimal error estimates for the pressure in $l^2(\mathbf{L}^2)$, the time step k was assumed to be small enough. Now, we prove that this constraint is not necessary and that the regularity hypotheses on the exact solution leading to non-local compatibility conditions can be avoided by using weighted norms.

The paper is organized as follows:

In Section 1, we state the time-discrete scheme and the following error estimates obtained in [15] verified by the time-discrete errors

$$\mathbf{e}^{m+1} := \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1}, \quad \mathbf{e}^{m+1/2} := \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1/2} \quad \text{and} \quad e_p^m := p(t_m) - p^m$$

(hereafter $\delta_t \mathbf{e}^{m+1} := (\mathbf{e}^{m+1} - \mathbf{e}^m)/k$):

- $O(k^{1/2})$ error estimates for $\mathbf{e}^{m+1} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1}$ and $\mathbf{e}^{m+1/2} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1/2}$ in $l^\infty(\mathbf{H}^1) \cap l^2(\mathbf{H}^2)$ and for $e_p^m = p(t_m) - p^m$ in $l^2(H^1)$.
- Estimates for $\mathbf{e}^{m+1}$ and $\mathbf{e}^{m+1/2}$ in $l^\infty(\mathbf{H}^2)$ and for e_p^m in $l^\infty(H^1)$ (in fact, this argument does not work if a direct argument between the exact solution of (P) and the fully discrete scheme is applied, because the problem is how to obtain the discrete version of these estimates, that is, estimates in $l^\infty(\mathbf{W}^{1,6})$ for both velocities).
- $O(k)$ for $\mathbf{e}^{m+1}$ in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$.
- $O(k)$ for $\sqrt{\sigma(t_{m-1})}\delta_t \mathbf{e}^{m+1}$ and for $\sqrt{\sigma(t_{m-1})}e_p^{m+1}$ in $l^2(\mathbf{L}^2)$.
- $O(k)$ for $\sigma(t_m)\delta_t \mathbf{e}^{m+1}$ in $l^\infty(\mathbf{L}^2)$, for $\sigma(t_{m-1})\delta_t \mathbf{e}^{m+1}$ in $l^2(\mathbf{H}^1)$ and for $\sigma(t_m)e_p^{m+1}$ in $l^\infty(L^2)$.

Moreover of previous results given in [15], in this paper we have to introduce some complementary estimates for the time-discrete scheme which did not appear in [15] (see (11)-(13) and (14) below), which will be used here to obtain the estimates for the fully discrete scheme.

In Section 2, we study the fully discrete scheme. First of all, we present the FE spaces and their approximation properties, describing the scheme and the problems verified by the discrete errors (comparing time-discrete scheme and fully discrete scheme):

$$\mathbf{e}_d^{m+1/2} := \mathbf{u}^{m+1/2} - \mathbf{u}_h^{m+1/2}, \quad \mathbf{e}_d^{m+1} := \mathbf{u}^{m+1} - \mathbf{u}_h^{m+1}, \quad e_{p,d}^{m+1} := p^{m+1} - p_h^{m+1},$$

where $h > 0$ is the mesh size parameter. Then, under the constraint

$$(\mathbf{H}) \qquad h \le C\, k$$

(that is, h small enough with respect to k) we will obtain the following error estimates:

- $O(h)$ for $\mathbf{e}_d^{m+1}$ and $\mathbf{e}_d^{m+1/2}$ in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$.
- Estimates in $l^\infty(\mathbf{W}^{1,6}(\Omega))$ for both discrete velocities.
- $O(h)$ for $\sqrt{\sigma(t_{m-1})}\,\delta_t \mathbf{e}_d^{m+1}$ in $l^2(\mathbf{L}^2)$ and for $\sqrt{\sigma(t_{m-1})}\, e_{p,d}^{m+1}$ in $l^2(L^2)$.
- $O(h)$ for $\sigma(t_m)\delta_t \mathbf{e}_d^{m+1}$ in $l^\infty(\mathbf{L}^2)$ which implies $O(h)$ for $\sigma(t_m)e_{p,d}^{m+1}$ in $l^\infty(L^2)$.

It should be noted that, for the decoupled scheme studied in this paper, it is not clear how to obtain error estimates via a "direct argument" (comparing directly the exact solution of (P) and the fully discrete scheme), where constraints of k small enough in function of h could be appear. Therefore, we do not known how to avoid constraint **(H)** to obtain optimal error estimates, contrary to the Euler's semi-implicit scheme [10] (see [16] for a detailed proof avoiding any constrain about the discrete parameters).

In this paper, the following discrete Gronwall's lemma will be used (see [18, page 369]):

Lemma 1. *(Discrete Gronwall inequality) Let k, B and a_m , b_m , c_m , γ_m be non-negative numbers such that*

$$a_{r+1} + k\sum_{m=0}^{r} b_m \leq k\sum_{m=0}^{r}\gamma_m a_m + k\sum_{m=0}^{r} c_m + B \qquad \forall r \geq 0.$$

Then, one has

$$a_{r+1} + k\sum_{m=0}^{r} b_m \leq \exp\left(k\sum_{m=0}^{r}\gamma_m\right)\left\{k\sum_{m=0}^{r} c_m + B\right\} \qquad \forall r \geq 0.$$

2. Time-discrete scheme

2.1. Description of the scheme. Given a (uniform) partition of the time interval $[0,T]$ with diameter $k = T/M$, $\{t_m = m\,k\}_{m=0}^{M}$, and $(\mathbf{f}^m)_{m=1}^{M}$ an approximation of $\mathbf{f}(t_m)$ we have to define $(\mathbf{u}^m, p^m)_{m=1}^{M}$ an approximation of the solution $(\mathbf{u}, p)$ of (P) at the time $t = t_m$.
Initialization : $\mathbf{u}^0 = \mathbf{u}_0$
Time step $m+1$**:**

Substep 1 : Given $\mathbf{u}^m$, to find $\mathbf{u}^{m+1/2}$ solution of

$$(S_1)^{m+1} \quad \begin{cases} \dfrac{1}{k}(\mathbf{u}^{m+1/2} - \mathbf{u}^m) + (\mathbf{u}^m \cdot \nabla)\mathbf{u}^{m+1/2} - \nu\Delta\mathbf{u}^{m+1/2} = \mathbf{f}^{m+1} \quad \text{in } \Omega, \\ \mathbf{u}^{m+1/2}|_{\partial\Omega} = 0. \end{cases}$$

Substep 2 : Give $\mathbf{u}^{m+1/2}$, to find $\mathbf{u}^{m+1}$ and p^{m+1} solution of

$$(S_2)^{m+1} \quad \begin{cases} \dfrac{1}{k}(\mathbf{u}^{m+1} - \mathbf{u}^{m+1/2}) - \nu\Delta(\mathbf{u}^{m+1} - \mathbf{u}^{m+1/2}) + \nabla p^{m+1} = 0 \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u}^{m+1} = 0 \quad \text{in } \Omega, \qquad \mathbf{u}^{m+1}|_{\partial\Omega} = 0. \end{cases}$$

2.2. Differential problems verified by the errors. For simplicity and without loss of generality, we fix the viscosity constant $\nu = 1$.

The errors verify the following problems:

$$(E_1)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{e}^{m+1/2} - \mathbf{e}^m) - \Delta\mathbf{e}^{m+1/2} = -\nabla\, p(t_{m+1}) + \mathcal{E}^{m+1} \quad \text{in } \Omega, \\ \mathbf{e}^{m+1/2}|_{\partial\Omega} = 0, \end{cases}$$

where $\mathcal{E}^{m+1}$ is the consistency error ([15]). On the other hand,

$$(E_2)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}) - \Delta(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}) - \nabla p^{m+1} = 0 \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{e}^{m+1} = 0 \quad \text{in } \Omega, \qquad \mathbf{e}^{m+1}|_{\partial\Omega} = 0. \end{cases}$$

Finally, adding $(E_1)^{m+1}$ and $(E_2)^{m+1}$, we arrive at

$$(E_3)^{m+1} \qquad \begin{cases} \delta_t\mathbf{e}^{m+1} - \Delta\mathbf{e}^{m+1} + \nabla e_p^{m+1} = \mathcal{E}^{m+1} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{e}^{m+1} = 0 \quad \text{in } \Omega, \qquad \mathbf{e}^{m+1}|_{\partial\Omega} = 0. \end{cases}$$

2.3. Main results for the time discrete scheme [15]**.** Let us to introduce the following Hilbert spaces:

$$\begin{aligned} \mathbf{H} &= \{\mathbf{v} \in \mathbf{L}^2(\Omega) \ : \ \nabla\cdot\mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v}\cdot\mathbf{n}_{\partial\Omega} = 0\}, \\ \mathbf{V} &= \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \ : \ \nabla\cdot\mathbf{v} = 0 \text{ in } \Omega\}, \end{aligned}$$

being $\mathbf{n}_{\partial\Omega}$ the normal outwards vector of $\partial\Omega$.

We denote $\mathbf{H}^{-1}(\Omega)$ and $\mathbf{V}'$ the dual space of $\mathbf{H}_0^1(\Omega)$ and $\mathbf{V}$ respectively. The norm and scalar product in $L^2(\Omega)$ will be denoted by $|\cdot|$ and $(\cdot,\cdot)$, whereas the norm in $H_0^1(\Omega)$ of the gradient in $L^2(\Omega)$ will be denoted by $\|\cdot\|$. Any other norm in a space X will be denoted by $\|\cdot\|_X$.

By C we will denote different constants, always independent of k (and h).

In the sequel, we will assume the following regularity hypothesis on Ω:

(H0) $\quad \Omega \subset I\!R^3$ such that the Stokes problem in Ω has the $\mathbf{H}^2 \times H^1$ regularity.

Now, we present the main estimates for the semi-discrete in time scheme, which were obtained in [15]. These results assume hypotheses for the exact solution which do not require to assume a non-local compatibility condition for $\mathbf{u}_0$ and $\mathbf{f}(0)$ (uncheckable in practice), related to the existence of $p_0 \in H^1$ (the initial pressure) solution of an over-determined elliptic problem ([17]).

Theorem 2. *Assuming the following regularity for the exact solution* $(\mathbf{u}, p)$ *of problem* (P), $\mathbf{u} \in L^\infty(\mathbf{H}^2 \cap \mathbf{V}), p \in L^\infty(H^1), \mathbf{u}_t \in L^\infty(\mathbf{L}^2) \cap L^2(\mathbf{H}^1), \sqrt{\sigma(t)}\,\mathbf{u}_{tt} \in L^2(\mathbf{L}^2)$, *then*

$$(1)\qquad \|\delta_t \mathbf{e}^{m+1}\|_{l^2(\mathbf{L}^2)} + \|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{H}^1)\cap l^2(\mathbf{H}^2)} + \|e_p^{m+1}\|_{l^2(H^1)} \le C\,k^{1/2},$$

$$(2)\qquad \|\mathbf{e}^{m+1} - \mathbf{e}^m\|_{l^2(\mathbf{H}^1)} \le C\,k,$$

$$(3)\qquad \|\mathbf{e}^{m+1}, e_p^{m+1}\|_{l^\infty(\mathbf{H}^2\times H^1)} \le C,$$

$$(4)\qquad \|\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{L}^2)} \le C\,k, \qquad \|\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{H}^1)} \le C\,k^{1/2},$$

$$(5)\qquad \|\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{H}^2)} \le C.$$

In particular, previous bounds of the errors (3) and (5) can be extended to the scheme:

$$(6)\qquad \|\mathbf{u}^{m+1}, p^{m+1}\|_{l^\infty(\mathbf{H}^2\times H^1)} + \|\mathbf{u}^{m+1/2}\|_{l^\infty(\mathbf{H}^2)} \le C.$$

Theorem 3. *Under hypotheses of Theorem 2 and assuming* $\mathbf{u}_{tt} \in L^2(\mathbf{V}')$, *then*

$$(7)\qquad \|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\,k.$$

As a consequence of $(4)_1$ and (7), we can also get

$$(8)\qquad \|\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{L}^2)} \le C\,k.$$

In the sequel, we have to introduce the weight function $\sigma(t)$ in order to avoid global compatibility conditions on the data.

Lemma 4. *Under hypotheses of Theorem 3, if we also assume* $\sqrt{\sigma(t)}\,p_t \in L^2(H^1)$, *then*

$$(9)\qquad \|\sqrt{\sigma^{m-1}}\,(\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^{m+1/2})\|^2_{l^\infty(\mathbf{H}^1)} \le C.$$

Proof. The idea is balancing $\Big(\delta_t (E_3)^{m+1}, \delta_t \delta_t \mathbf{e}^{m+1}\Big)$ and the regularity $\mathbf{H}^2 \times H^1$ of the Stokes problem $\delta_t (E_3)^{m+1}$ satisfied by $(\delta_t \mathbf{e}^{m+1}, \delta_t e_p^{m+1})$, obtaining:

$$(10)\qquad \|\sqrt{\sigma^m}\,\delta_t \mathbf{e}^{m+1}\|^2_{l^\infty(\mathbf{H}^1)} + k \sum_{m\ge 1} \|\sqrt{\sigma^{m-1}}\,(\delta_t \mathbf{e}^{m+1}, \delta_t e_p^{m+1})\|^2_{\mathbf{H}^2\times H^1} \le C.$$

Then, (9) can be deduced considering $\Big(\delta_t (E_2)^{m+1}, k\,(\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^{m+1/2})\Big)$, see (3.21)-(3.22) in [15]. $\square$

Now, we are going to deduce some estimates which did not appear in [15].

First, since $\sigma^m = k + \sigma^{m-1}$, we have

$$\sigma^m\,\|\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^{m+1/2}\|^2 = k\,\|\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^{m+1/2}\|^2 + \sigma^{m-1}\,\|\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^{m+1/2}\|^2$$

$$\leq \frac{C}{k}\Big(\|\mathbf{e}^{m+1}-\mathbf{e}^{m+1/2}\|^2+\|\mathbf{e}^{m-1/2}-\mathbf{e}^{m}\|^2\Big)+\sigma^{m-1}\,\|\delta_t\mathbf{e}^{m+1}-\delta_t\mathbf{e}^{m+1/2}\|^2\leq C.$$

Here, we have used $(4)_2$ and (9). Then,

$$\|\sqrt{\sigma^m}\,(\delta_t\mathbf{e}^{m+1}-\delta_t\mathbf{e}^{m+1/2})\|^2_{l^\infty(\mathbf{H}^1)}\leq C. \tag{11}$$

In particular, from (10) and (11), we obtain

$$\|\sqrt{\sigma^m}\,\delta_t\mathbf{e}^{m+1/2}\|^2_{l^\infty(\mathbf{H}^1)}\leq C. \tag{12}$$

On the other hand, from (10) and making $\Big(\delta_t(E_2)^{m+1}, k\,\Delta(\delta_t\mathbf{e}^{m+1}-\delta_t\mathbf{e}^{m+1/2})\Big)$, we obtain

$$k\sum_{m\geq 1}\|\sqrt{\sigma^{m-1}}\,\delta_t\mathbf{e}^{m+1/2}\|^2_{\mathbf{H}^2}\leq C. \tag{13}$$

The following two results are given in [15].

Theorem 5. *Under hypotheses of Lemma 4, if we also assume* $\sqrt{\sigma(t)}\,\mathbf{u}_t\in L^\infty(\mathbf{H}^1)\cap L^2(\mathbf{H}^2)$ *and* $\sqrt{\sigma(t)}\,\mathbf{u}_{ttt}\in L^2((\mathbf{H}^2\cap\mathbf{V})')$*, then*

$$\|\sqrt{\sigma^m}\,\delta_t\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{V}')}+\|\sqrt{\sigma^{m-1}}\,\delta_t\mathbf{e}^{m+1}\|_{l^2(\mathbf{L}^2)}\leq C\,k.$$

As a consequence, assuming $\sqrt{\sigma(t)}\,\mathbf{u}_{tt}\in L^2(\mathbf{H}^{-1})$*, we also arrive at*

$$\|\sqrt{\sigma^{m-1}}\,e_p^{m+1}\|_{l^2(L^2)}\leq C\,k.$$

Theorem 6. *Under hypotheses of Theorem 5, if we also assume* $\sigma(t)\,\mathbf{u}_{ttt}\in L^2(\mathbf{V}')$*, then*

$$\|\sigma^m\,\delta_t\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)}+\|\sigma^{m-1}\,\delta_t\mathbf{e}^{m+1}\|_{l^2(\mathbf{H}^1)}\leq C\,k.$$

As a consequence, assuming $\sigma(t)\,\mathbf{u}_{tt}\in L^\infty(\mathbf{H}^{-1})$*, we also arrive at*

$$\|\sigma^m\,(\mathbf{e}^{m+1},e_p^{m+1})\|_{l^\infty(\mathbf{H}^1\times L^2)}\leq C\,k.$$

Finally, if we also assume $\sigma(t)\,p_t\in L^\infty(L^2)$, we can obtain the following estimate which did not appear in [15]:

$$\|\sigma^m\delta_t p^{m+1}\|_{l^\infty(L^2)}\leq C. \tag{14}$$

Indeed, we can write

$$\sigma^m\,\delta_t e_p^{m+1}=\frac{\sigma^m e_p^{m+1}-\sigma^{m-1}e_p^m}{k}-\frac{(\sigma^m-\sigma^{m-1})\,e_p^m}{k}.$$

Then, by using Theorem 6 and $\sigma^m-\sigma^{m-1}\leq k$, one has

$$\|\sigma^m\,\delta_t e_p^{m+1}\|_{l^\infty(L^2)}\leq\|\frac{\sigma^m e_p^{m+1}}{k}\|_{l^\infty(L^2)}+\|\frac{\sigma^{m-1}e_p^m}{k}\|_{l^\infty(L^2)}+\|e_p^m\|_{l^\infty(L^2)}\leq C,$$

hence (14) holds, since $\|\sigma^m\delta_t p(t_{m+1})\|_{l^\infty(L^2)}\leq\|\sigma\,p_t\|_{l^\infty(L^2)}\leq C$.

Notice that regularity $\sigma(t)\,p_t\in L^\infty(L^2)$ does not imply compatibility conditions, because this regularity becomes from the already imposed regularity for velocity $\sigma(t)\,\mathbf{u}_{tt}\in L^\infty(\mathbf{H}^{-1})$ and $\sigma(t)\,\mathbf{u}_t\in L^\infty(\mathbf{H}^1)$.

3. Fully discrete scheme

3.1. Finite element approximation and fully discrete scheme. We consider a FE approximation of the time-discrete problems $(S_1)^{m+1}$ and $(S_2)^{m+1}$. We restrict ourselves to the case where Ω is a $2D$ polygon or a $3D$ polyhedron satisfying (**H0**). We consider three families of finite element spaces $\mathbf{X}_h$, $\mathbf{Y}_h \subset \mathbf{H}_0^1(\Omega)$ and $Q_h \subset L_0^2(\Omega)$ associated to a family of triangulations of the domain Ω of mesh size h, which it will be assumed regular and quasi-uniform (in the sense of Ciarlet [7]), because it will be necessary to use the inverse inequality $\|\mathbf{v}_h, q_h\|_{W^{1,6}\times L^6} \le C\, h^{-1}\|\mathbf{v}_h, q_h\|_{\mathbf{H}^1\times L^2}$ for each $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times Q_h$. The finite element functions in $\mathbf{X}_h$, $\mathbf{Y}_h$ and Q_h are locally polynomials of degree at least 1, 1 and 0, respectively. Moreover, the approximating spaces $\mathbf{Y}_h$ and Q_h are thus required to satisfy the standard "*inf* − *sup*" stability condition ([10]):

There exists $\beta > 0$ independent of h such that, for all $h > 0$,

$$\inf_{q_h\in Q_h\setminus\{0\}} \left(\sup_{\mathbf{v}_h\in \mathbf{Y}_h\setminus\{0\}} \frac{(q_h, \nabla\cdot \mathbf{v}_h)}{\|\mathbf{v}_h\|\,|q_h|} \right) \ge \beta.$$

In such a way, defining $(I_h, J_h) : \mathbf{H}_0^1(\Omega)\times L_0^2(\Omega) \to \mathbf{Y}_h \times Q_h$ as the Stokes projector, that is, $(I_h\mathbf{v}, J_h q) \in \mathbf{Y}_h \times Q_h$:

$$\left\{ \begin{array}{rcl} (\nabla(I_h\mathbf{v}-\mathbf{v}), \nabla\mathbf{v}_h) - (J_h q - q, \nabla\cdot\mathbf{v}_h) &=& 0 \quad \forall\, \mathbf{v}_h \in \mathbf{Y}_h, \\ (\nabla\cdot(I_h\mathbf{v}-\mathbf{v}), q_h) &=& 0 \quad \forall\, q_h \in Q_h \end{array} \right. \tag{15}$$

the following approximation and stability properties hold ([10]):

$$\|\mathbf{v}-I_h\mathbf{v}, q - J_h q\|_{\mathbf{H}^1\times L^2} + \frac{1}{h}\,|\mathbf{v}-I_h\mathbf{v}| \le C\, h\,\|\mathbf{v}, q\|_{\mathbf{H}^2\times H^1}$$

$$\|I_h\mathbf{u}, J_h p\|_{\mathbf{W}^{1,6}\times L^6} \le C\,\|\mathbf{u}, p\|_{\mathbf{H}^2\times H^1} \tag{16}$$

On the other hand, defining $K_h : \mathbf{H}_0^1(\Omega) \to \mathbf{X}_h$ as the (scalar) Poisson projector:

$$K_h\mathbf{v} \in \mathbf{X}_h, \quad (\nabla(K_h\mathbf{v}-\mathbf{v}), \nabla\mathbf{v}_h) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{X}_h \tag{17}$$

one has

$$\|\mathbf{v}-K_h\mathbf{v}\| + \frac{1}{h}\,|\mathbf{v}-K_h\mathbf{v}| \le C\, h\,\|\mathbf{v}\|_{\mathbf{H}^2} \tag{18}$$

$$\|K_h\mathbf{v}\|_{\mathbf{W}^{1,6}} \le C\,\|\mathbf{v}\|_{\mathbf{H}^2} \tag{19}$$

For instance, $(\mathbf{Y}_h, Q_h)$ can be chosen as $P_2\times P_1$ (Taylor-Hood), $P_1 b\times P_1$ (mini-element) or $P_2\times P_0$ (discontinuous discrete pressure) [10]. With respect to the choice of $\mathbf{X}_h$, we will see that in general this choice is not important to get optimal error estimates for the end-of-step velocity, see Remark 10 below, but to get optimal error estimates for the pressure we have to choice $\mathbf{X}_h = \mathbf{Y}_h$.

Lemma 7. *Stability properties (16) and (19) hold.*

Proof. We only give an idea about how to obtain (16), reasoning in a similar way, we can obtain (19). Indeed,

$$\|I_h\mathbf{u}, J_h p\|_{\mathbf{W}^{1,6}\times L^6} \le \|I_h\mathbf{u}-\overline{I}_h\mathbf{u}, J_h p-\overline{J}_h p\|_{\mathbf{W}^{1,6}\times L^6} + \|\overline{I}_h\mathbf{u}, \overline{J}_h p\|_{\mathbf{W}^{1,6}\times L^6} := I_1 + I_2$$

where $\overline{I}_h$ and $\overline{J}_h$ are adequate average local operators. We bound I_1 as follows

$$\begin{aligned} I_1 &\le \frac{C}{h}\,\|I_h\mathbf{u}-\overline{I}_h\mathbf{u}, J_h p - \overline{J}_h p\|_{\mathbf{H}^1\times L^2} \\ &\le \frac{C}{h}\Big(\|I_h\mathbf{u}-\mathbf{u}, J_h p - p\|_{\mathbf{H}^1\times L^2} + \|\overline{I}_h\mathbf{u}-\mathbf{u}, \overline{J}_h p - p\|_{\mathbf{H}^1\times L^2}\Big) \\ &\le C\,\|\mathbf{u}, p\|_{\mathbf{H}^2\times H^1}. \end{aligned}$$

Here, we have used the inverse inequality $\|\mathbf{v}_h, q_h\|_{W^{1,6}\times L^6} \le C\, h^{-1}\|\mathbf{v}_h, q_h\|_{\mathbf{H}^1\times L^2}$ for each $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times Q_h$ and the $O(h)$ approximation of the projector (I_h, J_h) and the interpolator $(\overline{I}_h, \overline{J}_h)$ in $\mathbf{H}^1 \times L^2$.

On the other hand, by using the stability of the average local operators $(\overline{I}_h, \overline{J}_h)$, one has

$$I_2 \le C\, \|\mathbf{u}, p\|_{\mathbf{W}^{1,6}\times L^6}.$$

Notice that a more precise stability estimate like (16), changing $\|\mathbf{u}, p\|_{\mathbf{H}^2\times H^1}$ by $\|\mathbf{u}, p\|_{\mathbf{W}^{1,6}\times L^6}$ was obtained in [9]. □

As usual, we will use the following skew-symmetric part of the trilinear form of the convective term and some equivalent expressions:

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}\int_\Omega \Big\{(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} - (\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}\Big\}$$

$$= \int_\Omega \Big\{(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} + \frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{w}\Big\} = -\int_\Omega \Big\{(\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v} + \frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{w}\Big\}$$

for any $\mathbf{u}\in\mathbf{H}^1_0$, $\mathbf{v}\in\mathbf{H}^1$, $\mathbf{w}\in\mathbf{H}^1$.

Previous equalities hold even in the discrete case, hence we can use, in the sequel, any of these three possibilities. Obviously, $c(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}$ whether $\nabla\cdot\mathbf{u} = 0$.

The trilinear form $c(\cdot,\cdot,\cdot)$ verifies

$$c(\mathbf{u},\mathbf{v},\mathbf{v}) = 0, \qquad \forall\, \mathbf{u}\in\mathbf{H}^1_0, \quad \forall\, \mathbf{v}\in\mathbf{H}^1, \tag{20}$$

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) \le C \left\{ \begin{array}{l} \|\mathbf{u}\|\, \|\mathbf{v}\|_{W^{1,3}\cap L^\infty}\, |\mathbf{w}| \\ \|\mathbf{u}\|_{L^3}\, \|\mathbf{v}\|\, \|\mathbf{w}\| \end{array} \right.$$

where the role of $\mathbf{u},\mathbf{v},\mathbf{w}$ can be interchanged, using the appropriate expression of $c(\cdot,\cdot,\cdot)$. The fully discrete scheme remains as follows:

Initialization: Let $\mathbf{u}_h^0 \in \mathbf{Y}_h$ be an approximation of $\mathbf{u}_0$

Step of time $m+1$:

Substep 1 : Given $\mathbf{u}_h^m \in \mathbf{Y}_h$, to compute $\mathbf{u}_h^{m+1/2} \in \mathbf{X}_h$ such that, for all $\mathbf{v}_h \in \mathbf{X}_h$

$$(S_1)_h^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{u}_h^{m+1/2} - \mathbf{u}_h^m, \mathbf{v}_h) + c(\mathbf{u}_h^m, \mathbf{u}_h^{m+1/2}, \mathbf{v}_h) + (\nabla\, \mathbf{u}_h^{m+1/2}, \nabla\, \mathbf{v}_h) \\ = (\mathbf{f}^{m+1}, \mathbf{v}_h). \end{array} \right.$$

Substep 2 : Given $\mathbf{u}_h^{m+1/2} \in \mathbf{X}_h$, to compute $(\mathbf{u}_h^{m+1}, p_h^{m+1}) \in \mathbf{Y}_h \times Q_h$, such that for all $(\mathbf{v}_h, q_h) \in \mathbf{Y}_h \times Q_h$

$$(S_2)_h^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{u}_h^{m+1} - \mathbf{u}_h^{m+1/2}, \mathbf{v}_h) + (\nabla(\mathbf{u}_h^{m+1} - \mathbf{u}_h^{m+1/2}), \nabla\mathbf{v}_h) \\ -(p_h^{m+1}, \nabla\cdot\mathbf{v}_h) = 0, \\ (\nabla\cdot\mathbf{u}_h^{m+1}, q_h) = 0. \end{array} \right.$$

In the first substep, a decoupled linear convection-diffusion scheme must be computed, whereas the second substep can be seen as a (generalized) Stokes problem.

Notice that, using (20), one can extend results of stability and convergence for the time discrete scheme enounceed in the previous section to this fully discrete scheme (see [1]). In fact, making $\Big((S_1)_h^{m+1}, \mathbf{u}_h^{m+1/2}\Big) + \Big((S_2)_h^{m+1}, \mathbf{u}_h^{m+1}\Big)$, one can obtain

$$\|\mathbf{u}_h^m\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\mathbf{u}_h^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C. \tag{21}$$

3.2. Problems related to the space discrete errors. We will present an error analysis for the fully discrete scheme $(\mathbf{u}_h^{m+1/2}, \mathbf{u}_h^{m+1}, p_h^{m+1})$ as an approximation of the time-discrete scheme $(\mathbf{u}^{m+1/2}, \mathbf{u}^{m+1}, p^{m+1})$. Consequently, we consider the following spatial errors:

$$\mathbf{e}_d^{m+1} = \mathbf{u}^{m+1} - \mathbf{u}_h^{m+1}, \qquad \mathbf{e}_d^{m+1/2} = \mathbf{u}^{m+1/2} - \mathbf{u}_h^{m+1/2}, \quad e_{p,d}^{m+1} = p^{m+1} - p_h^{m+1}.$$

These errors can be decomposed as follows (splitting the discrete and the interpolation parts):

$$\mathbf{e}_d^{m+1} = \mathbf{e}_h^{m+1} + \mathbf{e}_i^{m+1}, \qquad \mathbf{e}_d^{m+1/2} = \mathbf{e}_h^{m+1/2} + \mathbf{e}_i^{m+1/2}, \qquad e_{p,d}^{m+1} = e_{p,h}^{m+1} + e_{p,i}^{m+1},$$

being $\mathbf{e}_i$ interpolation errors and $\mathbf{e}_h$ space discrete errors, concretely

$$\mathbf{e}_h^{m+1} = I_h\mathbf{u}^{m+1} - \mathbf{u}_h^{m+1} \text{ and } \mathbf{e}_i^{m+1} = \mathbf{u}^{m+1} - I_h\mathbf{u}^{m+1},$$
$$\mathbf{e}_h^{m+1/2} = K_h\mathbf{u}^{m+1/2} - \mathbf{u}_h^{m+1/2} \text{ and } \mathbf{e}_i^{m+1/2} = \mathbf{u}^{m+1/2} - K_h\mathbf{u}^{m+1/2},$$
$$e_{p,h}^{m+1} = J_h p^{m+1} - p_h^{m+1} \text{ and } e_{p,i}^{m+1} = p^{m+1} - J_h p^{m+1}.$$

Comparing $(S_1)^{m+1}, (S_2)^{m+1}$ with $(S_1)_h^{m+1}, (S_2)_h^{m+1}$, and using the specific properties of the projectors $(\nabla\, \mathbf{e}_i^{m+1/2}, \nabla\, \mathbf{v}_h) = 0$ (owing to (17)) and $(\nabla\, \mathbf{e}_i^{m+1}, \nabla\, \mathbf{v}_h) + (e_{p,i}^{m+1}, \nabla\cdot \mathbf{v}_h) = 0$ (owing to (15)), we have the following variational problems verified by the space errors $\mathbf{e}_h^{m+1/2}$ and $(\mathbf{e}_h^{m+1}, e_{p,h}^{m+1})$ respectively:

$$(E_1)_h^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{e}_h^{m+1/2} - \mathbf{e}_h^m, \mathbf{v}_h) + (\nabla\, \mathbf{e}_h^{m+1/2}, \nabla\, \mathbf{v}_h) \\ = -\dfrac{1}{k}(\mathbf{e}_i^{m+1/2} - \mathbf{e}_i^m, \mathbf{v}_h) + \mathbf{NL}_h^{m+1}(\mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in \mathbf{X}_h, \end{cases}$$

where

$$\mathbf{NL}_h^{m+1}(\mathbf{v}_h) = -c(\mathbf{e}_d^m, \mathbf{u}^{m+1/2}, \mathbf{v}_h) - c(\mathbf{u}_h^m, \mathbf{e}_d^{m+1/2}, \mathbf{v}_h)$$

and, for all $(\mathbf{v}_h, q_h) \in \mathbf{Y}_h \times Q_h$,

$$(E_2)_h^{m+1} \quad \begin{cases} \dfrac{1}{k}(\mathbf{e}_h^{m+1} - \mathbf{e}_h^{m+1/2}, \mathbf{v}_h) + (\nabla\,(\mathbf{e}_h^{m+1} - \mathbf{e}_h^{m+1/2}), \nabla\, \mathbf{v}_h) \\ -(e_{p,h}^{m+1}, \nabla\cdot \mathbf{v}_h) = -\dfrac{1}{k}(\mathbf{e}_i^{m+1} - \mathbf{e}_i^{m+1/2}, \mathbf{v}_h) + (\nabla\, \mathbf{e}_i^{m+1/2}, \nabla\, \mathbf{v}_h), \\ (\nabla\cdot \mathbf{e}_h^{m+1}, q_h) = 0. \end{cases}$$

3.3. $O(h)$-error estimates for both velocities in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ under the constraint (H) $h \le C\,k$. The following constraint between the time step k and the mesh size h must be assumed:

$$(\mathbf{H}) \qquad h \le C\,k.$$

Theorem 8. *Assume hypotheses of Theorem 2, constraint* (**H**) *and* $|\mathbf{e}_h^0| \le C\,h$. *Then, the following error estimates hold*

$$\|\mathbf{e}_h^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\mathbf{e}_h^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\,h, \tag{22}$$

$$\|\mathbf{e}_h^{m+1/2} - \mathbf{e}_h^m\|_{l^2(\mathbf{L}^2)} + \|\mathbf{e}_h^{m+1} - \mathbf{e}_h^{m+1/2}\|_{l^2(\mathbf{L}^2)} \le C\,\sqrt{k}\,h. \tag{23}$$

Remark 9. *¿From (7), (8) and Theorem 8, we can bound the total velocity error as follows*

$$\|\mathbf{u}(t_{m+1}) - \mathbf{u}_h^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\mathbf{u}(t_{m+1}) - \mathbf{u}_h^{m+1/2}\|_{l^\infty(\mathbf{L}^2)} \le C(k+h),$$
$$\|\mathbf{u}(t_{m+1}) - \mathbf{u}_h^{m+1/2}\|_{l^2(\mathbf{H}^1)} \le C(\sqrt{k}+h).$$

Proof. (Of Theorem 8). Theorem 8 is announced in [4]. Here, for convenience's reader, we give an outline of the proof. The main idea is to make

$$2\,k\sum_{m=0}^{M-1}\left\{((E_1)_h^{m+1},\mathbf{e}_h^{m+1/2})+((E_2)_h^{m+1},\mathbf{e}_h^{m+1})\right\}.$$

In fact, making $2\,k\,((E_1)_h^{m+1},\mathbf{e}_h^{m+1/2})$, we get

$$\begin{aligned}&|\mathbf{e}_h^{m+1/2}|^2-|\mathbf{e}_h^m|^2+|\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^m|^2+2\,k\,\|\mathbf{e}_h^{m+1/2}\|^2\\&=\ 2(\mathbf{e}_i^{m+1/2}-\mathbf{e}_i^m,\mathbf{e}_h^{m+1/2})\\&+2\,k\,c(\mathbf{e}_h^m,\mathbf{u}^{m+1/2},\mathbf{e}_h^{m+1/2})+2\,k\,c(\mathbf{e}_i^m,\mathbf{u}^{m+1/2},\mathbf{e}_h^{m+1/2})\\&-2\,k\,c(\mathbf{u}_h^m,\mathbf{e}_h^{m+1/2},\mathbf{e}_h^{m+1/2})-2\,k\,c(\mathbf{u}_h^m,\mathbf{e}_i^{m+1/2},\mathbf{e}_h^{m+1/2}).\end{aligned}\tag{24}$$

Here $-2\,k\,c(\mathbf{u}_h^m,\mathbf{e}_h^{m+1/2},\mathbf{e}_h^{m+1/2})=0$ owing to (20). Also, we bound the term

$$\begin{aligned}&2(\mathbf{e}_i^{m+1/2}-\mathbf{e}_i^m,\mathbf{e}_h^{m+1/2})\le\varepsilon\,k\|\mathbf{e}_h^{m+1/2}\|^2+\tfrac{C}{k}\left(|\mathbf{e}_i^{m+1/2}|^2+|\mathbf{e}_i^m|^2\right)\\&\le\varepsilon\,k\|\mathbf{e}_h^{m+1/2}\|^2+C\,\frac{h^4}{k}\left(\|\mathbf{u}^{m+1/2}\|^2_{\mathbf{H}^2}+\|\mathbf{u}^m,p^m\|^2_{\mathbf{H}^2\times H^1}\right)\le\varepsilon\,k\|\mathbf{e}_h^{m+1/2}\|^2+C\,\frac{h^4}{k}\end{aligned}$$

where we have used (6) in the last inequality. We bound the main terms related to $\mathbf{e}_i^{m+1/2}$ of the RHS of (24) (using again (6)) :

$$\begin{aligned}2\,k\,c(\mathbf{u}_h^m,\mathbf{e}_i^{m+1/2},\mathbf{e}_h^{m+1/2})&\le C\,k\,\|\mathbf{u}_h^m\|^2\,\|\mathbf{e}_i^{m+1/2}\|^2_{L^3}+\varepsilon\,k\|\mathbf{e}_h^{m+1/2}\|^2\\&\le C\,k\,\|\mathbf{u}_h^m\|^2\,|\mathbf{e}_i^{m+1/2}|\,\|\mathbf{e}_i^{m+1/2}\|+\varepsilon\,k\|\mathbf{e}_h^{m+1/2}\|^2\\&\le C\,k\,h^3\,\|\mathbf{u}_h^m\|^2\ \|\mathbf{u}^{m+1/2}\|^2_{\mathbf{H}^2}+\varepsilon\,k\|\mathbf{e}_h^{m+1/2}\|^2\\&\le C\,k\,h^3\,\|\mathbf{u}_h^m\|^2+\varepsilon\,k\|\mathbf{e}_h^{m+1/2}\|^2.\end{aligned}$$

On the other hand, making $2\,k\,((E_2)_h^{m+1},\mathbf{e}_h^{m+1})$, we arrive at

$$\begin{aligned}&|\mathbf{e}_h^{m+1}|^2-|\mathbf{e}_h^{m+1/2}|^2+|\mathbf{e}_h^{m+1}-\mathbf{e}_h^{m+1/2}|^2\\&+2\,k\left\{\|\mathbf{e}_h^{m+1}\|^2-\|\mathbf{e}_h^{m+1/2}\|^2+\|\mathbf{e}_h^{m+1}-\mathbf{e}_h^{m+1/2}\|^2\right\}\\&=2(\mathbf{e}_i^{m+1}-\mathbf{e}_i^{m+1/2},\mathbf{e}_h^{m+1})-2\,k\left(\nabla\mathbf{e}_i^{m+1/2},\nabla\mathbf{e}_h^{m+1}\right)\end{aligned}\tag{25}$$

We bound the term

$$\begin{aligned}&2(\mathbf{e}_i^{m+1}-\mathbf{e}_i^{m+1/2},\mathbf{e}_h^{m+1})\le\varepsilon\,k\,\|\mathbf{e}_h^{m+1}\|^2+\tfrac{C}{k}\left(|\mathbf{e}_i^{m+1/2}|^2+|\mathbf{e}_i^{m+1}|^2\right)\\&\le\varepsilon\,k\,\|\mathbf{e}_h^{m+1}\|^2+C\,\frac{h^4}{k}\left(\|\mathbf{u}^{m+1/2}\|^2_{\mathbf{H}^2}+\|\mathbf{u}^{m+1},p^{m+1}\|^2_{\mathbf{H}^2\times H^1}\right)\le\varepsilon\,k\,\|\mathbf{e}_h^{m+1}\|^2+C\,\frac{h^4}{k}\end{aligned}$$

and, on the other side,

$$-2\,k\left(\nabla\mathbf{e}_i^{m+1/2},\nabla\mathbf{e}_h^{m+1}\right)\le\varepsilon\,k\,\|\mathbf{e}_h^{m+1}\|^2+C\,k\,\|\mathbf{e}_i^{m+1/2}\|^2\le\varepsilon\,k\,\|\mathbf{e}_h^{m+1}\|^2+C\,k\,h^2.$$

Then, adding (24) and (25) and taking into account the above bounds and (6), we have

$$\begin{aligned}&|\mathbf{e}_h^{m+1}|^2-|\mathbf{e}_h^m|^2+|\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^m|^2+|\mathbf{e}_h^{m+1}-\mathbf{e}_h^{m+1/2}|^2\\&+2\,k\left(\|\mathbf{e}_h^{m+1}\|^2+\|\mathbf{e}_h^{m+1}-\mathbf{e}_h^{m+1/2}\|^2\right)\le C\,\frac{h^4}{k}+C\,k\,h^3\,\|\mathbf{u}_h^m\|^2+C\,k\,h^2.\end{aligned}$$

Then, adding from $m=0$ to r (with any $r<M$), we can get

$$\begin{aligned}&\|\mathbf{e}_h^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)}+\|\mathbf{e}_h^{m+1}\|^2_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)}\\&\le C\,h^2(1+h^2/k^2)+C\,k\,h^3\sum_m\|\mathbf{u}_h^m\|^2.\end{aligned}\tag{26}$$

Finally, the first term of RHS of (26) is bounded by $C\,h^2$ using (**H**) and the last term is bounded by $C\,h^3$ using (21). □

Now, since

$$\|\mathbf{u}_h^{m+1/2}\|^2+\|\mathbf{u}_h^{m+1}\|^2\le C\Big(\|\mathbf{e}_h^{m+1/2}\|^2+\|\mathbf{e}_h^{m+1}\|^2+\|K_h\mathbf{u}^{m+1/2}\|^2+\|I_h\mathbf{u}^{m+1}\|^2\Big)$$
$$\le C\Big(\|\mathbf{e}_h^{m+1/2}\|^2+\|\mathbf{e}_h^{m+1}\|^2+\|\mathbf{u}^{m+1/2}\|^2+\|\mathbf{u}^{m+1},p^{m+1}\|^2_{\mathbf{H}^1\times L^2}\Big),$$

then, from (22) and constraint (**H**), we get

$$\|\mathbf{u}_h^{m+1/2}\|^2+\|\mathbf{u}_h^{m+1}\|^2\le C\left(\frac{h^2}{k}+1\right)\le C. \tag{27}$$

Notice that, in the above bound (27), it is only necessary the constraint $h^2\le C\,k$.

On the other hand, using the inverse inequality $\|\mathbf{u}_h\|^2_{\mathbf{W}^{1,3}\cap\mathbf{L}^\infty}\le C\dfrac{1}{h}\|\mathbf{u}_h\|^2$ (see [5]), we have

$$k\sum_{m=0}^{M-1}(\|\mathbf{e}_h^{m+1/2}\|^2_{\mathbf{W}^{1,3}\cap\mathbf{L}^\infty}+\|\mathbf{e}_h^{m+1}\|^2_{\mathbf{W}^{1,3}\cap\mathbf{L}^\infty})\le C\,\frac{k}{h}\sum_{m=0}^{M-1}\Big(\|\mathbf{e}_h^{m+1/2}\|^2+\|\mathbf{e}_h^{m+1}\|^2\Big)\le C\,h$$

here, we have used (22).

In particular, using constraint (**H**), we have

$$\|\mathbf{e}_h^{m+1/2}\|_{l^\infty(\mathbf{W}^{1,3}\cap\mathbf{L}^\infty)}+\|\mathbf{e}_h^{m+1}\|_{l^\infty(\mathbf{W}^{1,3}\cap\mathbf{L}^\infty)}\le C. \tag{28}$$

¿From here and using the stability properties of K_h and I_h given in (19) and (16) to deduce

$$\|K_h\mathbf{u}^{m+1/2}\|_{\mathbf{W}^{1,6}}\le C\,\|\mathbf{u}^{m+1/2}\|_{\mathbf{H}^2}\le C$$

and

$$\|I_h\mathbf{u}^{m+1}\|_{\mathbf{W}^{1,6}}\le C\,\|\mathbf{u}^{m+1},p^{m+1}\|_{\mathbf{H}^2\times H^1}\le C,$$

we arrive at

$$\|\mathbf{u}_h^{m+1/2}\|_{l^\infty(\mathbf{W}^{1,3}\cap\mathbf{L}^\infty)}+\|\mathbf{u}_h^{m+1}\|_{l^\infty(\mathbf{W}^{1,3}\cap\mathbf{L}^\infty)}\le C. \tag{29}$$

On the other hand, from (23) and using again (**H**), one has

$$\|\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^m\|^2_{l^2(\mathbf{L}^2)}+\|\mathbf{e}_h^{m+1}-\mathbf{e}_h^{m+1/2}\|^2_{l^2(\mathbf{L}^2)}\le C\,k^2$$

(again here it is only necessary that $h^2\le C\,k$), This estimate can be re-written as:

$$\frac{\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^m}{k}\quad\text{and}\quad\frac{\mathbf{e}_h^{m+1}-\mathbf{e}_h^{m+1/2}}{k}\quad\text{are bounded in } l^2(\mathbf{L}^2).$$

In particular,

$$\|\delta_t\mathbf{e}_h^{m+1}\|_{l^2(\mathbf{L}^2)}\le C. \tag{30}$$

Remark 10. *Theorem 8 is valid without constraints between* $\mathbf{X}_h$ *and* $\mathbf{Y}_h$*. But, in the sequel, it will be necessary to consider the same discrete space* $\mathbf{X}_h\equiv\mathbf{Y}_h$ *(which it will be denoted as* $\mathbf{X}_h$*).*

3.4. $O(h)$ **for** $\sqrt{\sigma^m}\,\mathbf{e}_h^{m+1}$ **in** $l^\infty(\mathbf{H}^1)$**, for** $\sqrt{\sigma^{m-1}}\,\delta_t\mathbf{e}_h^{m+1}$ **in** $l^2(\mathbf{L}^2)$ **and for** $\sqrt{\sigma^{m-1}}\,(\mathbf{e}_h^{m+1}, e_{p,h}^{m+1})$ **in** $l^2(\mathbf{W}^{1,6}\times L^6)$**.** First of all, we introduce an auxiliary result on the continuous dependence of discrete Stokes and Poisson problem.

Lemma 11. *Let* $\mathbf{g}\in\mathbf{L}^2$.

a) *If* $(\mathbf{u}_h,p_h)\in\mathbf{X}_h\times Q_h$ *is the solution of the discrete Stokes problem*

$$\left\{\begin{array}{l}(\nabla\,\mathbf{u}_h,\nabla\,\mathbf{v}_h)-(p_h,\nabla\cdot\,\mathbf{v}_h)=(\mathbf{g},\mathbf{v}_h)\qquad\forall\,\mathbf{v}_h\in\mathbf{X}_h\\(\nabla\cdot\mathbf{u}_h,q_h)=0\qquad\forall\,q_h\in Q_h\end{array}\right.\tag{31}$$

then there exists $K_s>0$ *such that*

$$\|\mathbf{u}_h,p_h\|^2_{\mathbf{W}^{1,6}\times L^6}\le K_s\,|\mathbf{g}|^2.$$

b) *If* $\mathbf{u}_h\in\mathbf{X}_h$ *is the solution of the discrete Poisson problem*

$$(\nabla\,\mathbf{u}_h,\nabla\,\mathbf{v}_h)=(\mathbf{g},\mathbf{v}_h)\quad\forall\,\mathbf{v}_h\in\mathbf{X}_h\tag{32}$$

then there exists $K_p>0$ *such that*

$$\|\mathbf{u}_h\|^2_{\mathbf{W}^{1,6}}\le K_p\,|\mathbf{g}|^2.\tag{33}$$

Proof. Let $(\mathbf{u},p)$ be the solution of Stokes Problem (or respectively $\mathbf{u}$ solution of Poisson Problem) with second member $\mathbf{g}$. From (**H0**), this solution verifies ([10])

$$\|\mathbf{u},p\|_{\mathbf{H}^2\times H^1}\le C\,|\mathbf{g}|\quad\text{(or respectively }\|\mathbf{u}\|_{H^2}\le C\,|\mathbf{g}|).$$

Moreover, since $(\mathbf{u}_h,p_h)$ is solution of (31) (respectively $\mathbf{u}_h$ solution of (32)) can be identified as $(\mathbf{u}_h,p_h)=(I_h\mathbf{u},J_hp)$ (respectively $\mathbf{u}_h=K_h\mathbf{u}$). Then, from the stability property of (I_h,J_h) (respectively K_h),

$$\|\mathbf{u}_h,p_h\|_{\mathbf{W}^{1,6}\times L^6}=\|I_h\mathbf{u},J_hp\|_{\mathbf{W}^{1,6}\times L^6}\le C\,\|\mathbf{u},p\|_{\mathbf{H}^2\times H^1}\le\;|\mathbf{g}|.$$

(respectively, $\|\mathbf{u}_h\|_{\mathbf{W}^{1,6}}=\|K_h\mathbf{u}\|_{\mathbf{W}^{1,6}}\le C\,\|\mathbf{u}\|_{\mathbf{H}^2}\le\;|\mathbf{g}|$). □

Theorem 12. *Under hypothesis of Lemma 4 and Theorem 8, the following error estimate holds*

$$\|\sqrt{\sigma^{m-1}}\,\delta_t\mathbf{e}_h^{m+1}\|_{l^2(\mathbf{L}^2)}+\|\sqrt{\sigma^m}\,\mathbf{e}_h^{m+1}\|_{l^\infty(\mathbf{H}^1)}+\|\sqrt{\sigma^{m-1}}\left(\mathbf{e}_h^{m+1},e_{p,h}^{m+1}\right)\|_{l^2(\mathbf{W}^{1,6}\times L^6)}\le C\,h.$$

Proof. Adding $(E_1)_h^{m+1}$ and $(E_2)_h^{m+1}$, one has for all $(\mathbf{v}_h,q_h)\in\mathbf{X}_h\times Q_h$,

$$(E_3)_h^{m+1}\qquad\left\{\begin{array}{l}(\delta_t\mathbf{e}_h^{m+1},\mathbf{v}_h)+(\nabla\,\mathbf{e}_h^{m+1},\nabla\mathbf{v}_h)-(e_{p,h}^{m+1},\nabla\cdot\mathbf{v}_h)\\=\mathbf{NL}_h^{m+1}(\mathbf{v}_h)-(\delta_t\mathbf{e}_i^{m+1},\mathbf{v}_h)\\(\nabla\cdot\mathbf{e}_h^{m+1},q_h)=0.\end{array}\right.$$

The key is to consider $\left((E_3)_h^{m+1},\delta_t\mathbf{e}_h^{m+1}\right)$ and to use the $(\mathbf{W}^{1,6}\times L^6)$-regularity of the discrete Stokes problem related to $(E_3)_h^{m+1}$. Indeed, due to $(\mathbf{W}^{1,6}\times L^6)$-regularity of the discrete Stokes problem $(E_3)_h^{m+1}$ given in Lemma 11, we have

$$\|\mathbf{e}_h^{m+1},e_{p,h}^{m+1}\|^2_{\mathbf{W}^{1,6}\times L^6}\le 4\,K_s\,(|\delta_t\mathbf{e}_h^{m+1}|^2+|\delta_t\mathbf{e}_i^{m+1}|^2+|\mathbf{NL}_h^{m+1}|^2)\tag{34}$$

On the other hand, by multiplying $(E_3)_h^{m+1}$ by $\mathbf{v}_h=\delta_t\mathbf{e}_h^{m+1}$, we obtain

$$\begin{aligned}\frac12\,|\delta_t\mathbf{e}_h^{m+1}|^2+\frac{1}{2k}\Big(\|\mathbf{e}_h^{m+1}\|^2-\|\mathbf{e}_h^m\|^2+\|\mathbf{e}_h^{m+1}-\mathbf{e}_h^m\|^2\Big)\\\le C\left(|\delta_t\mathbf{e}_i^{m+1}|^2+|\mathbf{NL}_h^{m+1}|^2\right)\end{aligned}\tag{35}$$

Then, combining the above estimates (34) and (35), multiplying (35) by $10\,K_s$ plus (34),

$$\begin{aligned}
&K_s\,|\delta_t \mathbf{e}_h^{m+1}|^2 + \tfrac{5\,K_s}{k}\Big(\|\mathbf{e}_h^{m+1}\|^2 - \|\mathbf{e}_h^m\|^2 + \|\mathbf{e}_h^{m+1} - \mathbf{e}_h^m\|^2\Big) + \|\mathbf{e}_h^{m+1}, e_{p,h}^{m+1}\|^2_{\mathbf{W}^{1,6}\times L^6}\\
&\le C\Big(|\delta_t \mathbf{e}_i^{m+1}|^2 + |\mathbf{NL}_h^{m+1}|^2\Big)\\
&\le C\,h^2\|\delta_t \mathbf{u}^{m+1}, \delta_t p^{m+1}\|^2_{\mathbf{H}^1\times L^2} + C\Big(\|\mathbf{u}^{m+1/2}\|^2_{\mathbf{W}^{1,3}\times\mathbf{L}^\infty}\|\mathbf{e}_d^m\|^2\\
&\quad + \|\mathbf{u}_h^m\|^2_{\mathbf{W}^{1,3}\times\mathbf{L}^\infty}\|\mathbf{e}_d^{m+1/2}\|^2\Big)\\
&\le C\,h^2\|\delta_t \mathbf{u}^{m+1}, \delta_t p^{m+1}\|^2_{\mathbf{H}^1\times L^2} + C\Big(\|\mathbf{e}_h^m\|^2 + \|\mathbf{e}_h^{m+1/2}\|^2 + h^2\Big).
\end{aligned}$$

Here, we have used estimates (6) and (29).

Now, we need to introduce a weight to avoid compatibility conditions. Multiplying by $\sigma^{m-1} = \sigma(t_{m-1})$, applying the equality

$$\sigma^{m-1}\,\delta_t a^{m+1} = \delta_t(\sigma^m\,a^{m+1}) - a^{m+1}\,\delta_t\sigma^m, \tag{36}$$

for $a^{m+1} = \|\mathbf{e}_h^{m+1}\|^2$ and bounding the residual term $\|\mathbf{e}_h^{m+1}\|^2\,\delta_t\sigma^m$ by $\|\mathbf{e}_h^{m+1}\|^2$ (since $\delta_t\sigma^m \le 1$), we obtain

$$\begin{aligned}
&\frac{5\,K_s}{k}\Big(\|\sqrt{\sigma^m}\,\mathbf{e}_h^{m+1}\|^2 - \|\sqrt{\sigma^{m-1}}\,\mathbf{e}_h^m\|^2 + \|\sqrt{\sigma^{m-1}}\,(\mathbf{e}_h^{m+1} - \mathbf{e}_h^m)\|^2\Big)\\
&+K_s\,|\sqrt{\sigma^{m-1}}\,\delta_t \mathbf{e}_h^{m+1}|^2 + \|\sqrt{\sigma^{m-1}}(\mathbf{e}_h^{m+1}, e_{p,h}^{m+1})\|^2_{\mathbf{W}^{1,6}\times L^6}\\
&\le \|\mathbf{e}_h^{m+1}\|^2 + C\,h^2\Big(\|\delta_t \mathbf{u}^{m+1}\|^2 + |\sqrt{\sigma^{m-1}}\,\delta_t p^{m+1}|^2\Big) + C\,(\|\mathbf{e}_h^m\|^2 + \|\mathbf{e}_h^{m+1/2}\|^2 + h^2).
\end{aligned}$$

Notice that we have introduced the weight to control the term $|\sqrt{\sigma^{m-1}}\,\delta_t p^{m+1}|^2$ without hypothesis requiring compatibility for the data. Then, this term will be bounded as

$$|\sqrt{\sigma^{m-1}}\,\delta_t p^{m+1}|^2 \le |\sqrt{\sigma^{m-1}}\,\delta_t e_p^{m+1}|^2 + |\sqrt{\sigma^{m-1}}\,\delta_t p(t_{m+1})|^2.$$

Now, multiplying by k and adding from $m=1$ to r (for any $r<M$), we have

$$\begin{aligned}
&5K_s\|\sqrt{\sigma^r}\,\mathbf{e}_h^{r+1}\|^2 + K_s\,k\textstyle\sum_{m=1}^r |\sqrt{\sigma^{m-1}}\,\delta_t \mathbf{e}_h^{m+1}|^2\\
&+k\textstyle\sum_{m=1}^r \|\sqrt{\sigma^{m-1}}(\mathbf{e}_h^{m+1}, e_{p,h}^{m+1})\|^2_{\mathbf{W}^{1,6}\times L^6}\\
&\le k\sum_{m=1}^r \|\mathbf{e}_h^{m+1}\|^2 + C\,k\sum_{m=1}^r \|\mathbf{e}_h^m\|^2 + C\,k\sum_{m=1}^r \|\mathbf{e}_h^{m+1/2}\|^2 + C\,h^2\\
&+C\,h^2\,k\sum_{m=1}^r\Big(\|\delta_t \mathbf{e}^{m+1}\|^2 + \|\delta_t \mathbf{u}(t_{m+1})\|^2 + |\sqrt{\sigma^{m-1}}\,\delta_t e_p^{m+1}|^2 + |\sqrt{\sigma^{m-1}}\,\delta_t p(t_{m+1})|^2\Big).
\end{aligned}$$

Then, bounding directly the RHS, thanks to the regularity of the exact solution $\mathbf{u}_t \in L^2(\mathbf{H}^1)$, $\sqrt{\sigma}p_t \in L^2(L^2)$, (2), (10) and (22), we obtain the desired estimates. □

In particular, we have the corresponding error estimates for the total error.

Corollary 13. *Under assumptions of Theorems 5 and 12, the following estimate holds*

$$\|\sqrt{\sigma^{m-1}}(p(t_{m+1}) - p_h^{m+1})\|_{l^2(L^2)} \le C\,(k+h).$$

Moreover, under assumptions of Theorem 6, one has

$$\|\sigma^m\,(\mathbf{u}(t_{m+1}) - \mathbf{u}_h^{m+1})\|_{l^\infty(\mathbf{H}^1)} \le C\,(k+h).$$

3.5. *$O(h)$* **for** $\sigma^m \delta_t \mathbf{e}_h^{m+1}$ **in** $l^\infty(\mathbf{L}^2)$ **and for** $\sigma^m\, e_{p,h}^{m+1}$ **in** $l^\infty(L^2)$**.** Making $\delta_t(E_1)_h^{m+1}$ and $\delta_t(E_2)_h^{m+1}$, one obtains for all $\mathbf{v}_h \in \mathbf{X}_h$ ($\forall m \geq 1$):

$$(D_1)_h^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\delta_t \mathbf{e}_h^{m+1/2} - \delta_t \mathbf{e}_h^m, \mathbf{v}_h) + (\nabla \delta_t \mathbf{e}_h^{m+1/2}, \nabla \mathbf{v}_h) \\ = -\dfrac{1}{k}(\delta_t \mathbf{e}_i^{m+1/2} - \delta_t \mathbf{e}_i^m, \mathbf{v}_h) + \delta_t\, \mathbf{NL}_h^{m+1}(\mathbf{v}_h). \end{array} \right.$$

and, for all $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times Q_h$,

$$(D_2)_h^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\delta_t\, \mathbf{e}_h^{m+1} - \delta_t\, \mathbf{e}_h^{m+1/2}, \mathbf{v}_h) + (\nabla\, (\delta_t \mathbf{e}_h^{m+1} - \delta_t \mathbf{e}_h^{m+1/2}), \nabla\, \mathbf{v}_h) \\ -(\delta_t e_{p,h}^{m+1}, \nabla \cdot \mathbf{v}_h) = -\dfrac{1}{k}(\delta_t\, \mathbf{e}_i^{m+1} - \delta_t\, \mathbf{e}_i^{m+1/2}, \mathbf{w}_h), \\ (\nabla \cdot \delta_t \mathbf{e}_h^{m+1}, q_h) = 0. \end{array} \right.$$

Finally, adding $(D_1)_h^{m+1}$ and $(D_2)_h^{m+1}$ we obtain, for all $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times Q_h$:

$$(D_3)_h^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\delta_t \mathbf{e}_h^{m+1} - \delta_t\, \mathbf{e}_h^m, \mathbf{v}_h) + (\nabla \delta_t \mathbf{e}_h^{m+1}, \nabla\, \mathbf{v}_h) - (\delta_t e_{p,h}^{m+1}, \nabla \cdot \mathbf{v}_h) \\ = \delta_t\, \mathbf{NL}_h^{m+1}(\mathbf{v}_h) - \dfrac{1}{k}(\delta_t \mathbf{e}_i^{m+1} - \delta_t\, \mathbf{e}_i^m, \mathbf{v}_h), \\ (\nabla \cdot \delta_t \mathbf{e}_h^{m+1}, q_h) = 0. \end{array} \right.$$

Theorem 14. *Under the hypotheses of Theorem 6, Theorem 12 and the regularity $\sigma(t)\, p_t \in L^\infty(L^2)$, it holds*

$$\|\sigma^m \delta_t \mathbf{e}_h^{m+1}\|_{l^\infty(\mathbf{L}^2)} \leq C\, h.$$

Proof. We prove the generic estimate for $\delta_t \mathbf{e}_h^{m+1}$ and $\delta_t \mathbf{e}_h^{m+1/2}$, for each $m \geq 1$.

Taking $2\, \delta_t \mathbf{e}_h^{m+1/2} \in \mathbf{X}_h$ as test function in $(D_1)_h^{m+1}$, one has

$$\begin{array}{l} \dfrac{1}{k} \left(|\delta_t \mathbf{e}_h^{m+1/2}|^2 - |\delta_t \mathbf{e}_h^m|^2 + |\delta_t \mathbf{e}_h^{m+1/2} - \delta_t \mathbf{e}_h^m|^2 \right) + 2\, \|\delta_t \mathbf{e}_h^{m+1/2}\|^2 \\ = \dfrac{2}{k} \left(\delta_t \mathbf{e}_i^{m+1/2} - \delta_t \mathbf{e}_i^m, \delta_t \mathbf{e}_h^{m+1/2} \right) + 2 \left\langle \delta_t \mathbf{NL}_h^{m+1}, \delta_t \mathbf{e}_h^{m+1/2} \right\rangle. \end{array} \tag{37}$$

On the other hand, taking $2\, \delta_t \mathbf{e}_h^{m+1} \in \mathbf{X}_h$ as test function in $(D_2)_h^{m+1}$, one has:

$$\begin{array}{l} \dfrac{1}{k}\Big(|\delta_t \mathbf{e}_h^{m+1}|^2 - |\delta_t \mathbf{e}_h^{m+1/2}|^2 + |\delta_t \mathbf{e}_h^{m+1} - \delta_t \mathbf{e}_h^{m+1/2}|^2 \Big) \\ + \Big\{ \|\delta_t \mathbf{e}_h^{m+1}\|^2 - \|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + \|\delta_t \mathbf{e}_h^{m+1} - \delta_t \mathbf{e}_h^{m+1/2}\|^2 \Big\} \\ = \dfrac{2}{k}(\delta_t \mathbf{e}_i^{m+1} - \delta_t \mathbf{e}_i^{m+1/2}, \delta_t \mathbf{e}_h^{m+1}). \end{array} \tag{38}$$

Making (37) + (38), we have

$$\begin{array}{l} \dfrac{1}{k} \left(|\delta_t \mathbf{e}_h^{m+1}|^2 - |\delta_t \mathbf{e}_h^m|^2 + |\delta_t \mathbf{e}_h^{m+1/2} - \delta_t \mathbf{e}_h^m|^2 + |\delta_t \mathbf{e}_h^{m+1} - \delta_t \mathbf{e}_h^{m+1/2}|^2 \right) \\ + \left\{ \|\delta_t \mathbf{e}_h^{m+1}\|^2 + \|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + \|\delta_t \mathbf{e}_h^{m+1} - \delta_t \mathbf{e}_h^{m+1/2}\|^2 \right\} \\ = \dfrac{2}{k} \left(\delta_t \mathbf{e}_i^{m+1/2} - \delta_t \mathbf{e}_i^m, \delta_t \mathbf{e}_h^{m+1/2} \right) \\ + \dfrac{2}{k}\, (\delta_t \mathbf{e}_i^{m+1} - \delta_t \mathbf{e}_i^{m+1/2}, \delta_t \mathbf{e}_h^{m+1}) + 2 \left\langle \delta_t \mathbf{NL}_h^{m+1}, \delta_t \mathbf{e}_h^{m+1/2} \right\rangle. \end{array} \tag{39}$$

Now, bounding as in Theorem 8, we have

$$\begin{array}{l} \dfrac{2}{k}(\delta_t \mathbf{e}_i^{m+1/2} - \delta_t \mathbf{e}_i^m, \delta_t \mathbf{e}_h^{m+1/2}) \leq \varepsilon\, \|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + \dfrac{C}{k^2} \Big(|\delta_t \mathbf{e}_i^{m+1/2}|^2 + |\delta_t \mathbf{e}_i^m|^2 \Big) \\ \leq \varepsilon\, \|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\, \dfrac{h^4}{k^2} \left(\|\delta_t \mathbf{u}^{m+1/2}\|_{\mathbf{H}^2}^2 + \|\delta_t \mathbf{u}^m, \delta_t p^m\|_{\mathbf{H}^2 \times H^1}^2 \right) \end{array}$$

and

$$\frac{2}{k}(\delta_t \mathbf{e}_i^{m+1} - \delta_t \mathbf{e}_i^{m+1/2}, \delta_t \mathbf{e}_h^{m+1}) \leq \varepsilon \, \|\delta_t \mathbf{e}_h^{m+1}\|^2 + \frac{C}{k^2}\Big(|\delta_t \mathbf{e}_i^{m+1/2}|^2 + |\delta_t \mathbf{e}_i^{m+1}|^2\Big)$$
$$\leq \varepsilon \|\delta_t \mathbf{e}_h^{m+1}\|^2 + C\,\frac{h^4}{k^2}\Big(\|\delta_t \mathbf{u}^{m+1/2}\|^2_{\mathbf{H}^2} + \|\delta_t \mathbf{u}^{m+1}, \delta_t p^{m+1}\|^2_{\mathbf{H}^2\times H^1}\Big).$$

By using constraint (**H**), we bound two previous terms by

$$\varepsilon(\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + \|\delta_t \mathbf{e}_h^{m+1}\|^2)$$
$$+C\,h^2\Big(\|\delta_t \mathbf{u}^{m+1/2}\|^2_{\mathbf{H}^2} + \|\delta_t \mathbf{u}^{m}, \delta_t p^{m}\|^2_{\mathbf{H}^2\times H^1} + \|\delta_t \mathbf{u}^{m+1}, \delta_t p^{m+1}\|^2_{\mathbf{H}^2\times H^1}\Big).$$

On the other hand, we have to bound the term $2\left\langle \delta_t \mathbf{NL}_h^{m+1}, \delta_t \mathbf{e}_h^{m+1/2}\right\rangle$ of the RHS of (39).

This term can be writen as follows:

$$2\left\langle \delta_t \mathbf{NL}_h^{m+1}, \delta_t \mathbf{e}_h^{m+1/2}\right\rangle$$
$$= 2\,c(\delta_t \mathbf{e}_d^{m},\, \mathbf{u}^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) + 2\,c(\delta_t\, \mathbf{u}_h^{m}, \mathbf{e}_d^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2})$$
$$+2\,c(\mathbf{e}_d^{m-1}, \delta_t\, \mathbf{u}^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) + 2\,c\,(\mathbf{u}_h^{m-1}, \delta_t \mathbf{e}_d^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2})$$
$$:= \sum_{i=1}^4 J_i$$

Bounding each J_i term (using (6)):

$$J_1 = 2\,c(\delta_t \mathbf{e}_h^{m},\, \mathbf{u}^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) + 2\,c(\delta_t \mathbf{e}_i^{m},\, \mathbf{u}^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) = J_{11} + J_{12}$$

$$J_{11} \leq \varepsilon\,\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\,\|\mathbf{u}^{m+1/2}\|^2_{\mathbf{H}^2}\,|\delta_t \mathbf{e}_h^{m}|^2 \leq \varepsilon\,\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\,|\delta_t \mathbf{e}_h^{m}|^2$$

$$J_{12} \leq \varepsilon\,\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\,\|\mathbf{u}^{m+1/2}\|^2_{\mathbf{H}^2}\,|\delta_t \mathbf{e}_i^{m}|^2 \leq \varepsilon\,\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\,h^2\,\|\delta_t \mathbf{u}^{m}, \delta_t p^{m}\|^2_{\mathbf{H}^1\times L^2}$$

$$J_2 = 2\,c(\delta_t\, \mathbf{u}_h^{m}, \mathbf{e}_h^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) + 2\,c(\delta_t\, \mathbf{u}_h^{m}, \mathbf{e}_i^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) = J_{21} + J_{22}$$

$$\begin{aligned} J_{21} &= 2\,c(\delta_t\, \mathbf{e}_h^{m}, \mathbf{e}_h^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) - 2\,c(I_h\,\delta_t \mathbf{u}^{m}, \mathbf{e}_h^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) \\ &\leq C\,|\delta_t \mathbf{e}_h^{m}|\,\|\mathbf{e}_h^{m+1/2}\|_{\mathbf{W}^{1,3}\cap\mathbf{L}^\infty}\,\|\delta_t \mathbf{e}_h^{m+1/2}\| + C\,\|I_h \delta_t \mathbf{u}^{m}\|\,\|\mathbf{e}_h^{m+1/2}\|_{\mathbf{L}^3}\,\|\delta_t \mathbf{e}_h^{m+1/2}\| \\ &\leq \varepsilon\,\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\,|\delta_t \mathbf{e}_h^{m}|^2 + C\,\|\delta_t \mathbf{u}^{m}, \delta_t p^{m}\|^2_{\mathbf{H}^1\times L^2}\,\|\mathbf{e}_h^{m+1/2}\|^2 \end{aligned}$$

here, we have used (28).

Now, using the inverse inequality $\|\mathbf{u}_h\|^2_{\mathbf{L}^3} \leq \dfrac{C}{h}|\mathbf{u}_h|^2$, we bound J_{22} as follows,

$$\begin{aligned} J_{22} &= -2\,c(\delta_t\, \mathbf{e}_h^{m}, \mathbf{e}_i^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) - 2\,c(I_h \delta_t\, \mathbf{u}^{m}, \mathbf{e}_i^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) \\ &\leq C\,\|\delta_t \mathbf{e}_h^{m}\|_{\mathbf{L}^3}\,\|\mathbf{e}_i^{m+1/2}\|\,\|\delta_t \mathbf{e}_h^{m+1/2}\| + C\,\|\delta_t \mathbf{e}_h^{m+1/2}\|\,\|\mathbf{e}_i^{m+1/2}\|\,\|I_h \delta_t \mathbf{u}^{m}\| \\ &\leq \varepsilon\,\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\,\frac{1}{h}\,|\delta_t \mathbf{e}_h^{m}|^2\,h^2 + C\,h^2\,\|\delta_t(\mathbf{u}^{m}, p^{m})\|^2_{\mathbf{H}^1\times L^2} \\ &\leq \varepsilon\,\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\,h\,|\delta_t \mathbf{e}_h^{m}|^2 + C\,h^2\,\|\delta_t(\mathbf{u}^{m}, p^{m})\|^2_{\mathbf{H}^1\times L^2} \end{aligned}$$

Now, we bound J_3:

$$\begin{aligned} J_3 &= 2\,c(\mathbf{e}_h^{m-1} + \mathbf{e}_i^{m-1}, \delta_t\, \mathbf{u}^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) \\ &\leq \varepsilon\,\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\,\|\delta_t \mathbf{u}^{m+1/2}\|^2_{\mathbf{L}^3}\,(\|\mathbf{e}_h^{m-1}\|^2 + \|\mathbf{e}_i^{m-1}\|^2) \\ &\leq \varepsilon\,\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\,\|\delta_t \mathbf{u}^{m+1/2}\|^2\,(\|\mathbf{e}_h^{m-1}\|^2 + h^2) \end{aligned}$$

$$J_4 = 2\,c(\mathbf{u}_h^{m-1}, \delta_t \mathbf{e}_h^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) + 2\,c(\mathbf{u}_h^{m-1}, \delta_t \mathbf{e}_i^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) = J_{41} + J_{42}$$
$$J_{41} = 0$$

$$J_{42} \leq C\;\|\mathbf{u}_h^{m-1}\|\,\|\delta_t \mathbf{e}_i^{m+1/2}\|_{\mathbf{L}^3}\,\|\delta_t \mathbf{e}_h^{m+1/2}\| \leq \varepsilon\,\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\,h^3\|\delta_t \mathbf{u}^{m+1/2}\|^2_{\mathbf{H}^2}$$

Here, we have used (27) and the error interpolation

$$\|\delta_t \mathbf{e}_i^{m+1/2}\|_{\mathbf{L}^3}^2 \le C\,|\delta_t \mathbf{e}_i^{m+1/2}|\,\|\delta_t \mathbf{e}_i^{m+1/2}\| \le C\,h^3 \|\delta_t \mathbf{u}^{m+1/2}\|_{\mathbf{H}^2}^2.$$

Then, taking into account the above bounds in (39) and choising ε small enough, we have

$$\begin{aligned}
&\frac{1}{k}\Big(|\delta_t \mathbf{e}_h^{m+1}|^2 - |\delta_t \mathbf{e}_h^{m}|^2 + |\delta_t \mathbf{e}_h^{m+1/2} - \delta_t \mathbf{e}_h^{m}|^2 + |\delta_t \mathbf{e}_h^{m+1} - \delta_t \mathbf{e}_h^{m+1/2}|^2\Big)\\
&+\Big\{\|\delta_t \mathbf{e}_h^{m+1}\|^2 + \|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + \|\delta_t \mathbf{e}_h^{m+1} - \delta_t \mathbf{e}_h^{m+1/2}\|^2\Big\}\\
&\le C\,h^2\Big(\|\delta_t \mathbf{u}^{m+1/2}\|_{\mathbf{H}^2}^2 + \|\delta_t(\mathbf{u}^m,p^m)\|_{\mathbf{H}^2\times H^1}^2 + \|\delta_t(\mathbf{u}^{m+1},p^{m+1})\|_{\mathbf{H}^2\times H^1}^2\Big)\\
&+C\,|\delta_t \mathbf{e}_h^{m}|^2 + C\,h^2\,\|\delta_t(\mathbf{u}^m,p^m)\|_{\mathbf{H}^1\times L^2}^2\\
&+C\|\delta_t(\mathbf{u}^m,p^m)\|_{\mathbf{H}^1\times L^2}^2\,\|\mathbf{e}_h^{m+1/2}\|^2\\
&+C\,\|\delta_t \mathbf{u}^{m+1/2}\|^2\,(\|\mathbf{e}_h^{m-1}\|^2 + h^2) + C\,h^3\|\delta_t \mathbf{u}^{m+1/2}\|_{\mathbf{H}^2}^2
\end{aligned}\tag{40}$$

Reasoning as in Theorem 12, now it will be necessary to introduce a stronger weight to avoid compatibility conditions. Then, multiplying (40) by $(\sigma^{m-1})^2 = (\sigma(t_{m-1}))^2$ and applying (36) for $a^{m+1} = |\delta_t \mathbf{e}_h^{m+1}|^2$,

$$(\sigma^{m-1})^2\,\frac{|\delta_t \mathbf{e}_h^{m+1}|^2 - |\delta_t \mathbf{e}_h^{m}|^2}{k} = \frac{|\sigma^m \delta_t \mathbf{e}_h^{m+1}|^2 - |\sigma^{m-1}\delta_t \mathbf{e}_h^{m}|^2}{k} - |\delta_t \mathbf{e}_h^{m+1}|^2\,\delta_t(\sigma^m)^2$$

and we bound the residual term as

$$|\delta_t \mathbf{e}_h^{m+1}|^2\,\delta_t(\sigma^m)^2 \le k\,|\delta_t \mathbf{e}_h^{m+1}|^2 + 2\,\sigma^{m-1}\,|\delta_t \mathbf{e}_h^{m+1}|^2.$$

Then, multiplying (40) by $k(\sigma^{m-1})^2$ and summing through $m = 1$ to r (for any $r < M$), since $\sigma^0 = 0$, we obtain

$$\begin{aligned}
&|\sigma^r\,\delta_t \mathbf{e}_h^{r+1}|^2\\
&+\sum_{m=1}^{r}\Big\{|\sigma^{m-1}\,(\delta_t \mathbf{e}_h^{m+1/2} - \delta_t \mathbf{e}_h^{m})|^2 + |\sigma^{m-1}\,(\delta_t \mathbf{e}_h^{m+1} - \delta_t \mathbf{e}_h^{m+1/2})|^2\Big\}\\
&+k\sum_{m=1}^{r}\Big\{\|\sigma^{m-1}\,\delta_t \mathbf{e}_h^{m+1}\|^2 + \|\sigma^{m-1}\,\delta_t \mathbf{e}_h^{m+1/2}\|^2\\
&\qquad\qquad + + \|\sigma^{m-1}\,(\delta_t \mathbf{e}_h^{m+1} - \delta_t \mathbf{e}_h^{m+1/2})\|^2\Big\}\\
&\le k^2\sum_{m=1}^{r}|\delta_t \mathbf{e}_h^{m+1}|^2 + 2\,k\sum_{m=1}^{r}\sigma^{m-1}\,|\delta_t \mathbf{e}_h^{m+1}|^2\\
&+C\,h^2\,k\sum_{m=1}^{r}\Big\{\|\sigma^{m-1}\,\delta_t \mathbf{u}^{m+1/2}\|_{\mathbf{H}^2}^2 + \|\sigma^{m-1}\delta_t(\mathbf{u}^m,p^m)\|_{\mathbf{H}^2\times H^1}^2\\
&\qquad\qquad +\|\sigma^{m-1}\delta_t(\mathbf{u}^{m+1},p^{m+1})\|_{\mathbf{H}^2\times H^1}^2\Big\}\\
&+C\,k\sum_{m=1}^{r}|\sigma^{m-1}\delta_t \mathbf{e}_h^{m}|^2 + C\,h^2\,k\sum_{m=1}^{r}\|\sigma^{m-1}\delta_t(\mathbf{u}^m,p^m)\|_{\mathbf{H}^1\times L^2}^2\\
&+C\,\|\sigma^{m-1}\delta_t(\mathbf{u}^m,p^m)\|_{l^\infty(\mathbf{H}^1\times L^2)}^2\,k\sum_{m=1}^{r}\|\mathbf{e}_h^{m+1/2}\|^2\\
&+C\,\|\sigma^{m-1}\,\delta_t \mathbf{u}^{m+1/2}\|_{l^\infty(\mathbf{H}^1)}^2\,k\sum_{m=1}^{r}(\|\mathbf{e}_h^{m-1}\|^2 + h^2)\\
&+C\,h^3\,k\sum_{m=1}^{r}\|\sigma^{m-1}\,\delta_t \mathbf{u}^{m+1/2}\|_{\mathbf{H}^2}^2
\end{aligned}\tag{41}$$

Now, using the estimates of Theorems 8 and 12, we have

$$k^2\sum|\delta_t \mathbf{e}_h^{m+1}|^2 + 2\,k\sum\sigma^{m-1}\,|\delta_t \mathbf{e}_h^{m+1}|^2 \le C\,h^2.$$

Therefore, it remains to see that the rest of terms of (41) can be bounded adequately.

In this sense, from (10), (12), (13) and (14), we have the estimates

$$k \sum_{m\geq 1} \Big(\|\sqrt{\sigma^{m-1}}\, \delta_t \mathbf{u}^{m+1/2}\|^2_{\mathbf{H}^2} + \|\sqrt{\sigma^{m-1}}\delta_t(\mathbf{u}^{m+1}, p^{m+1})\|^2_{\mathbf{H}^2\times H^1} \Big) \leq C \tag{42}$$

and

$$\|\sqrt{\sigma^m}\, \delta_t \mathbf{u}^{m+1/2}\|_{l^\infty(\mathbf{H}^1)} + \|\sqrt{\sigma^m}\delta_t(\mathbf{u}^{m+1}, p^{m+1})\|_{l^\infty(\mathbf{H}^1\times L^2)} \leq C. \tag{43}$$

On the other hand, we are going to prove

$$k \sum_{m\geq 2} \|\sigma^{m-1}\delta_t(\mathbf{u}^m, p^m)\|^2_{\mathbf{H}^2\times H^1} \leq C. \tag{44}$$

Indeed, we can write $\sigma^{m-1} \leq \sigma^{m-2} + k$ and then, we have

$$k \sum_{m\geq 2} \|\sigma^{m-1}\, \delta_t \mathbf{u}^m\|^2_{\mathbf{H}^2} \leq 2\, k \sum_{m\geq 2} \|\sigma^{m-2}\, \delta_t \mathbf{u}^m\|^2_{\mathbf{H}^2} + 2\, k \sum_{m\geq 2} \|k\, \delta_t \mathbf{u}^m\|^2_{\mathbf{H}^2} \leq C,$$

where we have used that $k \sum_{m\geq 2} \|k\, \delta_t \mathbf{u}^m\|^2_{\mathbf{H}^2} = k \sum_{m\geq 2} \|\mathbf{u}^m - \mathbf{u}^{m-1}\|^2_{\mathbf{H}^2} \leq C$. We can bound the other term of (44) of similar way.

Finally, applying Theorem 8, we have

$$k \sum_{m=1}^{r} \Big\{ \|\mathbf{e}_h^{m+1/2}\|^2 + \|\mathbf{e}_h^{m-1}\|^2 \Big\} \leq C\, h^2.$$

Then, by applying the Discrete Gronwall's Lemma to (41) and estimates (42), (43) and (44), the proof is finished.

□

Theorem 15. *Under assumptions of Theorem 14, the following estimate holds*

$$\|\sigma^m\, e_{p,h}^{m+1}\|_{l^\infty(L^2)} \leq C\, h.$$

Proof. By using the $(\mathbf{H}^1 \times L^2)$-regularity of the discrete Stokes problem $(E_3)_h^{m+1}$, and applying (6), (29) and Theorem 8,

$$\begin{aligned}
&\|\mathbf{e}_h^{m+1}\| + |e_{p,h}^{m+1}| \leq C\Big(\|\delta_t \mathbf{e}_h^{m+1}\|_{\mathbf{H}^{-1}} + \|\mathbf{NL}_h^{m+1}\|_{\mathbf{H}^{-1}} + \|\delta_t \mathbf{e}_i^{m+1}\|_{\mathbf{H}^{-1}} \Big) \\
&\leq C\Big(|\delta_t \mathbf{e}_h^{m+1}| + |\mathbf{e}_d^m|\, \|\mathbf{u}^{m+1/2}\|_{\mathbf{W}^{1,3}\cap \mathbf{L}^\infty} + \|\mathbf{u}_h^m\|_{\mathbf{W}^{1,3}\cap \mathbf{L}^\infty}\, |\mathbf{e}_d^{m+1/2}| + |\delta_t \mathbf{e}_i^{m+1}| \Big) \\
&\leq C\Big(|\delta_t \mathbf{e}_h^{m+1}| + h + h\, \|\delta_t(\mathbf{u}^{m+1}, p^{m+1})\|_{\mathbf{H}^1\times L^2} \Big).
\end{aligned}$$

Then, multiplying by σ^m, we have

$$|\sigma^m\, e_{p,h}^{m+1}| \leq C\Big(|\sigma^m\, \delta_t \mathbf{e}_h^{m+1}| + h + h\, \|\sigma^m \delta_t(\mathbf{u}^{m+1}, p^{m+1})\|_{\mathbf{H}^1\times L^2} \Big),$$

hence, using (43) and Theorem 14, we obtain the desired estimate. □

As a consequence of this theorem, we get optimal error estimate for the total error of the pressure.

Corollary 16. *Under assumptions of Theorems 6 and 15,*

$$\|\sigma^m(p(t_{m+1}) - p_h^{m+1})\|_{l^\infty(L^2)} \leq C\, (k + h).$$

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